



# Road pricing and spatial distribution of traffic flow in radial-arc network

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## Abstract

This paper discusses the effect of road pricing on the spatial distribution of traffic flow. The traffic flow density is derived for a circular city with a radial-arc network. The traffic flow density describes the amount of traffic as a function of position and allows us to identify the location of potential congestion areas. The analytical expression for the traffic flow density demonstrates how the size of the toll area and the toll level affect the spatial distribution of traffic flow. As the size of the toll area increases, the decrease in traffic flow inside the toll area becomes smaller. As the toll level increases, the increase in traffic flow at the boundary of the toll area becomes greater. The effect of the travel cost on the spatial distribution of traffic flow is also examined. These findings can be used to determine the size of the toll area and the toll level required to achieve a certain level of traffic congestion.

**Keywords** Transportation · Traffic flow density · Area pricing · Continuous approximation · Circular city

**Mathematics Subject Classification** 90B06 · 90B85

## 1 Introduction

Road pricing has been implemented in several cities such as London, Singapore, and Stockholm. The objectives of road pricing are to reduce the traffic flow in the city center and financially support infrastructure management. Although marginal cost pricing is the first-best pricing [23], the implementation is difficult because of practical restrictions. The second-best pricing such as cordon and area pricing is then widely adopted. Cordon pricing charges a toll to vehicles passing or entering

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a designated area, whereas area pricing charges a toll to all vehicles driving inside the area. Road pricing can affect both travel demand and travel routes. Examining how road pricing affects the spatial distribution of traffic flow is therefore useful for designing road pricing systems.

The optimal design of road pricing has been studied using discrete network models. The discrete models use detailed traffic data on actual road networks and aim to develop efficient algorithms to obtain exact solutions. May and Milne [8] compared cordon-based, distance-based, time-based, and delay-based pricing systems. Sumalee [14] obtained the optimal cordon location and toll level under the network equilibrium condition. Further studies on cordon pricing have considered multi-layered and multi-centered cordon [25] and time-dependent pricing [3, 13, 27]. Maruyama and Harata [6] and Maruyama and Sumalee [7] compared the performance of cordon and area pricing using a trip-chain equilibrium model. Zhang et al. [24] compared cordon and area pricing from the perspective of travel demand management. Takaki et al. [16] and Takaki et al. [17] obtained the optimal shape of the toll area and toll level for area pricing. Further studies on area pricing have considered time-dependent pricing [28], joint distance- and time-dependent pricing [2], heterogeneity of users [26], and travelers' degree of satisfaction during their trips [1].

As a complement to the discrete models reviewed above, continuous approximation models, which use approximated travel demand on a plane or idealized networks such as grid and radial-arc networks, aim to find fundamental relationships between variables. The continuous models often yield analytical solutions that help reveal managerial insights, thus supplementing discrete models. Mun et al. [10] obtained the optimal cordon location and toll level in a linear monocentric city. The model was extended by Mun et al. [11] to a non-monocentric city, Verhoef [22] to include land and labor markets, Li et al. [5] to consider the interaction between auto and bus, and Tsai and Lu [20] to multiple-cordon. Miyagawa [9] compared cordon and area pricing in a radial-arc network in terms of the traffic volume in the toll area and the toll revenue.

In this paper, we develop a model for analyzing the effect of road pricing on the spatial distribution of traffic flow. The model uses a continuous approximation where origins and destinations are uniformly distributed in a circular city with a radial-arc network. The radial-arc network can be found in many cities such as Tokyo, Paris, and Moscow. The model yields an analytical expression for the spatial distribution of traffic flow. The analytical expression leads to a fundamental understanding of the effect of road pricing, thus providing a basic framework for designing road pricing systems. The model focuses on area pricing rather than cordon pricing because area pricing is more effective than cordon pricing in reducing the traffic volume in the toll area. In addition, the derivation of the spatial distribution of traffic flow for area pricing provides a basis for that for cordon pricing. The total traffic volume inside the toll area in a circular city with a radial-arc network was derived by Miyagawa [9]. We extend the scope to traffic flow as a function of position, which allows us to examine the locational variation of traffic flow.

The spatial distribution of traffic flow in a circular city with a radial-arc network was derived by Vaughan [21]. The distribution was extended by Tanaka and Kurita [18] to the distribution in a sector-shaped city, Tanaka and Kurita [19] to incorporate the time

variation of traffic flow, and Suzuki and Miura [15] to consider the effect of routing systems. The spatial distribution of traffic flow in road pricing has not been derived previously.

The remainder of this paper is organized as follows. The next section develops a radial-arc network model. The following section derives the spatial distribution of traffic flow when no toll is charged. The penultimate section derives the spatial distribution of traffic flow in road pricing. The final section presents concluding remarks.

