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Area 2

On the circumradius condition for piecewise linear triangular elements

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Abstract We discuss the error analysis of linear interpolation on triangular elements. We claim that the circumradius condition is more essential than the well-known maximum angle condition for convergence of the finite element method, especially for the linear Lagrange finite element. Numerical experiments show that this condition is the best possible. We also point out that the circumradius condition is closely related to the definition of surface area.

Keywords Linear interpolation · The circumradius condition · The finite element method · Schwarz's example · The definition of surface area

Mathematics Subject Classification 65D05 · 65N30 · 26B15

1 Introduction

In numerical analysis, linear interpolation on triangular elements is one of the more fundamental conceptions. Specifically, as meshes become finer, it is an important tool in understanding why and how finite element approximations converge to an exact solution.

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Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Suppose that we would like to solve the Poisson equation of finding $u \in H_0^1(\Omega)$ for a given $f \in L^2(\Omega)$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega. \tag{1}$$

With a triangulation τ of Ω, we define the FEM solution u_h by

$$\int_\Omega \nabla u_h \cdot \nabla v_h \mathrm{d}x = \int_\Omega f v_h \mathrm{d}x, \qquad \forall v_h \in S_\tau, \tag{2}$$

where $\mathscr{P}_1$ is the set of all polynomials whose degree is at most 1 and

$$S_\tau := \left\{ v_h \in H_0^1(\Omega) \;\middle|\; v_h\big|_K \in \mathscr{P}_1, \forall K \in \tau \right\}.$$

Let $K \subset \mathbb{R}^2$ be a triangle with apices $\mathbf{x}_i$, $i = 1, 2, 3$. We shall always consider K to be a closed set in $\mathbb{R}^2$. For a continuous function $f \in C^0(K)$, the linear interpolation $I_K f \in \mathscr{P}_1$ is defined by

$$(I_K f)(\mathbf{x}_i) = f(\mathbf{x}_i), \qquad i = 1, 2, 3.$$

If $f \in C^0(\overline{\Omega})$, the linear interpolation $I_\tau f$ is defined by $(I_\tau f)|_K = I_K f$. Céa's lemma claims that the error $|u - u_h|_{1,2,\Omega}$ is estimated as

$$|u - u_h|_{1,2,\Omega} \le \inf_{v_h \in S_\tau} |u - v_h|_{1,2,\Omega} \le |u - I_\tau u|_{1,2,\Omega} = \left(\sum_{K \in \tau} |u - I_K u|_{1,2,K}^2 \right)^{1/2}.$$

Therefore, the interpolation error $|u - I_K u|_{1,2,K}$ provides an *upper bound* of $|u - u_h|_{1,2,\Omega}$.

It has been known that we need to impose a geometric condition to K to obtain an error estimation of $|u - I_K u|_{1,2,K}$. We mention the following well-known results. Let h_K be the diameter of K and ρ_K be the maximum radius of the inscribed circle in K.

- **The minimum angle condition**, Zlámal [21]. *Let θ_0, $0 < \theta_0 < \pi/3$, be a constant. If any angle θ of K satisfies $\theta \ge \theta_0$ and $h_K \le 1$, then there exists a constant $C = C(\theta_0)$ independent of h_K such that*

$$\|v - I_K v\|_{1,2,K} \le C h_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

- **The regularity (inscribed ball) condition**, see, for example, Ciarlet [6]. *Let $\sigma > 0$ be a constant. If $h_K/\rho_K \le \sigma$ and $h_K \le 1$, then there exists a constant $C = C(\sigma)$ independent of h_K such that*

$$\|v - I_K v\|_{1,2,K} \le C h_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

– **The maximum angle condition**, Babuška-Aziz [2], Jamet [9]. *Let* $\theta_1, 2\pi/3 \le \theta_1 < \pi$, *be a constant. If any angle* θ *of* K *satisfies* $\theta \le \theta_1$ *and* $h_K \le 1$, *then there exists a constant* $C = C(\theta_1)$ *independent of* h_K *such that*

$$\|v - I_K v\|_{1,2,K} \le C h_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

It is easy to show that the minimum angle condition is equivalent to the regularity condition [6, Exercise 3.1.3,p130]. Since its discovery, the maximum angle condition was believed to be the most essential condition for convergence of solutions of the finite element method.

