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Some optimal control problems for a two-phase field model of solidification

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Abstract In this paper we deal with some optimal control problems for a solidification phase field model of metallic alloys. The model allows crystallizations of two kinds, each one described by its own phase field. Accordingly, the state is the triplet (τ, u, v), where τ is the temperature and u and v are phase field functions. The optimality conditions for the optimal control problems considered in this work are obtained by using the Dubovitskii-Milyutin formalism.

Keywords Solidification · Phase field models · Parabolic partial differential equations · Optimal control · Dubovitskii-Milyutin formalism

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1 Introduction

Among the possibilities to model phase changes, phase field models are perhaps the most successful in the sense that for them it is rather natural to incorporate several

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physical phenomena influencing phase changes. They also allow for the occurrence of transition layers (mushy zones). For such models, numerical simulation is possible even in the case of formation of rather complex geometries, like dendrities, as interfaces separating different phases.

One of the first authors to use a phase field to model solidification and melting of simple materials was Fix [10]; after him, many other researchers have applied this approach in several important and more general situations. The mathematical analysis of models of this kind can be found for instance in the articles by Caginalp et al. [5–8] (see also the references therein).

In this paper we will deal with the following nonlinear partial differential model for the solidification of an alloy:

$$
\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f \quad \text{in } Q,\\
&u_t - k_1\Delta u = -a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1) \quad \text{in } Q,\\
&v_t - k_2\Delta v = -a_2 v(1-v-u)(1-2v-u+c_2\tau+d_2) \quad \text{in } Q,\\
&\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega\times(0,T),\\
&\tau=\tau_0, \quad u=u_0, \quad v=v_0 \quad \text{in } \Omega\times\{t=0\}.
\end{aligned}
\tag{1}
$$

Here, $\Omega\subset\mathbb{R}^3$ is a bounded C^2 domain, $T>0$ and $Q=\Omega\times(0,T)$; the function τ is the temperature and the phase field functions u and v are used to identify two different levels of solid crystallization; f is a density of heat sources and sinks; the constants l_1 and l_2 have the same sign and are related to the latent heats associated to each kind of crystallization; b, k_1, k_2, a_1 and a_2 are given positive constants; c_1, c_2, d_1 and d_2 are given constants; $n=n(x)$ denotes the outwards unit normal to $\partial\Omega$; the initial data τ_0, u_0 and v_0 are suitable given functions.

We remark that the previous system involves two phase fields and can be seen as a generalization of the model treated by Hoffman and Jiang in [12]; it is also related to the model presented in Steinbach et al. [17] and [16], since it has similar iteration potentials. Some basic existence, uniqueness and regularity results for this model have been obtained in [3].

We will analyze several optimal control problems for (1). The control will be f and the cost function will be given by

$$
\begin{aligned}
J(\tau,u,v,f) = {}& \frac{\alpha_1}{k}\iint_Q |\tau-\tau_d|^k + \frac{\alpha_2}{m}\iint_Q |u-u_d|^m\\
&+ \frac{\alpha_3}{m}\iint_Q |v-v_d|^m + \frac{N}{q}\iint_Q |f|^q,
\end{aligned}
\tag{2}
$$

where α_1, α_2, $\alpha_3 \geq 0$ and $N>0$ are constants, the exponents k, m and q are >1, $\tau_d\in L^k(Q)$ and $u_d, v_d\in L^m(Q)$. Several different constraints on the state and control variables will be considered; see Sects. 5 and 6 below.

In this paper, one of our main aims is to show that the *Dubovitskii-Milyutin formalism* can be used to obtain the associated first-order optimality systems. The same results can be obtained using other related but different techniques; for instance, see the results in [4] and [9]. However, we would like to emphasize two features of the strategy we have chosen:

- It provides a unified framework for many different control problems.
- It allows to identify and determine the admissible directions in each case.

Remark 1.1 An interesting question is the following: for instance, let us take $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in (2); assume that, for each $N > 0$, we are able to solve the optimal control problem

$$\begin{cases} \text{Minimize} & J(\tau, u, v, f), \\ \text{subject to} & f \in L^q(Q) \text{ and } (\tau, u, v, f) \text{ satisfies (2)} \end{cases}$$

and let us denote by (τ^N, u^N, v^N, f^N) the associated solutions. Then, what happens as $N \to \infty$? Do we have $(\tau^N, u^N, v^N) \to (\tau_d, u_d, v_d)$ in $L^m(Q) \times L^m(Q) \times L^k(Q)$?

The characterization of (τ^N, u^N, v^N, f^N) (furnished by the Dubovitskii-Milyutin formalism) serves to provide an answer to this question. This will be shown in a forthcoming paper.

The paper is organized as follows. In Sect. 2, we fix the notations and we recall certain results that will be used along the paper. In Sect. 3, we recall the Dubovitskii-Milyutin formalism.

Section 4 deals with a relatively simple (unconstrained) optimal control problem for (1)–(2). We will prove an existence result and then we will apply the Dubovitskii-Milyutin formalism to deduce the associated optimality system.

In Sects. 5 and 6 we consider some constrained optimal control problems for which we obtain similar results. In particular, Sect. 6 deals with constraints associated to the temperature τ and/or the temperature gradient $\nabla\tau$.[1]

2 Preliminaries and hypotheses

We will use standard notations; for convenience, we will recall in this section several spaces and properties that will be needed below.

For any given $p \in [1, +\infty]$ and any $r \in \mathbb{R}$, we will denote the usual associated Sobolev space by $W^r_p(\Omega)$. The main properties of $W^r_p(\Omega)$ can be found for instance in [1]; here, we will only mention the following result, that is a consequence of the well known *Sobolev Embedding Theorem* (see Theorem 5.4, p. 97 in [1]):

Lemma 2.1 *Assume that* $\Omega \subset \mathbb{R}^3$ *satisfies the cone property and* $2 \le 3p/5 < +\infty$. *Then* $W^2_{3p/5}(\Omega) \hookrightarrow W^{2-2/p}_q(\Omega)$ (*with a continuous embedding*) *for any* $q \in [3p/5, p)$.

[1] Of course, this is the most realistic, interesting and difficult situation. It is concerned with controlling and, simultaneously, taking care of the heating process.

The nonlinear system (1) will be studied in the functional spaces $W_q^{2,1}(Q)$, where

$$W_q^{2,1}(Q) = \{f \in L^q(Q) : D^\alpha f \in L^q(Q) \text{ for } 1 \le |\alpha| \le 2,\ f_t \in L^q(Q)\}.$$

For the main results concerning these spaces, we refer for instance to [13] and [15]. Let us however recall some results concerning the embedding of $W_q^{2,1}(Q)$ in L^p spaces (see [14], p. 15; see also Lemma 3.3, p. 80, in [13]):

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2 domain and let us set $Q = \Omega \times (0,T)$, where $T > 0$. Then the embedding $W_q^{2,1}(Q) \hookrightarrow L^q(0,T; W_q^1(\Omega))$ is continuous and compact. Furthermore, $W_q^{2,1}(Q) \hookrightarrow L^p(Q)$ with a continuous embedding, where*

$$p = \begin{cases} (\frac{1}{q} - \frac{2}{5})^{-1} & \text{if } 2 \le q < 5/2, \\ \text{any finite exponent} & \text{if } q = 5/2, \\ +\infty & \text{if } q > 5/2. \end{cases}$$

For $2 \le \tilde{p} < p$, the embedding $W_q^{2,1}(Q) \hookrightarrow L^{\tilde{p}}(Q)$ is compact. For any $q > 5/2$, one has

$$W_q^{2,1}(Q) \hookrightarrow L^\infty(Q), \tag{3}$$

with a continuous and compact embedding. Finally, one also has

$$W_q^{2,1}(Q) \hookrightarrow C^0([0,T]; W_q^{2-2/q}(\Omega)), \tag{4}$$

for all finite $q > 1$, again with a continuous and compact embedding.

Lemma 2.3 *In the conditions of Lemma 2.2, one has*

$$W_q^{2,1}(Q) \hookrightarrow L^p(0,T; W_p^1(\Omega))$$

with a continuous embedding, where

$$p = \begin{cases} (\frac{1}{q} - \frac{1}{5})^{-1} & \text{if } 2 \le q < 5, \\ \text{any finite exponent} & \text{if } q = 5, \\ +\infty & \text{if } q > 5. \end{cases}$$

For $2 \le \tilde{p} < p$, the embedding $W_q^{2,1}(Q) \hookrightarrow L^{\tilde{p}}(0,T; W_{\tilde{p}}{}^1(\Omega))$ is compact. Furthermore, for any $q > 5$, one has

$$W_q^{2,1}(Q) \hookrightarrow L^\infty(0,T; W_\infty^1(\Omega)), \tag{5}$$

with a continuous and compact embedding.

