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ORIGINAL PAPER **Area 1**

Block-centered finite difference methods for parabolic equation with time-dependent coefficient

Hongxing Rui · Hao Pan

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Abstract Two block-centered finite difference schemes are introduced and analyzed to solve parabolic equation with time-dependent diffusion coefficient. One scheme is Euler backward scheme with first order accuracy in time increment while the other is Crank–Nicolson scheme with second order accuracy in time increment. Second-order error estimates in spacial meshsize both for the original unknown and its derivatives in discrete L^2 norms are established on non-uniform rectangular grid. Numerical experiments using the schemes show that the convergence rates are in agreement with the theoretical analysis.

Keywords Block-centered finite difference · Parabolic equation · Time-dependent diffusion coefficient · Numerical analysis

Mathematics Subject Classification (2000) 65M06 · 65M12 · 65M15

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H. Rui (✉)
School of Mathematics, Shandong University, Jinan 250100, China
e-mail: hxrui@sdu.edu.cn

H. Pan
School of Information Science and Engineering, Shandong Agricultural University, Taian 271018, China
e-mail: pan_hao2003@163.com

1 Introduction

A standard numerical method used in petroleum reservoir simulation for solving second-order elliptic partial differential equations is block-centered finite differences [3,6]. Block-centered finite differences, sometimes called cell-centered finite differences, can be thought as the lowest order Raviart–Thomas mixed element method, see [7], with proper quadrature formulation [9]. In [10] a block-centered finite difference method for linear elliptic problem with diagonal diffusion coefficient was introduced, and second-order approximates both to velocity and to pressure can be obtained. Then in [1] and [2] cell-centered finite differences for linear elliptic problem with tensor diffusion coefficients were considered, and in [8] a block-centered finite difference method for the nonlinear Darcy–Forchheimer model was considered. In [11] a cell-centered finite difference method for elliptic problems on quadrilateral grids based on the lowest order Brezzi–Douglas–Marini mixed element [4] was considered.

In [10] a block-centered finite difference method for parabolic problem with backward Euler approximation in time variable was also introduced and analyzed. Then in [5] an explicit/implicite, conservative domain decomposition procedure for parabolic problems based on block-centered finite difference was introduced. But these papers just considered the case when the diffusion coefficients are independent of time variable, and their analysis can not been expanded straightforwardly to the case when the diffusion coefficients depends on time variable. This limits the application of block-centered finite difference method.

In this paper we consider the block-centered finite difference methods for parabolic equation with a time-dependent diffusion coefficient. We present two block-centered finite difference schemes, one is backward Euler scheme with first-order accuracy in time increment while the other is the Crank–Nicolson scheme with second-order accuracy in time increment. We demonstrate that the proposed schemes are second-order accuracy in space meshsize both for the original unknown, called pressure in porous media flow, and its derivatives, called velocity in porous media flow, in discrete L^2 norms on non-uniform rectangular grid. These error estimates are super-convergence. The key step to the super-convergence analysis, is to construct a proper relation between the velocity $\boldsymbol{u}$ and the difference of the pressure p, see Lemma 4.2 below for detail. Then we carry out some numerical examples to show the accuracy of the presented block-centered finite difference schemes.

The paper is organized as follows. In Sect. 2 we give the problem and some notations. In Sect. 3 we present the block-centered finite difference methods. In Sect. 4 we present the numerical analysis for the presented methods. In Sect. 5 some numerical experiments using the blocked centered finite difference methods are carried out.

Through out the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

2 The problem and some notations

In this section we consider the following parabolic problem with time-dependent diffusion coefficient in a two dimensional domain,

$$\begin{cases} \dfrac{\partial p}{\partial t} - \nabla \cdot (a(x,y,t)\nabla p) = f, & (x,y,t) \in \Omega \times (0,T), \\ p(x,y,t) = p_0(x,y), & (x,y) \in \Omega, \\ a(x,y,t)\nabla p \cdot \boldsymbol{n} = 0, & (x,y,t) \in \partial\Omega \times (0,T). \end{cases} \tag{2.1}$$

Introduce $\boldsymbol{u} = -a\nabla p$ it becomes

$$\begin{cases} \dfrac{\partial p}{\partial t} + \nabla \cdot \boldsymbol{u} = f, & (x,y,t) \in \Omega \times (0,T), \\ \boldsymbol{u} = -a(x,y,t)\nabla p, & \\ p(x,y,0) = p_0(x,y), & (x,y) \in \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & (x,y,t) \in \partial\Omega \times (0,T). \end{cases} \tag{2.2}$$

Here $\boldsymbol{n}$ represents the unit exterior normal vector to the boundary of Ω, $f(x,y,t) \in L^2(\Omega)$, a scalar function, represents the source and sink of the systems. $a = (a^x, a^y) = (a^x(x,y,t), a^y(x,y,t))$ represents the diffusion coefficient. We suppose that f and a are bounded smooth functions and there exist positive constants α and C such that

$$\alpha \le a^x \le C, \quad \alpha \le a^y \le C. \tag{2.3}$$

Usually for compressible flow in porous media, p represents the pressure while

$$\boldsymbol{u} = (u^x, u^y) = -a\nabla p = -\left(a^x \frac{\partial p}{\partial x}, a^y \frac{\partial p}{\partial y}\right) \tag{2.4}$$

represents the Darcy velocity of the fluid, so in this paper we call p pressure and $\boldsymbol{u}$ Darcy velocity.

We consider the block-centered finite difference method for the model problem. We use the partitions and notations like in [10]. For simplicity suppose $\Omega = (0,1) \times (0,1)$.

Let $N > 0$ be a positive integer. Set

$$\Delta t = T/N; \quad t^n = n\Delta t \quad \text{for } n \le T/N.$$

The two dimensional domain $\Omega = (0,1) \times (0,1)$ is partitioned by $\delta_x \times \delta_y$, where

$$\delta_x : 0 = x_{1/2} < x_{3/2} < \cdots < x_{N_x - 1/2} < x_{N_x + 1/2} = 1,$$
$$\delta_y : 0 = y_{1/2} < y_{3/2} < \cdots < y_{N_y - 1/2} < y_{N_y + 1/2} = 1.$$

For $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$, define

$$x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2},$$
$$h_i = x_{i+1/2} - x_{i-1/2}, \quad h = \max_i h_i,$$
$$h_{i+1/2} = \frac{h_{i+1} + h_i}{2} = x_{i+1} - x_i,$$

$$
\begin{aligned}
y_j &= \frac{y_{j-1/2} + y_{j+1/2}}{2}, \\
k_j &= y_{j+1/2} - y_{j-1/2}, \quad k = \max_j k_j, \\
k_{j+1/2} &= \frac{k_{j+1} + k_j}{2} = y_{j+1} - y_j, \\
\Omega_{i,j} &= (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}), \\
\Omega_{i+1/2,j} &= (x_i, x_{i+1}) \times (y_{j-1/2}, y_{j+1/2}), \\
\Omega_{i,j+1/2} &= (x_{i-1/2}, x_{i+1/2}) \times (y_j, y_{j+1}).
\end{aligned}
$$