However, Hannukainen–Korotov–Křížek pointed out that "the maximum angle condition is not necessary for convergence of the finite element method" by showing simple numerical examples [8]. We double checked the first numerical experiment in [8] with slightly different triangulations and obtained the same result for the error associated with the finite element approximations. Therefore, the question arises: "What is the essential condition to impose on triangulation for convergence of the finite element method?". One of the aims here is to give a partial answer to this question. Suppose that a sequence $\{\tau_n\}_{n=1}^{\infty}$ of triangulations of Ω is given. Let R_K be the circumradius of a triangle K and $R_{\tau_n} := \max_{K \in \tau_n} R_K$. We claim that the condition

$$\lim_{n \to \infty} R_{\tau_n} = 0 \tag{3}$$

is *more essential* than the maximum angle condition. The condition (3) is called the **circumradius condition**.

We moreover point out that the circumradius condition is closely related to the definition of surface area. In the 19th century, people believed that surface area could be defined as the limit of the area of inscribed polygonal surfaces. In the 1880s, Schwarz and Peano independently presented their famous example that refutes this expectation. See [5,15,16]. We shall observe in Sect. 3 that, in Schwarz's example, the limit of the inscribed polygonal surfaces is equal to the area of the cylinder if and only if the circumradius of triangles converges to 0.

We shall also show that the graph of $f \in W^{2,1}(\Omega)$ has finite area $A_L(f)$. Moreover, the areas of its inscribed polygonal surfaces converge to $A_L(f)$ if the sequence of triangulations satisfies the circumradius condition. See Theorem 6 in Sect. 3.

Let us summarize the notation and terminology to be used. The Lebesgue and Sobolev spaces on a domain $\Omega \subset \mathbb{R}^2$ are denoted by $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $m = 1, 2$, $1 \le p \le \infty$. As usual, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$. The norms and seminorms of $L^p(\Omega)$ and $W^{m,p}(\Omega)$ are denoted by $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$, $m = 0, 1, 2, 1 \le p \le \infty$. For a polygonal domain $\Omega \subset \mathbb{R}^2$, a *triangulation* τ is a set of triangles which satisfies the following properties: (recall that each K is a closed set.)

(i) $\bigcup_{K \in \tau} K = \overline{\Omega}$, and $\mathrm{int}K \cap \mathrm{int}K' = \emptyset$ for any $K, K' \in \tau$ with $K \ne K'$.

(ii) If $K \cap K' \ne \emptyset$ for $K, K' \in \tau$, $K \cap K'$ is either their apices or their edges.

For a triangulation τ, we define $|\tau| := \max_{K \in \tau} \mathrm{diam} K$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. By Sobolev's imbedding theorem, we have the continuous inclusion $W^{2,p}(\Omega) \subset C^0(\overline{\Omega})$ for any $p \in [1, \infty]$. Note that for $p = 1$ Morry's inequality is not applicable and the inclusion $W^{2,1}(\Omega) \subset C^0(\overline{\Omega})$ is not so obvious. For a proof of the critical imbedding, see [1, Theorem 4.12] and [4, Lemma 4.3.4].

2 Kobayashi's formula, the circumradius condition, and Schwarz's example

Recently, we made progress on the error analysis of linear interpolation on triangular elements. Liu–Kikuchi presented an explicit form of the constant C in the maximum angle condition [12]. Being inspired by Liu–Kikuchi's result, Kobayashi, one of the authors, obtained the following remarkable result with the assistance of numerical validated computation [10].