Proposition 2.1 *Assume that* $q > 1$, $f \in L^q(Q)$ *and* $v_0 \in W_q^{2-2/q}(\Omega)$ *satisfies* $\partial v_0/\partial n|_{\partial\Omega} = 0$. *Then the linear system*

$$\begin{aligned} v_t - \Delta v = f \quad &\text{in } Q, \\ \partial v/\partial n = 0 \quad &\text{on } \partial\Omega \times (0,T), \\ v = v_0 \quad &\text{in } \Omega \times \{t=0\} \end{aligned} \tag{6}$$

possesses exactly one strong solution $v \in W_q^{2,1}(Q)$. *Furthermore, the linear mapping* $(f, v_0) \mapsto v$ *is continuous, i.e. there exists* $C > 0$ *such that*

$$\|v\|_{W_q^{2,1}(Q)} \le C(\|v_0\|_{W_q^{2-2/q}} + \|f\|_{L^q(Q)}).$$

Next, for easy reference, we collect some hypotheses that will be assumed in the remainder of this work:

$$\begin{aligned} &\text{(i)} \quad \Omega \subset \mathbb{R}^3 \text{ is a bounded } C^2\text{-domain}, \ 0 < T < +\infty, \ Q = \Omega \times (0,T); \\ &\text{(ii)} \quad \tau_0, u_0, v_0 \in L^\infty(\Omega) \text{ and } u_0, v_0 \ge 0; \\ &\text{(iii)} \quad b, l_1, l_2, k_1, k_2, a_1, a_2, c_1, c_2, d_1, d_2 \text{ are real constants;} \\ &\qquad\ \ b, k_1, k_2, a_1, a_2 \text{ are positive.} \end{aligned} \tag{7}$$

Let us recall several results concerning the existence, uniqueness, regularity and stability and regularity of the solutions of (1). These results are proved in [3].

Theorem 2.1 *Let us assume that hypotheses* (7) *hold,* $f \in L^q(Q)$ *with* $q > 5/2$ *and* τ_0, u_0, $v_0 \in W_2^2(\Omega)$ *with* $\partial\tau_0/\partial n|_{\partial\Omega} = \partial u_0/\partial n|_{\partial\Omega} = \partial v_0/\partial n|_{\partial\Omega} = 0$. *There exists* $\kappa_0 > 0$, *depending on* Ω, T, *the constants in* (1) *and these data such that, if*

$$\max(|c_1|, |c_2|, |d_1|, |d_2|) \le \kappa_0, \tag{8}$$

then (1) *possesses exactly one solution* $(\tau, u, v) \in W_{\overline{q}}^{2,1}(Q) \times W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$ *with* $\overline{q} = \min\{10/3, q\}$ *satisfying*:

1. *The estimates*

$$\begin{aligned} &\|\tau\|_{W_2^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|v\|_{W_2^{2,1}(Q)} \\ &\quad \le C(\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^2(Q)}) \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\|\tau\|_{W_{\overline{q}}^{2,1}(Q)} + \|u\|_{W_{10/3}^{2,1}(Q)} + \|v\|_{W_{10/3}^{2,1}(Q)} \\ &\quad \le C(\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^q(Q)} \\ &\qquad + \|\tau_0\|_{W_2^2}^3 + \|u_0\|_{W_2^2}^3 + \|v_0\|_{W_2^2}^3 + \|f\|_{L^2(Q)}^3), \end{aligned} \tag{10}$$

where C *depends on* Ω, T *and the constants of the problem.*

2. *The estimates*

$$0 \le u, v \le K := \max\Big(\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}, \max_i |d_i| + 1 \Big). \tag{11}$$

3. *Whenever* $\tau_0, u_0, v_0 \in W^2_{3p/5}(\Omega)$ *with* $2 \le 3p/5 < +\infty$, *the estimate*

$$\begin{aligned} &\|\tau\|_{W^{2,1}_{\tilde{q}}(Q)} + \|u\|_{W^{2,1}_p(Q)} + \|v\|_{W^{2,1}_p(Q)} \\ &\quad \le C\big(\|\tau_0\|_{W^2_{3p/5}} + \|u_0\|_{W^2_{3p/5}} + \|v_0\|_{W^2_{3p/5}} + \|f\|_{L^q(Q)}\big), \end{aligned} \tag{12}$$

where C *only depends on* Ω, T, K *and the constants of the problem.*

Theorem 2.2 *Let the assumptions of Theorem 2.1 hold and let us set* $K_0 = \max_i |d_i| + 1$. *For* $i = 1, 2$, *consider initial conditions* τ_0^i, u_0^i, $v_0^i \in W_2^2(\Omega)$ *such that* $\partial\tau_0^i/\partial n|_{\partial\Omega} = \partial u_0^i/\partial n|_{\partial\Omega} = \partial v_0^i/\partial n|_{\partial\Omega} = 0$ *and* $0 \le u_0^i, v_0^i \le K_0$. *Also, for* $i = 1, 2$, *let* $f_i \in L^q(Q)$ *be given with* $q > 5/2$ *and let* $(\tau_i, u_i, v_i) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ *be a solution of* (1) *associated to* $(f_i, \tau_0^i, u_0^i, v_0^i)$. *Then* $(\tau_i, u_i, v_i) \in W^{2,1}_{\overline{q}}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$, *where* $\overline{q}$ *is given by* $\overline{q} = \min\{10/3, q\}$ *and the following estimate holds*:

$$\begin{aligned} &\|\tau_1 - \tau_2\|_{W^{2,1}_{\overline{q}}(Q)} + \|u_1 - u_2\|_{W^{2,1}_{10/3}(Q)} + \|v_1 - v_2\|_{W^{2,1}_{10/3}(Q)} \\ &\quad \le C\big[\|\tau_0^1 - \tau_0^2\|_{W_2^2} + \|u_0^1 - u_0^2\|_{W_2^2} + \|v_0^1 - v_0^2\|_{W_2^2} \\ &\qquad + \|f_1 - f_2\|_{L^q(Q)}\big]. \end{aligned}$$

Here, C *depends on* Ω, T, K_0 *and the constants in* (1). *Moreover, if* τ_0^i, u_0^i, $v_0^i \in W^2_{3p/5}(\Omega)$ *with* $2 \le 3p/5 < +\infty$, *then* $(\tau_i, u_i, v_i) \in W^{2,1}_{\tilde{q}}(Q) \times W^{2,1}_p(Q) \times W^{2,1}_p(Q)$ *with* $\tilde{q} = \min\{p, q\}$ *and we also have*:

$$\begin{aligned} &\|\tau_1 - \tau_2\|_{W^{2,1}_{\tilde{q}}(Q)} + \|u_1 - u_2\|_{W^{2,1}_p(Q)} + \|v_1 - v_2\|_{W^{2,1}_p(Q)} \\ &\quad \le C\big[\|\tau_0^1 - \tau_0^2\|_{W^2_{3p/5}} + \|u_0^1 - u_0^2\|_{W^2_{3p/5}} + \|v_0^1 - v_0^2\|_{W^2_{3p/5}} \\ &\qquad + \|f_1 - f_2\|_{L^q(Q)}\big], \end{aligned} \tag{13}$$

where C *is as before.*

3 The Dubovitskii-Milyutin formalism

In this section, we will recall the formalism of Dubovitskii and Milyutin as applied to a general constrained optimization problem. The details of this theory can be found for instance in references [11] and [18].

Our framework will be the following. Let X be a Banach space and let $J : X \mapsto \mathbb{R}$ be a given function. We will consider the following problem:

$$\begin{cases} \text{Minimize} & J(\xi), \\ \text{Subject to} & \xi \in \mathcal{Q} = \bigcap_{\ell=1}^{n+m} \mathcal{Q}_\ell, \end{cases} \tag{14}$$

where the $\mathcal{Q}_\ell$ ($\ell = 1, \ldots, n+m$) are by definition *the restriction sets.*

It will be assumed that

$$\operatorname{int} \mathcal{Q}_i \neq \emptyset \quad \forall i = 1, \ldots, n \tag{15}$$

and

$$\operatorname{int} \mathcal{Q}_j = \emptyset \quad \forall j = n+1, \ldots, n+m. \tag{16}$$

In particular, this is the situation if

- For any $i = 1, \ldots, n$, $\mathcal{Q}_i$ is an inequality restriction set of the form

  $$\mathcal{Q}_i = \{\xi \in X : p_i(\xi) \le a_i\},$$

  where $p_i : X \mapsto \mathbb{R}$ is a continuous seminorm and $a_i > 0$ and
- For each $j = n+1, \ldots, n+m$, $\mathcal{Q}_j$ is the equality restriction set

  $$\mathcal{Q}_j = \{\xi \in X : M_j(\xi) = 0\},$$

  where $M_j : X \mapsto Y_j$ is a differentiable mapping (Y_j is another Banach space).

However, there can be other situations where (15) and (16) still hold.

Let us recall the following

Definition 1 Let $\{K_i\}_{i=1}^m$ be a family of cones in a normed space Z. It will be said that they form a system of cones with the same sense if, for any $R > 0$, there exist positive numbers $R_1, \ldots, R_m$ with the following property: whenever $\zeta \in Z$, $\|\zeta\|_Z \le R$ and $\zeta = \sum_{i=1}^m \zeta_i$ for some $\zeta_i \in K_i$ ($i = 1, \ldots, m$), we necessarily have $\|\zeta_i\|_Z \le R_i$ for all i.