## 2 Radial-arc network model

Consider a circular city with radius  $a$ , as shown in Fig. 1. The city has a dense radial-arc network. Any point in the city is expressed as  $(r, \theta)$  ( $0 \leq r \leq a, 0 \leq \theta < 2\pi$ ) in the polar coordinate centered at the city center. A toll area is represented as a circle centered at the city center with radius  $b$ . All vehicles driving inside the toll area are charged a fixed toll  $t$ , irrespective of the travel distance within the area.

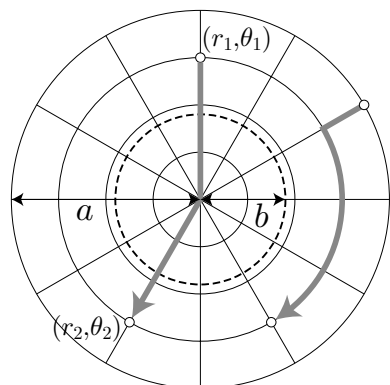
Origins and destinations of trips are assumed to be uniformly distributed in the city. That is, trips occur between any two points in the city. The uniform distribution serves as a basis for further analysis with more realistic distributions. The uniform distribution of origins and destinations was also used by Vaughan [21]. Although Vaughan [21] assumed that the travel demand between origins and destinations is independent of the travel cost, we assume that the travel demand decreases with the travel cost. The travel cost  $C$  for trips of length  $R$  is defined as

$$C = \alpha R + t, \tag{1}$$

where  $\alpha$  is the travel cost per unit distance. Every traveler is assumed to use the least cost route. The travel demand  $D$  is expressed as

$$D = D_0 e^{-\beta C}, \tag{2}$$

Fig. 1 Circular city with a radial-arc network



where  $D_0$  is the travel demand when  $C = 0$  and  $\beta (> 0)$  is a parameter for elasticity. The travel demand then decreases with the trip length  $R$  and the toll level  $t$ . The exponential function has been widely used in spatial interaction models [12].

Let  $f_r$  and  $f_a$  be the densities of traffic flow passing a point  $(r, \theta)$  along radial and arc roads, respectively. The amount of traffic flow passing the arc between two points  $(r, \theta_1)$  and  $(r, \theta_2)$  along radial roads, denoted by  $V_r$ , is given by

$$V_r = \int_{\theta_1}^{\theta_2} f_r r \, d\theta, \tag{3}$$

and the amount of traffic flow passing the segment between two points  $(r_1, \theta)$  and  $(r_2, \theta)$  along arc roads, denoted by  $V_a$ , is given by

$$V_a = \int_{r_1}^{r_2} f_a \, dr, \tag{4}$$

as shown in Fig. 2.

### 3 Traffic flow density without road pricing

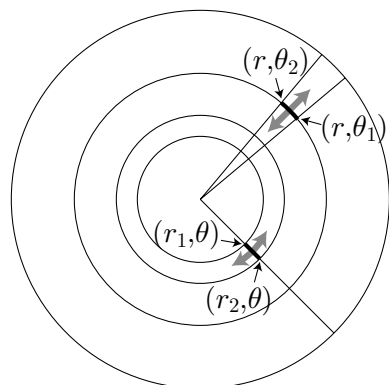
In this section, we derive the traffic flow density when no toll is charged, that is,  $t = 0$ . The traffic flow density for inelastic travel demand ( $\beta = 0$  in Eq. (2)) was derived by Vaughan [21]. We extend the analysis to incorporate elastic travel demand.

Let  $P_1(r_1, \theta_1)$  and  $P_2(r_2, \theta_2)$  be origin and destination of trips, respectively. The shortest distance between  $P_1(r_1, \theta_1)$  and  $P_2(r_2, \theta_2)$  is given by

$$R = \begin{cases} |r_1 - r_2| + \min\{r_1, r_2\}\varphi, & 0 \leq \varphi < 2, \\ r_1 + r_2, & 2 \leq \varphi \leq \pi, \end{cases} \tag{5}$$

where  $\varphi = \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}$  [4]. Both radial and arc roads are used if  $\varphi < 2$ , whereas only radial roads are used if  $\varphi \geq 2$ , as shown in Fig. 1.

**Fig. 2** Traffic flow on radial and arc roads



First, we derive the traffic flow density on radial roads. We can assume  $\theta = 0$  without loss of generality because of the symmetry of the circular city. The traffic passes the infinitesimal arc between two points  $(r, 0)$  and  $(r, d\theta)$  along radial roads if

$$P_1(r_1, \theta_1) \in \{(r_1, \theta_1) \mid r \leq r_1 \leq a, 0 \leq \theta_1 \leq d\theta\}, \tag{6}$$

$$P_2(r_2, \theta_2) \in \{(r_2, \theta_2) \mid 0 \leq r_2 \leq r, -2 \leq \theta_2 \leq 2\} \cup \{(r_2, \theta_2) \mid 0 \leq r_2 \leq a, 2 \leq \theta_2 \leq 2\pi - 2\}, \tag{7}$$

as shown in Fig. 3. Taking the round trip into account, we have the amount of traffic flow passing the arc

$$f_r r d\theta = \int_0^{d\theta} \int_r^a \int_0^2 \int_0^r 4D_0 \exp[-\alpha\beta\{r_1 - r_2 + r_2(\theta_2 - \theta_1)\}] r_1 r_2 dr_2 d\theta_2 dr_1 d\theta_1 + \int_0^{d\theta} \int_r^a \int_2^\pi \int_0^a 4D_0 \exp\{-\alpha\beta(r_1 + r_2)\} r_1 r_2 dr_2 d\theta_2 dr_1 d\theta_1. \tag{8}$$

