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Area 2

A stable and structure-preserving scheme for a non-local Allen–Cahn equation

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Abstract
We propose a stable and structure-preserving finite difference scheme for a non-local Allen–Cahn equation which describes a process of phase separation in a binary mixture. The proposed scheme inherits characteristic properties, the conservation of mass and the decrease of the global energy from the equation. We show the stability and unique existence of the solution of the scheme. We also prove the error estimate for the scheme. Numerical experiments demonstrate the effectiveness of the proposed scheme.

Keywords Non-local Allen–Cahn equation · Discrete variational derivative method

Mathematics Subject Classification 65M06

1 Introduction

Allen and Cahn introduced the Allen–Cahn equation as a model for antiphase domain coarsening in a binary alloy [1]. It has been applied to various problems, for example, phase transition [1,8], image analysis [4,11,21] and motion by mean curvature [2,3,12,13,17,18,22].

Let $T > 0$ be a finite time, and let $L > 0$ be the length of the one-dimensional material. In this paper, we study the following initial-boundary value problem for a non-local Allen–Cahn equation introduced by Rubinstein and Sternberg [23]:

$$\begin{cases} u_t = u_{xx} + \dfrac{2u}{\varepsilon^2}(1-u^2) + \lambda^\varepsilon & \text{in } (0, L) \times (0, T), \\ \lambda^\varepsilon = -\dfrac{1}{L}\displaystyle\int_0^L \dfrac{2u}{\varepsilon^2}(1-u^2)dx & \text{in } (0, T), \end{cases} \tag{1}$$

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under the Neumann boundary conditions:

$$u_x(0,t) = u_x(L,t) = 0 \tag{2}$$

for all $t > 0$. The unknown function u is an order parameter, which is the concentration of one of two components in a binary mixture. The parameter $0 < \varepsilon \ll 1$ is related to the thickness of the interface layer which can develop in parts of the solution with a steep gradient.

Rubinstein and Sternberg introduced the Eq. (1) as a model for a process of phase separation in a binary mixture which conserves the total mass of two species [23]. They introduced the non-local term λ^ε, which is a Lagrange multiplier, to ensure the mass conservation (6). Here, we remark that the classical Allen–Cahn equation, in which the non-local term λ^ε in (1) is absent, does not have the mass conservation. Bronsard and Stoth proved that the Eq. (1) converges, as $\varepsilon \to 0$, to the volume preserving mean curvature flow in a radial symmetry case [7]. Golovaty obtained a similar result to [7] for the Allen–Cahn equation with a different non-local term [14]. Chen et al. [9] obtained the convergence in the general case. Moreover, the Eq. (1) has been studied analytically and numerically [5,6,10,24,26,27]. However, compared with the number of studies of the classical Allen-Cahn equation, there are not many numerical results of the non-local Allen–Chan equation.

Brassel and Bretin [6] concluded that the following another non-local Allen–Cahn equation:

$$\begin{cases} u_t = u_{xx} + \dfrac{2u}{\varepsilon^2}(1-u^2) + \dfrac{1}{\varepsilon^2}\tilde{\lambda}^\varepsilon(1-u^2) & \text{in } (0,L)\times(0,T), \\ \tilde{\lambda}^\varepsilon = \dfrac{-\displaystyle\int_0^L 2u(1-u^2)dx}{\displaystyle\int_0^L (1-u^2)dx} & \text{in } (0,T), \end{cases} \tag{3}$$

has better volume-preserving properties than (1) in the sense that an error for the conservation of the volume is smaller. However, as Takasao [25] mentioned, (3) does not have the dissipative property of J such as (7). Kim et al. [19] proposed a practically unconditionally stable scheme for (3), and yet they did not give the proof of the stability and the error estimate for the scheme. Zhai et al. [27] compared three methods to approximate (3), including the Crank–Nicolson (CN) finite difference method, the finite difference operator splitting (OS) method, and the Fourier spectral operator splitting (FSOS) method. They checked that the convergence rates of the CN scheme and the OS scheme approach second as the mesh size becomes small and that the FSOS scheme is second order accurate in time through numerical experiments. Nevertheless, Lee [20] commented that their proposed scheme are not second-order accurate in time and/or do not satisfy the conservation of mass. In addition, Lee [20] discretized (3) by a Fourier spectral method in space and first-, second-, third-order implicit–explicit Runge–Kutta schemes in time. Although he checked the convergence of the schemes, the convergence rate and that the schemes are first-, second-, third-order accurate in time respectively through numerical experiments, he did not give the proof of them.

Then, we propose a structure-preserving scheme for (1) based on the discrete variational derivative method (DVDM) proposed by Furihata [15,16]. Our proposed scheme inherits characteristic properties, the conservation of mass (6) and the decrease of the global energy (7) from the original equation, whereas DVDM scheme inherits just one property in general. Furthermore, we prove that the solution of the scheme converges to the one of the target equation in the sense of discrete L^2-norm and that the convergence rate is $O(\Delta x^2 + \Delta t^2)$. Moreover, we prove the stability of the scheme, the unique existence of the solution of the scheme. Also, based on this study, we expect that we can design a structure-preserving scheme for another non-local Allen–Chan equation such as (3) by using DVDM. Here, we remark that there are not that many results of the application of DVDM to partial differential equations (PDEs) with a non-local term to the best of our knowledge.

In this paper, as mentioned above, we design a finite difference scheme for (1) based on DVDM so that the scheme inherits the conservative and dissipative properties such as (6) and (7) from the original Eq. (1) in the discrete sense. Here, let us define the "local energy" G and the "global energy" J, which characterize the Eq. (1):

$$G(u, u_x) := \frac{|u_x|^2}{2} + \frac{1}{\varepsilon^2}\frac{(1-u^2)^2}{2}, \tag{4}$$

$$J(u) := \int_0^L G(u, u_x)dx. \tag{5}$$

Then, the Eq. (1) has following properties:

$$\frac{d}{dt}\int_0^L u dx = 0, \tag{6}$$

$$\frac{d}{dt}J(u) \leq 0. \tag{7}$$

DVDM is a numerical method for designing numerical schemes for PDEs with conservative and dissipative properties such as (6) and (7), and the DVDM schemes inherit conservative/dissipative property from the original PDEs in a discrete sense. From the perspective of numerical computation, the properties often lead us to stable computation. Hence, if the designed schemes retain the properties in a discrete sense, then the schemes are expected stable.

Also, the following property holds for the global energy J:

$$\frac{d}{dt}J(u) = \int_0^L \frac{\delta G}{\delta u} u_t dx \tag{8}$$

under the boundary conditions (2). The notation $\delta G/\delta u$ is the (first) variational derivative of G concerning u. From the integration by parts and the boundary conditions (2), we can show

$$\frac{d}{dt}J(u) = \int_0^L \left\{ -u_{xx} - \frac{2}{\varepsilon^2}u(1-u^2) \right\} u_t dx.$$

Therefore, we have

$$\frac{\delta G}{\delta u} = -u_{xx} - \frac{2}{\varepsilon^2}u(1-u^2) \tag{9}$$

from (8). We can rewrite (1) as follows by using (9):

$$u_t = -\frac{\delta G}{\delta u} + \lambda^\varepsilon \quad \text{in } (0,L)\times(0,T). \tag{10}$$

Furthermore,

$$\lambda^\varepsilon = -\frac{1}{L}\int_0^L \left(-\frac{\delta G}{\delta u} - u_{xx}\right)dx = \frac{1}{L}\left(\int_0^L \frac{\delta G}{\delta u}dx + [u_x]_0^L\right) = \frac{1}{L}\int_0^L \frac{\delta G}{\delta u}dx \quad \text{in } (0,T), \tag{11}$$

by the boundary conditions (2). Namely, we can rewrite (1) as

$$\begin{cases} u_t = -\dfrac{\delta G}{\delta u} + \lambda^\varepsilon & \text{in } (0,L)\times(0,T), \\ \lambda^\varepsilon = \dfrac{1}{L}\displaystyle\int_0^L \dfrac{\delta G}{\delta u}dx & \text{in } (0,T). \end{cases} \tag{12}$$

Therefore, we can use DVDM and prove the conservative property (6) and the dissipative property (7) easily. In fact,

$$\begin{aligned} \frac{d}{dt}\int_0^L u dx &= \int_0^L \left(-\frac{\delta G}{\delta u} + \lambda^\varepsilon\right)dx = -\int_0^L \frac{\delta G}{\delta u}dx + \lambda^\varepsilon \int_0^L dx \\ &= -\int_0^L \frac{\delta G}{\delta u}dx + \frac{1}{L}\int_0^L \frac{\delta G}{\delta u}dx \cdot L = 0, \end{aligned}$$

where we have used (10) in the first equality, and (11) in the third equality. Moreover, from (8), (10) and the conservation of mass (6), we can show

$$\begin{aligned} \frac{d}{dt}J(u) &= \int_0^L \frac{\delta G}{\delta u}u_t dx = \int_0^L \left(-u_t + \lambda^\varepsilon\right)u_t dx = -\int_0^L (u_t)^2 dx + \lambda^\varepsilon \int_0^L u_t dx \\ &= -\int_0^L (u_t)^2 dx \le 0. \end{aligned}$$

The rest of this paper proceeds as follows. In Sect. 2, we propose a finite difference scheme for (12), whose solution satisfies the discrete version of the conservation property (6) and the dissipative property (7). In Sect. 3, we prove that the solution of the proposed scheme satisfies the global boundedness. In Sect. 4, we prove that the scheme has a unique solution under a specific condition. In Sect. 5, we prove the error estimate for the scheme. In Sect. 6, we show that the numerical examples demonstrate the effectiveness of the scheme.

2 Proposed scheme

In this section, we propose a scheme for (12) and show that it has two properties corresponding to (6) and (7).

We define $U_k^{(m)}$ $(k = -1, 0, 1, \ldots, K, K+1, m = 0, 1, 2, \ldots)$ to be the approximation to $u(x, t)$ at location $x = k\Delta x$ and time $t = m\Delta t$, where Δx is a space mesh size, i.e., $\Delta x := L/K$ and Δt is a time mesh size. We define some basic operators, the shift operators s_k^+, s_k^-, the average operators μ_k^+, μ_k^- and the difference operators δ_k^+, δ_k^-, $\delta_k^{\langle 1\rangle}$, $\delta_k^{\langle 2\rangle}$ concerning subscript k.

$$\begin{aligned}
s_k^+ f_k &:= f_{k+1}, \quad s_k^- f_k := f_{k-1},\\
\mu_k^+ f_k &:= \frac{f_k + f_{k+1}}{2}, \quad \mu_k^- f_k := \frac{f_k + f_{k-1}}{2},\\
\delta_k^+ f_k &:= \frac{f_{k+1} - f_k}{\Delta x}, \quad \delta_k^- f_k := \frac{f_k - f_{k-1}}{\Delta x},\\
\delta_k^{\langle 1\rangle} f_k &:= \frac{f_{k+1} - f_{k-1}}{2\Delta x}, \quad \delta_k^{\langle 2\rangle} f_k := \frac{f_{k+1} - 2f_k + f_{k-1}}{\Delta x^2}
\end{aligned}$$

for all $\{f_k\}_{k=0}^K \in \mathbb{R}^{K+1}$. As a discretization of the integral, we adopt the summation operator $\sum_{k=0}^{K}{}''$ defined by

$$\sum_{k=0}^{K}{}'' f_k := \frac{1}{2} f_0 + \sum_{n=1}^{K-1} f_k + \frac{1}{2} f_K \quad \text{for all } \{f_k\}_{k=0}^K \in \mathbb{R}^{K+1}.$$

The concrete form of the proposed scheme for (12) is, for $m = 0, 1, \ldots,$

$$\begin{cases}
\dfrac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} = -\dfrac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k} + \lambda_{\mathrm{d}}^{\varepsilon}\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right) \quad (k=0,\ldots,K),\\
\lambda_{\mathrm{d}}^{\varepsilon}\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right) = \dfrac{1}{L}\displaystyle\sum_{k=0}^{K}{}'' \dfrac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k}\Delta x,
\end{cases} \tag{13}$$

where

$$\frac{\delta G_{\mathrm{d}}}{\delta(\boldsymbol{U}, \boldsymbol{V})_k} = -\delta_k^{\langle 2\rangle}\left(\frac{U_k + V_k}{2}\right) - \frac{2}{\varepsilon^2}\left(\frac{U_k + V_k}{2}\right)\left(1 - \frac{U_k^2 + V_k^2}{2}\right) \quad (k = 0, \ldots, K). \tag{14}$$

The discrete boundary conditions are

$$\delta_k^{\langle 1\rangle} U_k^{(m)} = 0 \quad (k = 0, K, m = 0, 1, \ldots). \tag{15}$$

Note that the discrete boundary conditions (15) mean

$$U_{-1}^{(m)} = U_{1}^{(m)}, \quad U_{K+1}^{(m)} = U_{K-1}^{(m)} \quad (m = 0, 1, \ldots).$$

Let us define a discrete local energy $G_{\mathrm{d}} \colon \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$ by

$$G_{\mathrm{d},k}(\boldsymbol{U}) := \frac{1}{2}\frac{(\delta_k^{+}U_k)^2 + (\delta_k^{-}U_k)^2}{2} + \frac{(1-U_k^2)^2}{2\varepsilon^2} \quad (k = 0, \ldots, K). \tag{16}$$