For a function $g(x, y, t)$, let $g^n_{l,m}$ denote $g(x_l, y_m, t^n)$ where l may take values $i,\ i + 1/2$ for non-negative integers i, and m may take values $j,\ j + 1/2$ for non-negative integers j. For discrete functions with values at proper discrete points, define

$$
\begin{aligned}
[d_t g]^n_{l,m} &= \frac{g^n_{l,m} - g^{n-1}_{l,m}}{\Delta t}, \\
[d_x g]^n_{i+1/2,j} &= \frac{g^n_{i+1,j} - g^n_{i,j}}{h_{i+1/2}}, \\
[d_y g]^n_{i,j+1/2} &= \frac{g^n_{i,j+1} - g^n_{i,j}}{k_{j+1/2}}, \\
[D_x g]^n_{i,j} &= \frac{g^n_{i+1/2,j} - g^n_{i-1/2,j}}{h_i}, \\
[D_y g]^n_{i,j} &= \frac{g^n_{i,j+1/2} - g^n_{i,j-1/2}}{k_j},
\end{aligned}
$$

For discrete functions $\{\theta^n_{l,m}\}$ and $\{g^n_{l,m}\}$ define the discrete inner products, norms and semi-norms as follows,

$$
\begin{aligned}
&(\theta, g)_M = (\theta, g)_{M_x,M_y} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i k_j \theta_{i,j} g_{i,j}, \\
&\|f\|^2_M = (f, f)_{M_x,M_y}, \\
&(\theta, g)_x = (\theta, g)_{T_x,M_y} = \sum_{i=2}^{N_x} \sum_{j=1}^{N_y} h_{i-1/2} k_j \theta_{i-1/2,j} g_{i-1/2,j}, \\
&\|\theta\|^2_x = (\theta, \theta)_x, \\
&(\theta, g)_y = (\theta, g)_{M_x,T_y} = \sum_{i=1}^{N_x} \sum_{j=2}^{N_y} h_i k_{j-1/2} \theta_{i,j-1/2} g_{i,j-1/2}, \\
&\|\theta\|^2_y = (\theta, \theta)_y.
\end{aligned}
$$

Here for simplicity we omit the superscript n. For simplicity from now on we always omit the superscript n if the omission does not cause conflicts.

3 Block-centered finite difference methods

In this section we consider the block-centered finite difference methods for parabolic problem with time-dependent diffusion coefficient.

Denote by $\{W^{x,n}_{i+1/2,j}\}, \{W^{y,n}_{i,j+1/2}\}$ and $\{Z^n_{i,j}\}$ the block-centered finite difference approximations to $\{u^x(x_{i+1/2}, y_j, t^n)\}, \{u^y(x_i, y_{j+1/2}, t^n)\}$ and $\{p(x_i, y_j, t^n)\}$, respectively. Their values are defined by the backward Euler scheme and the Crank–Nicolson scheme, respectively.

Set the boundary condition and the initial approximation as follows,

$$\begin{cases} W^{x,n}_{1/2,j} = 0,\ W^{x,n}_{N_x+1/2,j} = 0,\ j = 1, \ldots, N_y, \\ W^{y,n}_{i,1/2} = 0,\ W^{y,n}_{i,N_y+1/2} = 0,\ i = 1, \ldots, N_x. \end{cases} \tag{3.1}$$

$$Z^0_{i,j} = p^0_{i,j}, \quad W^{x,0}_{i+1/2,j} = u^{x,0}_{i+1/2,j}, \quad W^{y,0}_{i,j+1/2} = u^{y,0}_{i,j+1/2}, \tag{3.2}$$

for $i = 1, \ldots, N_x,\ j = 1, \ldots, N_y$. Here

$$u^{x,0} = -\left[a^x \frac{\partial p}{\partial x}\right]^0, \quad u^{y,0} = -\left[a^y \frac{\partial p}{\partial y}\right]^0.$$

The schemes are as follows.

Scheme I For $n \geq 1$ find $\{W^{x,n}_{i+1/2,j}\}, \{W^{y,n}_{i,j+1/2}\}$ and $\{Z^n_{i,j}\}$ such that

$$[d_t Z]^n_{i,j} + [D_x W^x]^n_{i,j} + [D_y W^y]^n_{i,j} = f^n_{i,j}, \tag{3.3}$$

$$W^{x,n}_{i+1/2,j} = -[a^x d_x Z]^n_{i+1/2,j}, \tag{3.4}$$

$$W^{y,n}_{i,j+1/2} = -[a^y d_y Z]^n_{i,j+1/2}. \tag{3.5}$$

Scheme II For $n \geq 1$ find $\{W^{x,n}_{i+1/2,j}\}, \{W^{y,n}_{i,j+1/2}\}$ and $\{Z^n_{i,j}\}$ such that

$$[d_t Z]^n_{i,j} + \frac{[D_x W^x]^n_{i,j} + [D_x W^x]^{n-1}_{i,j}}{2} + \frac{[D_y W^y]^n_{i,j} + [D_y W^y]^{n-1}_{i,j}}{2} = f^{n-1/2}_{i,j}, \tag{3.6}$$

$$W^{x,n}_{i+1/2,j} = -[a^x d_x Z]^n_{i+1/2,j}, \tag{3.7}$$

$$W^{y,n}_{i,j+1/2} = -[a^y d_y Z]^n_{i,j+1/2}. \tag{3.8}$$

It is clear that the approximate solution of Scheme I or Scheme II exists uniquely.

4 Numerical analysis

In this section we verify that if the analytical solution u and p are sufficiently smooth, then Z and $W = (W^x, W^y)$ are second-order approximations to p and u, respectively.

For simplicity we sometimes omit the superscript n if the omission does not cause conflicts, and use the notation $p_{l,m}(t)$ or $p_{l,m}$ to denote $p(x_l, y_m, t)$ for suitable (l, m).

We give some lemmas first.

Lemma 4.1 *If p is sufficiently smooth then there holds*

$$\begin{cases} \frac{\partial p_{i+1/2,j}}{\partial x} = [d_x p]_{i+1/2,j} - \frac{1}{8}\left[d_x\left(h^2\frac{\partial^2 p}{\partial x^2}\right)\right]_{i+1/2,j} + \epsilon^x_{i+1/2,j}(p), \\ \frac{\partial p_{i,j+1/2}}{\partial y} = [d_y p]_{i,j+1/2} - \frac{1}{8}\left[d_y\left(k^2\frac{\partial^2 p}{\partial y^2}\right)\right]_{i,j+1/2} + \epsilon^y_{i,j+1/2}(p), \end{cases} \tag{4.1}$$

with the following approximate properties

$$\epsilon^x_{i+1/2,j}(p) = O(h^2), \quad \epsilon^y_{i,j+1/2}(p) = O(k^2). \tag{4.2}$$

Here h_i and k_j are looked as discrete functions,

$$\begin{aligned} \left[d_x\left(h^2\frac{\partial^2 p}{\partial x^2}\right)\right]_{i+1/2,j} &= \frac{1}{h_{i+1/2}}\left(h^2_{i+1}\frac{\partial^2 p_{i+1,j}}{\partial x^2} - h^2_i\frac{\partial^2 p_{i,j}}{\partial x^2}\right), \\ \left[d_y\left(k^2\frac{\partial^2 p}{\partial y^2}\right)\right]_{i,j+1/2} &= \frac{1}{k_{j+1/2}}\left(k^2_{j+1}\frac{\partial^2 p_{i,j+1}}{\partial y^2} - k^2_j\frac{\partial^2 p_{i,j}}{\partial y^2}\right). \end{aligned}$$