The traffic flow density on radial roads is then given by

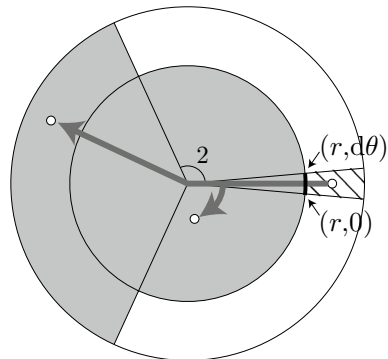
$$f_r = \frac{1}{r} \int_r^a \int_0^2 \int_0^r 4D_0 \exp\{-\alpha\beta(r_1 - r_2 + r_2\theta_2)\} r_1 r_2 dr_2 d\theta_2 dr_1 + \frac{1}{r} \int_r^a \int_2^\pi \int_0^a 4D_0 \exp\{-\alpha\beta(r_1 + r_2)\} r_1 r_2 dr_2 d\theta_2 dr_1 \tag{9}$$

$$= \frac{4D_0}{\alpha^4 \beta^4 r} [(e^{\alpha\beta r} - 1)^2 \{(\alpha\beta r + 1)e^{-2\alpha\beta r} - (\alpha\beta a + 1)e^{-\alpha\beta(a+r)}\} + (\pi - 2)(\alpha\beta a + 1 - e^{\alpha\beta a}) \{(\alpha\beta a + 1)e^{-2\alpha\beta a} - (\alpha\beta r + 1)e^{-\alpha\beta(a+r)}\}].$$

Next, we derive the traffic flow density on arc roads. The traffic passes the infinitesimal segment between two points  $(r, 0)$  and  $(r + dr, 0)$  along arc roads if

$$P_1(r_1, \theta_1) \in \{(r_1, \theta_1) \mid r \leq r_1 \leq a, -2 \leq \theta_1 \leq 0\}, \tag{10}$$

**Fig. 3** Traffic flow on radial roads



$$P_2(r_2, \theta_2) \in \{(r_2, \theta_2) \mid r \leq r_2 \leq r + dr, 0 \leq \theta_2 \leq \theta_1 + 2\}, \tag{11}$$

as shown in Fig. 4a, or

$$P_1(r_1, \theta_1) \in \{(r_1, \theta_1) \mid r \leq r_1 \leq r + dr, -2 \leq \theta_1 \leq 0\}, \tag{12}$$

$$P_2(r_2, \theta_2) \in \{(r_2, \theta_2) \mid r \leq r_2 \leq a, 0 \leq \theta_2 \leq \theta_1 + 2\}, \tag{13}$$

as shown in Fig. 4b. Note that the amounts of traffic flow for the above two cases are the same. Taking the round trip into account, we have the amount of traffic flow passing the segment

$$f_a dr = \int_{-2}^0 \int_r^a \int_0^{\theta_1+2} \int_r^{r+dr} 4D_0 \exp[-\alpha\beta\{r_1 - r_2 + r_2(\theta_2 - \theta_1)\}] r_1 r_2 dr_2 d\theta_2 dr_1 d\theta_1. \tag{14}$$

The traffic flow density on arc roads is then given by

$$\begin{aligned} f_a &= r \int_{-2}^0 \int_r^a \int_0^{\theta_1+2} 4D_0 \exp[-\alpha\beta\{r_1 - r + r(\theta_2 - \theta_1)\}] r_1 d\theta_2 dr_1 d\theta_1 \\ &= \frac{4D_0}{\alpha^4 \beta^4 r} (2\alpha\beta r + 1 - e^{2\alpha\beta r}) \{(\alpha\beta a + 1)e^{\alpha\beta r} - (\alpha\beta r + 1)e^{\alpha\beta a}\} e^{-\alpha\beta(a+2r)}. \end{aligned} \tag{15}$$

The traffic flow densities on radial and arc roads are shown in Fig. 5, where  $a = 1, D_0 = 1, \alpha = 1$ . Note that the traffic flow density on radial roads  $f_r$  diverges to infinity at the city center and decreases with the distance from the city center, whereas the traffic flow density on arc roads  $f_a$  has a maximum around  $r = a/2$ . The traffic flow densities for  $\beta = 0$  (Fig. 5a) are identical with those derived by Vaughan [21]. Introducing the travel demand function (2) allows us to examine how the travel cost and the elasticity of demand affect the spatial distribution of traffic flow.