The relation between G_{d} of (16) and $\delta G_{\mathrm{d}}/\delta(\cdot,\cdot)$ of (14) is given by

$$\begin{aligned}
&\sum_{k=0}^{K}{}'' G_{\mathrm{d},k}(\boldsymbol{U})\, \Delta x - \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}(\boldsymbol{V})\, \Delta x \\
&\quad = \sum_{k=0}^{K}{}'' \frac{\delta G_{\mathrm{d}}}{\delta(\boldsymbol{U},\boldsymbol{V})_k} (U_k - V_k)\, \Delta x \\
&\qquad + \left[\frac{\delta_k^{+}\left(\frac{U_k+V_k}{2}\right)\mu_k^{+}(U_k - V_k) + \delta_k^{-}\left(\frac{U_k+V_k}{2}\right)\mu_k^{-}(U_k - V_k)}{2} \right]_0^K .
\end{aligned}$$

In the calculation above, we have used the following general identity (second-order summation by parts formula):

$$\begin{aligned}
&\sum_{k=0}^{K}{}'' \frac{\left(\delta_k^{+} f_k\right)\left(\delta_k^{+} g_k\right) + \left(\delta_k^{-} f_k\right)\left(\delta_k^{-} g_k\right)}{2}\Delta x \\
&\quad = -\sum_{k=0}^{K}{}'' \left(\delta_k^{\langle 2\rangle} f_k\right) g_k \Delta x + \left[\frac{\left(\delta_k^{+} f_k\right)\left(\mu_k^{+} g_k\right) + \left(\delta_k^{-} f_k\right)\left(\mu_k^{-} g_k\right)}{2} \right]_0^K
\end{aligned} \tag{17}$$

for all $\{f_k\}_{k=0}^{K}, \{g_k\}_{k=0}^{K} \in \mathbb{R}^{K+1}$. Hence,

$$\begin{aligned}
&\sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m+1)}\right)\Delta x - \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m)}\right)\Delta x \\
&\quad = \sum_{k=0}^{K}{}'' \frac{\delta G_{\mathrm{d}}}{\delta(\boldsymbol{U}^{(m+1)},\boldsymbol{U}^{(m)})_k}\left(U_k^{(m+1)} - U_k^{(m)}\right)\Delta x \\
&\qquad + \frac{1}{4}\Big[\delta_k^{+}\left(U_k^{(m+1)} + U_k^{(m)}\right)\mu_k^{+}\left(U_k^{(m+1)} - U_k^{(m)}\right) \\
&\qquad + \delta_k^{-}\left(U_k^{(m+1)} + U_k^{(m)}\right)\mu_k^{-}\left(U_k^{(m+1)} - U_k^{(m)}\right)\Big]_0^K
\end{aligned}$$

for $m = 0, 1, \ldots$. Here, we show

$$\Big[\delta_k^+ \left(U_k^{(m+1)} + U_k^{(m)}\right) \mu_k^+ \left(U_k^{(m+1)} - U_k^{(m)}\right) \\ +\delta_k^- \left(U_k^{(m+1)} + U_k^{(m)}\right) \mu_k^- \left(U_k^{(m+1)} - U_k^{(m)}\right)\Big]_0^K = 0 \quad (m = 0, 1, \ldots). \tag{18}$$

Since

$$\left(\frac{\delta_k^+ + \delta_k^-}{2}\right) U_k^{(m)} = \delta_k^{\langle 1\rangle} U_k^{(m)} = 0 \quad (m = 0, 1, \ldots)$$

from the discrete boundary conditions (15), $\delta_k^+ U_k^{(m)} = -\delta_k^- U_k^{(m)}$ $(m = 0, 1, \ldots)$. Namely, $\delta_k^+ (U_k^{(m+1)} + U_k^{(m)}) = -\delta_k^- (U_k^{(m+1)} + U_k^{(m)})$ $(m = 0, 1, \ldots)$. Furthermore,

$$\left(\frac{\mu_k^+ - \mu_k^-}{\Delta x}\right) U_k^{(m)} = \delta_k^{\langle 1\rangle} U_k^{(m)} = 0 \quad (m = 0, 1, \ldots),$$

since

$$\frac{\mu_k^+ - \mu_k^-}{\Delta x} = \frac{1 + s_k^+ - (1 - s_k^-)}{2\Delta x} = \frac{s_k^+ - s_k^-}{2\Delta x} = \delta_k^{\langle 1\rangle}.$$

That is, $\mu_k^+ U_k^{(m)} = \mu_k^- U_k^{(m)}$, i.e., $\mu_k^+ (U_k^{(m+1)} + U_k^{(m)}) = \mu_k^- (U_k^{(m+1)} + U_k^{(m)})$ for $m = 0, 1, \ldots$. Hence, (18) holds. Therefore,

$$\sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m+1)}\right) \Delta x - \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m)}\right) \Delta x \\ = \sum_{k=0}^{K}{}'' \frac{\delta G_{\mathrm{d}}}{\delta(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)})_k} \left(U_k^{(m+1)} - U_k^{(m)}\right) \Delta x$$

for $m = 0, 1, \ldots$ The proposed scheme (13) has properties corresponding to (6) and (7), i.e.,

Theorem 1 *The solution of the scheme* (13) *under the discrete boundary conditions* (15) *satisfies the following equality and inequality.*

$$\sum_{k=0}^{K}{}'' U_k^{(m)} \Delta x = \sum_{k=0}^{K}{}'' U_k^{(0)} \Delta x \quad (m = 0, 1, \ldots), \tag{19}$$

$$J_{\mathrm{d}}\left(\boldsymbol{U}^{(m+1)}\right) \le J_{\mathrm{d}}\left(\boldsymbol{U}^{(m)}\right) \quad (m = 0, 1, \ldots), \tag{20}$$

where

$$J_{\mathrm{d}}\left(\boldsymbol{U}^{(m)}\right) := \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m)}\right)\Delta x$$

for $m = 0, 1, \ldots$

We call (19) the discrete conservation of mass and (20) the discrete decrease of the total energy.

Proof First, we can show the discrete conservation of mass (19) as follows:

$$\begin{aligned}
&\frac{1}{\Delta t}\left\{\sum_{k=0}^{K}{}'' U_k^{(m+1)}\Delta x - \sum_{k=0}^{K}{}'' U_k^{(m)}\Delta x\right\} \\
&= \sum_{k=0}^{K}{}''\left\{-\frac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k} + \lambda_{\mathrm{d}}^{\varepsilon}\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)\right\}\Delta x \\
&= -\sum_{k=0}^{K}{}''\frac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k}\Delta x + \lambda_{\mathrm{d}}^{\varepsilon}\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)\sum_{k=0}^{K}{}''\Delta x \\
&= -\sum_{k=0}^{K}{}''\frac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k}\Delta x + \frac{1}{L}\sum_{k=0}^{K}{}''\frac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k}\Delta x \cdot L = 0
\end{aligned}$$

for $m = 0, 1, \ldots$.

Next, the discrete decrease of the global energy (20) can be shown as

$$\begin{aligned}
&\frac{J_{\mathrm{d}}(\boldsymbol{U}^{(m+1)}) - J_{\mathrm{d}}(\boldsymbol{U}^{(m)})}{\Delta t} \\
&= \frac{1}{\Delta t}\sum_{k=0}^{K}{}''\left\{G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m+1)}\right) - G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m)}\right)\right\}\Delta x \\
&= \sum_{k=0}^{K}{}''\frac{\delta G_{\mathrm{d}}}{\delta\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)_k}\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}\Delta x \\
&= \sum_{k=0}^{K}{}''\left\{-\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} + \lambda_{\mathrm{d}}^{\varepsilon}\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)\right\}\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}\Delta x \\
&= -\sum_{k=0}^{K}{}''\left(\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}\right)^2\Delta x + \lambda_{\mathrm{d}}^{\varepsilon}\left(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}\right)\sum_{k=0}^{K}{}''\left(\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}\right)\Delta x \\
&= -\sum_{k=0}^{K}{}''\left(\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}\right)^2\Delta x \le 0 \quad (m = 0, 1, \ldots),
\end{aligned}$$

where we have used (19) in the fifth equality. □

3 Stability of the proposed scheme

In this section, we show that, if the proposed scheme has a solution, then the maximum norm of it is bounded. The proof consists of two lemmas. The first lemma shows that the discrete Sobolev norm of the solution of the proposed scheme is bounded. The second (the discrete Sobolev lemma) shows that if the discrete Sobolev norm of a discrete function is bounded, then the maximum norm of the function is bounded.

Lemma 1 *The solution of the scheme* (13) *under the discrete boundary conditions* (15) *satisfies the following inequality.*

$$\left\|\boldsymbol{U}^{(m)}\right\|_{H^1}^2 \leq \frac{1}{\min\left\{\dfrac{2}{\varepsilon^2}, \dfrac{1}{2}\right\}}\left\{\sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(0)}\right)\Delta x + \frac{4}{\varepsilon^2}L\right\} \quad (m = 0, 1, \ldots), \quad (21)$$

where $\|\cdot\|_{H^1}$ *is a discrete Sobolev norm which is defined as*

$$\|\boldsymbol{f}\|_{H^1} := \left(\sum_{k=0}^{K}{}''|f_k|^2\Delta x + \sum_{k=0}^{K-1}|\delta_k^+ f_k|^2\Delta x\right)^{\frac{1}{2}}$$

for all $\boldsymbol{f} = \{f_k\}_{k=0}^{K} \in \mathbb{R}^{K+1}$.

Proof From the decrease of the global energy (20), we can show

$$\begin{aligned}
&\sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(0)}\right)\Delta x \geq \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m)}\right)\Delta x\\
&= \sum_{k=0}^{K}{}''\left\{\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon^2}\left(U_k^{(m)}\right)^2 + \frac{1}{2}\cdot\frac{1}{\varepsilon^2}\cdot\left(U_k^{(m)}\right)^4 + \frac{1}{2}\frac{\left(\delta_k^+ U_k^{(m)}\right)^2 + \left(\delta_k^- U_k^{(m)}\right)^2}{2}\right\}\Delta x\\
&\geq \sum_{k=0}^{K}{}''\left\{\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon^2}\left(U_k^{(m)}\right)^2 - \frac{9}{2}\cdot\frac{1}{\varepsilon^2} + \frac{1}{2}\frac{\left(\delta_k^+ U_k^{(m)}\right)^2 + \left(\delta_k^- U_k^{(m)}\right)^2}{2}\right\}\Delta x\\
&\geq \min\left\{\frac{2}{\varepsilon^2}, \frac{1}{2}\right\}\sum_{k=0}^{K}{}''\left\{\left(U_k^{(m)}\right)^2 + \frac{\left(\delta_k^+ U_k^{(m)}\right)^2 + \left(\delta_k^- U_k^{(m)}\right)^2}{2}\right\}\Delta x + \left(\frac{1}{2\varepsilon^2} - \frac{9}{2\varepsilon^2}\right)\cdot L\\
&= \min\left\{\frac{2}{\varepsilon^2}, \frac{1}{2}\right\}\left\|\boldsymbol{U}^{(m)}\right\|_{H^1}^2 - \frac{4}{\varepsilon^2}L \quad (m = 0, 1, \ldots),
\end{aligned}$$

where we have used the following inequality:

$$-rY^2 + \frac{1}{2}rY^4 \geq 2rY^2 - \frac{9}{2}r$$

for all $Y \in \mathbb{R}$ and $r > 0$, in the second inequality, and the following equality in the last equality.

$$\sum_{k=0}^{K}{}'' \frac{\left(\delta_k^+ U_k^{(m)}\right)^2 + \left(\delta_k^- U_k^{(m)}\right)^2}{2} \Delta x = \sum_{k=0}^{K-1} \left(\delta_k^+ U_k^{(m)}\right)^2 \Delta x \tag{22}$$

In fact, we show the equality (22) by using the discrete boundary conditions (15). Therefore, (21) holds. □

Lemma 2 (Discrete Sobolev Lemma)

$$\max_{0 \le k \le K} |f_k| \le 2 \max \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{L}{2}} \right\} \|\boldsymbol{f}\|_{H^1} \quad \text{for all } \boldsymbol{f} = \{f_k\}_{k=0}^K \in \mathbb{R}^{K+1}. \tag{23}$$

Proof We can obtain the inequality (23) from the proof by Furihata and Matsuo [16]. □

Applying Lemma 2 to (21), we can obtain the following inequality.

Theorem 2 *The solution of the scheme* (13) *under the discrete boundary conditions* (15) *satisfies*

$$\max_{0 \le k \le K} \left| U_k^{(m)} \right| \le 2 \left[\frac{\max\left\{ \frac{1}{L}, \frac{L}{2} \right\}}{\min\left\{ \frac{2}{\varepsilon^2}, \frac{1}{2} \right\}} \left\{ \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(0)}\right) \Delta x + \frac{4}{\varepsilon^2} L \right\} \right]^{\frac{1}{2}} \quad (m = 0, 1, \ldots).$$

4 Unique existence of the solution of the proposed scheme

In this section, we prove, through the fixed-point theorem for a contraction mapping, that the proposed scheme (13) has a unique solution under a specific condition on Δt and Δx.