Theorem 1 (Kobayashi's formula) *Let A, B, C be the lengths of the three edges of K and S be the area of K. Define the constant $C(K)$ by*

$$C(K) := \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}\right)},$$

then the following estimate holds:

$$|v - I_K v|_{1,2,K} \le C(K)|v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

Let R_K be the radius of the circumcircle of K. From the formula $R_K = ABC/4S$, we realize $C(K) < R_K$ and obtain a corollary of Kobayashi's formula.

Corollary 2 *For any triangle $K \subset \mathbb{R}^2$, the following estimate holds*:

$$|v - I_K v|_{1,2,K} \le R_K|v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

Let $\theta_K \ge \pi/3$ be the maximum angle of K. By the law of sines, we have $h_k = 2R_K \sin\theta_K$. Therefore, if there is a constant θ_1, $2\pi/3 \le \theta_1 < \pi$ such that $\theta_K \le \theta_1$, then $h_K \ge (2\sin\theta_1)R_K$ and $\lim_{h_K \to 0} R_K = 0$. This means that, under the assumption $h_K \to 0$, (i) *the maximum angle condition implies the circumradius condition.*

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and an arbitrary $v \in H^2(\Omega)$ is taken and fixed. Consider now the isosceles triangle $K \subset \Omega$ depicted in Fig. 1. If $0 < h < 1$ and $\alpha > 1$, then $h^\alpha < h$ and the circumradius of K is $R_K = h^\alpha/2 + h^{2-\alpha}/8$. Hence, Kobayashi's formula and its corollary yield that, if $1 < \alpha < 2$, $|v - I_K v|_{1,2,K} \le R_K|v|_{2,2,K}$ and $R_K \to 0$ as $h \to 0$, whereas the maximum angle of K approaches π. This means

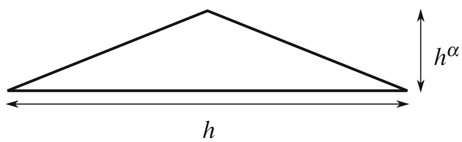

Fig. 1 An example of triangles which violates the maximum angle condition but $R_K \to 0$ as $h \to 0$

that, when $h_K \to 0$, (ii) *the circumradius condition does not necessarily imply the maximum angle condition*.

Gathering from (i) and (ii), we infer that the circumradius of a triangle is a more important indicator than its minimum and maximum angles.

Without the assistance of numerical validated computation, the authors then proved for arbitrary $p \in [1, \infty]$ the following theorem.

Theorem 3 (The circumradius condition [11]) *For an arbitrary triangle K with $R_K \leq 1$, there exists a constant C_p independent of K such that the following estimate holds*:

$$\|v - I_K v\|_{1,p,K} \leq C_p R_K |v|_{2,p,K}, \qquad \forall v \in W^{2,p}(K), \quad 1 \leq p \leq \infty. \tag{4}$$

For the case $p = 2$, the estimate (4) was shown by Rand [17, Theorem 7.10] in his Ph.D. dissertation but it was not published in a research paper.

Combining Céa's lemma and Corollary 2 or Theorem 3, we immediately obtain the following estimation.

Theorem 4 *Let u be the exact solution of* (1) *and u_h be the FEM solution of* (2). *Suppose that $u \in H^2(\Omega)$. Then we have, for $R_\tau \leq 1$,*

$$\|u - u_h\|_{1,2,\Omega} \leq C R_\tau |u|_{2,2,\Omega}, \qquad R_\tau := \max_{K \in \tau} R_K, \tag{5}$$

where the positive constant C depends only on C_2 and Ω.

Note that it follows from Corollary 2 that $C_2 = 1$. However, proving this without using validated numerical computation is not easy.

The isosceles triangle in Fig. 1 reminded the authors of Schwarz's example. As is well understood, the length of a curve is defined as the limit of the length of the inscribed polygonal edges. Hence, one might think that the area of a surface could be defined in a similar manner. Actually, mathematicians in the 19th century believed that the area of surface is the limit of the areas of inscribed polygonal surfaces.