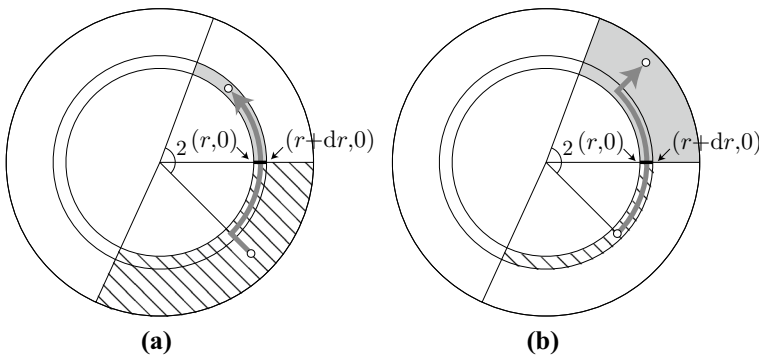


Fig. 4 Traffic flow on arc roads

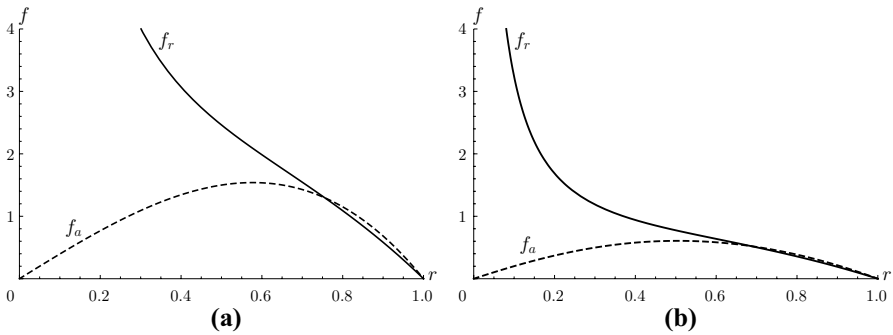


Fig. 5 Traffic flow density: a  $\beta = 0$ ; b  $\beta = 1$

### 4 Traffic flow density in road pricing

In this section, we derive the traffic flow density in road pricing and examine how the size of the toll area and the toll level affect the spatial distribution of traffic flow.

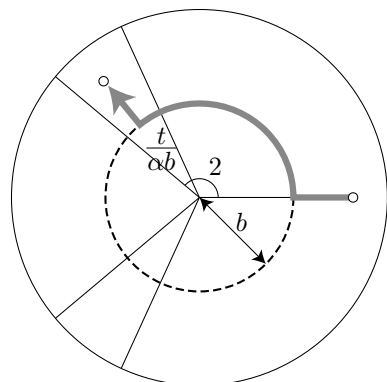
Road pricing affects not only travel demand but also travel routes. If both origin and destination are outside the toll area ( $r_1, r_2 \geq b$ ) and  $2 \leq \varphi \leq t/(ab) + 2$ , the traveler makes a detour around the toll area, as shown in Fig. 6. This is because the travel cost of making a detour is smaller than that of passing the toll area, that is,

$$ab\varphi \leq 2ab + t. \tag{16}$$

It follows that road pricing can increase travel distances, which is an adverse effect of road pricing. The travel distance between two points  $P_1(r_1, \theta_1)$  and  $P_2(r_2, \theta_2)$  is then rewritten as

$$R = \begin{cases} |r_1 - r_2| + \min\{r_1, r_2\}\varphi, & 0 \leq \varphi < 2, \\ r_1 + r_2 - 2b + b\varphi, & r_1, r_2 \geq b, 2 \leq \varphi \leq \min\{t/(ab) + 2, \pi\}, \\ r_1 + r_2, & \text{otherwise.} \end{cases} \tag{17}$$

Fig. 6 Detour around the toll area



Note that if  $t/(ab) + 2 \geq \pi \Leftrightarrow t \geq (\pi - 2)ab$ , travelers whose origin and destination are outside the toll area do not pass the city center.

The traffic flow density on radial roads is obtained by considering the traffic passing the infinitesimal arc between two points  $(r, 0)$  and  $(r, d\theta)$  along radial roads, as shown in Fig. 7. If  $0 \leq r < b, 0 \leq t < (\pi - 2)ab$  (Fig. 7a, b),

$$\begin{aligned}
 f_r &= \frac{1}{r} \int_r^a \int_0^2 \int_0^r 4D_0 \exp[-\beta\{\alpha(r_1 - r_2 + r_2\theta_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 &+ \frac{1}{r} \int_r^b \int_2^\pi \int_0^a 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 &+ \frac{1}{r} \int_b^a \int_2^{t/(ab)+2} \int_0^b 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 &+ \frac{1}{r} \int_b^a \int_{t/(ab)+2}^\pi \int_0^a 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1.
 \end{aligned}
 \tag{18}$$