To prove the unique existence of the solution of the proposed scheme, we rewrite the scheme (13) as follows:

$$\begin{aligned}
&\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} \\
&= \delta_k^{\langle 2 \rangle} \left(\frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) + \frac{1}{\varepsilon^2} \left(U_k^{(m+1)} + U_k^{(m)} \right) \left\{ 1 - \frac{\left(U_k^{(m+1)}\right)^2 + \left(U_k^{(m)}\right)^2}{2} \right\} \\
&\quad - \frac{1}{L} \sum_{k=0}^{K}{}'' \left[\delta_k^{\langle 2 \rangle} \left(\frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) + \frac{1}{\varepsilon^2} \left(U_k^{(m+1)} + U_k^{(m)} \right) \left\{ 1 - \frac{\left(U_k^{(m+1)}\right)^2 + \left(U_k^{(m)}\right)^2}{2} \right\} \right] \Delta x
\end{aligned}$$

$$
\begin{aligned}
&= \delta_k^{\langle 2\rangle}\left(\frac{U_k^{(m+1)}+U_k^{(m)}}{2}\right)+\frac{1}{\varepsilon^2}\left(U_k^{(m+1)}+U_k^{(m)}\right)\left\{1-\frac{\left(U_k^{(m+1)}\right)^2+\left(U_k^{(m)}\right)^2}{2}\right\}\\
&\quad -\frac{1}{L}\left[\delta_k^{\langle 1\rangle}\left(\frac{U_k^{(m+1)}+U_k^{(m)}}{2}\right)\right]_0^K\\
&\quad -\frac{1}{L}\sum_{k=0}^{K}{}''\left[\frac{1}{\varepsilon^2}\left(U_k^{(m+1)}+U_k^{(m)}\right)\left\{1-\frac{\left(U_k^{(m+1)}\right)^2+\left(U_k^{(m)}\right)^2}{2}\right\}\right]\Delta x\\
&= \delta_k^{\langle 2\rangle}\left(\frac{U_k^{(m+1)}+U_k^{(m)}}{2}\right)\\
&\quad +\frac{1}{\varepsilon^2}\left(U_k^{(m+1)}+U_k^{(m)}\right)-\frac{1}{2\varepsilon^2}\left(U_k^{(m+1)}+U_k^{(m)}\right)\left\{\left(U_k^{(m+1)}\right)^2+\left(U_k^{(m)}\right)^2\right\}\\
&\quad -\frac{1}{L}\sum_{k=0}^{K}{}''\left[\frac{1}{\varepsilon^2}\left(U_k^{(m+1)}+U_k^{(m)}\right)-\frac{1}{2\varepsilon^2}\left(U_k^{(m+1)}+U_k^{(m)}\right)\left\{\left(U_k^{(m+1)}\right)^2+\left(U_k^{(m)}\right)^2\right\}\right]\Delta x,
\end{aligned}
$$

where we have used the following general identity (summation of a difference) in the second equality.

$$
\sum_{k=0}^{K}{}''\delta_k^{\langle 2\rangle} f_k \Delta x = \left[\delta_k^{\langle 1\rangle} f_k\right]_0^K
$$

for all $\{f_k\}_{k=0}^K \in \mathbb{R}^{K+1}$. Namely,

$$
\begin{aligned}
&\frac{1}{\Delta t}U_k^{(m+1)}-\frac{1}{2}\delta_k^{\langle 2\rangle}U_k^{(m+1)}\\
&\quad=\frac{1}{\Delta t}U_k^{(m)}+\frac{1}{2}\delta_k^{\langle 2\rangle}U_k^{(m)}+\frac{1}{\varepsilon^2}U_k^{(m+1)}+\frac{1}{\varepsilon^2}U_k^{(m)}+\left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{U}^{(m+1)}\right\}_k\\
&\qquad -\frac{1}{L\varepsilon^2}\sum_{k=0}^{K}{}''U_k^{(m+1)}\Delta x-\frac{1}{L\varepsilon^2}\sum_{k=0}^{K}{}''U_k^{(m)}\Delta x-\frac{1}{L}\sum_{k=0}^{K}{}''\left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{U}^{(m+1)}\right\}_k\Delta x,
\end{aligned}
$$

where the mapping $F_{\boldsymbol{U}^{(m)}}\colon \mathbb{R}^{K+1}\to\mathbb{R}^{K+1}$ is defined as

$$
\left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\}_k := -\frac{1}{2\varepsilon}\left(V_k+U_k^{(m)}\right)\left\{(V_k)^2+\left(U_k^{(m)}\right)^2\right\}\quad (k=0,\ldots,K)
$$

for all $\boldsymbol{V} = \{V_k\}_{k=0}^{K} \in \mathbb{R}^{K+1}$, i.e.,

$$\left(\frac{1}{\Delta t} - \frac{1}{2}\delta_k^{\langle 2\rangle}\right)U_k^{(m+1)} = \left(\frac{1}{\Delta t} + \frac{1}{\varepsilon^2} + \frac{1}{2}\delta_k^{\langle 2\rangle}\right)U_k^{(m)} - \frac{1}{L\varepsilon^2}\sum_{k=0}^{K}{}'' U_k^{(m)}\Delta x + \frac{1}{\varepsilon^2}\left(U_k^{(m+1)} - \frac{1}{L}\sum_{k=0}^{K}{}'' U_k^{(m+1)}\Delta x\right) + \left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{U}^{(m+1)}\right\}_k - \frac{1}{L}\sum_{k=0}^{K}{}'' \left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{U}^{(m+1)}\right\}_k \Delta x. \tag{24}$$

In connection with the scheme (24), we define a mapping $\mathcal{T}_{\boldsymbol{U}^{(m)}} \colon \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$ using the following equation:

$$\left(\frac{1}{\Delta t} - \frac{1}{2}\delta_k^{\langle 2\rangle}\right)\left\{\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\}_k = \left(\frac{1}{\Delta t} + \frac{1}{\varepsilon^2} + \frac{1}{2}\delta_k^{\langle 2\rangle}\right)U_k^{(m)} - \frac{1}{L\varepsilon^2}\sum_{k=0}^{K}{}'' U_k^{(m)}\Delta x + \frac{1}{\varepsilon^2}\left(V_k - \frac{1}{L}\sum_{k=0}^{K}{}'' V_k\Delta x\right) + \left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\}_k - \frac{1}{L}\sum_{k=0}^{K}{}'' \left\{F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\}_k \Delta x \quad (k = 0, \ldots, K)$$

for all $\boldsymbol{V} = \{V_k\}_{k=0}^{K} \in \mathbb{R}^{K+1}$. Here, the operator in the equation above is defined under the discrete boundary conditions (15), i.e., $\{\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\}_{-1} = \{\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\}_1$, $\{\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\}_{K+1} = \{\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\}_{K-1}$, $U_{-1}^{(m)} = U_1^{(m)}$, and $U_{K+1}^{(m)} = U_{K-1}^{(m)}$. If the mapping $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ has a fixed-point $\boldsymbol{V}^*$, then $\boldsymbol{V}^*$ is the solution $\boldsymbol{U}^{(m+1)}$ of the proposed scheme (13) under the discrete boundary condition (15).

The matrix expression of $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ is given by

$$\left(\frac{1}{\Delta t}I - \frac{1}{2}D_2\right)\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V} = \left\{\left(\frac{1}{\Delta t} + \frac{1}{\varepsilon^2}\right)I + \frac{1}{2}D_2\right\}\boldsymbol{U}^{(m)} - \frac{1}{L\varepsilon^2}S\boldsymbol{U}^{(m)} + \frac{1}{\varepsilon^2}\left(I - \frac{1}{L}S\right)\boldsymbol{V} + \left(I - \frac{1}{L}S\right)F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}$$

for all $\boldsymbol{V} \in \mathbb{R}^{K+1}$, where I is the identity matrix of order $K+1$, further, S and D_2 are square matrices of order $K+1$ as

$$S := \Delta x \begin{pmatrix} \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \end{pmatrix},$$

$$D_2 := \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 2 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 2 & -2 \end{pmatrix}$$

under the discrete boundary conditions (15).

The following lemma implies that the mapping $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ is well-defined for all $\boldsymbol{U}^{(m)} \in \mathbb{R}^{K+1}$.

Lemma 3 *The* $(K+1) \times (K+1)$ *matrix* $(1/\Delta t)I - (1/2)D_2$ *is nonsingular.*

Proof Eigenvalues of D_2 are

$$\lambda_k := \frac{2}{\Delta x^2} \left\{ \cos\left(\frac{k}{K}\pi\right) - 1 \right\} \quad (k = 0, \ldots, K), \tag{25}$$

and the eigenvector $\boldsymbol{x}_k$ corresponding to the eigenvalue λ_k is

$$\boldsymbol{x}_k = \left(\cos\left(\frac{0 \cdot k}{K}\pi\right), \cos\left(\frac{1 \cdot k}{K}\pi\right), \ldots, \cos\left(\frac{K \cdot k}{K}\pi\right) \right)^\top \quad (k = 0, \ldots, K). \tag{26}$$

Since $D_2 \boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k$ $(k = 0, \ldots, K)$,

$$\left(\frac{1}{\Delta t} I - \frac{1}{2} D_2 \right) \boldsymbol{x}_k = \frac{1}{\Delta t} \boldsymbol{x}_k - \frac{1}{2} D_2 \boldsymbol{x}_k = \frac{1}{\Delta t} \boldsymbol{x}_k - \frac{1}{2} \lambda_k \boldsymbol{x}_k = \left(\frac{1}{\Delta t} - \frac{1}{2} \lambda_k \right) \boldsymbol{x}_k$$

for $k = 0, \ldots, K$. Hence, eigenvalues of $(1/\Delta t)I - (1/2)D_2$ are

$$\frac{1}{\Delta t} - \frac{1}{2}\lambda_k = \frac{1}{\Delta t} + \frac{1}{\Delta x^2} \left\{ 1 - \cos\left(\frac{k}{K}\pi\right) \right\} \geq \frac{1}{\Delta t} > 0 \quad (k = 0, \ldots, K).$$

Therefore, the positiveness of the eigenvalues implies the nonsingularity of $(1/\Delta t)I - (1/2)D_2$. □

Next, we prove the existence and uniqueness of the solution of the proposed scheme by the fixed-point theorem for a contraction mapping.

Theorem 3 *If*

$$\Delta t < \frac{L\varepsilon^2 \Delta x}{\left(L+\sqrt{(K+1)(2K-1)}\Delta x\right)} \min\left\{\frac{1}{9\left(\Delta x+65M_{\mathrm{d}}^2\right)}, \frac{1}{4\left(\Delta x+209M_{\mathrm{d}}^2\right)}\right\}, \quad (27)$$

then the mapping $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ *has a unique fixed-point in the closed ball* B*, where*

$$\begin{aligned} M_{\mathrm{d}} &:= \left\|\boldsymbol{U}^{(m)}\right\|_{L_{\mathrm{d}}^2}, \\ B &:= \left\{\boldsymbol{v} \in \mathbb{R}^{K+1};\ \|\boldsymbol{v}\|_{L_{\mathrm{d}}^2} \le 8M_{\mathrm{d}}\right\}, \\ \|\boldsymbol{v}\|_{L_{\mathrm{d}}^2} &:= \sqrt{\sum_{k=0}^{K}{}'' |v_k|^2 \Delta x}. \end{aligned}$$

Proof By the fixed-point theorem for a contraction mapping, it suffices to show that $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ is a contraction mapping on B.

First, we prove that $\mathcal{T}_{\boldsymbol{U}^{(m)}} B \subseteq B$. By Lemma 3, we have

$$\begin{aligned} \mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V} &= \left(\frac{1}{\Delta t}I - \frac{1}{2}D_2\right)^{-1}\left\{\left(\frac{1}{\Delta t}+\frac{1}{\varepsilon^2}\right)I + \frac{1}{2}D_2\right\}\boldsymbol{U}^{(m)} \\ &\quad + \left(\frac{1}{\Delta t}I - \frac{1}{2}D_2\right)^{-1}\left\{-\frac{1}{L\varepsilon^2}S\boldsymbol{U}^{(m)} + \left(I-\frac{1}{L}S\right)\left(\frac{1}{\varepsilon^2}\boldsymbol{V} + F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right)\right\} \end{aligned}$$

for all $\boldsymbol{V} \in B$. We diagonalize the matrix D_2 as

$$D_2 = X\Lambda X^{-1},$$

where X and Λ are square matrices of order $K+1$ as

$$\begin{aligned} X &:= (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_K), \\ \Lambda &:= \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_K), \end{aligned}$$

with $\boldsymbol{x}_k$ given by (26) and λ_k given by (25). Since $I = XX^{-1} = XIX^{-1}$,

$$\frac{1}{\Delta t}I - \frac{1}{2}D_2 = \frac{1}{\Delta t}XIX^{-1} - \frac{1}{2}X\Lambda X^{-1} = X\left(\frac{1}{\Delta t}I - \frac{1}{2}\Lambda\right)X^{-1}. \quad (28)$$

Similarly,

$$\left(\frac{1}{\Delta t}+\frac{1}{\varepsilon^2}\right)I + \frac{1}{2}D_2 = X\left\{\left(\frac{1}{\Delta t}+\frac{1}{\varepsilon^2}\right)I + \frac{1}{2}\Lambda\right\}X^{-1}.$$

By (28), we have

$$\left(\frac{1}{\Delta t}I-\frac{1}{2}D_2\right)^{-1}=\left(X^{-1}\right)^{-1}\left(X\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)\right)^{-1}=X\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)^{-1}X^{-1}.$$