We will need the following generalized version of the *Dubovitskii-Milyutin principle:*

Theorem 3.1 *Let $\xi_0 \in \bigcap_{\ell=1}^{n+m} Q_i$ be a local minimum of problem* (14). *Let DC_0 be the decreasing cone of the cost functional J at ξ_0, let FC_i be the feasible* (*or admissible*) *cone of $\mathcal{Q}_i$ at ξ_0 for $i = 1, \ldots, n$ and let TC_j be the tangent cone to $\mathcal{Q}_j$ at ξ_0 for $j = n+1, \ldots, n+m$. Suppose that*

1. *The cones DC_0 and FC_i ($i = 1, \ldots, n$) are open and convex.*
2. *The cones TC_j ($j = n+1, \ldots, n+m$) are closed and convex.*
3. $\bigcap_{j=n+1}^{n+m} TC_j \subset \tilde{K}$, *where $\tilde{K}$ is the tangent cone to $\bigcap_{j=n+1}^{n+m} Q_j$ at ξ_0.*
4. *The $[TC_j]^*$, $j = n+1, \ldots, n+m$, form a system of cones with the same sense.*

Then

$$DC_0 \cap \left(\bigcap_{i=1}^{n} FC_i \right) \cap \left(\bigcap_{j=n+1}^{n+m} TC_j \right) \neq \emptyset.$$

Consequently, there exist $G_0 \in [DC_0]^*$, $G_i \in [FC_i]^*$ *for* $i = 1, \dots, n$ *and* $G_j \in [TC_j]^*$ *for* $j = n+1, \dots, n+m$, *not all zero, such that*

$$G_0 + \sum_{i=1}^{n} G_i + \sum_{j=n+1}^{n+m} G_j = 0.$$

Recall that, for any set $B \subset X$, the associated *dual cone* is the set

$$B^* = \{h \in X' : \langle h, \xi \rangle \geq 0 \ \forall \xi \in B\};$$

in particular, if B is a subspace, one has:

$$B^* = B^{\perp} := \{h \in X' : \langle h, \xi \rangle = 0 \ \forall \xi \in B\}.$$

In order to identify the previous decreasing, feasible and tangent cones, we will use the following well known results:

- Assume that $J : X \mapsto \mathbb{R}$ is Fréchet-differentiable. Then, for any $\xi \in X$, the decreasing cone of J at ξ is open and convex and is given by

$$DC = \{\eta \in X : \langle J'(\xi), \eta \rangle < 0\},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual duality product associated to X and X'.
- Suppose that the set $\mathcal{Q}$ is given by

$$\mathcal{Q} = \{\xi \in X : p(\xi) \leq a\},$$

where $p : X \mapsto \mathbb{R}$ and $a \in \mathbb{R}$. Assume that $\xi \in \mathcal{Q}$, p is Fréchet-differentiable at ξ and $p'(\xi) \neq 0$. Then the feasible cone of $\mathcal{Q}$ at ξ is also open and convex and is given by

$$FC = \{\eta \in X : \langle p'(\xi), \eta \rangle < 0\}.$$

- Now, suppose that the set $\mathcal{Q}$ is given by

$$\mathcal{Q} = \{\xi \in X : M(\xi) = 0\}$$

where $M : X \mapsto Y$ is given. Assume that $\xi \in \mathcal{Q}$, M is strictly differentiable at ξ and $R(M'(\xi)) = Y$. Then M maps a neighborhood of ξ onto a neighborhood of $M(\xi)$ and the tangent cone to $\mathcal{Q}$ at ξ is the following closed subspace:

$$TC = N(M'(\xi)) = \{\eta \in X : M'(\xi)\eta = 0\}.$$

This is the well known *Lyusternik Theorem*; see for instance [2].

4 An unconstrained optimal control problem

In this section, we will consider a first (simple) optimal control problem for (1). We will prove the existence of optimal solutions and we will use the Dubovitskii-Milyutin formalism to obtain the associated optimality system.

Let the finite exponent q be given, with $q > 5/2$. The following function spaces will be considered:

$$E := \big(W_q^{2,1}(Q) \times W_{10/3}^{2,1}(Q)^2 \times L^q(Q)\big) \cap \{(\tau, u, v, f) : \partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0\} \tag{17}$$

and

$$\widetilde{E} := L^{\overline{q}}(Q) \times L^{10/3}(Q)^2 \times W_{\overline{q}}^{2-2/\overline{q}}(\Omega) \times W_{10/3}^{7/5}(\Omega)^2, \tag{18}$$

where

$$\overline{q} = \max\{q, 10/3\}. \tag{19}$$

Our cost functional $J : E \mapsto \mathbb{R}$ will be given (2), with $\alpha_i \geq 0$, $N > 0$, $k, m > 1$, $\tau_d \in L^k(Q)$ and $u_d, v_d \in L^m(Q)$. In the sequel, we use $\langle \cdot\,, \cdot\rangle$ to denote various duality products; we will denote by C a generic positive constant.

Let τ_0, u_0, v_0 be given in $W_2^2(\Omega)$. In our first optimal control problem, the state and control variables are only constrained to satisfy the state equation (1). The problem is the following:

$$\begin{cases} \text{Minimize} & J(\tau, u, v, f), \\ \text{subject to} & (\tau, u, v, f) \in \mathcal{Q}. \end{cases} \tag{20}$$

Here, $\mathcal{Q}$ is the equality constraint set

$$\mathcal{Q} = \{(\tau, u, v, f) \in E : M(\tau, u, v, f) = 0\} \tag{21}$$

and $M : E \mapsto \widetilde{E}$ is defined by

$$M(\tau, u, v, f) = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6), \tag{22}$$

with

$$\begin{aligned} &\tau_t - b\Delta\tau - l_1 u_t - l_2 v_t - f = \varphi_1 \quad \text{in } Q, \\ &u_t - k_1\Delta u + a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1) = \varphi_2 \quad \text{in } Q, \\ &v_t - k_2\Delta v + a_2 v(1-v-u)(1-2v-u+c_2\tau+d_2) = \varphi_3 \quad \text{in } Q, \\ &\tau - \tau_0 = \varphi_4, \quad u - u_0 = \varphi_5, \quad v - v_0 = \varphi_6 \quad \text{in } \Omega \times \{t = 0\}. \end{aligned} \tag{23}$$

In view of (17)–(18), the embeddings $W_q^{2,1}(Q) \hookrightarrow L^\infty(Q)$ and $W_{10/3}^{2,1}(Q) \hookrightarrow L^\infty(Q)$ and Lemma 2.2, the mapping $M : E \mapsto \widetilde{E}$ is well defined. Observe that one has $M(\tau, u, v, f) = 0$ if and only if (τ, u, v, f) solves (1).

Let us introduce the admissible set for (20):

$$E_{ad} = \{(\theta, w, z, h) \in \mathcal{Q} : J(\theta, w, z, h) < +\infty\}. \tag{24}$$

Then we have:

Lemma 4.1 *Assume that* (7) *holds. Also, assume that* $q > 5/2$,

$$\tau_0, u_0, v_0 \in W_2^2(\Omega) \tag{25}$$

and $\partial\tau_0/\partial n|_{\partial\Omega} = \partial u_0/\partial n|_{\partial\Omega} = \partial v_0/\partial n|_{\partial\Omega} = 0$. *Finally, assume that one has* (8), *where* κ_0 *is the (possibly small) constant associated to* Ω, T, *the constants in* (1), *the previous initial data and* $f = 0$. *Then* $E_{ad} \neq \emptyset$.

Proof We have to prove that there exists $(\tau, u, v, f) \in E$ satisfying

$$M(\tau, u, v, f) = 0 \quad \text{and} \quad J(\tau, u, v, f) < +\infty,$$

where M and J are respectively given by (23) and (2).

In view of Theorem 2.1, there exists a unique solution (τ, u, v) of (1) with $f = 0$ that belongs to the space $W_{\bar{q}}^{2,1}(Q) \times W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$.

Then, obviously $(\tau, u, v, 0) \in E$ and $M(\tau, u, v, 0) = 0$, i.e. $(\tau, u, v, 0) \in \mathcal{Q}$. Since $10/3 > 5/2$, we have $\tau, u, v \in L^\infty(Q)$. Therefore, $(\tau, u, v, 0) \in L^k(Q) \times L^m(Q)^2 \times L^q(Q)$ and $J(\tau, u, v, 0) < +\infty$. This proves that $(\tau, u, v, 0) \in E_{ad}$. □

Next, we prove the existence of an optimal solution of the control problem (20):

Theorem 4.1 *Under the assumptions of Lemma* 4.1, *the optimal control problem* (20), (21) *possesses at least one solution.*

Proof The proof is standard. For completeness and further reference, we present the whole argument.

First of all, observe that we have $E_{ad} \neq \emptyset$ by Lemma 4.1.

Let us consider a minimizing sequence $\{(\tau_n, u_n, v_n, f_n)\}$. Then one has $J(\tau_n, u_n, v_n, f_n) \le C$ and, consequently,

$$\|f_n\|_{L^q(Q)} \le C.$$

From Theorem 2.1 we deduce that

$$\|\tau_n\|_{W_{\bar{q}}^{2,1}(Q)} + \|u_n\|_{W_{10/3}^{2,1}(Q)} + \|v_n\|_{W_{10/3}^{2,1}(Q)} \le C.$$

Therefore, we can find subsequences (indexed again by n) such that

$$\begin{aligned} f_n &\to f \quad \text{weakly in } L^q(Q), \\ \tau_n &\to \tau \quad \text{weakly in } W_{\bar{q}}^{2,1}(Q), \\ u_n &\to u \quad \text{weakly in } W_{10/3}^{2,1}(Q), \\ v_n &\to v \quad \text{weakly in } W_{10/3}^{2,1}(Q). \end{aligned}$$

We have $W_{\overline{q}}^{2,1}(Q) \hookrightarrow L^\infty(Q)$ and $W_{10/3}^{2,1}(Q) \hookrightarrow L^\infty(Q)$ with compact embeddings. Therefore, (τ_n, u_n, v_n) converges strongly in the L^∞ norm. This implies $M(\tau, u, v, f) = 0$, i.e. (τ, u, v, f) solves (1). Obviously, $J(\tau, u, v, f) < +\infty$, whence $(\tau, u, v, f) \in E_{ad}$.

Since $\{(\tau_n, u_n, v_n, f_n)\}$ is a minimizing sequence, if we check that

$$\liminf_{n\to\infty} J(\tau_n, u_n, v_n, f_n) \geq J(\tau, u, v, f),$$

we will have proved that (τ, u, v, f) is an optimal solution. But the cost function J is obviously convex and continuous in $L^k(Q) \times L^m(Q)^2 \times L^r(Q)$, whence it is sequentially weakly lower semi-continuous in the same space.