Note that no traffic passes the arc if

$$P_1(r_1, \theta_1) \in \{(r_1, \theta_1) \mid b \leq r_1 \leq a, 0 \leq \theta_1 \leq d\theta\},
 \tag{19}$$

$$P_2(r_2, \theta_2) \in \{(r_2, \theta_2) \mid b \leq r_2 \leq a, 2 \leq \theta_2 \leq t/(ab) + 2\},
 \tag{20}$$

because the traveler makes a detour around the toll area. If  $0 \leq r < b, t \geq (\pi - 2)ab$ ,

$$\begin{aligned}
 f_r &= \frac{1}{r} \int_r^a \int_0^2 \int_0^r 4D_0 \exp[-\beta\{\alpha(r_1 - r_2 + r_2\theta_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 &+ \frac{1}{r} \int_r^b \int_2^\pi \int_0^a 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 &+ \frac{1}{r} \int_b^a \int_2^\pi \int_0^b 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1.
 \end{aligned}
 \tag{21}$$

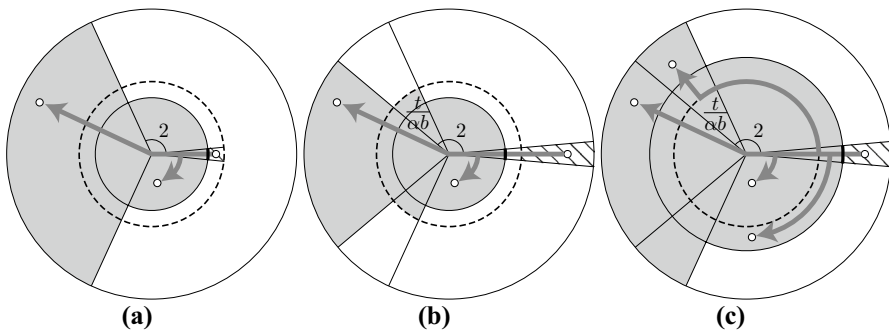


Fig. 7 Traffic flow on radial roads in road pricing



Note that no traffic passes the arc if both origin and destination are outside the toll area. If  $b \leq r \leq a, 0 \leq t < (\pi - 2)\alpha b$  (Fig. 7c),

$$\begin{aligned}
 f_r = & \frac{1}{r} \int_r^a \int_0^2 \int_0^b 4D_0 \exp[-\beta\{\alpha(r_1 - r_2 + r_2\theta_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_0^2 \int_b^r 4D_0 \exp\{-\alpha\beta(r_1 - r_2 + r_2\theta_2)\} r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_2^{t/(ab)+2} \int_0^b 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_2^{t/(ab)+2} \int_b^a 4D_0 \exp\{-\alpha\beta(r_1 + r_2 - 2b + b\theta_2)\} r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_{t/(ab)+2}^\pi \int_0^a 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1. \tag{22}
 \end{aligned}$$

Note that no traffic passes the toll area if

$$P_1(r_1, \theta_1) \in \{(r_1, \theta_1) \mid r \leq r_1 \leq a, 0 \leq \theta_1 \leq d\theta\}, \tag{23}$$

$$P_2(r_2, \theta_2) \in \{(r_2, \theta_2) \mid b \leq r_2 \leq a, 2 \leq \theta_2 \leq t/(ab) + 2\}. \tag{24}$$

If  $b \leq r \leq a, t \geq (\pi - 2)\alpha b$ ,

$$\begin{aligned}
 f_r = & \frac{1}{r} \int_r^a \int_0^2 \int_0^b 4D_0 \exp[-\beta\{\alpha(r_1 - r_2 + r_2\theta_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_0^2 \int_b^r 4D_0 \exp\{-\alpha\beta(r_1 - r_2 + r_2\theta_2)\} r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_2^\pi \int_0^b 4D_0 \exp[-\beta\{\alpha(r_1 + r_2) + t\}] r_1 r_2 \, dr_2 d\theta_2 dr_1 \\
 & + \frac{1}{r} \int_r^a \int_2^\pi \int_b^a 4D_0 \exp\{-\alpha\beta(r_1 + r_2 - 2b + b\theta_2)\} r_1 r_2 \, dr_2 d\theta_2 dr_1. \tag{25}
 \end{aligned}$$

Note that no traffic passes the toll area if both origin and destination are outside the toll area.