Then, the matrix expression of $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ is given by

$$\begin{aligned}\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}&=X\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)^{-1}\left\{\left(\frac{1}{\Delta t}+\frac{1}{\varepsilon^2}\right)I+\frac{1}{2}\Lambda\right\}X^{-1}\boldsymbol{U}^{(m)}\\&\quad+X\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)^{-1}X^{-1}\left\{-\frac{1}{L\varepsilon^2}S\boldsymbol{U}^{(m)}+\left(I-\frac{1}{L}S\right)\left(\frac{1}{\varepsilon^2}\boldsymbol{V}+F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right)\right\}\end{aligned}\tag{29}$$

for all $\boldsymbol{V}\in B$. Hence,

$$\begin{aligned}&\left\|\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\|_{L^2_{\mathrm{d}}}\\&\le\|X\|_{L^2_{\mathrm{d}}}\left\|\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)^{-1}\left\{\left(\frac{1}{\Delta t}+\frac{1}{\varepsilon^2}\right)I+\frac{1}{2}\Lambda\right\}\right\|_{L^2_{\mathrm{d}}}\left\|X^{-1}\right\|_{L^2_{\mathrm{d}}}\left\|\boldsymbol{U}^{(m)}\right\|_{L^2_{\mathrm{d}}}\\&\quad+\|X\|_{L^2_{\mathrm{d}}}\left\|\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)^{-1}\right\|_{L^2_{\mathrm{d}}}\left\|X^{-1}\right\|_{L^2_{\mathrm{d}}}\\&\quad\cdot\left\{\frac{1}{L\varepsilon^2}\|S\|_{L^2_{\mathrm{d}}}\left\|\boldsymbol{U}^{(m)}\right\|_{L^2_{\mathrm{d}}}+\left\|I-\frac{1}{L}S\right\|_{L^2_{\mathrm{d}}}\left(\frac{1}{\varepsilon^2}\|\boldsymbol{V}\|_{L^2_{\mathrm{d}}}+\left\|F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\|_{L^2_{\mathrm{d}}}\right)\right\}\\&\le4\left(1+\frac{\Delta t}{\varepsilon^2}\right)\left\|\boldsymbol{U}^{(m)}\right\|_{L^2_{\mathrm{d}}}+4\Delta t\left\{\frac{1}{L\varepsilon^2}\sqrt{(K+1)(2K-1)}\Delta x\left\|\boldsymbol{U}^{(m)}\right\|_{L^2_{\mathrm{d}}}\right.\\&\quad\left.+\left(1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right)\left(\frac{1}{\varepsilon^2}\|\boldsymbol{V}\|_{L^2_{\mathrm{d}}}+\left\|F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\|_{L^2_{\mathrm{d}}}\right)\right\}\\&\le4\left(1+\frac{\Delta t}{\varepsilon^2}\right)M_{\mathrm{d}}+4\Delta t\left\{\frac{1}{L\varepsilon^2}\sqrt{(K+1)(2K-1)}\Delta x\cdot M_{\mathrm{d}}\right.\\&\quad\left.+\left(1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right)\left(\frac{8}{\varepsilon^2}M_{\mathrm{d}}+\frac{585}{\varepsilon^2\Delta x}M_{\mathrm{d}}^3\right)\right\}\\&=\left[\left(1+\frac{\Delta t}{\varepsilon^2}\right)+\frac{\Delta t}{\varepsilon^2}\left\{\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right.\right.\\&\quad\left.\left.+\left(1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right)\left(8+\frac{585}{\Delta x}M_{\mathrm{d}}^2\right)\right\}\right]\cdot4M_{\mathrm{d}}\\&=\left[1+\frac{9\Delta t}{\varepsilon^2}\left(1+\frac{65}{\Delta x}M_{\mathrm{d}}^2\right)\left\{1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right\}\right]\cdot4M_{\mathrm{d}}\end{aligned}\tag{30}$$

for all $\boldsymbol{V} \in B$. Now, note that for all $(K+1) \times (K+1)$ matrix A,

$$\|A\|_{L^2_{\rm d}} = \sup_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|A\boldsymbol{x}\|_{L^2_{\rm d}}}{\|\boldsymbol{x}\|_{L^2_{\rm d}}}.$$

Here we have used the following estimates:

$$\|\mathrm{diag}(d_0, d_1, \ldots, d_{K-1}, d_K)\|_{L^2_{\rm d}} = \max_{0 \le k \le K} |d_k|,$$

$$\max_{0 \le k \le K} \left| \frac{1}{\frac{1}{\Delta t} - \frac{1}{2}\lambda_k} \right| = \Delta t, \tag{31}$$

$$\max_{0 \le k \le K} \left| \frac{\frac{1}{\Delta t} + \frac{1}{\varepsilon^2} + \frac{1}{2}\lambda_k}{\frac{1}{\Delta t} - \frac{1}{2}\lambda_k} \right| \le 1 + \frac{1}{\varepsilon^2}\Delta t, \tag{32}$$

$$\|S\|_{L^2_{\rm d}} \le \sqrt{(K+1)(2K-1)}\Delta x, \tag{33}$$

$$\left\| I - \frac{1}{L}S \right\|_{L^2_{\rm d}} \le 1 + \frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x, \tag{34}$$

$$\|X\|_{L^2_{\rm d}} \le 2\sqrt{K}, \tag{35}$$

$$\|X^{-1}\|_{L^2_{\rm d}} \le \frac{2}{\sqrt{K}}, \tag{36}$$

$$\left\| F_{\boldsymbol{U}^{(m)}} \boldsymbol{V} \right\|_{L^2_{\rm d}} \le \frac{585}{\varepsilon^2 \Delta x} M_{\rm d}^3 \tag{37}$$

that hold under the conditions $\left\|\boldsymbol{U}^{(m)}\right\|_{L^2_{\rm d}} = M_{\rm d}$ and $\|\boldsymbol{V}\|_{L^2_{\rm d}} \le 8M_{\rm d}$. We show how to obtain the estimates above. Firstly, we obtain the equality (31), since $\lambda_k \le 0$ $(k = 0, \ldots, K)$ and $\lambda_0 = 0$. In addition, by using (31), the estimate (32) holds. Secondly, we show the evaluation of the matrix norm (33). From the definition of S, we have

$$S^\top S = \Delta x^2 \begin{pmatrix} \frac{K+1}{4} & \frac{K+1}{2} & \cdots & \frac{K+1}{2} & \frac{K+1}{4} \\ \frac{K+1}{2} & K+1 & \cdots & K+1 & \frac{K+1}{2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{K+1}{2} & K+1 & \cdots & K+1 & \frac{K+1}{2} \\ \frac{K+1}{4} & \frac{K+1}{2} & \cdots & \frac{K+1}{2} & \frac{K+1}{4} \end{pmatrix} = \frac{K+1}{4}\Delta x^2 \begin{pmatrix} 1 & 2 & \cdots & 2 & 1 \\ 2 & 4 & \cdots & 4 & 2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 2 & 4 & \cdots & 4 & 2 \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Let P be the matrix of order $K+1$ as

$$P := \begin{pmatrix} 1 & 2 & \cdots & 2 & 1 \\ 2 & 4 & \cdots & 4 & 2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 2 & 4 & \cdots & 4 & 2 \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Then,

$$S^\top S = \frac{K+1}{4} \Delta x^2 P.$$

Let μ be an eigenvalue of P. The characteristic polynomial of P is

$$\det(P - \mu I) = (-1)^{K-1} \mu^K (\mu - 4K + 2).$$

Then, we obtain the eigenvalues $\mu = 0, 4K - 2$. So, the largest eigenvalue of $S^\top S$ is

$$\frac{K+1}{4} \Delta x^2 \cdot (4K - 2) = \frac{(K+1)(2K-1)}{2} \Delta x^2,$$

since $K \geq 1$. Hence, we have

$$\|S\|_{L^2_{\mathrm{d}}} \leq \sqrt{2} \|S\|_2 = \sqrt{2} \cdot \sqrt{\frac{(K+1)(2-1)}{2} \Delta x^2} = \sqrt{(K+1)(2K-1)} \Delta x$$

by using the following inequality.

$$\|A\|_{L^2_{\mathrm{d}}} \leq \sqrt{2}\, \|A\|_2 \quad \text{for all } (K+1) \times (K+1) \text{ matrix } A, \tag{38}$$

where $\|\cdot\|_2$ is the matrix 2-norm induced by the euclidean vector. Moreover, by using the estimate (33) and the triangle inequality, we obtain the inequality (34). Thirdly, we show the estimates (35) and (36). Let Q be the diagonal matrix of order $K+1$ as

$$Q := \begin{pmatrix} \frac{1}{\sqrt{2}} & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and Z be the square matrix of order $K+1$ as

$$Z := \sqrt{\frac{2}{K}} QXQ.$$

Then, Z is an orthogonal matrix, i.e., $Z^{-1} = Z^\top$. In fact, let $Z = (z_0, \dots, z_K)$, then

$$z_k = \begin{cases} \dfrac{1}{\sqrt{K}} \boldsymbol{y}_k, & (k = 0, K), \\ \sqrt{\dfrac{2}{K}} \boldsymbol{y}_k, & (k = 1, \dots, K-1), \end{cases}$$

where

$$\boldsymbol{y}_k := \begin{pmatrix} \dfrac{1}{\sqrt{2}} \cos\left(\frac{k}{K}\pi \cdot 0\right) \\ \cos\left(\dfrac{k}{K}\pi \cdot 1\right) \\ \vdots \\ \cos\left(\dfrac{k}{K}\pi \cdot (K-1)\right) \\ \dfrac{1}{\sqrt{2}} \cos\left(\dfrac{k}{K}\pi \cdot K\right) \end{pmatrix} \quad (k = 0, \dots, K).$$

Since for all $m, n \in \mathbb{Z}$ such that $0 \le m, n \le K$,

$$\boldsymbol{y}_m^\top \cdot \boldsymbol{y}_n = \sum_{k=0}^{K}{}'' \cos\Bigl(\frac{m}{K}\pi \cdot k\Bigr) \cos\Bigl(\frac{n}{K}\pi \cdot k\Bigr) = \begin{cases} K, & (m = n = 0, m = n = K), \\ \dfrac{1}{2}K, & (1 \le m = n \le K-1), \\ 0 & (m \ne n), \end{cases}$$

$\{z_k\}_{k=0}^K$ is an orthonormal basis of $\mathbb{R}^{K+1}$. Thus, Z is an orthogonal matrix. Hence, $\|Z\|_2 = 1$, since $Z^\top Z = Z^{-1} Z = I$. Also, $\|Q^{-1}\|_2 = \sqrt{2}$. Therefore, we obtain

$$\|X\|_{L^2_\mathrm{d}} \le \sqrt{2} \cdot \|X\|_2 = \sqrt{2} \cdot \sqrt{\frac{K}{2}} \left\| Q^{-1} Z Q^{-1} \right\|_2 \le 2\sqrt{K}.$$

by using the inequality (38). Similarly, we have

$$\|X^{-1}\|_{L^2_\mathrm{d}} \le \sqrt{2} \cdot \|X^{-1}\|_2 = \sqrt{2} \cdot \sqrt{\frac{2}{K}} \left\| Q Z^\top Q \right\|_2 \le \frac{2}{\sqrt{K}}$$

from the equalities $\|Z^\top\|_2 = 1$ and $\|Q\|_2 = 1$. Finally, we show the evaluation of the nonlinear term (37). By using the following inequality:

$$\sum_{k=0}^{K}{}'' a_k b_k \Delta x \le \frac{2}{\Delta x} \sum_{k=0}^{K}{}'' a_k \Delta x \sum_{k=0}^{K}{}'' b_k \Delta x$$

for all $\{a_k\}_{k=0}^K$, $\{b_k\}_{k=0}^K$ such that $a_k, b_k \geq 0$ $(k = 0, \ldots, K)$, we have

$$\begin{aligned}
&\left\| F_{\boldsymbol{U}^{(m)}} \boldsymbol{V} \right\|_{L_\mathrm{d}^2}^2 \\
&= \left(\frac{1}{2\varepsilon^2}\right)^2 \sum_{k=0}^{K}{}'' \left(V_k + U_k^{(m)}\right)^2 \left\{V_k^2 + \left(U_k^{(m)}\right)^2\right\}^2 \Delta x \\
&\leq \left(\frac{1}{2\varepsilon^2}\right)^2 \frac{2}{\Delta x} \sum_{k=0}^{K}{}'' \left(V_k + U_k^{(m)}\right)^2 \Delta x \sum_{k=0}^{K}{}'' \left\{V_k^2 + \left(U_k^{(m)}\right)^2\right\}^2 \Delta x \\
&\leq \left(\frac{1}{2\varepsilon^2}\right)^2 \left(\frac{2}{\Delta x}\right)^2 \sum_{k=0}^{K}{}'' \left\{V_k^2 + \left(U_k^{(m)}\right)^2\right\} \Delta x \sum_{k=0}^{K}{}'' \left\{V_k^2 + \left(U_k^{(m)}\right)^2\right\} \Delta x \\
&\quad \cdot \sum_{k=0}^{K}{}'' \left\{V_k^2 + 2V_k U_k^{(m)} + \left(U_k^{(m)}\right)^2\right\} \Delta x \\
&= \left(\frac{1}{2\varepsilon^2}\right)^2 \left(\frac{2}{\Delta x}\right)^2 \left(\|\boldsymbol{V}\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2}^2\right)^2 \left(\|\boldsymbol{V}\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2}^2 + 2\sum_{k=0}^{K}{}'' V_k U_k^{(m)} \Delta x\right)
\end{aligned}$$

for all $\boldsymbol{V} \in B$. Moreover, by using Schwarz inequality, we obtain

$$\left|\sum_{k=0}^{K}{}'' V_k U_k^{(m)} \Delta x\right| \leq \sqrt{\sum_{k=0}^{K}{}'' V_k^2 \Delta x} \sqrt{\sum_{k=0}^{K}{}'' \left(U_k^{(m)}\right)^2 \Delta x} = \|\boldsymbol{V}\|_{L_\mathrm{d}^2} \left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2}$$

for all $\boldsymbol{V} \in B$. Hence, we have the following estimate:

$$\begin{aligned}
\left\| F_{\boldsymbol{U}^{(m)}} \boldsymbol{V} \right\|_{L_\mathrm{d}^2}^2 &\leq \left(\frac{1}{2\varepsilon^2}\right)^2 \left(\frac{2}{\Delta x}\right)^2 \left(\|\boldsymbol{V}\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2}^2\right)^2 \\
&\quad \cdot \left(\|\boldsymbol{V}\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2}^2 + 2\|\boldsymbol{V}\|_{L_\mathrm{d}^2} \left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2}\right) \\
&\leq \left(\frac{1}{2\varepsilon^2}\right)^2 \left(\frac{2}{\Delta x}\right)^2 \left\{(8M_\mathrm{d})^2 + M_\mathrm{d}^2\right\}^2 \left\{(8M_\mathrm{d})^2 + M_\mathrm{d}^2 + 2 \cdot 8M_\mathrm{d} \cdot M_\mathrm{d}\right\} \\
&= \left(\frac{1}{2\varepsilon^2}\right)^2 \left(\frac{2}{\Delta x}\right)^2 65^2 \cdot 81 \cdot M_\mathrm{d}^6.
\end{aligned}$$

that holds under the conditions $\left\|\boldsymbol{U}^{(m)}\right\|_{L_\mathrm{d}^2} = M_\mathrm{d}$ and $\|\boldsymbol{V}\|_{L_\mathrm{d}^2} \leq 8M_\mathrm{d}$. Therefore, we obtain the estimate (37).