This ends the proof. □

Assume that $(\tau, u, v, f) \in E_{ad}$. It will be said that (τ, u, v, f) is a local optimal solution of the control problem (20), (21) if there exists $\varepsilon > 0$ such that

$$J(\tau, u, v, f) \leq J(\varphi, w, z, h) \tag{26}$$

for all $(\varphi, w, z, h) \in E_{ad}$ satisfying

$$\|\tau - \varphi\|_{W_{\overline{q}}^{2,1}(Q)} + \|u - w\|_{W_{10/3}^{2,1}(Q)} + \|v - z\|_{W_{10/3}^{2,1}(Q)} + \|f - h\|_{L^q(Q)} \leq \varepsilon. \tag{27}$$

We are now going to deduce the first-order optimality conditions for (20), (21), that is, the necessary conditions that have to be satisfied by any local optimal solution.

Thus, let $(\tau, u, v, f) \in E$ be a local optimal solution. Since J is convex and Fréchet-differentiable at any point, from the results recalled in Sect. 3 we obtain the following:

Lemma 4.2 *The decreasing cone of J at $(\tau, u, v, f) \in E$ is the set*

$$DC(J, (\tau, u, v, f)) = \{(\varphi, w, z, h) : \langle J'(\tau, u, v, f), (\varphi, w, z, h)\rangle < 0\}.$$

Consequently, the associated dual cone is

$$[DC(J, (\tau, u, v, f))]^* = \left\{-\lambda J'(\tau, u, v, f) : \lambda \in \mathbb{R},\ \lambda \geq 0\right\}.$$

For later use, we observe that the derivative of J at (τ, u, v, f) in the direction (φ, w, z, h) is given by

$$\begin{aligned}
&\langle J'(\tau, u, v, f), (\varphi, w, z, h)\rangle \\
&\quad = \alpha_1 \iint_Q |\tau - \tau_d|^{k-2}(\tau - \tau_d)\,\varphi \\
&\qquad + \alpha_2 \iint_Q |u - u_d|^{m-2}(u - u_d) w \\
&\qquad + \alpha_2 \iint_Q |v - v_d|^{m-2}(v - v_d) z \\
&\qquad + N \iint_Q |f|^{q-2} f\, h.
\end{aligned} \tag{28}$$

In order to compute the tangent cone to $\mathcal{Q}$ at a point (τ, u, v, f), we need the following result:

Lemma 4.3 *The mapping $M : E \mapsto \widetilde{E}$ is continuously differentiable and its Fréchet derivative is given by*

$$M'(\tau, u, v, f)(\varphi, w, z, h) = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6), \tag{29}$$

where

$$\begin{aligned}
&\varphi_t - b\Delta\varphi - l_1 w_t - l_2 z_t - h = \psi_1 \quad \text{in } Q,\\
&w_t - k_1\Delta w - F_{1,\tau}\varphi - F_{1,u}w - F_{1,v}z = \psi_2 \quad \text{in } Q,\\
&z_t - k_2\Delta z - F_{2,\tau}\varphi - F_{2,u}w - F_{2,v}z = \psi_3 \quad \text{in } Q,\\
&\varphi = \psi_4, \quad w = \psi_5, \quad z = \psi_6 \quad \text{in } \Omega \times \{t = 0\}.
\end{aligned} \tag{30}$$

Here, we have used the notation

$$F_1(\tau, u, v) = -a_1 u(1 - u - v)(1 - 2u - v + c_1\tau + d_1), \tag{31}$$

$$F_2(\tau, u, v) = -a_2 v(1 - v - u)(1 - 2v - u + c_2\tau + d_2) \tag{32}$$

and $F_{i,\tau}$, $F_{i,u}$ and $F_{i,v}$ denote the partial derivatives of F_i.

Furthermore, for any $(\tau, u, v, f) \in E$, $M'(\tau, u, v, f)$ maps E onto $\tilde{E}$.

Proof Let us denote by $M'(\tau, u, v, f)$ the linear mapping defined by (29)–(30). First, it can be shown that M is Gâteaux-differentiable and its Gâteaux-derivative at (τ, u, v, f) is given by $M'(\tau, u, v, f)$.

Indeed, it suffices to check that

$$\begin{aligned}
&\lim_{\kappa\to 0}\frac{1}{\kappa}(M(\tau + \kappa\theta, u + \kappa w, v + \kappa z, f + \kappa h) - M(\tau, u, v, f))\\
&\quad = M'(\tau, u, v, f)(\theta, w, z, h),
\end{aligned}$$

where the limit must be understood in the $\widetilde{E}$-sense. In fact, this is an almost immediate consequence of (22)–(23). In particular, we see that the Gâteaux derivative $M'(\tau, u, v, f)$ is a bounded linear mapping for any (τ, u, v, f).

To conclude that M is continuously differentiable, it is enough to prove that the mapping $(\tau, u, v, f) \mapsto M'(\tau, u, v, f)$ is continuous.

To this end, let us choose (τ_1, u_1, v_1, f_1) and (τ_2, u_2, v_2, f_2) in E. Then, we have for any $(\varphi, w, z, h) \in E$ the following:

$$\begin{aligned}
&\|M'(\tau_2, u_2, v_2, f_2)(\varphi, w, z, h) - M'(\tau_1, u_1, v_1, f_1)(\varphi, w, z, h)\|_{\widetilde{E}}\\
&\quad \le C\big(\|(F_1'(\tau_2, u_2, v_2) - F_1'(\tau_1, u_1, v_1))(\varphi, w, z)\|_{L^{10/3}(Q)}\\
&\qquad + \|(F_2'(\tau_2, u_2, v_2) - F_2'(\tau_1, u_1, v_1))(\varphi, w, z)\|_{L^{10/3}(Q)}\big)\\
&\quad \le C\|(\tau_2, u_2, v_2) - (\tau_1, u_1, v_1)\|_{W^{2,1}_{\overline{q}}(Q)\times W^{2,1}_{10/3}(Q)^2}\|(\varphi, w, z, h)\|_E,
\end{aligned}$$

where C depends on the norms of (τ_1, u_1, v_1, f_1) and (τ_2, u_2, v_2, f_2) in E. Thus, $(\tau, u, v, f) \mapsto M'(\tau, u, v, f)$ is certainly continuous and M is continuously differentiable.

From Proposition 2.1, the *Leray-Schauder Principle* and a simple bootstrapping argument, it is easy to see that, for all $(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6) \in \widetilde{E}$, the linear system (30) possesses exactly one solution in E. Therefore, $M'(\tau, u, v, f)$ maps E onto $\tilde{E}$.

This ends the proof. □

As a consequence of Lemma 4.3 and the results in Sect. 3, we get the following:

Lemma 4.4 *Let $\mathcal{Q}$ be given by* (21). *The tangent cone to $\mathcal{Q}$ at (τ, u, v, f) is the space*

$$TC(\mathcal{Q}, (\tau, u, v, f)) = \{(\varphi, w, z, h) \in E : M'(\tau, u, v, f)(\varphi, w, z, h) = 0\}.$$

In the next result, we deduce the first-order optimality conditions for (20), (21):

Theorem 4.2 *Let the assumptions of Theorem 4.1 be satisfied. Let us assume that (τ, u, v, f) is a local optimal solution of the control problem* (20), (21). *Then, there exist functions $(\theta, p, q) \in W^{2,1}_{k'}(Q) \times W^{2,1}_{m'}(Q) \times W^{2,1}_{m'}(Q)$ solving the so called adjoint problem*

$$\begin{aligned} -\theta_t - b\Delta\theta &= F_{1,\tau} p + F_{2,\tau} q + \alpha_1 |\tau - \tau_d|^{k-2}(\tau - \tau_d) \quad \text{in } Q, \\ -p_t - k_1 \Delta p &= -l_1 \theta_t + F_{1,u} p + F_{2,u} q + \alpha_2 |u - u_d|^{m-2}(u - u_d) \quad \text{in } Q, \\ -q_t - k_2 \Delta q &= -l_2 \theta_t + F_{1,v} p + F_{2,v} q + \alpha_3 |v - v_d|^{m-2}(v - v_d) \quad \text{in } Q, \\ \partial\theta/\partial n &= \partial p/\partial n = \partial q/\partial n = 0 \quad \text{on } \partial\Omega \times (0, T), \\ \theta &= p = q = 0 \quad \text{in } \Omega \times \{t = T\} \end{aligned} \tag{33}$$

(*where F_1 and F_2 are respectively given by* (31) *and* (32) *and $F_{i,\tau}$, $F_{i,u}$ and $F_{i,v}$ denote the partial derivatives of F_i*), *such that*

$$N|f|^{q-2} f + \theta = 0 \quad \text{a.e. in } Q. \tag{34}$$

Proof Since J attains a local minimum at $(\tau, u, v, f) \in \mathcal{Q}$, by Theorem 3.1 there exist $G_0 \in [DC(J, (\tau, u, v, f))]^*$ and $G_1 \in [TC(\mathcal{Q}, (\tau, u, v, f))]^*$, not vanishing simultaneously, such that

$$G_0 + G_1 = 0. \tag{35}$$

Let $h \in L^q(Q)$ be given and let $(\varphi, w, z, h) \in E$ be the unique solution of the associated linear system

$$\begin{aligned}
\varphi_t - b\Delta\varphi &= l_1 w_t + l_2 z_t + h \quad \text{in } Q,\\
w_t - k_1\Delta w &= F_{1,\tau}\varphi + F_{1,u}w + F_{1,v}z \quad \text{in } Q,\\
z_t - k_2\Delta z &= F_{2,\tau}\varphi + F_{2,u}w + F_{2,v}z \quad \text{in } Q,\\
\partial\varphi/\partial n &= \partial w/\partial n = \partial z/\partial n = 0 \quad \text{on } \partial\Omega \times (0,T),\\
\varphi &= w = z = 0 \quad \text{in } \Omega \times \{t=0\}.
\end{aligned} \tag{36}$$