In the 1880s, Schwarz and Peano independently showed, however, that this definition does not work [5,15,16,20]. Let Ω be a rectangle of height H and width $2\pi r$. Let m, n be positive integers. Suppose that this rectangle is divided into m equal strips, each of height H/m. Each strip is then divided into isosceles triangles whose base length is $2\pi r/n$, as depicted in Fig. 2. Then, the piecewise linear map $\varphi_\tau : \Omega \to \mathbb{R}^3$

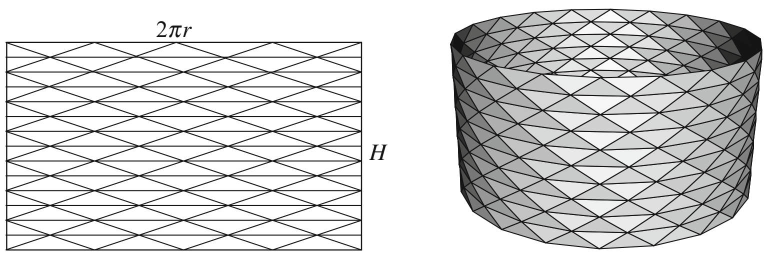

Fig. 2 Schwarz's example

is defined by “rolling up this rectangle” so that all vertexes are on the cylinder of height H and radius r. Then, the cylinder is approximated by the inscribed polygonal surface which consists of $2mn$ congruent isosceles triangles. Because the height of each triangle is $\sqrt{(H/m)^2 + r^2(1-\cos(\pi/n))^2}$ and the base length is $2r\sin(\pi/n)$, the area A_E of the inscribed polygonal surface is[1]

$$\begin{aligned} A_E &= 2mnr \sin\frac{\pi}{n}\sqrt{\left(\frac{H}{m}\right)^2 + r^2\left(1-\cos\frac{\pi}{n}\right)^2} \\ &= 2\pi r \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\sqrt{H^2 + \frac{\pi^4 r^2}{4}\left(\frac{m}{n^2}\right)^2\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^4}. \end{aligned}$$

If $m, n \to \infty$, we observe

$$\lim_{m,n\to\infty} A_E = 2\pi r\sqrt{H^2 + \frac{\pi^4 r^2}{4}\lim_{m,n\to\infty}\left(\frac{m}{n^2}\right)^2},$$

in particular,

$$\lim_{m,n\to\infty} A_E = 2\pi r H \quad \text{if and only if} \quad \lim_{m,n\to\infty}\frac{m}{n^2} = 0.$$

As we are now aware that the circumradius is an important factor, we compute the circumradius R of the isosceles triangle in Schwarz’s example. By a straightforward computation, we find that

$$R = \frac{\frac{H^2}{m} + \pi^2 r^2 \frac{m}{n^2}\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^2}{2\sqrt{H^2 + \frac{\pi^4 r^2}{4}\left(\frac{m}{n^2}\right)^2\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^4}}$$

and immediately realize that

$$\lim_{m,n\to\infty} A_E = 2\pi r H \Longleftrightarrow \lim_{m,n\to\infty}\frac{m}{n^2} = 0 \Longleftrightarrow \lim_{m,n\to\infty} R = 0. \tag{6}$$

This fact strongly suggests that the circumradius of triangles in a triangulation is essential for error estimations of linear interpolations.

With (6) in mind, we perform a numerical experiment similar to the one in [8]. Let $\Omega := (-1,1)\times(-1,1)$, $f(x,y) := a^2/(a^2-x^2)^{3/2}$, and $g(x,y) := (a^2-x^2)^{1/2}$ with $a := 1.1$. Then we consider the following Poisson equation: Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{7}$$

[1] The subscript ‘E’ of A_E stands for ‘Elementary’.