Proof Using Taylor's expansion we have that for any $t < T$

$$\begin{aligned} p_{i+1,j}(t) &= p_{i+1/2,j}(t) + \frac{h_{i+1}}{2}\frac{\partial p_{i+1/2,j}(t)}{\partial x} + \frac{h^2_{i+1}}{8}\frac{\partial^2 p_{i+1/2,j}(t)}{\partial x^2} \\ &\quad + \frac{1}{2}\int_{x_{i+1/2}}^{x_{i+1}} (x - x_{i+1})^2\frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx \\ &= p_{i+1/2,j}(t) + \frac{h_{i+1}}{2}\frac{\partial p_{i+1/2,j}(t)}{\partial x} \\ &\quad + \frac{h^2_{i+1}}{8}\left[\frac{\partial^2 p_{i+1,j}(t)}{\partial x^2} - \int_{x_{i+1/2}}^{x_{i+1}} \frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx\right] \\ &\quad + \frac{1}{2}\int_{x_{i+1/2}}^{x_{i+1}} (x - x_{i+1})^2\frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx. \end{aligned} \tag{4.3}$$

Similarly

$$p_{i,j}(t) = p_{i+1/2,j}(t) - \frac{h_i}{2}\frac{\partial p_{i+1/2,j}(t)}{\partial x} + \frac{h_i^2}{8}\left[\frac{\partial^2 p_{i,j}(t)}{\partial x^2} - \int_{x_{i+1/2}}^{x_i} \frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx\right]$$
$$+\frac{1}{2}\int_{x_{i+1/2}}^{x_i} (x - x_i)^2 \frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx. \tag{4.4}$$

Using Eqs. (4.3) and (4.4) and by direct calculation we have that

$$\frac{\partial p_{i+1/2,j}(t)}{\partial x} = [d_x p(t)]_{i+1/2,j} - \frac{1}{8}\left[d_x\left(h^2\frac{\partial^2 p(t)}{\partial x^2}\right)\right]_{i+1/2,j} + \epsilon^x_{i+1/2,j}(p(t)), \tag{4.5}$$

where

$$\begin{aligned}\epsilon^x_{i+1/2,j}(p(t)) &= \frac{1}{2h_{i+1/2}}\int_{x_{i+1/2}}^{x_{i+1}}\left(\frac{h_{i+1}^2}{4} - (x - x_{i+1})^2\right)\frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx\\ &\quad -\frac{1}{2h_{i+1/2}}\int_{x_{i+1/2}}^{x_i}\left(\frac{h_i^2}{4} - (x - x_i)^2\right)\frac{\partial^3 p}{\partial x^3}(x, y_j, t)\,dx\\ &= O(h^2).\end{aligned} \tag{4.6}$$

We complete the proof of the first approximation in Eq. (4.1).

Similarly, setting

$$\begin{aligned}\epsilon^y_{i,j+1/2}(p(t)) &= \frac{1}{2k_{j+1/2}}\int_{y_{j+1/2}}^{y_{j+1}}\left(\frac{k_{j+1}^2}{4} - (y - y_{j+1})^2\right)\frac{\partial^3 p}{\partial x^3}(x_i, y, t)\,dx\\ &\quad -\frac{1}{2}\int_{y_{j+1/2}}^{y_j}\left(\frac{k_j^2}{4} - (y - y_j)^2\right)\frac{\partial^3 p}{\partial x^3}(x_i, y, t)\,dx\\ &= O(k^2),\end{aligned} \tag{4.7}$$

completes the proof of the second one in Eq. (4.1). □

Define

$$\delta^n_{i,j} = \left[\frac{h^2}{8}\frac{\partial^2 p}{\partial x^2} + \frac{k^2}{8}\frac{\partial^2 p}{\partial y^2}\right]^n_{i,j} = \left[\frac{h_i^2}{8}\frac{\partial^2 p^n_{i,j}}{\partial x^2} + \frac{k_j^2}{8}\frac{\partial^2 p^n_{i,j}}{\partial y^2}\right]. \tag{4.8}$$

Lemma 4.2 *If p is sufficiently smooth then there holds*

$$\begin{cases} \left[\frac{1}{a^x}u^x\right]_{i+1/2,j} = -[d_x(p-\delta)]_{i+1/2,j} - \tilde{\epsilon}^x_{i+1/2,j}(p) \\ \left[\frac{1}{a^y}u^y\right]_{i,j+1/2} = -[d_y(p-\delta)]_{i,j+1/2} - \tilde{\epsilon}^y_{i,j+1/2}(p), \end{cases} \tag{4.9}$$

with the following approximate properties

$$\tilde{\epsilon}^x_{i+1/2,j}(p) = O(h^2+k^2), \quad \tilde{\epsilon}^y_{i,j+1/2}(p) = O(h^2+k^2). \tag{4.10}$$

Proof From the first equation of (4.1) we have that

$$\begin{aligned} \left[\frac{1}{a^x}u^x\right]_{i+1/2,j} &= -\frac{\partial p_{i+1/2,j}}{\partial x} \\ &= -[d_x p]_{i+1/2,j} + \frac{1}{8}\left[d_x\left(h^2\frac{\partial^2 p}{\partial x^2}\right)\right]_{i+1/2,j} - \epsilon^x_{i+1/2,j}(p) \\ &= -[d_x(p-\delta)]_{i+1/2,j} - \left[d_x\left(\frac{k^2}{8}\frac{\partial^2 p}{\partial y^2}\right)\right]_{i+1/2,j} - \epsilon^x_{i+1/2,j}(p). \end{aligned}$$

Set

$$\tilde{\epsilon}^x_{i+1/2,j}(p) = \epsilon^x_{i+1/2,j}(p) + \left[d_x\left(\frac{k^2}{8}\frac{\partial^2 p}{\partial y^2}\right)\right]_{i+1/2,j}. \tag{4.11}$$

Since

$$\left[d_x\left(k^2\frac{\partial^2 p}{\partial y^2}\right)\right]_{i+1/2,j} = k_j^2\left[d_x\left(\frac{\partial^2 p}{\partial y^2}\right)\right]_{i+1/2,j} = O(k^2),$$

we get the first equation of (4.9) with the estimate of $\tilde{\epsilon}^x_{i+1/2,j}(p)$.

The other part can be proven by set

$$\tilde{\epsilon}^y_{i,j+1/2}(p) = \epsilon^y_{i,j+1/2}(p) + \left[d_y\left(\frac{h^2}{8}\frac{\partial^2 p}{\partial x^2}\right)\right]_{i,j+1/2}. \tag{4.12}$$

□

The next lemma can be proven similar to [10].