Then $M'(\tau, u, v, f)(\varphi, w, z, h) = 0$ and consequently (φ, w, z, h) belongs to $TC(Q, (\tau, u, v, f))$. Thus, $\langle G_1, (\varphi, w, z, h)\rangle = 0$, which also implies

$$\langle G_0, (\varphi, w, z, h)\rangle = 0.$$

Since $G_0 \in [DC(J, (\tau, u, v, f))]^*$, by Lemma 4.2 there exists $\lambda \geq 0$ such that $G_0 = -\lambda J'(\tau, u, v, f)$. Hence,

$$\begin{aligned}
0 &= -\langle G_0, (\varphi, w, z, h)\rangle\\
&= \lambda\alpha_1 k \iint_Q |\tau - \tau_d|^{2k-2}(\tau - \tau_d)\varphi\\
&\quad + \lambda\alpha_2 \iint_Q |u - u_d|^{m-2}(u - u_d)w\\
&\quad + \lambda\alpha_2 \iint_Q |v - v_d|^{m-2}(v - v_d)z\\
&\quad + \lambda N \iint_Q |f|^{q-2} f h.
\end{aligned} \tag{37}$$

Observe that $\lambda \neq 0$; otherwise, we would have $G_0 = 0$ and, by (35), we would also have $G_1 = 0$, in contradiction with Theorem 3.1. Then, by multiplying (37) by $1/\lambda$, we easily get

$$\begin{aligned}
N &\iint_Q |f|^{q-2} f h\\
&= -\alpha_1 \iint_Q |\tau - \tau_d|^{k-2}(\tau - \tau_d)\varphi\\
&\quad - \alpha_2 \iint_Q |u - u_d|^{m-2}(u - u_d)w\\
&\quad - \alpha_2 \iint_Q |v - v_d|^{m-2}(v - v_d)z.
\end{aligned} \tag{38}$$

Let (θ, p, q) be the solution of the adjoint system (33) and let (φ, w, z, h) solve the linear problem (36). Then, by multiplying the first equation of (33) by φ, the second one by w, the third one by z, integrating each equality on $\Omega \times (0, T)$ and performing the usual integrations by parts, after addition, we find that the right hand side of (38) is equal to

$$\begin{aligned}
&\iint_Q (-\varphi_t + b\Delta\varphi + l_1 w_t + l_2 z_t)\,\theta \\
&\quad + \iint_Q (-w_t + k_1\Delta w + F_{1,\tau}\varphi + F_{1,u} w + F_{1,v} z)\,p \\
&\quad + \iint_Q (-z_t + k_2\Delta z + F_{2,\tau}\varphi + F_{2,u} w + F_{2,v} z) q \\
&= -\iint_Q h\,\theta.
\end{aligned}$$

Finally, taking into account (38), we see that

$$N \iint_Q |f|^{q-2} f\, h = -\iint_Q \theta\, h.$$

Since h is arbitrary in $L^r(Q)$, we obtain (34).
This ends the proof. □

Remark 4.1 Notice that (34) is equivalent to

$$f = -\left(\frac{1}{N}|\theta|\right)^{1/(q-1)} \operatorname{Sign}\theta \quad \text{a.e. in } Q.$$

5 Problems with constraints on the control

5.1 L^q constraints on the control

Let J and M be as before and let us consider the control problem (20), where we have now

$$\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2, \tag{39}$$

$$\mathcal{Q}_1 = \{(\tau, u, v, f) \in E : \|f\|_{L^q(Q)} \leq A_1\} \tag{40}$$

and

$$\mathcal{Q}_2 = \{(\tau, u, v, f) \in E : M(\tau, u, v, f) = 0\}. \tag{41}$$

Arguing exactly as in the proof of Lemma 4.1, we can prove again that the associated admissible set E_{ad}, given by (24), is nonempty:

Lemma 5.1 *Under the assumptions of Lemma* 4.1, *for the control problem* (20), (39), (40), (41), *one has* $E_{ad} \neq \emptyset$.

Furthermore, it is also easy to prove that optimal solutions exist:

Theorem 5.1 *Under the assumptions of Theorem 4.1, the control problem (20), (39), (40), (41) possesses at least one optimal solution.*

The proof of this result is very similar to the proof of Theorem 4.1. We just need to observe that, in this case, the elements of the minimizing sequence $\{(\tau_n, u_n, v_n, f_n)\}$ satisfy $\|f_n\|_{L^q(Q)} \le A_1$, for all n and $f_n \to f$ weakly in $L^q(Q)$. But these two facts imply that $\|f\|_{L^q(Q)} \le A_1$. Then the limit of the minimizing sequence belongs to E_{ad}.

We will now find first-order optimality conditions for (20), (39), (40), (41). We will use again the Dubovitskii-Milyutin formalism.

Thus, let $(\tau, u, v, f) \in E_{ad}$ be a local optimal solution. Of course, this means that, for some $\varepsilon > 0$, one has (26) whenever $(\theta, w, z, h) \in E_{ad}$ and (27) holds.

Recall that the decreasing cone of J at (τ, u, v, f) has been given before, in Lemma 4.2; the tangent cone of $\mathcal{Q}_2$ at (τ, u, v, f) was obtained in Lemma 4.4.

Lemma 5.2 *Let $(\tau, u, v, f) \in E$ be given, with $\|f\|_{L^q(Q)} \le A_1$. The feasible cone to $\mathcal{Q}_1$ at (τ, u, v, f) and its dual cone are respectively given by*

$$FC(\mathcal{Q}_1, (\tau, u, v, f)) = \{(\varphi, w, z, \lambda(h - f)) : \lambda > 0, \quad (\varphi, w, z, h) \in E,\ \|h\|_{L^q(Q)} < A_1\} \tag{42}$$

and

$$[FC(\mathcal{Q}_1, (\tau, u, v, f))]^* = \Big\{(0, 0, 0, g) : g \in L^{q'}(Q), \quad \iint_Q gh \ge \iint_Q gf\ \forall h \in L^q(Q) \text{ satisfying } \|h\|_{L^q(Q)} \le A_1\Big\}. \tag{43}$$

Proof Let us first assume that $f \neq 0$. Then, from the results in Sect. 3, we know that the feasible cone to $\mathcal{Q}_1$ at (τ, u, v, f) is given by

$$FC(\mathcal{Q}_1, (\tau, u, v, f)) = \Big\{(\varphi, w, z, k) \in E : \iint_Q |f|^{q-2} f k < 0\Big\}.$$

It is not difficult to check that this is just the set in (42). On the other hand, (43) is a direct consequence of (42).

If $f = 0$, (42) and (43) are immediate. □

Next, we establish the optimality conditions for the control problem (20), (39), (40), (41):

Theorem 5.2 *Let the assumptions of Theorem 4.1 be satisfied. Let us assume that (τ, u, v, f) is a local optimal solution of the control problem (20), (39), (40), (41).*

Then, there exist functions $(\theta, p, q) \in W^{2,1}_{k'}(Q) \times W^{2,1}_{m'}(Q) \times W^{2,1}_{m'}(Q)$ *satisfying* (33) *and*

$$\begin{cases} \iint_Q (N|f|^{q-2}f + \theta)(h - f) \geq 0 \\ \quad \forall h \in L^q(Q) \text{ with } \|h\|_{L^q(Q)} \leq A_1. \end{cases} \tag{44}$$

Proof From the Dubovitskii-Milyutin Theorem (Theorem 3.1), there exist

$$G_0 \in [DC(J, (\tau, u, v, f))]^*, \qquad G_1 \in [FC(\mathcal{Q}_1, (\tau, u, v, f))]^* \quad \text{and}$$
$$G_2 \in [TC(\mathcal{Q}_2, (\tau, u, v, f))]^*,$$

not simultaneously zero, such that

$$G_0 + G_1 + G_2 = 0. \tag{45}$$

Let $h \in L^q(Q)$ be an arbitrary control and let $(\varphi, w, z, h) \in E$ be the associated unique solution of (36). Then $M'(\tau, u, v, f)(\varphi, w, z, h) = 0$, whence $(\varphi, w, z, h) \in TC(Q_2, (\tau, u, v, f))$ and

$$\langle G_2, (\varphi, w, z, h)\rangle = 0. \tag{46}$$

Since $G_0 \in [DC(J, (\tau, u, v, f))]^*$, by Lemma 4.2 there exists $\lambda \geq 0$ such that $G_0 = -\lambda J'(\tau, u, v, f)$. Therefore, from (28), (45) and (46) we have

$$\begin{aligned} \langle G_1, (\varphi, w, z, h)\rangle &= -\langle G_0, (\varphi, w, z, h)\rangle \\ &= \lambda\alpha_1 \iint_Q |\tau - \tau_d|^{k-2}(\tau - \tau_d)\varphi \\ &\quad + \lambda\alpha_2 \iint_Q |u - u_d|^{m-2}(u - u_d)w \\ &\quad + \lambda\alpha_2 \iint_Q |v - v_d|^{m-2}(v - v_d)z \\ &\quad + \lambda N \iint_Q |f|^{q-2} f h. \end{aligned} \tag{47}$$

Let us prove that $\lambda \neq 0$.