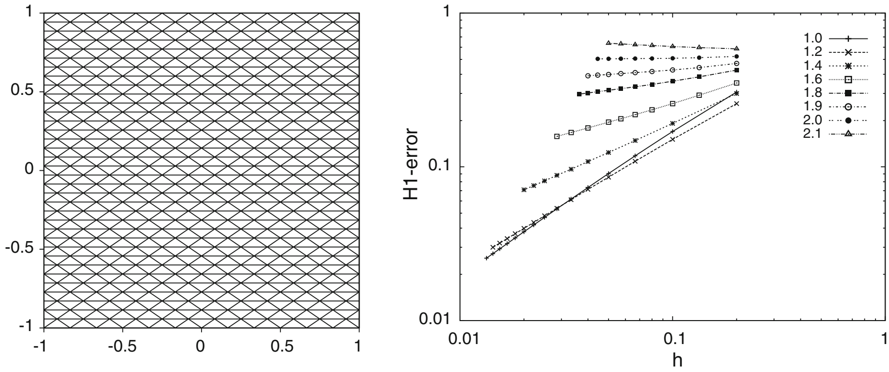

Fig. 3 The triangulation of Ω with $N = 12$ and $\alpha = 1.6$ and the errors for FEM solutions in H^1-norm. The *horizontal axis* represents the maximum diameter of the triangles and the *vertical axis* represents H^1-norm of the errors of the FEM solutions. The *number* next to the *symbol* indicates the value of α

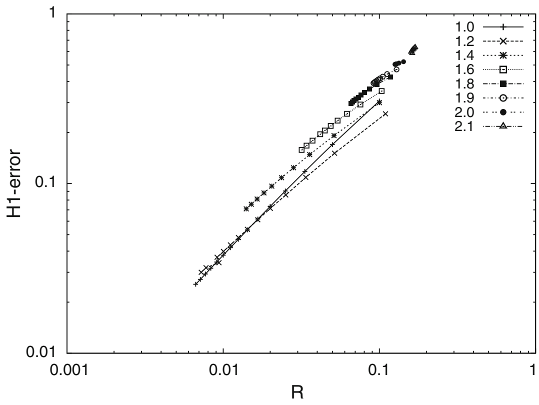

Fig. 4 Replotted data: the errors in H^1-norm of FEM solutions measured using the circumradius. The *horizontal axis* represents the maximum circumradius of the triangles

The exact solution of (7) is $u(x, y) = g(x, y)$ and its graph is a part of the cylinder. For a given positive integer N and $\alpha > 1$, we consider the isosceles triangle with base length $h := 2/N$ and height $2/\lfloor 2/h^\alpha \rfloor \approx h^\alpha$, as depicted in Fig. 1. For comparison, we also consider the isosceles triangle with base length h and height $h/2$ for $\alpha = 1$. We triangulate Ω with this triangle, as shown in Fig. 3. The behavior of the error is given in Fig. 3. The horizontal axis represents the mesh size measured by the maximum diameter of triangles in the meshes and the vertical axis represents the error associated with FEM solutions in H^1-norm. The graph clearly shows that the convergence rates worsen as α approaches 2.0. For $\alpha = 2.1$, the FEM solutions even diverge. We replot the same data in Fig. 4, in which the horizontal axis represents the maximum of the circumradius of triangles in the meshes. Figure 4 shows convergence rates are almost the same in all cases if we measure these with the circumradius.

From the results of the numerical experiments, we draw the following conclusions: suppose that we consider the Poisson equation (7).

- In our example, although the triangulation does not satisfy the maximum angle condition, the FEM solutions converge to the exact solution and the error behaves exactly as the estimation (5) predicts. If the triangulation does not satisfy the circumradius condition, the FEM solutions diverge even if meshes become finer with respect to the maximum diameter of the triangles. From this observation, we infer that, for convergence of the FEM solutions, the circumradius solution is *more essential* than the maximum angle condition and is the *best possible* as a geometric condition for triangulation.[2]
- The numerical experiments in [8] show that, in certain combinations of an exact solution and triangulation, FEM solutions can converge to an exact solution, although triangulation does not satisfy the maximum angle condition. We notice that their triangulations do not satisfy the circumradius condition either. Hence, the circumradius condition is not necessary for convergence of the finite element method.