Lemma 4.3 *Let* $\{V^x_{i+1/2,j}\}, \{V^y_{i,j+1/2}\}, \{W^x_{i+1/2,j}\}, \{W^y_{i,j+1/2}\}$ *and* $\{q^x_{i,j}\}, \{q^y_{i,j}\}$ *be discrete functions with* $W^x_{1/2,j} = W^x_{N_x+1/2,j} = W^y_{i,1/2} = W^y_{i,N_y+1/2} = 0$. *Then there holds*

$$\begin{cases} (d_x q^x, W^x)_x = -(q^x, D_x W^x)_M, \\ (d_y q^y, W^y)_y = -(q^y, D_y W^y)_M. \end{cases} \tag{4.13}$$

We consider the numerical analysis now. Define

$$\begin{cases} e^{p,n}_{i,j} = (Z-p)^n_{i,j}, \\ e^{x,n}_{i+1/2,j} = (W^x-u^x)^n_{i+1/2,j}, \quad e^{y,n}_{i,j+1/2} = (W^y-u^y)^n_{i,j+1/2}. \end{cases} \tag{4.14}$$

From Lemma 4.2, (3.4) and (3.5) we have that

$$\left[\frac{1}{a^x}(W^x-u^x)\right]_{i+1/2,j} = -[d_x(Z-p+\delta)]_{i+1/2,j} + \tilde{\epsilon}^x_{i+1/2,j}(p).$$
$$\left[\frac{1}{a^y}(W^y-u^y)\right]_{i,j+1/2} = -[d_y(Z-p+\delta)]_{i,j+1/2} + \tilde{\epsilon}^y_{i,j+1/2}(p).$$

That is

$$\left[\frac{1}{a^x}e^x\right]_{i+1/2,j} = -[d_x(e^p+\delta)]_{i+1/2,j} + \tilde{\epsilon}^x_{i+1/2,j}(p). \tag{4.15}$$

$$\left[\frac{1}{a^y}e^y\right]_{i,j+1/2} = -[d_y(e^p+\delta)]_{i,j+1/2} + \tilde{\epsilon}^y_{i,j+1/2}(p). \tag{4.16}$$

From Eq. (2.2) we have that

$$d_t p^n_{i,j} + [D_x u^x]^n_{i,j} + [D_y u^y]^n_{i,j} = f^n_{i,j} + \epsilon^{1,n}_{i,j}, \tag{4.17}$$

where

$$\begin{aligned} \epsilon^{1,n}_{i,j} &= d_t p^n_{i,j} - \frac{\partial p^n_{i,j}}{\partial t} + [D_x u^x]^n_{i,j} - \frac{\partial u^{x,n}_{i,j}}{\partial x} + [D_y u^y]^n_{i,j} - \frac{\partial u^{y,n}_{i,j}}{\partial y} \\ &= O(\Delta t + h^2 + k^2). \end{aligned} \tag{4.18}$$

Here we have used the fact that x_i is the midpoint of $(x_{i-1/2}, x_{i+1/2})$ and y_j is the midpoint of $(y_{j-1/2}, y_{j+1/2})$.

From (4.17) and (3.6) we have that

$$d_t(Z-p)^n_{i,j} + [D_x(W^x-u^x)]^n_{i,j} + [D_y(W^y-u^y)]^n_{i,j} = -\epsilon^{1,n}_{i,j}. \tag{4.19}$$

Denote by

$$\epsilon^{2,n}_{i,j} = d_t\delta^n_{i,j} = d_t\left(\frac{h^2}{8}\frac{\partial^2 p}{\partial x^2} + \frac{k^2}{8}\frac{\partial^2 p}{\partial y^2}\right)^n_{i,j} = \frac{h_i^2}{8}d_t\left(\frac{\partial^2 p}{\partial x^2}\right)^n_{i,j} + \frac{k_j^2}{8}d_t\left(\frac{\partial^2 p}{\partial y^2}\right)^n_{i,j}. \tag{4.20}$$

When p is sufficiently smooth, it is clear that

$$\epsilon^{2,n}_{i,j} = O(h^2+k^2). \tag{4.21}$$

From (4.19) we have that

$$d_t(e^p+\delta)^n_{i,j}+[D_xe^x]^n_{i,j}+[D_ye^y]^n_{i,j}=-\epsilon^{1,n}_{i,j}+\epsilon^{2,n}_{i,j}. \tag{4.22}$$

Multiplying (4.22) by $(e^p+\delta)^n_{i,j}h_ik_j$ and making summation on i, j for $1\le i\le N_x$, $1\le j\le N_y$ we have that

$$\begin{aligned}&(d_t(e^p+\delta)^n,(e^p+\delta)^n)_M+(D_xe^{x,n},(e^p+\delta)^n)_M+(D_ye^{y,n},(e^p+\delta)^n)_M\\&=(-\epsilon^{1,n}+\epsilon^{2,n},(e^p+\delta)^n)_M.\end{aligned} \tag{4.23}$$

By Lemma 4.3 we have that

$$\begin{aligned}&(d_t(e^p+\delta)^n,(e^p+\delta)^n)_M-(e^{x,n},d_x(e^p+\delta)^n)_x-(e^{y,n},d_y(e^p+\delta)^n)_y\\&=(-\epsilon^{1,n}+\epsilon^{2,n},(e^p+\delta)^n)_M.\end{aligned} \tag{4.24}$$

Noting

$$(d_t(e^p+\delta)^n,(e^p+\delta)^n)_M=\frac{1}{2}d_t\|(e^p+\delta)^n\|^2_M+\frac{\Delta t}{2}\|d_t(e^p+\delta)^n\|^2_M$$

and using (4.15) and (4.16) we have that

$$\begin{aligned}&\frac{1}{2}d_t\|(e^p+\delta)^n\|^2_M+\frac{\Delta t}{2}\|d_t(e^p+\delta)^n\|^2_M+\left(e^{x,n},\frac{1}{a^{x,n}}e^{x,n}\right)_x+\left(e^{y,n},\frac{1}{a^{y,n}}e^{y,n}\right)_y\\&=(-\epsilon^{1,n}+\epsilon^{2,n},(e^p+\delta)^n)_M+(e^{x,n},\tilde{\epsilon}^{x,n}(p))_x+(e^{y,n},\tilde{\epsilon}^{y,n}(p))_y.\end{aligned} \tag{4.25}$$

By Schwarz's inequality we have that

$$\begin{aligned}&\frac{1}{2}d_t\|(e^p+\delta)^n\|^2_M+\frac{\Delta t}{2}\|d_t(e^p+\delta)^n\|^2_M+\left\|\frac{1}{\sqrt{a^{x,n}}}e^{x,n}\right\|^2_x+\left\|\frac{1}{\sqrt{a^{y,n}}}e^{y,n}\right\|^2_y\\&=(-\epsilon^{1,n}+\epsilon^{2,n},(e^p+\delta)^n)_M+(e^{x,n},\tilde{\epsilon}^{x,n}(p))_x+(e^{y,n},\tilde{\epsilon}^{y,n}(p))_y\\&\le\frac{1}{2}\|(e^p+\delta)^n\|^2_M+\frac{1}{2}\left(\|\frac{1}{\sqrt{a^{x,n}}}e^{x,n}\|^2_x+\|\frac{1}{\sqrt{a^{y,n}}}e^{y,n}\|^2_y\right)\\&\quad+C_1(\|\epsilon^{1,n}+\epsilon^{2,n}\|^2_M+\|\tilde{\epsilon}^{x,n}(p)\|^2_x+\|\tilde{\epsilon}^{y,n}(p)\|^2_y).\end{aligned} \tag{4.26}$$