Indeed, let us assume that $\lambda = 0$. Then, by (45), we have $G_1 + G_2 = 0$. Since $G_1 \in [FC(\mathcal{Q}_1, (\tau, u, v, f))]^*$, we also have

$$\langle G_2, (\varphi, w, z, h)\rangle = -\iint_Q gh \quad \forall(\varphi, w, z, h) \in E$$

for some $g \in L^{q'}(Q)$. On the other hand, for each $h \in L^q(Q)$ there exists a unique $(\varphi, w, z) \in W^{2,1}_{q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$ satisfying (36). Then $(\varphi, w, z, h) \in TC(\mathcal{Q}_2, (\tau, u, v, f))$ and $\langle G_2, (\varphi, w, z, h)\rangle = 0$. Consequently,

$$\iint_Q gh = 0 \quad \forall h \in L^q(Q)$$

and $g = 0$, which is in contradiction with the Dubovitskii-Milyutin Theorem.

Since $\lambda \neq 0$, it can be assumed that $\lambda = 1$ and then

$$\begin{aligned}
&\alpha_1 \iint_Q |\tau - \tau_d|^{k-2}(\tau - \tau_d)\varphi \\
&\quad + \alpha_2 \iint_Q |u - u_d|^{m-2}(u - u_d)w \\
&\quad + \alpha_2 \iint_Q |v - v_d|^{m-2}(v - v_d)z \\
&\quad + N \iint_Q |f|^{q-2} f h \\
&= \langle G_1, (\varphi, w, z, h)\rangle = \iint_Q gh \qquad (48)
\end{aligned}$$

for some $g \in L^{q'}(Q)$ satisfying

$$\iint_Q g\,(h - f) \geq 0 \quad \forall h \in L^q(Q) \text{ such that } \|h\|_{L^q(Q)} \leq A_1. \qquad (49)$$

Let $h \in L^q(Q)$ be given. Let (θ, p, q) be the solution of the adjoint problem (33) and let (φ, w, z, h) be the solution of (36). Multiplying the first equation of (33) by φ, the second one by w, the third one by z, integrating on $\Omega \times (0, T)$ and performing the usual integrations by parts, in view of (48) the following is found:

$$\begin{aligned}
&\iint_Q gh - N \iint_Q |f|^{q-2} f h \\
&= \iint_Q (\varphi_t - b\Delta\varphi - l_1 w_t - l_2 z_t)\theta \\
&\quad + \iint_Q (w_t - k_1\Delta w - F_{1,\tau}\varphi - F_{1,u} w - F_{1,v} z) p \\
&\quad + \iint_Q (z_t - k_2\Delta z - F_{2,\tau}\varphi - F_{2,u} w - F_{2,v} z) q \\
&= \iint_Q \theta h.
\end{aligned}$$

Since h is arbitrary in $L^q(Q)$, this shows that $g = N|f|^{q-2} f + \theta$. Consequently, we get (44) and the proof is achieved. □

Remark 5.1 Let us introduce the set $U_{ad} = \{f \in L^r(Q) : \|f\|_{L^r(Q)} \leq A_1\}$. Then Theorem 5.2 can also be stated as follows:

Let the assumptions of Theorem 4.1 be satisfied. Let (τ, u, v, f) be a local optimal solution of the control problem (20), (39), (40), (41) and let (θ, p, q)

satisfy (33). *Then the unique global minimum in* U_{ad} *of the function*

$$h \mapsto \frac{N}{q} \iint_Q |h|^q + \iint_Q \theta h$$

is attained at f.

5.2 Pointwise constraints on the control

In this section, we will deal with the control problem (20), where

$$\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2, \tag{50}$$

$$\mathcal{Q}_1 = \{(\tau, u, v, f) \in E : |f| \leq B_1 \text{ a.e. in } Q\} \tag{51}$$

and again

$$\mathcal{Q}_2 = \{(\tau, u, v, f) \in E : M(\tau, u, v, f) = 0\}. \tag{52}$$

As before, the associated admissible set will be denoted by E_{ad}. It is given by (24). The following results hold:

Lemma 5.3 *Under the assumptions of Lemma* 4.1, *for the control problem* (20), (50), (51), (52), *one has* $E_{ad} \neq \emptyset$.

Theorem 5.3 *Under the assumptions of Theorem* 4.1, *the control problem* (20), (50), (51), (52) *possesses at least one optimal solution*.

Proof It suffices to argue as in the proof of Theorem 4.1 using Lemma 5.3 instead of Lemma 4.1 and noticing that the function f obtained in this proof satisfies $|f| \leq B_1$ a.e. in Q. □

We will now establish first-order optimality conditions. Notice that, in this case, int $\mathcal{Q}_1 = \emptyset$. Consequently, we have to identify the tangent cone to $\mathcal{Q}_1$ at a local optimal solution.

For any $(\tau, u, v, f) \in \mathcal{Q}_1$, we will denote by $N_+(f)$ any set of points $(x, t) \in Q$ such that

$$f(x,t) = B_1 \quad \text{a.e. in } N_+(f) \quad \text{and} \quad f(x,t) < B_1 \quad \text{a.e. outside } N_+(f).$$

Similarly, we will denote by $N_-(f)$ any set of points such that

$$f(x,t) = -B_1 \quad \text{a.e. in } N_-(f) \quad \text{and} \quad f(x,t) > -B_1 \quad \text{a.e. outside } N_-(f).$$

Finally, $N(f)$ will stand for any subset of Q with the following property:

$$|f(x,t)| < B_1 \quad \text{a.e. in } N(f) \quad \text{and} \quad |f(x,t)| = B_1 \quad \text{a.e. outside } N(f).$$

Then we have the following

Lemma 5.4 *Let us assume that* $(\tau, u, v, f) \in \mathcal{Q}_1$. *The tangent cone to* $\mathcal{Q}_1$ *at* (τ, u, v, f) *is given by*

$$TC(\mathcal{Q}_1, (\tau, u, v, f)) \\ = \{(\varphi, w, z, h) \in E : h \leq 0 \text{ a.e. in } N_+(f) \text{ and } h \geq 0 \text{ a.e. in } N_-(f)\}. \quad (53)$$

Consequently, we have

$$[TC(\mathcal{Q}_1, (\tau, u, v, f))]^* \\ = \{(0, 0, 0, g) : g \in L^{r'}(Q),\ g \leq 0 \text{ a.e. in } N_+(f), g \geq 0 \text{ a.e. in } N_-(f), \\ g = 0 \text{ a.e. in } N(f)\}. \quad (54)$$

Proof By definition we have $(\varphi, w, z, h) \in TC(\mathcal{Q}_1, (\tau, u, v, f))$ if and only if $(\varphi, w, z, h) \in E$ and there exists ε_0 such that, for each $0 < \varepsilon \leq \varepsilon_0$, we can find points $a(\varepsilon) \in E$ with $\|a(\varepsilon)\|_E = o(\varepsilon)$ satisfying

$$(\tau, u, v, f) + \varepsilon(\varphi, w, z, h) + a(\varepsilon) \in \mathcal{Q}_1,$$

that is to say,

$$|f + \varepsilon h + a_4(\varepsilon)| \leq B_1,$$

where $a_4(\varepsilon)$ denotes the fourth component of $a(\varepsilon)$.

In view of the definitions of the sets $N_+(f)$ and $N_-(f)$, it is clear from this that (53) holds. On the other hand, it is immediate to deduce (54) from (53). □

Theorem 5.4 *Let the assumptions of Theorem* 4.1 *be satisfied. Let us assume that* (τ, u, v, f) *is a local optimal solution of the control problem* (20), (50), (51), (52). *Then, there exist functions* $(\theta, p, q) \in W^{2,1}_{k'}(Q) \times W^{2,1}_{m'}(Q) \times W^{2,1}_{m'}(Q)$ *satisfying* (33) *and*

$$N|f|^{q-2}f + \theta \begin{cases} \leq 0 & \text{a.e. in } N_+(f), \\ = 0 & \text{a.e. in } N(f), \\ \geq 0 & \text{a.e. in } N_-(f). \end{cases} \quad (55)$$

Proof Recall that the tangent cone to $\mathcal{Q}_1$ at (τ, u, v, f) and its associated dual cone are respectively given by (53) and (54). Obviously, the cone $TC(\mathcal{Q}_1, (\tau, u, v, f))$ is closed and convex.

In view of the results in Sects. 4 and 5.1, in order to apply Theorem 3.1 in this context, we have to check the following:

- $TC(\mathcal{Q}_1, (\tau, u, v, f)) \cap TC(\mathcal{Q}_2, (\tau, u, v, f)) \subset TC(\mathcal{Q}_1 \cap \mathcal{Q}_2, (\tau, u, v, f))$.
- $[TC(\mathcal{Q}_1, (\tau, u, v, f))]^*$ and $[TC(\mathcal{Q}_2, (\tau, u, v, f))]^*$ form a system of cones with the same sense.

Let us first prove that any

$$(\varphi, w, z, h) \in TC(\mathcal{Q}_1, (\tau, u, v, f)) \cap TC(\mathcal{Q}_2, (\tau, u, v, f)) \tag{56}$$

must necessarily belong to $TC(\mathcal{Q}_1 \cap \mathcal{Q}_2, (\tau, u, v, f))$.