These conclusions answer, partially but not completely, the question which Hannukainen–Korotov–Křížek posed. We infer from the numerical experiments that matching between exact solutions and geometry of triangulation seems important. Further and deeper understanding of how FEM solutions converge to an exact solution is strongly desired.

3 The circumradius condition and the definition of surface area

At the present time, the most general definition of surface area is that of Lebesgue. Let $\Omega := (a, b) \times (c, d) \subset \mathbb{R}^2$ be a rectangle and τ_n be a sequence of triangulation of Ω such that $\lim_{n\to\infty} |\tau_n| = 0$. Let $f \in C^0(\overline{\Omega})$ be a given continuous function. Let $f_n \in S_{\tau_n}$ be such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $\overline{\Omega}$. Note that the graph of $z = f_n(x, y)$ is a set of triangles and its area is defined as a sum of these triangular areas. We denote this area by $A_E(f_n)$ and have

$$A_E(f_n) = \int_{\Omega} \sqrt{1 + |\nabla f_n|^2} \mathrm{d}x.$$

Let Φ_f be the set of all such sequences $\{(f_n, \tau_n)\}_{n=1}^{\infty}$. Then the area $A_L(f) = A_L(f; \Omega)$ of the graph $z = f(x, y)$ is defined by

$$A_L(f) = A_L(f; \Omega) := \inf_{\{(f_n,\tau_n)\}\in\Phi_f} \liminf_{n\to\infty} A_E(f_n).$$

This $A_L(f)$ is called the **surface area of** $z = f(x, y)$ **in the Lebesgue sense**. For a fixed f, $A_L(f; \Omega)$ is additive and continuous with respect to the rectangular domain Ω. Tonelli then presented the following theorem.

[2] By the statement (i) given after Corollary 2, we realize that if the circumradius condition does not hold then the maximum angle condition does not hold either.

For a continuous function $f \in C^0(\overline{\Omega})$, we define $W_1(x)$, $W_2(y)$ by

$$W_1(x) := \sup_{\tau(y)} \sum_i \left| f(x, y_{i-1}) - f(x, y_i) \right|, \quad x \in (a, b),$$
$$W_2(y) := \sup_{\tau(x)} \sum_j \left| f(x_{j-1}, y) - f(x_j, y) \right|, \quad y \in (c, d),$$

where $\tau(y)$, $\tau(x)$ are subdivisions $c = y_0 < y_1 < \cdots < y_N = d$ and $a = x_0 < x_1 < \cdots < x_M = b$, respectively and 'sup' are taken for all such subdivisions. Then, a function f has **bounded variation in the Tonelli sense** if

$$\int_a^b W_1(x)\mathrm{d}x + \int_c^d W_2(y)\mathrm{d}y < \infty.$$

Also, a function f is called **absolutely continuous in the Tonelli sense** if, for almost all $y \in (c, d)$ and $x \in (a, b)$, the functions $g(x) := f(x, y)$ and $h(y) := f(x, y)$ are absolutely continuous on (a, b) and (c, d), respectively.

Theorem 5 (Tonelli) *For a continuous function $f \in C(\overline{\Omega})$ defined on a rectangular domain Ω, its graph $z = f(x, y)$ has finite area $A_L(f) < \infty$ if and only if f has bounded variation in the Tonelli sense. If this is the case, we have*

$$A_L(f) \geq \int_\Omega \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}\mathbf{x}. \tag{8}$$

In the above inequality, the equality holds if and only if f is absolutely continuous in the Tonelli sense.