If

$$\Delta t \leq \frac{L\varepsilon^2 \Delta x}{9\left(\Delta x + 65M_\mathrm{d}^2\right)\left(L + \sqrt{(K+1)(2K-1)}\Delta x\right)},$$

then

$$1+\frac{9\Delta t}{\varepsilon^2}\left(1+\frac{65}{\Delta x}M_{\mathrm{d}}^2\right)\left\{1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right\}\leq 2, \tag{39}$$

since

$$\frac{L\varepsilon^2\Delta x}{9\left(\Delta x+65M_{\mathrm{d}}^2\right)\left(L+\sqrt{(K+1)(2K-1)}\Delta x\right)} = \frac{\varepsilon^2}{9\left(1+\frac{65}{\Delta x}M_{\mathrm{d}}^2\right)\left\{1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right\}}.$$

From (30) and (39), we see that $\left\|\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\right\|_{L_{\mathrm{d}}^2}\leq 8M_{\mathrm{d}}$, i.e., $\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}\in B$. Hence, $\mathcal{T}_{\boldsymbol{U}^{(m)}}B\subseteq B$.

Next, we prove that $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ is contractive. Using (29) and the estimates above, we can show

$$\begin{aligned}
&\left\|\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}-\mathcal{T}_{\boldsymbol{U}^{(m)}}\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}\\
&\leq \|X\|_{L_{\mathrm{d}}^2}\left\|\left(\frac{1}{\Delta t}I-\frac{1}{2}\Lambda\right)^{-1}\right\|_{L_{\mathrm{d}}^2}\left\|X^{-1}\right\|_{L_{\mathrm{d}}^2}\left\|I-\frac{1}{L}S\right\|_{L_{\mathrm{d}}^2}\\
&\quad\cdot\left(\frac{1}{\varepsilon^2}\left\|\boldsymbol{V}-\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}+\left\|F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}-F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}\right)\\
&\leq 4\Delta t\left(1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right)\left(\frac{1}{\varepsilon^2}\left\|\boldsymbol{V}-\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}+\left\|F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}-F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}\right)\\
&\leq\frac{4\Delta t}{\varepsilon^2}\left(1+\frac{209M_{\mathrm{d}}^2}{\Delta x}\right)\left(1+\frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right)\left\|\boldsymbol{V}-\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2},
\end{aligned}$$

because

$$\left\|F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}-F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}\leq\frac{209M_{\mathrm{d}}^2}{\varepsilon^2\Delta x}\left\|\boldsymbol{V}-\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2} \tag{40}$$

for all $\boldsymbol{V},\boldsymbol{V}'\in B$. In fact, we show the estimate (40). For all $\boldsymbol{V},\boldsymbol{V}'\in B$,

$$\begin{aligned}
&\left\|F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}-F_{\boldsymbol{U}^{(m)}}\boldsymbol{V}'\right\|_{L_{\mathrm{d}}^2}^2\\
&=\left(\frac{1}{2\varepsilon^2}\right)^2\sum_{k=0}^{K}{}''\left\{V_k^3+V_k\left(U_k^{(m)}\right)^2+V_k^2U_k^{(m)}+\left(U_k^{(m)}\right)^3\right.\\
&\quad\left.-\left(V_k'\right)^3-V_k'\left(U_k^{(m)}\right)^2-\left(V_k'\right)^2U_k^{(m)}-\left(U_k^{(m)}\right)^3\right\}^2\Delta x
\end{aligned}$$

$$= \left(\frac{1}{2\varepsilon^2}\right)^2 \sum_{k=0}^{K}{}'' \left[\left\{ V_k^3 - (V_k')^3 \right\} + (V_k - V_k')\left(U_k^{(m)}\right)^2 + \left\{ V_k^2 - (V_k')^2 \right\} U_k^{(m)} \right]^2 \Delta x$$

$$= \left(\frac{1}{2\varepsilon^2}\right)^2 \sum_{k=0}^{K}{}'' \left\{ V_k^2 + V_k V_k' + (V_k')^2 + \left(U_k^{(m)}\right)^2 + (V_k + V_k')\, U_k^{(m)} \right\}^2 (V_k - V_k')^2 \Delta x. \tag{41}$$

Moreover,

$$\max_{0\le k\le K} |V_k| \le \sqrt{\frac{2}{\Delta x}} \sqrt{\sum_{k=0}^{K}{}'' V_k^2 \Delta x} = \sqrt{\frac{2}{\Delta x}}\, \| \boldsymbol{V} \|_{L_{\mathrm{d}}^2} \le \sqrt{\frac{2}{\Delta x}} \cdot 8 M_{\mathrm{d}} \tag{42}$$

for all $\boldsymbol{V} \in B$. Similarly,

$$\max_{0\le k\le K} \left| U_k^{(m)} \right| \le \sqrt{\frac{2}{\Delta x}} \cdot M_{\mathrm{d}}. \tag{43}$$

Therefore, by using (41), (42) and (43),

$$\begin{aligned}
&\left\| F_{\boldsymbol{U}^{(m)}} \boldsymbol{V} - F_{\boldsymbol{U}^{(m)}} \boldsymbol{V}' \right\|_{L_{\mathrm{d}}^2}^2 \\
&\le \left(\frac{1}{2\varepsilon^2}\right)^2 \sum_{k=0}^{K}{}'' \left\{ \frac{2}{\Delta x} \cdot 64 M_{\mathrm{d}}^2 + \frac{2}{\Delta x} \cdot 64 M_{\mathrm{d}}^2 + \frac{2}{\Delta x} \cdot 64 M_{\mathrm{d}}^2 + \frac{2}{\Delta x} M_{\mathrm{d}}^2 \right. \\
&\quad \left. + \left(\sqrt{\frac{2}{\Delta x}} \cdot 8 M_{\mathrm{d}} + \sqrt{\frac{2}{\Delta x}} \cdot 8 M_{\mathrm{d}} \right) \sqrt{\frac{2}{\Delta x}} M_{\mathrm{d}} \right\}^2 \left(V_k - V_k'\right)^2 \Delta x \\
&= \left(\frac{1}{2\varepsilon^2}\right)^2 \sum_{k=0}^{K}{}'' \left(\frac{2}{\Delta x} \cdot 192 M_{\mathrm{d}}^2 + \frac{2}{\Delta x} M_{\mathrm{d}}^2 + \frac{2}{\Delta x} \cdot 16 M_{\mathrm{d}}^2 \right)^2 \left(V_k - V_k'\right)^2 \Delta x \\
&= \left(\frac{209 M_{\mathrm{d}}^2}{\varepsilon^2 \Delta x} \right)^2 \left\| \boldsymbol{V} - \boldsymbol{V}' \right\|_{L_{\mathrm{d}}^2}^2 \quad \text{for all } \boldsymbol{V}, \boldsymbol{V}' \in B.
\end{aligned}$$

Hence, the estimate (40) holds.

If

$$\Delta t < \frac{L\varepsilon^2 \Delta x}{4\left(\Delta x + 209 M_{\mathrm{d}}^2\right)\left(L + \sqrt{(K+1)(2K-1)}\Delta x\right)}$$

then

$$0 \le \frac{4\Delta t}{\varepsilon^2} \left(1 + \frac{209 M_{\mathrm{d}}^2}{\Delta x} \right) \left(1 + \frac{1}{L} \sqrt{(K+1)(2K-1)} \Delta x \right) < 1,$$

since

$$\frac{L\varepsilon^2 \Delta x}{4\left(\Delta x + 209 M_{\mathrm{d}}^2\right)\left(L + \sqrt{(K+1)(2K-1)}\Delta x\right)}$$
$$= \frac{\varepsilon^2}{4\left(1 + \frac{209 M_{\mathrm{d}}^2}{\Delta x}\right)\left(1 + \frac{1}{L}\sqrt{(K+1)(2K-1)}\Delta x\right)}.$$

Therefore, $\mathcal{T}_{\boldsymbol{U}^{(m)}}$ is contractive. This completes the proof. □

5 Error estimate for the proposed scheme

In this section, we show an error estimate of the numerical solution of the proposed scheme. Let $\Delta t := T/M$. We define the error as

$$e_k^{(m)} := U_k^{(m)} - u_k^{(m)} \quad (k = -1, 0, \ldots, K, K+1, m = 0, 1, \ldots, M),$$

where $u_k^{(m)} := u(k\Delta x, m\Delta t)$ and u is the solution of the target Eq. (12). We define an extension of u by

$$u(-\Delta x, t) := u(\Delta x, t), \quad u((K+1)\Delta x, t) := u((K-1)\Delta x, t) \tag{44}$$

for all $t \in [0, T]$. In what follows, we use the following special time-difference and -averaging operators:

$$\delta_m^{\langle 1\rangle} f^{(m)} := \frac{f^{\left(m+\frac{1}{2}\right)} - f^{\left(m-\frac{1}{2}\right)}}{\Delta t}, \quad s_m^{\langle 1\rangle} f^{(m)} := \frac{f^{\left(m+\frac{1}{2}\right)} + f^{\left(m-\frac{1}{2}\right)}}{2}.$$

Moreover, for simplicity, we use the expression:

$$\frac{\partial}{\partial x} f(a) = \left.\frac{\partial}{\partial x} f(x)\right|_{x=a}.$$

Lemma 4 *For $m = 0, 1, \ldots, M-1$, the error $\boldsymbol{e}^{(m)}$ satisfies*

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2 - \left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2\right) \\
&\quad \le \frac{1}{2}\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2 + \left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2\right) + \left\|\boldsymbol{\xi}_1^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 + \left\|\boldsymbol{\xi}_2^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 \\
&\qquad + 4\left\|\tilde{\phi}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 + 4L\left\{\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right\}^2.
\end{aligned}$$

where

$$\lambda^{\varepsilon,\left(m+\frac{1}{2}\right)} := \lambda^{\varepsilon}\left(\left(m+\frac{1}{2}\right)\Delta t\right) = -\frac{1}{L}\int_0^L \frac{2}{\varepsilon^2}(u-u^3)dx\Big|_{t=\left(m+\frac{1}{2}\right)\Delta t},$$

$$\tilde{\phi}(f_k, g_k) := \frac{2}{\varepsilon^2}\left\{\frac{f_k+g_k}{2} - \frac{(f_k)^3+(f_k)^2 g_k + f_k(g_k)^2+(g_k)^3}{4}\right\},$$

$$\phi_k^{\left(m+\frac{1}{2}\right)} := \frac{2}{\varepsilon^2}\left\{u_k^{\left(m+\frac{1}{2}\right)} - \left(u_k^{\left(m+\frac{1}{2}\right)}\right)^3\right\},$$

$$\xi_{1,k}^{\left(m+\frac{1}{2}\right)} := 2\left(\frac{\partial}{\partial t} - \delta_m^{\langle 1\rangle}\right)u_k^{\left(m+\frac{1}{2}\right)},$$

$$\xi_{2,k}^{\left(m+\frac{1}{2}\right)} := 2\left(\delta_k^{\langle 2\rangle}s_m^{\langle 1\rangle} - \frac{\partial^2}{\partial x^2}\right)u_k^{\left(m+\frac{1}{2}\right)}$$

for $k = 0, 1, \ldots, K$.