Let $\varepsilon > 0$ be given. We have to find $c(\varepsilon) \in E$ with $\|c(\varepsilon)\|_E = o(\varepsilon)$ such that

$$(\tau, u, v, f) + \varepsilon(\varphi, w, z, h) + c(\varepsilon) \in \mathcal{Q}_1 \cap \mathcal{Q}_2.$$

In view of (56), there exist $a(\varepsilon), b(\varepsilon) \in E$, with $\|a(\varepsilon)\|_E, \|b(\varepsilon)\|_E = o(\varepsilon)$ for $i = 1, 2$, such that

$$(\varphi_\varepsilon, w_\varepsilon, z_\varepsilon, h_\varepsilon) := (\tau, u, v, f) + \varepsilon(\varphi, w, z, h) + a(\varepsilon) \in \mathcal{Q}_1 \quad \text{and}$$
$$(\widetilde{\varphi}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{z}_\varepsilon, \widetilde{h}_\varepsilon) := (\tau, u, v, f) + \varepsilon(\varphi, w, z, h) + b(\varepsilon) \in \mathcal{Q}_2.$$

Let $(\varphi'_\varepsilon, w'_\varepsilon, z'_\varepsilon)$ be the unique solution in $W^{2,1}_q(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$ of (36) with $h = h_\varepsilon$. Then $M'(\tau, u, v, f)(\varphi'_\varepsilon, w'_\varepsilon, z'_\varepsilon, h_\varepsilon) = 0$ and $(\varphi'_\varepsilon, w'_\varepsilon, z'_\varepsilon, h_\varepsilon) \in \mathcal{Q}_1 \cap \mathcal{Q}_2$.

Let us introduce

$$c(\varepsilon) = (\varphi'_\varepsilon, w'_\varepsilon, z'_\varepsilon, h_\varepsilon) - (\tau, u, v, f) - \varepsilon(\varphi, w, z, h).$$

We have to show that $\|a(\varepsilon)\|_E = o(\varepsilon)$. But, taking into account that for the linear system (36) the mapping $h \mapsto (\varphi, w, z)$ is continuous, the following is found:

$$\begin{aligned}
\|c(\varepsilon)\|_E &\le \|(\varphi'_\varepsilon, w'_\varepsilon, z'_\varepsilon, h_\varepsilon) - (\widetilde{\varphi}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{z}_\varepsilon, \widetilde{h}_\varepsilon)\|_E \\
&\qquad + \|(\widetilde{\varphi}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{z}_\varepsilon, \widetilde{h}_\varepsilon) - (\tau, u, v, f) - \varepsilon(\varphi, w, z, h)\|_E \\
&\le C\|h_\varepsilon - \widetilde{h}_\varepsilon\|_{L^q(Q)} + \|b(\varepsilon)\|_E \\
&\le C\|(\varphi_\varepsilon, w_\varepsilon, z_\varepsilon, h_\varepsilon) - (\widetilde{\varphi}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{z}_\varepsilon, \widetilde{h}_\varepsilon)\|_E + \|b(\varepsilon)\|_E \\
&\le C\|(\varphi_\varepsilon, w_\varepsilon, z_\varepsilon, h_\varepsilon) - (\tau, u, v, f) - \varepsilon(\varphi, w, z, h)\|_E \\
&\qquad + (C+1)\|b(\varepsilon)\|_E \\
&= C\|a(\varepsilon)\|_E + (C+1)\|b(\varepsilon)\|_E = o(\varepsilon).
\end{aligned}$$

Consequently, $(\varphi, w, z, h) \in TC(\mathcal{Q}_1 \cap \mathcal{Q}_2, (\tau, u, v, f))$.

Let us now prove that the $[TC(\mathcal{Q}_i, (\tau, u, v, f))]^*$, $i = 1, 2$, form a system of cones with the same sense.

Let $R > 0$ and $G_i \in [TC(\mathcal{Q}_i, (\tau, u, v, f))]^*$ $(i = 1, 2)$ be given and assume that $\|G_1 + G_2\|_{E'} \le R$. We have to show that there exist $R_1, R_2 > 0$ such that $\|G_i\|_{E'} \le R_i$, for $i = 1, 2$.

Since $G_1 \in [TC(\mathcal{Q}_1, (\tau, u, v, f))]^*$, we have $G_1 = (0, 0, 0, g_1)$, for some $g_1 \in L^{q'}(Q)$. On the other hand, we must have $G_2 = (\phi_1, \phi_2, \phi_3, \phi_4)$ for some ϕ_j satisfying

$$\phi_1 \in (W^{2,1}_q(Q))', \qquad \phi_2, \phi_3 \in (W^{2,1}_{10/3}(Q))', \qquad \phi_4 \in L^{q'}(Q).$$

Consequently,

$$\begin{aligned} R &\geq \|G_1 + G_2\|_{E'} \\ &\geq C\big[\|\phi_1\|_{(W_{\overline{q}}^{2,1}(Q))'} + \|\phi_2\|_{(W_{10/3}^{2,1}(Q))'} + \|\phi_3\|_{(W_{10/3}^{2,1}(Q))'} + \|g_1 + \phi_4\|_{L^{q'}(Q)}\big] \end{aligned}$$

and we get

$$\|\phi_1\|_{(W_{\overline{q}}^{2,1}(Q))'}, \|\phi_2\|_{(W_{10/3}^{2,1}(Q))'}, \|\phi_3\|_{(W_{10/3}^{2,1}(Q))'} \leq \frac{R}{C}.$$

Let $h \in L^q(Q)$ be given. As before, there exists (φ, w, z) such that $(\varphi, w, z, h) \in E$ and $M'(\tau, u, v, f)(\varphi, w, z, h) = 0$. Then $\langle G_2, (\varphi, w, z, h)\rangle = 0$ and

$$\begin{aligned} \left|\iint_Q \phi_4 h\right| &\leq |\langle \phi_1, \varphi\rangle| + |\langle \phi_2, w\rangle| + |\langle \phi_3, z\rangle| \\ &\leq \frac{R}{C}\big[\|\varphi\|_{W_{\overline{q}}^{2,1}(Q)} + \|w\|_{W_{10/3}^{2,1}(Q)} + \|z\|_{W_{10/3}^{2,1}(Q)}\big] \leq CR\|h\|_{L^r(Q)}. \end{aligned}$$

Since h is arbitrary in $L^q(Q)$, this implies $\|\phi_4\|_{L^{q'}(Q)} \leq CR$.

It is now immediate to check that $\|G_1\|_{E'} \leq R_1$ and $\|G_2\|_{E'} \leq R_2$ for some constants R_i of the form $R_i = CR$.

Therefore, in view of Theorem 3.1, there exist

$$G_0 \in [DC(J, (\tau, u, v, f))]^*, \qquad G_1 \in [TC(\mathcal{Q}_1, (\tau, u, v, f))]^* \quad \text{and}$$
$$G_2 \in [TC(\mathcal{Q}_2, (\tau, u, v, f))]^*,$$

not simultaneously zero, such that

$$G_0 + G_1 + G_2 = 0.$$

Proceeding as we did in the proof of Theorem 5.2, we can now deduce the stated result. □

Remark 5.2 Let us introduce the set

$$U_{ad} = \{f \in L^q(Q) : |f| \leq B_1 \text{ a.e. in } Q\}.$$

Then (55) is equivalent to

$$\iint_Q (N|f|^{q-2} f + \theta)(h - f) \geq 0 \quad \forall h \in U_{ad}.$$

Consequently, Theorem 5.4 can also be stated as follows:

Let the assumptions of Lemma 4.1 be satisfied. Let (τ, u, v, f) be a local optimal solution of the control problem (20), (50), (51), (52) and let (θ, p, q) satisfy (33). Then the unique global minimum in U_{ad} of the function

$$h \mapsto \iint_Q \left(\frac{N}{q}|h|^q + \theta h\right)$$

is attained at f.

6 Problems with constraints on the state

6.1 Pointwise constraints on the temperature

We will now deal with the control problem (20), where

$$\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2, \tag{57}$$

$\mathcal{Q}_1$ is given by

$$\mathcal{Q}_1 = \{(\tau, u, v, f) \in E : C_1 \le \tau \le C_2 \text{ a.e. in } Q\} \tag{58}$$

for some C_i with $0 < C_1 < C_2$ and $\mathcal{Q}_2$ is given by

$$\mathcal{Q}_2 = \{(\tau, u, v, f) \in E : M(\tau, u, v, f) = 0\}. \tag{59}$$

We will denote again by E_{ad} the corresponding admissible set.
We must consider the auxiliary problem

$$\begin{aligned} &u_t - k_1 \Delta u = -a_1 u(1-u-v)(1-2u-v-2m_1) \quad \text{in } Q,\\ &v_t - k_2 \Delta v = -a_2 v(1-v-u)(1-2v-u-2m_2) \quad \text{in } Q,\\ &\partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega \times (0,T),\\ &u = u_0, \qquad v = v_0 \quad \text{in } \Omega \times \{t=0\}, \end{aligned} \tag{60}$$

where k_1, k_2, a_1 and a_2 are positive constants and $m_1, m_2 \in L^\infty(Q)$.
Let us set

$$K' = 1 + \max_i \|m_i\|_{L^\infty(Q)}. \tag{61}$$

Then, arguing as in [3], the following can be proved:

Theorem 6.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2 domain. Let us assume that k_1, k_2, a_1 and a_2 are positive constants, m_1, $m_2 \in L^\infty(Q)$ and $u_0, v_0 \in W_2^2(\Omega)$ satisfy $\partial u_0/\partial n|_{\partial\Omega} = \partial v_0/\partial n|_{\partial\Omega} = 0$ and $0 \le u_0, v_0 \le K'$. Then (60) possesses exactly one solution $(u, v) \in W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$ satisfying*

$$\begin{aligned} &\|u\|_{W^{2,1}_{10/3}(Q)} + \|v\|_{W^{2,1}_{10/3}(Q)} \le C\big(\|u_0\|_{W_2^2(\Omega)} + \|v_0\|_{W_2^2(\Omega)}\big),\\ &0 \le u, v \le K'. \end{aligned}$$

Here, C depends on Ω, T, k_1, k_2, a_1, a_2, m_1 and m_2.