Summing (4.26) for n from 1 to m, $m\le N$ we have that

$$\begin{aligned}&\|(e^p+\delta)^m\|^2_M+\sum_{n=1}^m\Delta t^2\|d_t(e^p+\delta)^n\|^2_M\\&\quad+\sum_{n=1}^m\Delta t\left(\|\frac{1}{\sqrt{a^{x,n}}}e^{x,n}\|^2_x+\|\frac{1}{\sqrt{a^{y,n}}}e^{y,n}\|^2_y\right)\end{aligned}$$

$$\leq \sum_{n=1}^{m} \Delta t \|(e^p+\delta)^n\|_M^2 + \|(e^p+\delta)^0\|_M^2$$

$$+2C_1 \sum_{n=1}^{n} \Delta t (\|\epsilon^{1,n}+\epsilon^{2,n}\|_M^2 + \|\tilde{\epsilon}^{x,n}(p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(p)\|_y^2). \quad (4.27)$$

Using the estimates of $\epsilon^{1,n}$, $\epsilon^{2,n}$, $\tilde{\epsilon}^{x,n}(p)$, $\tilde{\epsilon}^{y,n}(p)\|_y$, by Gronwall's inequality we have that

$$\|(Z-p+\delta)^m\|_M^2 + \sum_{n=1}^{m} \Delta t^2 \|(d_t(Z-p+\delta)^n\|_M^2$$
$$+\sum_{n=1}^{m} \Delta t \left(\| \frac{1}{\sqrt{a^{x,n}}} (W^x-u^x)^n\|_x^2 + \| \frac{1}{\sqrt{a^{y,n}}} (W^y-u^y)^n\|_y^2 \right)$$
$$\leq C\|(Z-p+\delta)^0\|_M^2 + C\sum_{n=1}^{m} \Delta t (\|\epsilon^{1,n}+\epsilon^{2,n}\|_M^2 + \|\tilde{\epsilon}^{x,n}(p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(p)\|_y^2)$$
$$\leq C(\Delta t^2 + h^4 + k^4). \quad (4.28)$$

Theorem 4.4 *Suppose the analytical solution is sufficiently smooth. For Scheme I there exists a positive constant C independent of h, k and Δt such that*

$$\|(Z-p)^m\|_M + \left(\sum_{n=1}^{m} \Delta t (\|(W^x-u^x)^n\|_x^2 + \|(W^y-u^y)^n\|_y^2) \right)^{\frac{1}{2}}$$
$$\leq C(\Delta t + h^2 + k^2), \quad m \leq N. \quad (4.29)$$

Proof Combining (4.28) with the estimate for δ completes the proof. □

Next we estimate $(W^y - u^x)$ and $(W^y - u^y)$.

Multiplying (4.22) by $d_t(Z-p+\delta)_{i,j}^n h_i k_j$ and making summation on i, j for $1 \leq i \leq N_x$, $1 \leq j \leq N_y$ we have that

$$\|d_t(e^p+\delta)^n\|_M^2 + (D_x e^{x,n}, d_t(e^p+\delta)^n)_M + (D_y e^{y,n}, d_t(e^p+\delta)^n)_M$$
$$= (-\epsilon^{1,n} + \epsilon^{2,n}, d_t(e^p+\delta)^n)_M. \quad (4.30)$$

From Lemma 4.3 we have that

$$(D_x e^{x,n}, d_t(e^p+\delta)^n)_M = -(e^{x,n}, d_x d_t(e^p+\delta)^n)_x$$
$$= -(e^{x,n}, d_t d_x(e^p+\delta)^n)_x$$
$$= \left(e^{x,n}, d_t \left(\frac{1}{a^x} e^x \right)^n \right)_x - (e^{x,n}, d_t(\tilde{\epsilon}^{x,n}(p))_x$$

$$
\begin{aligned}
&= \frac{1}{2} d_t \left\| \left(\frac{1}{\sqrt{a^x}} e^x \right)^n \right\|_x^2 + \frac{1}{2} \left(d_t \left(\frac{1}{a^x} \right)^n e^{x,n}, e^{x,n} \right)_x \\
&\quad -(e^{x,n}, \epsilon^{\tilde{x},n}(d_t p))_x \\
&\le \frac{1}{2} d_t \left\| \frac{1}{\sqrt{a^{x,n}}} e^{x,n} \right\|_x^2 + C(\|e^{x,n}\|_x^2 + \|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2).
\end{aligned} \tag{4.31}
$$

Similarly

$$
(D_y e^{y,n}, d_t(e^p + \delta)^n)_M \le \frac{1}{2} d_t \left\| \frac{1}{\sqrt{a^{y,n}}} e^{y,n} \right\|_y^2 + C(\|e^{y,n}\|_y^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2). \tag{4.32}
$$

Then from Eqs. (4.30), (4.31) and (4.32) we have that

$$
\begin{aligned}
&\|d_t(e^p + \delta)^n\|_M^2 + \frac{1}{2} d_t \left[\left\| \frac{1}{\sqrt{a^{x,n}}} e^{x,n} \right\|_x^2 + \left\| \frac{1}{\sqrt{a^{y,n}}} e^{y,n} \right\|_y^2 \right] \\
&\quad \le \frac{1}{2} \|\epsilon^{1,n} + \epsilon^{2,n}\|_M^2 + \frac{1}{2} \|d_t(e^p + \delta)^n)\|_M^2 \\
&\qquad + C_2(\|e^{x,n}\|_x^2 + \|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2 + \|e^{y,n}\|_y^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2).
\end{aligned} \tag{4.33}
$$

Multiplying 2 in two sides of the equation we get that

$$
\begin{aligned}
&\|d_t(e^p + \delta)^n\|_M^2 + d_t \left[\left\| \frac{1}{\sqrt{a^{x,n}}} e^{x,n} \right\|_x^2 + \left\| \frac{1}{\sqrt{a^{y,n}}} e^{y,n} \right\|_y^2 \right] \\
&\quad \le 2C_2(\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) + \|\epsilon^{1,n} + \epsilon^{2,n}\|_M^2 \\
&\qquad + 2C_2(\|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2).
\end{aligned} \tag{4.34}
$$

Summing for n from1 to m, $m \le N$ we have that

$$
\begin{aligned}
&\sum_{n=1}^{m} \Delta t \|d_t(e^p + \delta)^n\|_M^2 + \left\| \frac{1}{\sqrt{a^{x,m}}} e^{x,m} \right\|_x^2 + \left\| \frac{1}{\sqrt{a^{y,m}}} e^{y,m} \right\|_y^2 \\
&\quad \le 2C_2 \sum_{n=1}^{m} \Delta t(\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) + \sum_{n=1}^{m} \Delta t(\|\epsilon^{1,m} + \epsilon^{2,m}\|_M^2) \\
&\qquad + 2C_2 \sum_{n=1}^{n} \Delta t(\|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2) \\
&\qquad + \left\| \frac{1}{\sqrt{a^{x,0}}} e^{x,0} \right\|_x^2 + \left\| \frac{1}{\sqrt{a^{y,0}}} e^{y,0} \right\|_y^2.
\end{aligned} \tag{4.35}
$$

Using the estimates of ϵ^1, ϵ^2, $\tilde{\epsilon}^x(p)$, $\tilde{\epsilon}^y(p)$ and the estimate of $d_t\delta$, by Gronwall's inequality we have the follow theorem.

Theorem 4.5 *Suppose the analytical solution is sufficiently smooth. For Scheme I there exists a positive constant C independent of h, k and Δt such that*

$$\left(\sum_{n=1}^{m}\Delta t\|d_t(Z-p+\delta)^n\|_M^2\right)^{\frac{1}{2}}+\|(W^x-u^x)^m\|_x+\|(W^y-u^y)^m\|_y \leq C(\Delta t+h^2+k^2),\quad m\leq N. \tag{4.36}$$

Now we consider the convergence analysis for Scheme II.