The traffic flow density on arc roads is obtained by considering the traffic passing the infinitesimal segment between two points  $(r, 0)$  and  $(r + dr, 0)$  along arc roads, as shown in Fig. 8. If  $0 \leq r < b$  (Fig. 8a),

$$f_a = r \int_{-2}^0 \int_r^a \int_0^{\theta_1+2} 4D_0 \exp[-\beta\{\alpha\{r_1 - r + r(\theta_2 - \theta_1)\} + t\}] r_1 \, d\theta_2 dr_1 d\theta_1. \tag{26}$$

If  $r = b, 0 \leq t < (\pi - 2)\alpha b$  (Fig. 8b),

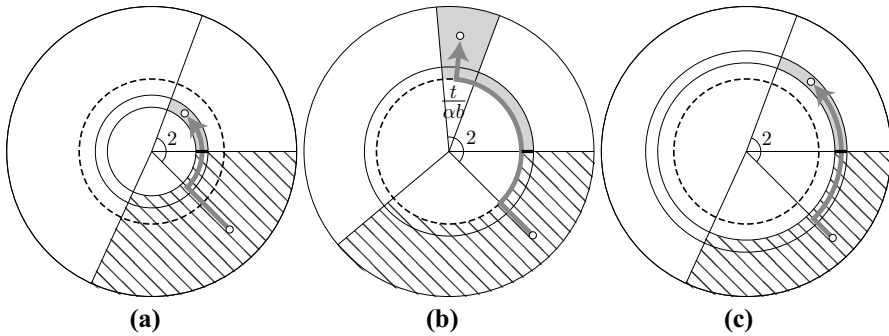


Fig. 8 Traffic flow on arc roads in road pricing

$$\begin{aligned}
 f_a = & b \int_{-2}^0 \int_b^a \int_0^{\theta_1+2} 4D_0 \exp[-\alpha\beta\{r_1 - b + b(\theta_2 - \theta_1)\}]r_1 d\theta_2 dr_1 d\theta_1 \\
 & + \int_{-t/(\alpha b)-2}^0 \int_b^a \int_{\theta_1+2}^{\theta_1+t/(\alpha b)+2} \int_b^a 2D_0 \exp[-\alpha\beta\{r_1 + r_2 - 2b + b(\theta_2 - \theta_1)\}]r_1 r_2 dr_2 d\theta_2 dr_1 d\theta_1.
 \end{aligned}
 \tag{27}$$

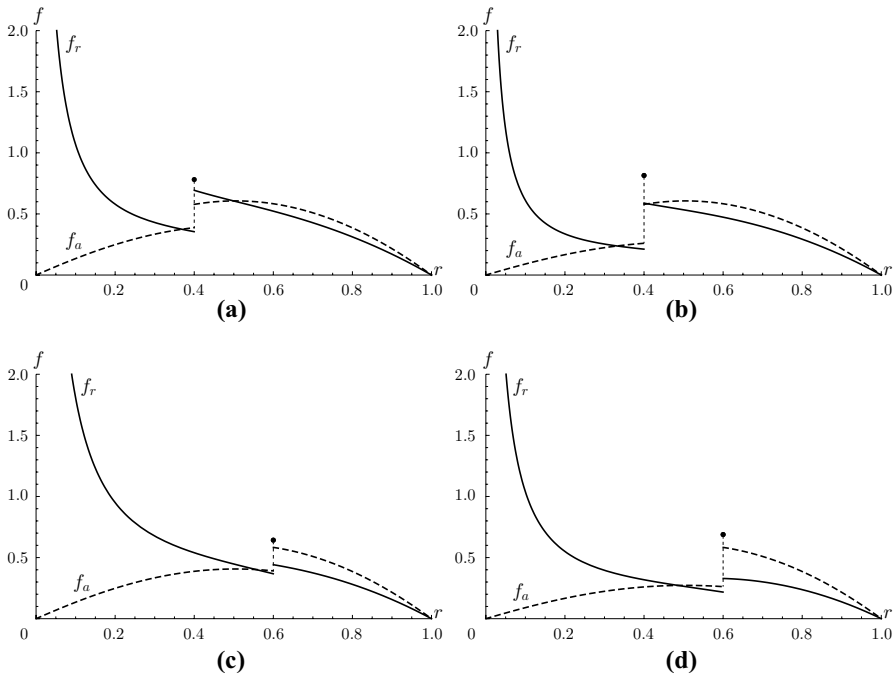
If  $r = b, t \geq (\pi - 2)\alpha b$ ,

$$\begin{aligned}
 f_a = & b \int_{-2}^0 \int_b^a \int_0^{\theta_1+2} 4D_0 \exp[-\alpha\beta\{r_1 - b + b(\theta_2 - \theta_1)\}]r_1 d\theta_2 dr_1 d\theta_1 \\
 & + \int_{-\pi}^0 \int_b^a \int_{\theta_1+2}^{\theta_1+\pi} \int_b^a 2D_0 \exp[-\alpha\beta\{r_1 + r_2 - 2b + b(\theta_2 - \theta_1)\}]r_1 r_2 dr_2 d\theta_2 dr_1 d\theta_1.
 \end{aligned}
 \tag{28}$$

Note that in the above two cases, the traffic making the detour around the toll area also passes the segment. If  $b < r \leq a$  (Fig. 8c),

$$f_a = r \int_{-2}^0 \int_r^a \int_0^{\theta_1+2} 4D_0 \exp[-\alpha\beta\{r_1 - r + r(\theta_2 - \theta_1)\}]r_1 d\theta_2 dr_1 d\theta_1. \tag{29}$$