Proof For $m = 0, 1, \ldots, M-1$, subtracting the following original equation:

$$\frac{\partial}{\partial t}u_k^{\left(m+\frac{1}{2}\right)} = \frac{\partial^2}{\partial x^2}u_k^{\left(m+\frac{1}{2}\right)} + \phi_k^{\left(m+\frac{1}{2}\right)} + \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}$$

from the following proposed scheme:

$$\delta_m^{\langle 1\rangle}U_k^{\left(m+\frac{1}{2}\right)} = \delta_k^{\langle 2\rangle}s_m^{\langle 1\rangle}U_k^{\left(m+\frac{1}{2}\right)} + \tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) + \lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)})$$

at $t = (m+1/2)\Delta t$ for $k = 0, 1, \ldots, K$, we obtain

$$\begin{aligned}\delta_m^{\langle 1\rangle}e_k^{\left(m+\frac{1}{2}\right)} &= \delta_k^{\langle 2\rangle}s_m^{\langle 1\rangle}e_k^{\left(m+\frac{1}{2}\right)} + \frac{1}{2}\xi_{1,k}^{\left(m+\frac{1}{2}\right)} + \frac{1}{2}\xi_{2,k}^{\left(m+\frac{1}{2}\right)} \\ &\quad + \left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right) + \left(\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right).\end{aligned} \tag{45}$$

Hence, we obtain the following equality from (45):

$$\begin{aligned}&\frac{1}{\Delta t}\sum_{k=0}^{K}{}'' \left\{\left(e_k^{(m+1)}\right)^2 - \left(e_k^{(m)}\right)^2\right\}\Delta x \\ &= \sum_{k=0}^{K}{}'' \left(\delta_m^{\langle 1\rangle}e_k^{\left(m+\frac{1}{2}\right)}\right)\left(s_m^{\langle 1\rangle}e_k^{\left(m+\frac{1}{2}\right)}\right)\Delta x\end{aligned}$$

$$= \sum_{k=0}^{K}{}'' \left(\delta_k^{\langle 2\rangle} s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)\left(s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right) \Delta x + \sum_{k=0}^{K}{}'' \left(s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)\left\{ \frac{1}{2}\xi_{1,k}^{\left(m+\frac{1}{2}\right)} + \frac{1}{2}\xi_{2,k}^{\left(m+\frac{1}{2}\right)} \right.$$
$$\left. + \left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right) + \left(\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right)\right\} \Delta x.$$

Here, we obtain the following inequality from the summation by parts (17):

$$\sum_{k=0}^{K}{}'' \left(\delta_k^{\langle 2\rangle} s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)\left(s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right) \Delta x$$
$$= \left[\left(\delta_k^{\langle 1\rangle} s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)\left(s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)\right]_0^K - \frac{1}{2}\sum_{k=0}^{K}{}'' \left\{\left(\delta_k^{+} s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)^2 + \left(\delta_k^{-} s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)^2\right\} \Delta x$$
$$\leq 0$$

since $\delta_k^{\langle 1\rangle} s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)} = 0$ $(k = 0, K)$ under the discrete boundary conditions (15) and the definition of the extension (44). From the above, we obtain the following inequality:

$$\frac{1}{\Delta t}\sum_{k=0}^{K}{}'' \left\{\left(e_k^{(m+1)}\right)^2 - \left(e_k^{(m)}\right)^2\right\} \Delta x$$
$$\leq \sum_{k=0}^{K}{}'' \left(s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)\left\{ \frac{1}{2}\xi_{1,k}^{\left(m+\frac{1}{2}\right)} + \frac{1}{2}\xi_{2,k}^{\left(m+\frac{1}{2}\right)} \right.$$
$$\left. + \left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right) + \left(\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right)\right\} \Delta x$$
$$\leq \frac{1}{2}\sum_{k=0}^{K}{}'' \left(s_m^{\langle 1\rangle} e_k^{\left(m+\frac{1}{2}\right)}\right)^2 \Delta x + \frac{1}{2}\sum_{k=0}^{K}{}'' \left\{ \frac{1}{2}\xi_{1,k}^{\left(m+\frac{1}{2}\right)} + \frac{1}{2}\xi_{2,k}^{\left(m+\frac{1}{2}\right)} \right.$$
$$\left. + \left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right) + \left(\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right)\right\}^2 \Delta x$$
$$\leq \frac{1}{4}\sum_{k=0}^{K}{}'' \left\{\left(e_k^{(m+1)}\right)^2 + \left(e_k^{(m)}\right)^2\right\} \Delta x + 2\sum_{k=0}^{K}{}'' \left\{ \frac{1}{4}\left(\xi_{1,k}^{\left(m+\frac{1}{2}\right)}\right)^2 + \frac{1}{4}\left(\xi_{2,k}^{\left(m+\frac{1}{2}\right)}\right)^2 \right.$$
$$\left. + \left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right)^2 + \left(\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right)^2\right\} \Delta x$$
$$\leq \frac{1}{4}\sum_{k=0}^{K}{}'' \left(e_k^{(m+1)}\right)^2 \Delta x + \frac{1}{4}\sum_{k=0}^{K}{}'' \left(e_k^{(m)}\right)^2 \Delta x + \frac{1}{2}\sum_{k=0}^{K}{}'' \left(\xi_{1,k}^{\left(m+\frac{1}{2}\right)}\right)^2 \Delta x$$

$$+\frac{1}{2}\sum_{k=0}^{K}{}'' \left(\xi_{2,k}^{\left(m+\frac{1}{2}\right)}\right)^2 \Delta x + 2\sum_{k=0}^{K}{}'' \left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right)^2 \Delta x$$
$$+ 2L\left(\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right)^2$$

where we have used the inequality $ab \le (1/2)(a^2+b^2)$ for all $a, b \in \mathbb{R}$ in the second inequality and the inequality

$$(a_1 + \cdots + a_n)^2 \le n(a_1^2 + \cdots + a_n^2) \quad \text{for all } a_1, \ldots, a_n \in \mathbb{R} \tag{46}$$

in the third inequality. This completes the proof. □

Lemma 5

$$\left\{\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right\}^2 \le \frac{2}{L}\left\|\tilde{\phi}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 + \frac{C^2}{8\varepsilon^4}\Delta x^4, \tag{47}$$

where

$$C := \sup_{\substack{0\le x\le L\\ 0\le t\le T}} \left|\frac{\partial^2}{\partial x^2}(u-u^3)\right|.$$

Proof By using the inequality (46), we have

$$\left\{\lambda_{\mathrm{d}}^{\varepsilon}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \lambda^{\varepsilon,\left(m+\frac{1}{2}\right)}\right\}^2$$
$$= \left\{-\frac{1}{L}\sum_{k=0}^{K}{}''\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right)\Delta x + \frac{1}{L}\int_0^L \frac{2}{\varepsilon^2}(u-u^3)dx\bigg|_{t=\left(m+\frac{1}{2}\right)\Delta t}\right\}^2$$
$$= \frac{1}{L^2}\left[\left\{\sum_{k=0}^{K}{}''\left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right)\Delta x\right\}\right.$$
$$\left. + \left\{\sum_{k=0}^{K}{}''\phi_k^{\left(m+\frac{1}{2}\right)}\Delta x - \int_0^L \frac{2}{\varepsilon^2}(u-u^3)dx\bigg|_{t=\left(m+\frac{1}{2}\right)\Delta t}\right\}\right]^2$$
$$\le \frac{2}{L^2}\left\{\sum_{k=0}^{K}{}''\left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right)\Delta x\right\}^2$$
$$+ \frac{2}{L^2}\left\{\sum_{k=0}^{K}{}''\phi_k^{\left(m+\frac{1}{2}\right)}\Delta x - \int_0^L \frac{2}{\varepsilon^2}(u-u^3)dx\bigg|_{t=\left(m+\frac{1}{2}\right)\Delta t}\right\}^2.$$

Since

$$\left(\sum_{k=0}^{K}{}'' f_k \Delta x\right)^2 \leq L \sum_{k=0}^{K}{}'' f_k^2 \Delta x$$

for all $\{f_k\}_{k=0}^{K} \in \mathbb{R}^{K+1}$, we obtain

$$\frac{2}{L^2}\left\{\sum_{k=0}^{K}{}''\left(\tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \phi_k^{\left(m+\frac{1}{2}\right)}\right)\Delta x\right\}^2$$
$$\leq \frac{2}{L^2}\cdot L\sum_{k=0}^{K}{}''\left(\tilde{\phi}(U_k^{(m+1)}, U_k^{(m)}) - \phi_k^{\left(m+\frac{1}{2}\right)}\right)^2\Delta x = \frac{2}{L}\left\|\tilde{\phi}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2.$$

Here, we define

$$\Phi^{\left(m+\frac{1}{2}\right)}(x) := u\left(x, \left(m+\frac{1}{2}\right)\Delta t\right) - \left\{u\left(x, \left(m+\frac{1}{2}\right)\Delta t\right)\right\}^3 \quad \text{for all } x \in [0, L].$$

Since $u(\cdot, t) \in C^2([0, L])$ for any fixed $t \in [0, T]$, we obtain $\Phi^{\left(m+\frac{1}{2}\right)} \in C^2([0, L])$. Therefore, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{K}{}''\phi_k^{\left(m+\frac{1}{2}\right)}\Delta x - \int_0^L \frac{2}{\varepsilon^2}(u-u^3)dx\bigg|_{t=\left(m+\frac{1}{2}\right)\Delta t}\right| \\
&= \frac{2}{\varepsilon^2}\left|\sum_{k=0}^{K}{}''\Phi^{\left(m+\frac{1}{2}\right)}(k\Delta x)\Delta x - \int_0^L \Phi^{\left(m+\frac{1}{2}\right)}(x)dx\right| \\
&\leq \frac{2}{\varepsilon^2}\cdot\frac{1}{8}\Delta x^2\int_0^L\left|\frac{\partial^2}{\partial x^2}\Phi^{\left(m+\frac{1}{2}\right)}(x)\right|dx \\
&= \frac{1}{4\varepsilon^2}\Delta x^2\int_0^L\left|\frac{\partial^2}{\partial x^2}(u-u^3)\bigg|_{t=\left(m+\frac{1}{2}\right)\Delta t}\right|dx \leq \frac{CL}{4\varepsilon^2}\Delta x^2
\end{aligned}$$

from the Euler–Maclaurin summation formula. Thus, we obtain

$$\frac{2}{L^2}\left\{\sum_{k=0}^{K}{}''\phi_k^{\left(m+\frac{1}{2}\right)}\Delta x - \int_0^L\frac{2}{\varepsilon^2}(u-u^3)dx\bigg|_{t=\left(m+\frac{1}{2}\right)\Delta t}\right\}^2 \leq \frac{2}{L^2}\cdot\frac{C^2L^2}{16\varepsilon^4}\Delta x^4 = \frac{C^2}{8\varepsilon^4}\Delta x^4.$$

From the above, we have (47). □

Lemma 6

$$\left\| \tilde{\phi}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)} \right\|_{L^2_{\mathrm{d}}}^2 \le \frac{4}{\varepsilon^4}\left(1+3C_2^2\right)^2 \left(\left\| \boldsymbol{e}^{(m+1)} \right\|_{L^2_{\mathrm{d}}}^2 + \left\| \boldsymbol{e}^{(m)} \right\|_{L^2_{\mathrm{d}}}^2 \right) + \frac{1}{12} \left\| \boldsymbol{\xi}_3^{\left(m+\frac{1}{2}\right)} \right\|_{L^2_{\mathrm{d}}}^2 + \frac{1}{12} \left\| \boldsymbol{\xi}_4^{\left(m+\frac{1}{2}\right)} \right\|_{L^2_{\mathrm{d}}}^2,$$

where

$$\begin{aligned} C_2 &:= \max_{0\le l\le M} \left\{ \max_{0\le k\le K} \left| U_k^{(l)} \right|, \sup_{x\in[0,L]} |u(x, l\Delta t)| \right\}, \\ \xi_{3,k}^{\left(m+\frac{1}{2}\right)} &:= \frac{2\sqrt{3}}{\varepsilon^2} C_2 \left(u_k^{(m+1)} - u_k^{(m)} \right)^2, \\ \xi_{4,k}^{\left(m+\frac{1}{2}\right)} &:= \frac{8\sqrt{3}}{\varepsilon^2}\left(1+3C_2^2\right) \left\{ \left(s_m^{\langle 1\rangle} - 1 \right) u_k^{\left(m+\frac{1}{2}\right)} \right\}, \end{aligned}$$

for $k = 0, 1, \ldots, K$.

Remark 1 Note that C_2 is finite since the proposed scheme is numerically stable and the solution $u(\cdot, t) \in C^0([0, L])$ for any fixed $t \in [0, T]$.