From Eq. (2.2) we have that

$$d_t p_{i,j}^n+\frac{1}{2}([D_xu^x]_{i,j}^n+[D_xu^x]_{i,j}^{n-1}+[D_yu^y]_{i,j}^n+[D_yu^y]_{i,j}^{n-1})=f_{i,j}^{n-1/2}+\epsilon_{i,j}^{3,n}, \tag{4.37}$$

where, like (4.18),

$$\begin{aligned}\epsilon_{i,j}^{3,n}&=d_tp_{i,j}^n-\frac{\partial p_{i,j}^{n-1/2}}{\partial t}+\frac{1}{2}[D_xu^x]_{i,j}^n+\frac{1}{2}[D_xu^x]_{i,j}^{n-1}-\frac{\partial u_{i,j}^{x,n-1/2}}{\partial x}\\&\quad+\frac{1}{2}[D_yu^y]_{i,j}^n+\frac{1}{2}[D_yu^y]_{i,j}^{n-1}-\frac{\partial u_{i,j}^{y,n-1/2}}{\partial y}\\&=O(\Delta t^2+h^2+k^2).\end{aligned} \tag{4.38}$$

From (4.37) and (3.6) we have that

$$\begin{aligned}&d_t(Z-p)_{i,j}^n+\frac{1}{2}([D_x(W^x-u^x)]_{i,j}^n+[D_x(W^x-u^x)]_{i,j}^{n-1})\\&\quad+\frac{1}{2}([D_y(W^y-u^y)]_{i,j}^n+[D_y(W^y-u^y)]_{i,j}^{n-1})=-\epsilon_{i,j}^{3,n}.\end{aligned} \tag{4.39}$$

Using the notations as before, from (4.39) we have that

$$\begin{aligned}&d_t(e^p+\delta)_{i,j}^n+\frac{1}{2}([D_xe^x]_{i,j}^n+[D_xe^x]_{i,j}^{n-1})+\frac{1}{2}([D_ye^y]_{i,j}^n+[D_ye^y]_{i,j}^{n-1})\\&=-\epsilon_{i,j}^{3,n}+\epsilon_{i,j}^{2,n}.\end{aligned} \tag{4.40}$$

First we estimate e^x and e^y.

Multiplying (4.40) by $d_t(e^p+\delta)_{i,j}^n h_ik_j$ and making summation on i, j for $1\leq i\leq N_x$, $1\leq j\leq N_y$ we have that

$$\begin{aligned}&\|d_t(e^p+\delta)^n\|_M^2+\frac{1}{2}(D_xe^{x,n}+D_xe^{x,n-1},d_t(e^p+\delta)^n)_M\\&\quad+\frac{1}{2}(D_ye^{y,n}+D_ye^{y,n-1},d_t(e^p+\delta)^n)_M\\&=(-\epsilon^{3,n}+\epsilon^{2,n},d_t(e^p+\delta)^n)_M.\end{aligned} \tag{4.41}$$

From Lemmas 4.2 and 4.3 we have that

$$\begin{aligned}
&\frac{1}{2}(D_x e^{x,n} + D_x e^{x,n-1}, d_t(e^p+\delta)^n)_M \\
&\quad = -\frac{1}{2}(e^{x,n} + e^{x,n-1}, d_t d_x (e^p+\delta)^n)_x \\
&\quad = \frac{1}{2}\left(e^{x,n} + e^{x,n-1}, d_t\left(\frac{1}{a^x}e^x\right)^n\right)_x - \frac{1}{2}(e^{x,n}+e^{x,n-1}, d_t(\epsilon^{\tilde{x},n}(p))_x \\
&\quad = \frac{1}{2}d_t\| \left(\frac{1}{\sqrt{a^x}}e^x\right)^n \|_x^2 + \frac{1}{2}\left(d_t\left(\frac{1}{a^{x,n}}\right)e^{x,n}, e^{x,n-1}\right)_x \\
&\qquad -\frac{1}{2}(e^{x,n}+e^{x,n-1}, \epsilon^{\tilde{x},n}(d_t p))_x \\
&\quad \geq \frac{1}{2}d_t\left\|\frac{1}{\sqrt{a^{x,n}}}e^{x,n}\right\|_x^2 - C(\|e^{x,n}\|_x^2 + \|e^{x,n-1}\|_x^2 + \|\epsilon^{\tilde{x},n}(d_t p)\|_x^2). \qquad (4.42)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\frac{1}{2}(e^{y,n} + e^{y,n-1}, d_t(e^p+\delta)^n)_M \\
&\quad \geq \frac{1}{2}d_t\|\frac{1}{\sqrt{a^{y,n}}}e^{y,n}\|_y^2 - C(\|e^{y,n}\|_y^2 + \|e^{y,n-1}\|_y^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2). \qquad (4.43)
\end{aligned}$$

Then from (4.41), (4.42) and (4.43) we have that

$$\begin{aligned}
&\|d_t(e^p+\delta)^n\|_M^2 + \frac{1}{2}d_t\left[\left\|\frac{1}{\sqrt{a^{x,n}}}e^{x,n}\right\|_x^2 + \left\|\frac{1}{\sqrt{a^{y,n}}}e^{y,n}\right\|_y^2\right] \\
&\quad \leq \frac{1}{2}\|\epsilon^{3,n}+\epsilon^{2,n}\|_M^2 + \frac{1}{2}\|d_t(e^p+\delta)^n)\|_M^2 + C_3(\|e^{x,n}\|_x^2 + \|e^{x,n-1}\|_x^2 \\
&\qquad +\|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2) \\
&\qquad +C_3(\|e^{y,n}\|_y^2 + \|e^{y,n-1}\|_y^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2). \qquad (4.44) \\
&\|d_t(e^p+\delta)^n\|_M^2 + d_t\left[\left\|\frac{1}{\sqrt{a^{x,n}}}e^{x,n}\right\|_x^2 + \left\|\frac{1}{\sqrt{a^{y,n}}}e^{y,n}\right\|_y^2\right] \\
&\quad \leq \|\epsilon^{3,n}+\epsilon^{2,n}\|_M^2 + 2C_3(\|e^{x,n}\|_x^2 + \|e^{x,n-1}\|_x^2 + \|e^{y,n}\|_y^2 + \|e^{y,n-1}\|_y^2) \\
&\qquad +2C_3(\|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2). \qquad (4.45)
\end{aligned}$$

Summing for n from 1 to m, $m \leq N$ we have that

$$\begin{aligned}
&\sum_{n=1}^{m}\Delta t\|d_t(e^p+\delta)^n\|_M^2 + \left\|\frac{1}{\sqrt{a^{x,n}}}e^{x,m}\right\|_x^2 + \left\|\frac{1}{\sqrt{a^{y,m}}}e^{y,m}\right\|_y^2 \\
&\quad \leq 4C_3\sum_{n=1}^{m}\Delta t(\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) + \sum_{n=1}^{m}\Delta t(\|\epsilon^{3,n}+\epsilon^{2,n}\|_M^2)
\end{aligned}$$

$$+2C_3\sum_{n=1}^{m}\Delta t((\|\tilde{\epsilon}^{x,n}(d_t p)\|_x^2+\|\tilde{\epsilon}^{y,n}(d_t p)\|_y^2)$$
$$+(1+2C_3\Delta t)\left\|\frac{e^{x,0}}{\sqrt{a^{x,0}}}\right\|_x^2+(1+2C_3\Delta t)\left\|\frac{e^{y,0}}{\sqrt{a^{y,0}}}\right\|_y^2. \quad (4.46)$$