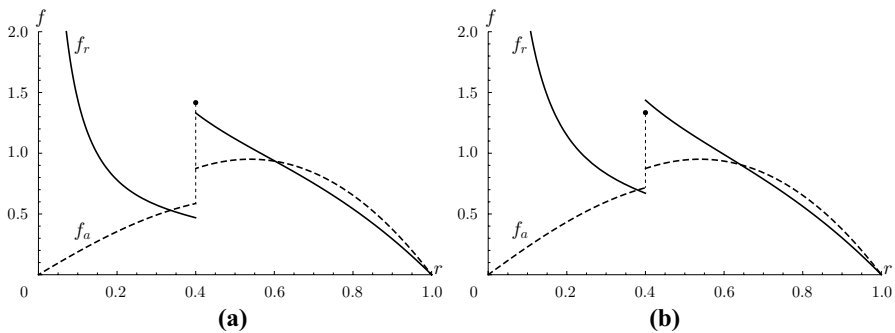
The traffic flow densities on radial and arc roads are shown in Fig. 9, where  $a = 1, D_0 = 1, \alpha = 1, \beta = 1$ . By comparing with Fig. 5a, we can see how road pricing affects the spatial distribution of traffic flow. The traffic flow density on radial roads decreases both inside and outside the toll area. The traffic flow density on arc roads, in contrast, decreases inside the toll area, increases at the boundary of the toll area, and is constant outside the toll area. The reason why the traffic flow density jumps at the boundary of the toll area is that some traffic makes a detour around the toll area, as shown in Fig. 6. As the toll level increases, the increase in traffic flow at the boundary of the toll area becomes greater. As the size of the toll area increases, the decrease in traffic flow inside the toll area becomes smaller. These findings help planners determine the size of the toll area and the toll level. For example, to reduce



**Fig. 9** Traffic flow density in road pricing: **a**  $b = 0.4, t = 0.4$ ; **b**  $b = 0.4, t = 0.8$ ; **c**  $b = 0.6, t = 0.4$ ; **d**  $b = 0.6, t = 0.8$

the amount of traffic flow near the city center, the toll area should be small and the toll level should be high. On the other hand, to ease the congestion caused by the increase in traffic flow at the boundary of the toll area, the toll should not be too high.

The effect of the travel cost on the traffic flow density is shown in Fig. 10, where  $a = 1, D_0 = 1, b = 0.4, t = 0.4$ . Comparing with Fig. 9a shows that reducing the traffic flow density inside the toll area is much easier when both the unit travel cost  $\alpha$



**Fig. 10** Traffic flow density in road pricing: **a**  $\alpha = 0.5, \beta = 1$ ; **b**  $\alpha = 1, \beta = 0.5$

and the elasticity of demand  $\beta$  are high. Note that if the unit travel cost is low, the increase in traffic flow density at the boundary of the toll area can be large. These findings also help planners assess the effectiveness of road pricing and identify the location of potential congestion areas.

## 5 Conclusions

This paper has developed a continuous approximation model for analyzing the effect of road pricing on the spatial distribution of traffic flow. The traffic flow density has been derived for a circular city with a radial-arc network. The model provides a fundamental understanding of the effect of road pricing, thus supplementing discrete network models for empirical analysis.

The model gives an insight into the design of road pricing systems as follows. First, the analytical expression for the traffic flow density demonstrates how the size of the toll area and the toll level affect the spatial distribution of traffic flow. As the size of the toll area increases, the decrease in traffic flow inside the toll area becomes smaller. As the toll level increases, the increase in traffic flow at the boundary of the toll area becomes greater. Note that finding these relationships by using discrete models requires the computation of traffic flow for various combinations of the parameters. These relationships help planners determine the size of the toll area and the toll level required to achieve a certain level of traffic congestion. Second, the model shows how the travel cost affects the spatial distribution of traffic flow. The amount of traffic flow inside the toll area can be easily reduced when the unit travel cost and the elasticity of demand are high. This result is useful for assessing the effectiveness of road pricing. Finally, the traffic flow density can be used to identify the location of potential congestion areas. Road pricing increases the amount of traffic flow at the boundary of the toll area. This should be considered when introducing road pricing and estimating sufficient road capacity to accommodate traffic flow.

Promising directions for future research are as follows. First, not only a radial-arc network but also a grid network has been frequently used in continuous approximation models. Second, cordon pricing in which vehicles passing or entering the toll area are charged a toll has also been implemented. Finally, the effect of road pricing on the total travel distance in the city should be examined.