As a consequence, we have:

Lemma 6.1 *Assume that the hypotheses of Lemma 4.1 hold and, moreover,*

$$C_1 \le \tau_0 \le C_2 \quad \textit{a.e. in } \Omega.$$

Then the admissible set E_{ad} *associated to the control problem* (20), (57), (58), (59) *satisfies* $E_{ad} \neq \emptyset$.

Proof Let $\tau \in W_q^{2,1}(Q)$ be the unique solution of

$$\begin{aligned} \tau_t - b\Delta\tau = 0 \quad & \text{in } Q, \\ \partial\tau/\partial n = 0 \quad & \text{on } \partial\Omega \times (0,T), \\ \tau = \tau_0 \quad & \text{in } \Omega \times \{t = 0\}. \end{aligned}$$

By the *maximum principle* for the heat equation, we have $C_1 \le \tau \le C_2$ a.e. in Q.

Now, let us introduce m_1 and m_2, with $m_i = -\frac{1}{2}(c_i\tau + d_i)$ for $i = 1, 2$. Obviously, $m_i \in L^\infty(Q)$.

Let (u, v) be the unique solution in $W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$ of the corresponding problem (60) (furnished by Theorem 6.1) and let us set $f = \tau_t - b\Delta\tau - l_1 u_t - l_2 v_t$. Then, it is easy to check that $(\tau, u, v, f) \in E$, $M(\tau, u, v, f) = 0$ and $J(\tau, u, v, f) < +\infty$. Consequently, $(\tau, u, v, f) \in E_{ad}$.

This ends the proof. □

Theorem 6.2 *Under the assumptions of Lemma* 6.1, *the control problem* (20), (57), (58), (59) *possesses at least one optimal solution.*

For the proof, once more, it suffices to argue as in the proof of Theorem 4.1. This time, we have to use Lemma 6.1 instead of Lemma 4.1. The key point is that the temperature τ obtained in this proof satisfies $C_1 \le \tau \le C_2$ a.e. in Q.

In order to establish first-order optimality conditions in this case, we must identify the feasible cone to $\mathcal{Q}_1$ at a local optimal solution. Notice that, since E is given by (17) and $W_{\overline{q}}^{2,1}(Q) \hookrightarrow L^\infty(Q)$ by Lemma 2.2, one has:

$$\operatorname{int} \mathcal{Q}_1 = \Big\{(\varphi, w, z, h) \in E : C_1 < \operatorname*{ess\,inf}_Q \varphi \le \operatorname*{ess\,sup}_Q \varphi < C_2\Big\}. \tag{62}$$

A direct consequence is the following

Lemma 6.2 *Assume that* $(\tau, u, v, f) \in \mathcal{Q}_1$. *Then*

$$\begin{aligned} & FC(\mathcal{Q}_1, (\tau, u, v, f)) \\ & \quad = \Big\{(\lambda(\varphi - \tau), w, z, h) : \lambda > 0,\ (\varphi, w, z, h) \in E, \\ & \qquad C_1 < \operatorname*{ess\,inf}_Q \varphi \le \operatorname*{ess\,sup}_Q \varphi < C_2\Big\} \end{aligned} \tag{63}$$

and

$$\begin{aligned} & [FC(\mathcal{Q}_1, (\tau, u, v, f))]^* \\ & \quad = \{(\zeta, 0, 0, 0) : \zeta \in (W_q^{2,1}(Q))',\ \langle\zeta, \varphi\rangle \ge \langle\zeta, \tau\rangle\ \forall\varphi \in W_q^{2,1}(Q) \textit{ with} \\ & \qquad C_1 \le \varphi \le C_2 \textit{ a.e.}\}. \end{aligned} \tag{64}$$

Arguing as in the proof of Theorem 5.2 (but using Lemma 6.2 instead of Lemma 5.2), we get now:

Theorem 6.3 *Let the assumptions of Theorem 6.2 be satisfied. Let us assume that* (τ, u, v, f) *is a local optimal solution of the control problem* (20), (57), (58), (59). *Then there exist* $(\theta, p, q) \in W^{2,1}_{k'}(Q) \times W^{2,1}_{m'}(Q) \times W^{2,1}_{m'}(Q)$ *and* $\zeta \in (W^{2,1}_q(Q))'$ *satisfying* (33) *and*

$$\langle \zeta, \varphi - \tau \rangle \geq 0 \quad \forall \varphi \in W^{2,1}_q(Q) \text{ with } C_1 \leq \varphi \leq C_2 \text{ a.e.}, \tag{65}$$

such that

$$\begin{cases} \iint_Q (N|f|^{q-2} f + \theta) h = \langle \zeta, \varphi \rangle \\ \quad \forall (\varphi, w, z, h) \in E \text{ satisfying (36)}. \end{cases}$$

6.2 Pointwise constraints on the temperature gradient

We will finally consider the optimal control problem (20), where

$$\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \tag{66}$$

and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are respectively given by

$$\mathcal{Q}_1 = \{(\tau, u, v, f) \in E : |\nabla \tau| \leq D_1 \text{ a.e. in } Q\}, \tag{67}$$

and

$$\mathcal{Q}_2 = \{(\tau, u, v, f) \in E : M(\tau, u, v, f) = 0\}. \tag{68}$$

Once more, the associated admissible set will be denoted by E_{ad}.

Lemma 6.3 *Assume that the hypotheses of Lemma 4.1 hold and, furthermore,*

$$\tau_0 \in W^2_q(\Omega), \quad |\nabla \tau_0| \leq D_1 \quad \text{a.e. in } \Omega.$$

Then, for the control problem (20), (66), (67), (68), *one has* $E_{ad} \neq \emptyset$.

Proof Let us introduce the function τ, with $\tau(x,t) = \tau_0(x)$ a.e. in Q. Then $\tau \in W^{2,1}_q(Q)$ and

$$|\nabla \tau| \leq D_1 \quad \text{a.e. in } Q.$$

Let us set $m_i = -\frac{1}{2}(c_i \tau + d_i)$ for $i = 1, 2$. Let (u, v) be the unique solution in $W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$ of the corresponding problem (60) (furnished by Theorem 6.1) and, finally, let us set $f = -b\Delta\tau - l_1 u_t - l_2 v_t \in L^q(Q)$.

It is then clear that $(\tau, u, v, f) \in \mathcal{Q}_1$, $M(\tau, u, v, f) = 0$ and $J(\tau, u, v, f) < +\infty$. In other words, $(\tau, u, v, f) \in E_{ad}$. □

From this lemma, we can deduce easily the existence of optimal solutions:

Theorem 6.4 *Under the assumptions of Lemma 6.3, the control problem (20), (66), (67), (68) possesses at least one optimal solution.*

Next, we will present first-order optimality conditions for this control problem. Here, for technical reasons, we will assume that $q > 5$.

We have to identify the feasible cone to $\mathcal{Q}_1$ at a point (τ, u, v, f). To this end, we first notice that, since E is given by (17) and $W_q^{2,1}(Q) \hookrightarrow L^\infty(0,T; W_\infty^1(\Omega))$ by Lemma 2.3, one has

$$\operatorname{int} \mathcal{Q}_1 = \{(\varphi, w, z, h) \in E : \|\nabla\varphi\|_{L^\infty(Q)} < D_1\}.$$

Lemma 6.4 *Assume that* $(\tau, u, v, f) \in \mathcal{Q}_1$ *and* $q > 5$. *Then*

$$\begin{aligned} &FC(\mathcal{Q}_1, (\tau, u, v, f)) \\ &\quad = \{(\lambda(\varphi - \tau), w, z, h) : \lambda > 0, (\varphi, w, z, h) \in E, \\ &\qquad \|\nabla\varphi\|_{L^\infty(Q)} < D_1\} \end{aligned} \tag{69}$$

and

$$\begin{aligned} &[FC(\mathcal{Q}_1, (\tau, u, v, f))]^* \\ &\quad = \{(\zeta, 0, 0, 0) : \zeta \in (W_q^{2,1}(Q))', \langle\zeta, \varphi\rangle \geq \langle\zeta, \tau\rangle\ \forall\theta \in W_q^{2,1}(Q) \text{ with} \qquad (70) \\ &\qquad \|\nabla\varphi\|_{L^\infty(Q)} \leq D_1\}. \qquad (71) \end{aligned}$$

Arguing as before, we can now deduce the optimality system for (20), (66), (67), (68):

Theorem 6.5 *Let the assumptions of Theorem 6.4 be satisfied and suppose that* $q > 5$. *Let* (τ, u, v, f) *be a local optimal solution of the control problem (20), (66), (67), (68). Then there exist* $(\theta, p, q) \in W_{k'}^{2,1}(Q) \times W_{m'}^{2,1}(Q) \times W_{m'}^{2,1}(Q)$ *and* $\zeta \in (W_q^{2,1}(Q))'$ *satisfying (33) and*

$$\langle\zeta, \varphi - \tau\rangle \geq 0 \quad \forall\varphi \in W_q^{2,1}(Q) \text{ with } \|\nabla\varphi\|_{L^\infty(Q)} \leq D_1 \tag{72}$$

such that

$$\begin{cases} \iint_Q (N|f|^{q-2}f + \theta)h = \langle\zeta, \varphi\rangle \\ \quad \forall(\varphi, w, z, h) \in E \text{ satisfying } (36). \end{cases}$$

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