Using the estimates of $\epsilon^{3,n}$, $\epsilon^{2,n}$, $\tilde{\epsilon}^{x,n}(p)$, $\tilde{\epsilon}^{y,n}(p)$ and the estimate of $d_t\delta$, by Gronwall's inequality we have that

Theorem 4.6 *Suppose the analytical solution is sufficiently smooth. For Scheme II there exists a positive constant C independent of h, k and Δt such that*

$$\left(\sum_{n=1}^{m}\Delta t\|d_t(Z-p+\delta)^n\|_M^2\right)^{\frac{1}{2}}+\|(W^x-u^x)^m\|_x+\|(W^y-u^y)^m\|_y$$
$$\le C(\Delta t^2+h^2+k^2),\quad m\le N. \quad (4.47)$$

Now we consider the error estimate of $(Z-p)$ of Scheme II.

Multiplying (4.40) by $(e^p+\delta)^n_{i,j}h_ik_j$ and making summation on i, j for $1\le i\le N_x$, $1\le j\le N_y$ we have that

$$(d_t(e^p+\delta)^n,(e^p+\delta)^n)_M+\frac{1}{2}(D_x(e^{x,n}+e^{x,n-1}),(e^p+\delta)^n)_M$$
$$+\frac{1}{2}(D_y(e^{y,n}+e^{y,n-1}),(e^p+\delta)^n)_M=(-\epsilon^{3,n}+\epsilon^{2,n},(e^p+\delta)^n)_M. \quad (4.48)$$

By Lemma 4.3 we have that

$$(d_t(e^p+\delta)^n,(e^p+\delta)^n)_M-\frac{1}{2}(e^{x,n}+e^{x,n-1},d_x(e^p+\delta)^n)_x$$
$$-\frac{1}{2}(e^{y,n}+e^{y,n-1},d_y(e^p+\delta)^n)_y=(-\epsilon^{3,n}+\epsilon^{2,n},(e^p+\delta)^n)_M. \quad (4.49)$$

Using (4.15) and (4.16) we have that

$$\frac{1}{2}d_t\|(e^p+\delta)^n\|_M^2+\frac{\Delta t}{2}\|(d_t(e^p+\delta)^n\|_M^2$$
$$=-\frac{1}{2}\left(e^{x,n}+e^{x,n-1},\frac{1}{a^{x,n}}e^{x,n}\right)_x-\frac{1}{2}\left(e^{y,n}+e^{y,n-1},\frac{1}{a^{y,n}}e^{y,n}\right)_y$$
$$+(-\epsilon^{3,n}+\epsilon^{2,n},(e^p+\delta)^n)_M+(e^{x,n},\tilde{\epsilon}^{x,n}(p))_x+(e^{y,n},\tilde{\epsilon}^{y,n}(p))_y. \quad (4.50)$$

By Schwarz's inequality we have that

$$\frac{1}{2}d_t\|(e^p+\delta)^n\|_M^2+\frac{\Delta t}{2}\|(d_t(e^p+\delta)^n\|_M^2$$
$$\le\|(e^p+\delta)^n\|_M^2+C_3(\|e^{x,n}\|_x^2+\|e^{x,n-1}\|_x^2+\|e^{y,n}\|_y^2+\|e^{y,n-1}\|_y^2)$$
$$+C_3(\|\epsilon^{3,n}+\epsilon^{2,n}\|_M^2+\|\tilde{\epsilon}^{x,n}(p)\|_x^2+\|\tilde{\epsilon}^{y,n}(p)\|_y^2). \quad (4.51)$$

Summing for n from 1 to m, $m \le N$ we have that

$$\begin{aligned}
&\|(e^p+\delta)^m\|_M^2 + \sum_{n=1}^m \Delta t^2 \|(d_t(e^p+\delta)^n\|_M^2 \\
&\le \sum_{n=1}^m \Delta t \|(e^p+\delta)^n\|_M^2 + 4C_3 \sum_{n=1}^m \Delta t (\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) \\
&\quad +2C_3 \sum_{n=1}^m \Delta t (\|\epsilon^{3,n}+\epsilon^{2,n}\|_M^2 + \|\tilde{\epsilon}^{x,n}(p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(p)\|_y^2) + \|(e^p+\delta)^0\|_M^2.
\end{aligned} \tag{4.52}$$

Using Theorem 4.6, the estimates of ϵ^3, ϵ^2, $\tilde{\epsilon}^x(p)$, $\tilde{\epsilon}^y(p)$, and by Gronwall's inequality we have that

$$\begin{aligned}
&\|(Z-p+\delta)^m\|_M^2 + \sum_{n=1}^m \Delta t^2 \|(d_t(Z-p+\delta)^n\|_M^2 \\
&\le C\|(Z-p+\delta)^0\|_M^2 + C\sum_{n=1}^m \Delta t (\|\epsilon^{3,n}+\epsilon^{2,n}\|_M^2 + \|\tilde{\epsilon}^{x,n}(p)\|_x^2 + \|\tilde{\epsilon}^{y,n}(p)\|_y^2) \\
&\quad +C\sum_{n=1}^m \Delta t (\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) \\
&\le C(\Delta t^4 + h^4 + k^4).
\end{aligned} \tag{4.53}$$

Using the estimate of δ we have the following theorem.

Theorem 4.7 *Suppose the analytical solution is sufficiently smooth. For Scheme II there exists a positive constant C independent of h, k and Δt such that*

$$\|(Z-p)^m\|_M \le C(\Delta t^2 + h^2 + k^2), \quad m \le N. \tag{4.54}$$

5 Numerical examples

In this section we carry out some numerical experiments using the block-centered finite difference schemes. For simplicity, the region are selected as an unit square, $\Omega = [0, 1] \times [0, 1]$, while the time interval is chosen as [0, 1].

We test Examples 1 and 2 to verify the convergence rates of the presented schemes. The initial spatial partition is a 5×5 grid. And then the grid is refined 3 times, each time we divided every rectangular element into 4 uniform rectangular elements. The initial temporal step is $1/20$. The initial grid with degree of freedom is plotted in Fig. 1.

The a priori error in discrete L^2- and L^∞-norms at the last time step is computed. The time step is refined as $\Delta t = h^2$ to show the convergence for implicit Euler scheme and $\Delta t = h$ for Crank–Nicolson scheme. The numerical results are listed in Figs. 2 and 3 and Tables 1, 2, 3 and 4.