For a proof of this theorem, see [18, Chapter V, pp.163–185]. It follows from Tonelli's theorem that if $f \in W^{1,\infty}(\Omega)$ then the area $A_L(f)$ is finite and the equality holds in (8). In the following theorem we consider the case $f \in W^{2,1}(\Omega)$.

Theorem 6 *Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain. If $f \in W^{2,1}(\Omega)$, then its graph has finite area, that is, $A_L(f) < \infty$, and the equality holds in (8). Moreover, if a sequence $\{\tau_n\}_{n=1}^\infty$ of triangulations of Ω satisfies the circumradius condition, then we have*

$$\lim_{n \to \infty} \int_\Omega \sqrt{1 + |\nabla I_{\tau_n} f|^2} \mathrm{d}x = A_L(f) = \int_\Omega \sqrt{1 + |\nabla f|^2}\, \mathrm{d}\mathbf{x}.$$

Proof At first, we notice f is of bounded variation and absolutely continuous in the Tonelli sense. Let $\omega := \{x\} \times (c, d)$ for $x \in (a, b)$. We consider the trace operator $\gamma : W^{2,1}(\Omega) \to W^{1,1}(\omega)$ defined by $(\gamma f)(y) := f(x, y)$. Then, γ is a bounded linear operator and it is easy to see that

$$\sum_{\tau(y)} \left| (\gamma f)(y_{i-1}) - (\gamma f)(y_i) \right| \leq \int_c^d \left| (\gamma f)'(y) \right| \mathrm{d}y, \quad W_1(x) \leq \int_c^d \left| f_y(x, y) \right| \mathrm{d}y.$$

Similarly, we obtain $W_2(y) \le \int_a^b |f_x(x, y)|\mathrm{d}x$ and Fubini's theorem implies that f has bounded variation in the Tonelli sense. Hence, Theorem 5 yields $A_L(f) < \infty$ and (8) holds. We show that f is absolutely continuous in the Tonelli sense in exactly the same manner. Therefore, the equality holds in (8).

For the piecewise linear interpolation $I_{\tau_n} f$, we have

$$A_E(I_{\tau_n} f) = A_L(I_{\tau_n} f) = \int_\Omega \sqrt{1 + (I_{\tau_n} f)_x^2 + (I_{\tau_n} f)_y^2}\, \mathrm{d}\mathbf{x}.$$

Hence, $|A_L(f) - A_E(I_{\tau_n} f)|$ is estimated as

$$\begin{aligned}|A_L(f) - A_E(I_{\tau_n} f)| &\le \int_\Omega \left|\sqrt{1 + f_x^2 + f_y^2} - \sqrt{1 + (I_{\tau_n} f)_x^2 + (I_{\tau_n} f)_y^2}\right| \mathrm{d}\mathbf{x} \\ &\le \int_\Omega \frac{\left|\left(f_x + (I_{\tau_n} f)_x\right)\left(f_x - (I_{\tau_n} f)_x\right) + \left(f_y + (I_{\tau_n} f)_y\right)\left(f_y - (I_{\tau_n} f)_y\right)\right|}{\sqrt{1 + f_x^2 + f_y^2} + \sqrt{1 + (I_{\tau_n} f)_x^2 + (I_{\tau_n} f)_y^2}} \mathrm{d}\mathbf{x} \\ &\le |f - I_{\tau_n} f|_{1,1,\Omega} \\ &\le C_1 R_{\tau_n} |f|_{2,1,\Omega} \to 0 \qquad \text{as } R_{\tau_n} \to 0,\end{aligned}$$

because

$$\frac{\left|f_x + (I_{\tau_n} f)_x\right|}{\sqrt{1 + f_x^2 + f_y^2} + \sqrt{1 + (I_{\tau_n} f)_x^2 + (I_{\tau_n} f)_y^2}} \le 1,$$
$$\frac{\left|f_y + (I_{\tau_n} f)_y\right|}{\sqrt{1 + f_x^2 + f_y^2} + \sqrt{1 + (I_{\tau_n} f)_x^2 + (I_{\tau_n} f)_y^2}} \le 1.$$