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## References

1. Chen, Y., Zheng, N., Vu, H.L.: A novel urban congestion pricing scheme considering travel cost perception and level of service. *Transp. Res. C* **125**, 103042 (2021)
2. Gu, Z., Shafiei, S., Liu, Z., Saberi, M.: Optimal distance-and time-dependent area-based pricing with the network fundamental diagram. *Transp. Res. C* **95**, 1–28 (2018)
3. Kristoffersson, I.: Impacts of time-varying cordon pricing: validation and application of mesoscopic model for Stockholm. *Transp. Policy* **28**, 51–60 (2013)
4. Kurita, O.: Theory of road patterns for a circular disk city: distributions of Euclidean, recti-linear, circular-radial distances. *J. City Plann. Inst. Japan* **36**, 859–864 (2001). (in Japanese)
5. Li, Z.-C., Lam, W.H.K., Wong, S.C.: Modeling intermodal equilibrium for bimodal transportation system design problems in a linear monocentric city. *Transp. Res. B* **46**, 30–49 (2012)
6. Maruyama, T., Harata, N.: Difference between area-based and cordon-based congestion pricing: Investigation by trip-chain-based network equilibrium model with nonadditive path costs. *Transp. Res. Record* **1964**, 1–8 (2006)
7. Maruyama, T., Sumalee, A.: Efficiency and equity comparison of cordon- and area-based road pricing schemes using a trip-chain equilibrium model. *Transp. Res. A* **41**, 655–671 (2007)
8. May, A.D., Milne, D.S.: Effects of alternative road pricing systems on network performance. *Transp. Res. A* **34**, 407–436 (2000)
9. Miyagawa, M.: Cordon and area road pricing in radial-arc network. *J. Operat. Res. Soc. Japan* **62**, 121–131 (2019)
10. Mun, S., Konishi, K., Yoshikawa, K.: Optimal cordon pricing. *J. Urban Econ.* **54**, 21–38 (2003)
11. Mun, S., Konishi, K., Yoshikawa, K.: Optimal cordon pricing in a non-monocentric city. *Transp. Res. A* **39**, 723–736 (2005)
12. Roy, J.R.: *Spatial interaction modelling*. Springer-Verlag, Berlin (2010)
13. Simoni, M.D., Pel, A.J., Waraich, R.A., Hoogendoorn, S.P.: Marginal cost congestion pricing based on the network fundamental diagram. *Transp. Res. C* **56**, 221–238 (2015)
14. Sumalee, A.: Optimal road user charging cordon design: a heuristic optimization approach. *Comput-Aided Civil Infrastruct. Eng.* **19**, 377–392 (2004)
15. Suzuki, T., Miura, H.: Spatial distributions of flow density and crossing density and the dependency on routing system. *J. City Plann. Inst. Japan* **51**, 909–914 (2016)
16. Takaki, R., Maruyama, T., Mizokami, S.: The optimal area-based network congestion pricing problem: development and applications of algorithm for determining optimal toll level and charging boundary. *J. Japan Soc. Civil Eng.* **67**, 1233–1242 (2011)
17. Takaki, R., Maruyama, T., Mizokami, S.: Optimal network congestion pricing design problem controlling the shape of charging boundary: algorithm development and applications. *J. Japan Soc. Civil Eng.* **70**, 88–101 (2014)
18. Tanaka, K., Kurita, O.: Spatial distribution of traffic in a sector-shaped city under radial-arc routing system: an analysis on the optimal assignment of road area for designing a congestion-free city. *J. City Plann. Inst. Japan* **36**, 865–870 (2001)
19. Tanaka, K., Kurita, O.: Mathematical analysis of the radial-ring road network in the light of the traffic flow density: spatial and temporal distribution of traffic during the peak commuting period. *Trans. Japan Soc. Industr. Appl. Math.* **13**, 321–352 (2003)
20. Tsai, J.-F., Lu, S.-Y.: Reducing traffic externalities by multiple-cordon pricing. *Transportation* **45**, 597–622 (2018)
21. Vaughan, R.J.: *Urban spatial traffic patterns*. Pion, London (1987)
22. Verhoef, E.T.: Second-best congestion pricing schemes in the monocentric city. *J. Urban Econ.* **58**, 367–388 (2005)
23. Walters, A.A.: The theory and measurement of private and social cost of highway congestion. *Econometrica* **29**, 676–699 (1961)
24. Zhang, L., Liu, H., Sun, D.: Comparison and optimization of cordon and area pricings for managing travel demand. *Transport* **29**, 248–259 (2014)
25. Zhang, X., Yang, H.: The optimal cordon-based network congestion pricing problem. *Transp. Res. B* **38**, 517–537 (2004)
26. Zheng, N., Geroliminis, N.: Area-based equitable pricing strategies for multimodal urban networks with heterogeneous users. *Transp. Res. A* **136**, 357–374 (2020)

27. Zheng, N., Waraich, R.A., Axhausen, K.W., Geroliminis, N.: A dynamic cordon pricing scheme combining the macroscopic fundamental diagram and an agent-based traffic model. *Transp. Res. A* **46**, 1291–1303 (2012)
28. Zheng, N., R erat, G., Geroliminis, N.: Time-dependent area-based pricing for multimodal systems with heterogeneous users in an agent-based environment. *Transp. Res. C* **62**, 133–148 (2016)

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