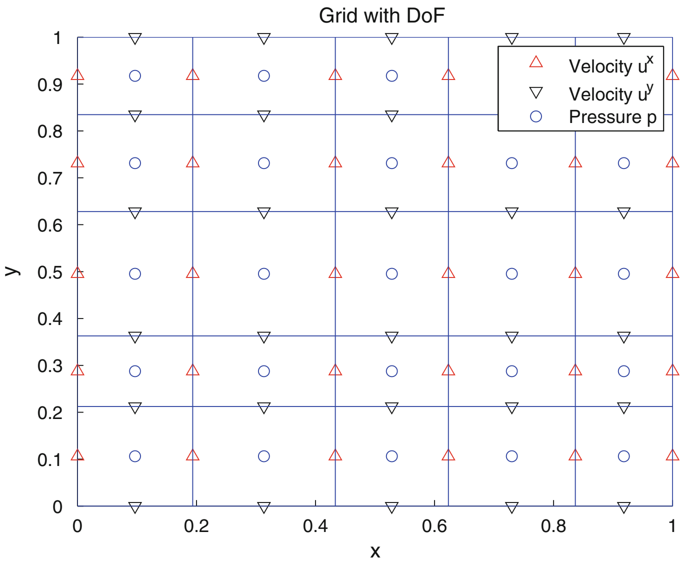

Fig. 1 Grid with degree of freedom of first level

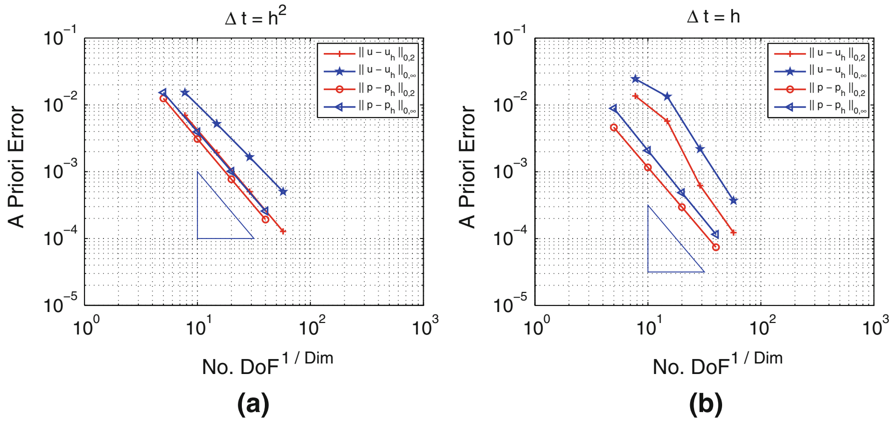

Fig. 2 Convergence rates of Example 1. *The tangent of the triangle is 2*

Example 1 A numerical example with Neumann border condition is considered as below. The the boundary condition and the right hand side of the equations are computed according to the analytic solution given as below.

$$\begin{cases} p(x, y) = \dfrac{2}{\pi} \arctan \dfrac{x + y - t - 1}{0.5}, \\ \boldsymbol{u}(x, y) = (\sin \pi t \sin \pi x \cos \pi y, \sin \pi t \cos \pi x \sin \pi y)^T, \\ a(x, y, t) = \begin{pmatrix} 2e^y (1 + 0.4 \sin x \sin y)(1 + \sin t) & 0 \\ 0 & e^x (1 + \cos t) \end{pmatrix}. \end{cases}$$

The numerical results are listed in Fig. 2 and Tables 1, 2.

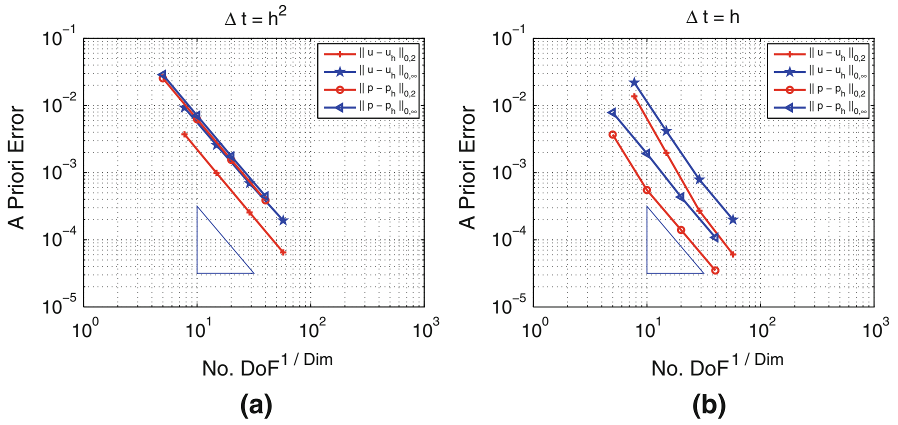

Fig. 3 Convergence rates of Example 2 *The tangent of the triangle is 2*

Table 1 Error and convergence rates of Scheme I for Example 1 ($\Delta t = h^2$)

Partition	Velocity $\|\boldsymbol{v}\|_{0,2}$ Error	Rate	Pressure $\|p\|_{0,2}$ Error	Rate
5×5	6.99E−3	–	1.25E−2	–
10×10	1.93E−3	−1.85	3.09E−3	−2.01
20×20	5.03E−4	−1.94	7.70E−4	−2.00
40×40	1.27E−4	−1.97	1.92E−4	−2.00

Table 2 Error and convergence rates of Scheme II for Example 1 ($\Delta t = h$)

Partition	Velocity $\|\boldsymbol{v}\|_{0,2}$ Error	Rate	Pressure $\|p\|_{0,2}$ Error	Rate
5×5	1.36E−2	–	4.60E−3	–
10×10	5.74E−3	−1.24	1.16E−3	−1.98
20×20	6.22E−4	−3.20	2.95E−4	−1.97
40×40	1.23E−4	−2.33	7.41E−5	−1.99

Table 3 Error and convergence rates of Scheme I for Example 2 ($\Delta t = h^2$)

Partition	Velocity $\|\boldsymbol{v}\|_{0,2}$ Error	Rate	Pressure $\|p\|_{0,2}$ Error	Rate
5×5	3.73E−3	–	2.54E−2	–
10×10	9.88E−4	−1.91	6.23E−3	−2.02
20×20	2.55E−4	−1.95	1.55E−3	−2.00
40×40	6.47E−5	−1.97	3.87E−4	−2.00

Example 2 A numerical example with Neumann border condition is considered as below. The boundary condition and the right hand side of the equations are computed according to the analytic solution given as below.

Table 4 Error and convergence rates of Scheme II for Example 2 ($\Delta t = h$)

Partition	Velocity	$\|\boldsymbol{v}\|_{0,2}$	Pressure	$\|p\|_{0,2}$
	Error	Rate	Error	Rate
5×5	1.37E−2	–	3.70E−3	–
10×10	1.97E−3	−2.79	5.50E−4	−2.74
20×20	2.70E−4	−2.86	1.40E−4	−1.97
40×40	6.04E−5	−2.16	3.49E−5	−2.00

$$\begin{cases} p(x,y,t) = \tanh \dfrac{x+y-t-1}{0.5}, \quad \boldsymbol{u}(x,y,t) = (-y,x)^T, \\ a(x,y,t) = \begin{pmatrix} e^{x+y+t}+1 & 0 \\ 0 & 0.6\sin(x+2y-t)+2 \end{pmatrix}. \end{cases}$$

The numerical results are listed in Fig. 3 and Tables 3 and 4.

From Figs. 2 and 3 and Tables 1, 2, 3 and 4, we can see that the block-centered finite difference approximations for pressure and velocity have the $(\Delta t + h^2)$ accuracy in discrete L^2-norms for implicit-Euler scheme, while they have the $(\Delta t^2 + h^2)$ accuracy in discrete L^2-norms for Crank–Nicolson scheme. These results are in consistent with the error estimates in Theorems 4.4–4.7.

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