Thus, Theorem 6 is proved. □

Note that, from Schwarz's example, Theorem 6 is the *best possible* with respect to the geometric condition for triangulation. At this point, one might be tempted to define the surface area using the circumradius condition in the following way:

Definition 7 Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Suppose that a sequence $\{\tau_n\}$ of triangulation of Ω satisfies the circumradius condition. Then, for a continuous function $f \in C^0(\overline{\Omega})$, the area $A_{CR}(f)$ of the surface $z = f(x, y)$ is defined by

$$A_{CR}(f) := \lim_{n\to\infty} A_E(I_{\tau_n} f).$$

Theorem 6 claims that, for $f \in W^{2,1}(\Omega)$, $A_{CR}(f)$ is well-defined and $A_{CR}(f) = A_L(f) < \infty$. The example given by Besicovitch shows that $A_{CR}(f)$ is not well-defined in $C^0(\overline{\Omega})$ in general [3]. That is, there exists $f \in C^0(\overline{\Omega})$ and two triangulation sequences $\{\tau_n\}_{n=1}^\infty$, $\{\mu_n\}_{n=1}^\infty$ of Ω which satisfy the circumradius condition such that $A_L(f) < \infty$ and

$$\lim_{n\to\infty} A_E(I_{\tau_n} f) \ne \lim_{n\to\infty} A_E(I_{\mu_n} f).$$

Therefore, we present the following problem. Let X be a Banach space such that $W^{2,1}(\Omega) \subset X \subset L^1(\Omega)$.

Problem 8 (1) Determine the largest function space X such that $A_{CR}(f)$ is well-defined for any $f \in C^0(\overline{\Omega}) \cap X$.
(2) With X defined in (1), *prove* or *disprove* whether $A_{CR}(f) = A_L(f)$ for any $f \in C^0(\overline{\Omega}) \cap X$ with $A_L(f) < \infty$.

4 Concluding remarks: history repeats itself

We have shown that the circumradius condition is more essential than the maximum angle condition for convergence of FEM solutions. Also, we have pointed out a close relationship between the circumradius condition and the definition of surface area. In concluding, we draw readers' attention to the similarity of two histories. After Schwarz and Peano found their counter example, mathematicians naturally tried to find a *proper* definition of surface area. The authors are unfamiliar with the history behind that quest. Instead, we suggest that readers look at [16, Chapter I] from which we mention the following remarks.

Let $S \subset \mathbb{R}^3$ be a general parametric surface. If there exists a Liptschitz map $\varphi : \Omega \to \mathbb{R}^3$ defined on a domain $\Omega \subset \mathbb{R}^2$ such that $S = \varphi(\Omega)$, S is called **rectifiable**. Let Ω be a rectangle and $\varphi : \Omega \to \mathbb{R}^3$ be a rectifiable surface. Suppose that we have a sequence $\{\tau_n\}_{n=1}^{\infty}$ of triangulation of Ω such that $|\tau_n| \to 0$ as $n \to \infty$. Then, the rectifiable surface φ has linear interpolations $I_{\tau_n}\varphi$. Rademacher showed [13,14] that if $\{\tau_n\}_{n=1}^{\infty}$ satisfies the minimum angle condition we have $\lim_{n\to\infty} A_E(I_{\tau_n}\varphi) = A_L(\varphi)$. Then, Young showed [19] that if $\{\tau_n\}_{n=1}^{\infty}$ satisfies the maximum angle condition we have $\lim_{n\to\infty} A_E(I_{\tau_n}\varphi) = A_L(\varphi)$. See also the comment by Fréchet [7] on Young's result.[3]

This means that the minimum and maximum angle conditions were already found about 50 years before they were rediscovered by FEM exponents. This is an interesting example of the proverb *History repeats itself.*

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[3] In [16, p.12], Radó wrote wrongly that the second result was by Fréchet.

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