Proof We denote

$$\tilde{\phi}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)} = \sum_{i=1}^{4} \boldsymbol{I}_i$$

where $\boldsymbol{I}_i = \{I_{i,k}\}_{k=0}^K$ with

$$\begin{aligned} I_{1,k} &:= \tilde{\phi}\left(U_k^{(m+1)}, U_k^{(m)}\right) - \tilde{\phi}\left(u_k^{(m+1)}, U_k^{(m)}\right), \\ I_{2,k} &:= \tilde{\phi}\left(u_k^{(m+1)}, U_k^{(m)}\right) - \tilde{\phi}\left(u_k^{(m+1)}, u_k^{(m)}\right), \\ I_{3,k} &:= \tilde{\phi}\left(u_k^{(m+1)}, u_k^{(m)}\right) - \frac{2}{\varepsilon^2}\left\{ s_m^{\langle 1\rangle} u_k^{\left(m+\frac{1}{2}\right)} - \left(s_m^{\langle 1\rangle} u_k^{\left(m+\frac{1}{2}\right)} \right)^3 \right\}, \\ I_{4,k} &:= \frac{2}{\varepsilon^2}\left\{ s_m^{\langle 1\rangle} u_k^{\left(m+\frac{1}{2}\right)} - \left(s_m^{\langle 1\rangle} u_k^{\left(m+\frac{1}{2}\right)} \right)^3 \right\} - \phi_k^{\left(m+\frac{1}{2}\right)}. \end{aligned}$$

Then, we obtain the following estimates:

$$|I_{1,k}| \le \frac{1}{\varepsilon^2}\left(1+3C_2^2\right)\left| e_k^{(m+1)} \right|,$$

$$
\begin{aligned}
|I_{2,k}| &\leq \frac{1}{\varepsilon^2}\left(1+3C_2^2\right)\left|e_k^{(m)}\right|, \\
|I_{4,k}| &\leq \frac{2}{\varepsilon^2}\left(1+3C_2^2\right)\left|(s_m^{\langle 1\rangle}-1)u_k^{\left(m+\frac{1}{2}\right)}\right|.
\end{aligned}
$$

The estimate for $\boldsymbol{I}_3$:

$$
|I_{3,k}| \leq \frac{1}{2\varepsilon^2}C_2\left(u_k^{(m+1)}-u_k^{(m)}\right)^2
$$

is obtained by

$$
\frac{a^3+a^2b+ab^2+b^3}{2}-\frac{(a+b)^3}{4}=\frac{(a+b)(a-b)^2}{4} \quad \text{for all } a,b\in\mathbb{R}.
$$

From the above estimates, we obtain

$$
\begin{aligned}
&\left\|\tilde{\phi}(\boldsymbol{U}^{(m+1)},\boldsymbol{U}^{(m)})-\boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 \\
&\quad\leq 4\sum_{i=1}^{4}\|\boldsymbol{I}_i\|_{L_{\mathrm{d}}^2}^2 \\
&\quad\leq 4\left\{\frac{1}{\varepsilon^4}(1+3C_2^2)^2\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2+\left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2\right)\right. \\
&\qquad\left.+\frac{1}{4\varepsilon^4}C_2^2\left\|\left(u_k^{(m+1)}-u_k^{(m)}\right)^2\right\|_{L_{\mathrm{d}}^2}^2+\frac{4}{\varepsilon^4}\left(1+3C_2^2\right)^2\left\|(s_m^{\langle 1\rangle}-1)u_k^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2\right\} \\
&\quad=\frac{4}{\varepsilon^4}\left(1+3C_2^2\right)^2\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2+\left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2\right)+\frac{1}{12}\left\|\frac{2\sqrt{3}}{\varepsilon^2}C_2\left(u_k^{(m+1)}-u_k^{(m)}\right)^2\right\|_{L_{\mathrm{d}}^2}^2 \\
&\qquad+\frac{1}{12}\left\|\frac{8\sqrt{3}}{\varepsilon^2}\left(1+3C_2^2\right)\left(s_m^{\langle 1\rangle}-1\right)u_k^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 \\
&\quad=\frac{4}{\varepsilon^4}\left(1+3C_2^2\right)^2\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2+\left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2\right)+\frac{1}{12}\left\|\boldsymbol{\xi}_3^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2+\frac{1}{12}\left\|\boldsymbol{\xi}_4^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2.
\end{aligned}
$$

This completes the proof. □

Lemma 7

$$
(1-\Delta t C_3)\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2 \leq \left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2+\Delta t\left\{\left(\sum_{i=1}^{4}\left\|\boldsymbol{\xi}_i^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2\right)+\frac{C^2L}{2\varepsilon^4}\Delta x^4\right\},
$$

where $C_3 := 1 + (96/\varepsilon^4)(1 + 3C_2^2)^2$.

Proof From Lemmas 4, 5 and 6, we have

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_\mathrm{d}^2}^2 - \left\|\boldsymbol{e}^{(m)}\right\|_{L_\mathrm{d}^2}^2\right)\\
&\quad\le \frac{1}{2}\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{e}^{(m)}\right\|_{L_\mathrm{d}^2}^2\right) + \left\|\boldsymbol{\xi}_1^{\left(m+\frac{1}{2}\right)}\right\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{\xi}_2^{\left(m+\frac{1}{2}\right)}\right\|_{L_\mathrm{d}^2}^2\\
&\qquad + 12\left\|\tilde{\phi}(\boldsymbol{U}^{(m+1)}, \boldsymbol{U}^{(m)}) - \boldsymbol{\phi}^{\left(m+\frac{1}{2}\right)}\right\|_{L_\mathrm{d}^2}^2 + \frac{C^2 L}{2\varepsilon^4}\Delta x^4\\
&\quad\le \left\{\frac{1}{2} + \frac{48}{\varepsilon^4}\left(1+3C_2^2\right)^2\right\}\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_\mathrm{d}^2}^2 + \left\|\boldsymbol{e}^{(m)}\right\|_{L_\mathrm{d}^2}^2\right) + \left(\sum_{i=1}^{4}\left\|\boldsymbol{\xi}_i^{\left(m+\frac{1}{2}\right)}\right\|_{L_\mathrm{d}^2}^2\right) + \frac{C^2 L}{2\varepsilon^4}\Delta x^4.
\end{aligned}$$

Hence, we obtain the following inequality:

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_\mathrm{d}^2}^2 - \left\|\boldsymbol{e}^{(m)}\right\|_{L_\mathrm{d}^2}^2\right)\\
&\quad\le \left\{\frac{1}{2} + \frac{48}{\varepsilon^4}\left(1+3C_2^2\right)^2\right\}\cdot 2\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_\mathrm{d}^2}^2 + \left(\sum_{i=1}^{4}\left\|\boldsymbol{\xi}_i^{\left(m+\frac{1}{2}\right)}\right\|_{L_\mathrm{d}^2}^2\right) + \frac{C^2 L}{2\varepsilon^4}\Delta x^4\\
&\quad= C_3\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_\mathrm{d}^2}^2 + \left(\sum_{i=1}^{4}\left\|\boldsymbol{\xi}_i^{\left(m+\frac{1}{2}\right)}\right\|_{L_\mathrm{d}^2}^2\right) + \frac{C^2 L}{2\varepsilon^4}\Delta x^4.
\end{aligned}$$

This completes the proof. □

Theorem 4 *Assume that the target Eq.* (12) *has a solution such that* $u \in C^4([0, L] \times [0, T])$. *If* Δt *satisfies the condition* (27) *and* $\Delta t < 1/(2C_3)$, *then there exists a constant* C_4 *such that*

$$\left\{\sum_{k=0}^{K}{}''\left(U_k^{(m)} - u(k\Delta x, m\Delta t)\right)^2 \Delta x\right\}^{\frac{1}{2}} \le C_4\sqrt{LT}e^{C_3 T}(\Delta x^2 + \Delta t^2)$$

for $m = 1, \ldots, M$.

This theorem means that the solution of the scheme (13) converges to the solution of the target Eq. (12) in the sense of discrete L^2-norm and that the convergence rate is $O(\Delta x^2 + \Delta t^2)$.

Proof If the target Eq. (12) has a solution such that $u \in C^4([0, L] \times [0, T])$, then by using Taylor's theorem,

$$\xi_{1,k}^{\left(m+\frac{1}{2}\right)} = -\frac{\Delta t^2}{12}\frac{\partial^3 u}{\partial t^3}\bigg|_{(x,t)=(k\Delta x, t_1)},$$

$$
\begin{aligned}
\xi_{2,k}^{\left(m+\frac{1}{2}\right)} &= \frac{\Delta x^2}{6}\frac{\partial^4 u}{\partial x^4}\bigg|_{(x,t)=\left(x_1,\left(m+\frac{1}{2}\right)\Delta t\right)} + \frac{\Delta t^2}{4}\frac{\partial^4 u}{\partial x^2 \partial t^2}\bigg|_{(x,t)=(x_2,t_2)},\\
\xi_{3,k}^{\left(m+\frac{1}{2}\right)} &= \frac{2\sqrt{3}}{\varepsilon^2}C_2\Delta t^2\left(\frac{\partial u}{\partial t}\bigg|_{(x,t)=(k\Delta x,t_3)}\right)^2,\\
\xi_{4,k}^{\left(m+\frac{1}{2}\right)} &= \frac{\sqrt{3}}{\varepsilon^2}(1+3C_2^2)\Delta t^2\frac{\partial^2 u}{\partial t^2}\bigg|_{(x,t)=(k\Delta x,t_4)},
\end{aligned}
$$

where $t_1, t_2, t_3, t_4 \in [m\Delta t, (m+1)\Delta t]$ and $x_1, x_2 \in [(k-1)\Delta x, (k+1)\Delta x]$ for $m = 0, 1, \ldots, M-1$ and $k = 0, \ldots, K$. From these results, we obtain

$$
\sum_{i=1}^{4}\left\|\boldsymbol{\xi}_i^{\left(m+\frac{1}{2}\right)}\right\|_{L_{\mathrm{d}}^2}^2 + \frac{C^2L}{2\varepsilon^4}\Delta x^4 \le C_4^2 L(\Delta x^2+\Delta t^2)^2 \quad (m = 0, 1, \ldots, M-1), \tag{48}
$$

where

$$
\begin{aligned}
C_4 := \sup_{\substack{0\le x\le L\\ 0\le t\le T}} \max\Bigg\{ &\frac{1}{12}\left|\frac{\partial^3 u}{\partial t^3}\right|, \frac{1}{6}\left|\frac{\partial^4 u}{\partial x^4}\right|, \frac{1}{4}\left|\frac{\partial^4 u}{\partial t^2 \partial x^2}\right|,\\
&\frac{2\sqrt{3}}{\varepsilon^2}C_2\left(\frac{\partial u}{\partial t}\right)^2, \frac{\sqrt{3}}{\varepsilon^2}(1+3C_2^2)\left|\frac{\partial^2 u}{\partial t^2}\right|, \frac{1}{2\varepsilon^4}\left|\frac{\partial^2}{\partial x^2}(u-u^3)\right|\Bigg\}.
\end{aligned}
$$

If $\Delta t < 1/(2C_3)$, then $0 < 1 - 2\Delta t C_3 \le 1 - \Delta t C_3 \le 1$ and

$$
\frac{1}{1-\Delta t C_3} \le 1 + 2\Delta t C_3 =: \tilde{C}_3. \tag{49}
$$

Hence, by using Lemma 7, (48) and (49), we have

$$
\left\|\boldsymbol{e}^{(m+1)}\right\|_{L_{\mathrm{d}}^2}^2 \le \tilde{C}_3\left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2 + \tilde{C}_3 \cdot \Delta t C_4^2 L(\Delta x^2+\Delta t^2)^2
$$

for $m = 0, 1, \ldots, M-1$. Therefore, by using this inequality iteratively, for $m = 0, 1, \ldots, M$,

$$
\begin{aligned}
\left\|\boldsymbol{e}^{(m)}\right\|_{L_{\mathrm{d}}^2}^2 &\le \tilde{C}_3\left\|\boldsymbol{e}^{(m-1)}\right\|_{L_{\mathrm{d}}^2}^2 + \tilde{C}_3 \cdot \Delta t C_4^2 L(\Delta x^2+\Delta t^2)^2\\
&\le (\tilde{C}_3)^2\left\|\boldsymbol{e}^{(m-2)}\right\|_{L_{\mathrm{d}}^2}^2 + ((\tilde{C}_3)^2+\tilde{C}_3)\cdot \Delta t C_4^2 L(\Delta x^2+\Delta t^2)^2\\
&\le \cdots \quad \cdots \quad \cdots\\
&\le (\tilde{C}_3)^m\left\|\boldsymbol{e}^{(0)}\right\|_{L_{\mathrm{d}}^2}^2 + \Delta t C_4^2 L(\Delta x^2+\Delta t^2)^2\sum_{j=1}^{m}(\tilde{C}_3)^j.
\end{aligned}
$$

Here, $\|\boldsymbol{e}^{(0)}\|_{L^2_{\mathrm{d}}}^2 = 0$ since $e_k^{(0)} = 0$ $(k = 0, \ldots, K)$. Moreover, by using $1 + x \leq e^x$ for all $x \geq 0$,

$$\begin{aligned}\sum_{j=1}^{m}(\tilde{C}_3)^j &= \sum_{j=1}^{m}(1 + 2\Delta t C_3)^j \leq \sum_{j=1}^{m}\exp(j \cdot 2\Delta t C_3) \leq \exp(M \cdot 2\Delta t C_3)\sum_{j=1}^{M} 1 \\ &= M\exp\left(M \cdot 2\frac{T}{M}C_3\right) \\ &= M\exp(2C_3 T).\end{aligned}$$

Hence, we obtain

$$\left\|\boldsymbol{e}^{(m)}\right\|_{L^2_{\mathrm{d}}}^2 \leq \Delta t C_4^2 L(\Delta x^2 + \Delta t^2)^2 \cdot M e^{2C_3 T} = C_4^2 L T e^{2C_3 T}(\Delta x^2 + \Delta t^2)^2$$

for $m = 1, \ldots, M$. This completes the proof. □

6 Numerical experiments

In this section, we demonstrate through numerical experiments that the proposed scheme is stable and that the numerical solution of the proposed scheme is efficient. Moreover, we compare the proposed scheme with the Crank–Nicolson (CN) scheme. The concrete form of the CN scheme for (12) is, for $m = 0, 1, \ldots,$

$$\begin{aligned}\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} &= \delta_k^{\langle 2\rangle}\left(\frac{U_k^{(m+1)} + U_k^{(m)}}{2}\right) + \tilde{F}\left(U_k^{(m+1)}, U_k^{(m)}\right) \\ &\quad - \frac{1}{L}\sum_{k=0}^{K}{}''\tilde{F}\left(U_k^{(m+1)}, U_k^{(m)}\right)\Delta x \quad (k = 0, \ldots, K),\end{aligned}$$

where

$$\tilde{F}\left(U_k^{(m+1)}, U_k^{(m)}\right) := \frac{2}{\varepsilon^2}\left[\left(\frac{U_k^{(m+1)} + U_k^{(m)}}{2}\right) - \left\{\frac{\left(U_k^{(m+1)}\right)^3 + \left(U_k^{(m)}\right)^3}{2}\right\}\right].$$

We simulate all our numerical computations by using Julia language.

6.1 Numerical solutions

The left figures show the numerical solution obtained by the proposed scheme. The right ones show that obtained by the CN scheme.

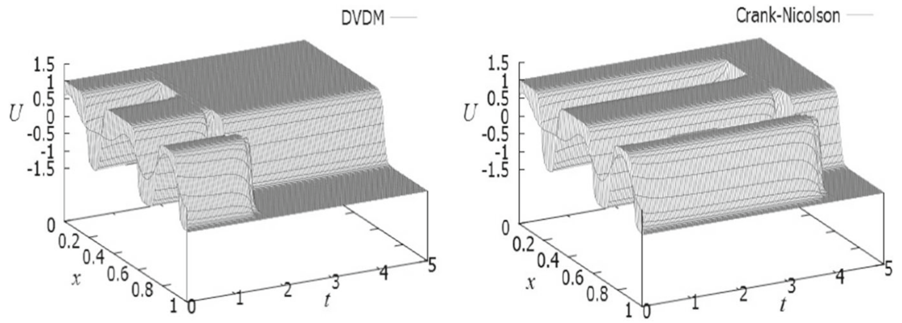

Fig. 1 Numerical solutions of (1) ($\varepsilon = 0.02$) obtained by the proposed scheme and the CN scheme with $\Delta x = 1/100$ and $\Delta t = 1/5000$

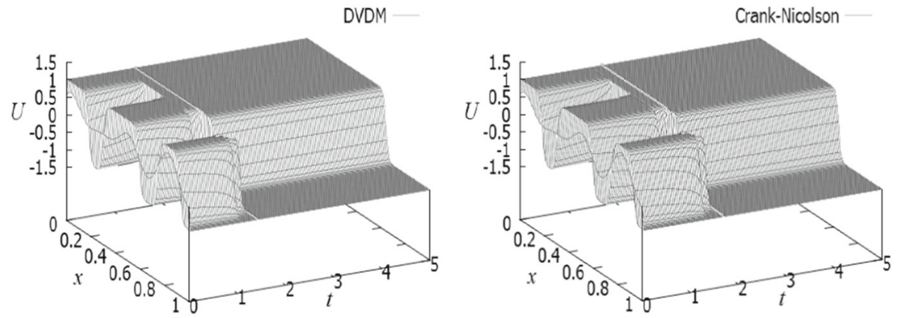

Fig. 2 Numerical solutions of (1) ($\varepsilon = 0.02$) obtained by the proposed scheme and the CN scheme with $\Delta x = 1/200$ and $\Delta t = 1/5000$

• Case 1

Fig. 1 shows numerical results for $\varepsilon = 0.02$ obtained by DVDM and the CN method with $\Delta x = 1/100$ and $\Delta t = 1/5000$. The initial data in Fig. 1 is

$$u(x, 0) = 0.26 + 0.07\cos(8\pi x) + 0.41\sin\left(\frac{11}{2}\pi x\right) + 0.24\cos(7\pi x). \quad (50)$$

The solution by the proposed scheme arrives at the steady state around at $t = 1.5$, whereas the one by the CN scheme is stable around at $t = 4$, namely, a little late time. In order to analyze the difference of these results, we refine the space mesh size.

• Case 2

In Fig. 2, we take Δx by half, i.e., $\Delta x = 1/200$. The result of the CN scheme improves. Both solutions arrive at the steady state around at $t = 1.5$. Furthermore, when we take smaller space mesh size, both solutions also arrive at the steady state around at $t = 1.5$. Hence, we expect that the solution by the proposed scheme is more reliable than that by the CN scheme when the space mesh size is coarse.

When we change the initial data into another one, the results are also different from each other. We remark that the direction of the time evolution is reverse to the previous one.

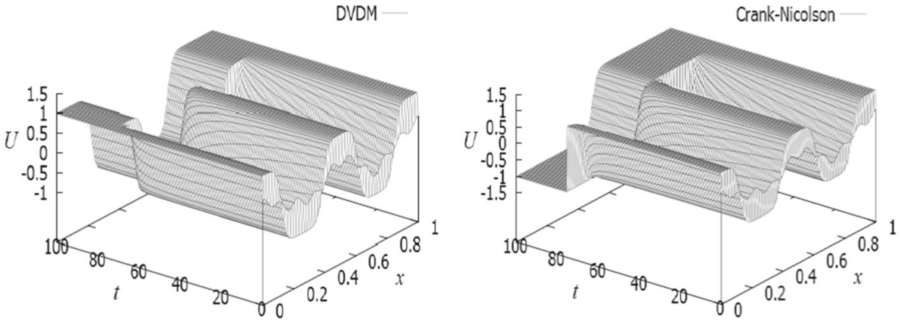

Fig. 3 Numerical solutions of (1) ($\varepsilon = 0.03$) obtained by the proposed scheme and the CN scheme with $\Delta x = 1/100$ and $\Delta t = 1/1000$

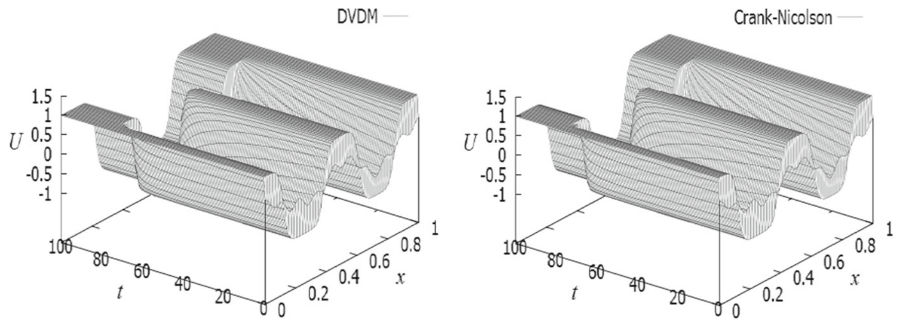

Fig. 4 Numerical solutions of (1) ($\varepsilon = 0.03$) obtained by the proposed scheme and the CN scheme with $\Delta x = 1/100$ and $\Delta t = 1/2000$

• Case 3
Figure 3 shows numerical results for $\varepsilon = 0.03$ obtained by DVDM and the CN method with $\Delta x = 1/100$ and $\Delta t = 1/1000$. The initial data in Fig. 3 is

$$u(x, 0) = 0.01 + 0.3\cos(4\pi x) + 0.08\sin\left(\frac{13}{2}\pi x\right)(\cos(4\pi x) - 1) + 0.11\cos(18\pi x). \tag{51}$$

Both solutions arrive at the steady state around at $t = 80$. However, the steady state of the solution by the CN scheme is different from that by the proposed scheme. As with previous numerical experiments, in order to analyze the difference of these results, we refine the time mesh size.
• Case 4
In Fig. 4, we take Δt by half, i.e., $\Delta t = 1/2000$. The result of the CN scheme improves. The steady state of the solution by the CN scheme coincides with that by the proposed scheme. In addition, when we take smaller time mesh size, the steady state of the solution by the CN scheme also coincides with that by the proposed scheme. Therefore, we also expect that the solution by the proposed scheme is more reliable than that by the CN scheme when the time mesh size is coarse.

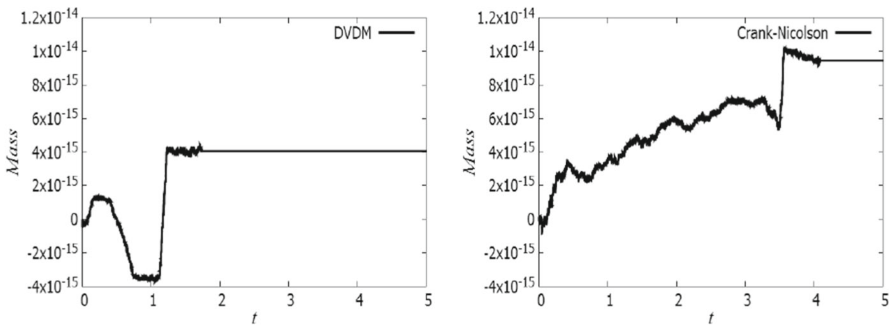

Fig. 5 The difference between the volume of the numerical solution in Fig. 1 and one of the initial data (50)

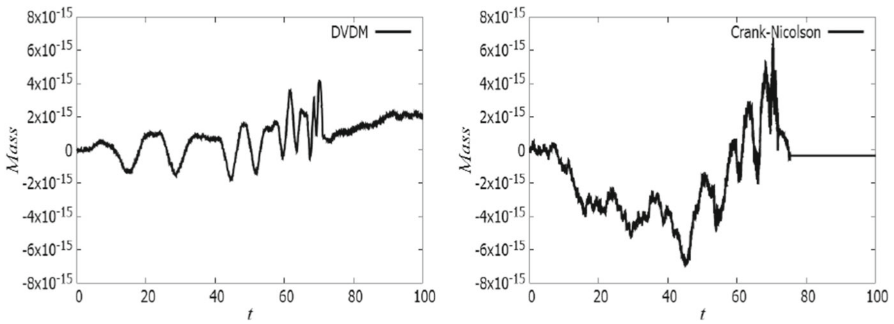

Fig. 6 The difference between the volume of the numerical solution in Fig. 3 and one of the initial data (51)

6.2 Conservative property

Next, we check the conservative property. The left figures show the results obtained by the proposed scheme. The right ones show those obtained by the CN scheme.

• Case 1

Figure 5 shows the following discrepancies:

$$\sum_{k=0}^{K}{}'' U_k^{(m)} \Delta x - \sum_{k=0}^{K}{}'' U_k^{(0)} \Delta x \quad (m = 0, 1, \ldots).$$

in Fig. 1. Theoretically, this value should be conserved. These graphs show that the mass is conserved numerically.

• Case 3

Figure 6 shows the discrepancies in Fig. 3. These graphs also show that the mass is conserved numerically.

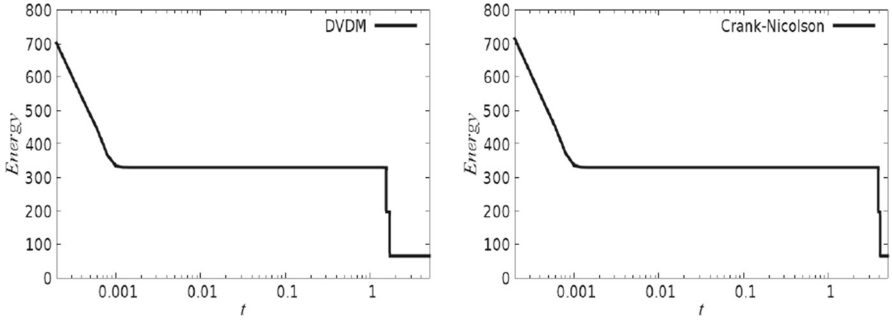

Fig. 7 The discrete global energy of the numerical solution in Fig. 1. The time axis is in log-scale

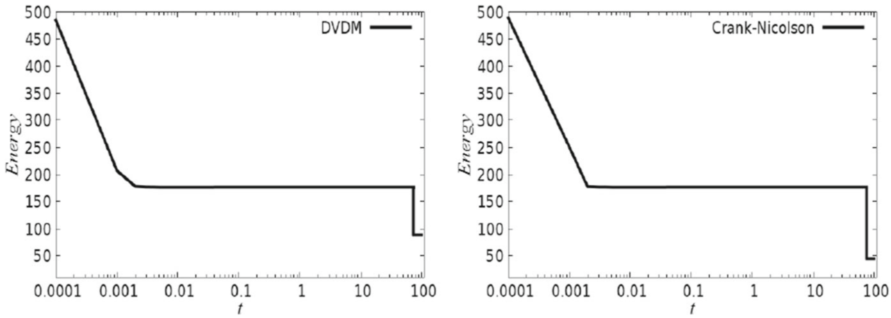

Fig. 8 The discrete global energy of the numerical solution in Fig. 3. The time axis is in log-scale

6.3 Dissipative property

Lastly, we check the dissipative property of energy. The left figures show the results obtained by the proposed scheme. The right ones show those obtained by the CN scheme.

• Case 1

Figure 7 shows the discrete global energies:

$$J_{\mathrm{d}}\left(\boldsymbol{U}^{(m)}\right) = \sum_{k=0}^{K}{}'' G_{\mathrm{d},k}\left(\boldsymbol{U}^{(m)}\right) \Delta x \quad (m = 0, 1, \ldots).$$

in Fig. 1. Theoretically, this value should decrease. These graphs show that the energy decreases numerically.

• Case 3

Figure 8 shows the discrete global energies in Fig. 3. In analogy with the Case 1, these graphs show that the decrease of the global energy is preserved numerically.

From the above, we can obtain the expected results. Additionally, the results of our scheme are better than those of the CN scheme when the mesh size is coarse.

7 Conclusion

We proposed a finite difference scheme to obtain numerical solutions of a non-local Allen–Cahn equation. The solution of the proposed scheme satisfies the discrete conservation of mass and the discrete decrease of the global energy. Moreover, the scheme is stable and has a unique solution. We also prove the error estimate for the scheme. Numerical experiments demonstrated that the proposed scheme is efficient and that our proposed scheme is more reliable than the Crank–Nicolson scheme when the mesh size is coarse.

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