

Shortest bibbranchings and valuated matroid intersection

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Abstract For a digraph $D = (V, A)$ and a partition $\{S, T\}$ of V , an arc set $B \subseteq A$ is called an S - T bibbranching if each vertex in T is reachable from S and each vertex in S reaches T in the subgraph (V, B) . Bibbranchings commonly generalize bipartite edge covers and arborescences. A totally dual integral linear system determining the S - T bibbranching polytope is provided by Schrijver, and the shortest S - T bibbranching problem, whose objective is to find an S - T bibbranching of minimum total arc weight, can be solved in polynomial time by the ellipsoid method or a faster combinatorial algorithm due to Keijsper and Pendavingh. The valuated matroid intersection problem, introduced by Murota, is a weighted generalization of the independent matching problem, including the independent assignment problem and the weighted matroid intersection problem. The valuated matroid intersection problem can be solved efficiently with polynomially many value oracles by extending classical combinatorial algorithms for the weighted matroid intersection problem. In this paper, we show that the shortest S - T bibbranching problem is polynomially reducible to the valuated matroid intersection problem. This reduction suggests one answer to why the shortest S - T bibbranching problem is tractable, and implies new combinatorial algorithms for the shortest S - T bibbranching problem based on the valuated matroid intersection algorithm, where a value oracle corresponds to computing a minimum-weight arborescence.

Keywords Shortest bibbranching · Arborescence · Valuated matroid intersection · Discrete convex function

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1 Introduction

Let $D = (V, A)$ be a directed graph (digraph) with vertex set V and arc set A , and let $w \in \mathbf{R}_+^A$ be a nonnegative arc-weight vector. For a partition $\{S, T\}$ of V , an arc subset $B \subseteq A$ is called an S - T *bibranching* if each vertex in T is reachable from S and each vertex in S reaches T in the subgraph (V, B) .

The notion of bibranchings was introduced by Schrijver [17]. Bibranchings commonly generalize bipartite edge covers and arborescences.

Bipartite edge cover. In an undirected graph $G = (V, E)$, an edge set $F \subseteq E$ is called an *edge cover* if every vertex is incident to at least one edge in F .

Suppose that G is a bipartite graph with color classes S and T . Then, obtain a digraph D from G by orienting every edge toward T . It is straightforward to see that an edge subset in G is an edge cover if and only if its corresponding arc subset in D is an S - T bibranching.

Arborescence. For a digraph $D = (V, A)$ and a vertex $r \in V$, an arc set $F \subseteq A$ is an r -*arborescence* if F contains no undirected cycle and each vertex $v \in V \setminus \{r\}$ has exactly one entering edge in F while r has no entering edge in F . Here, consider a partition $\{S, T\}$ of V , where $S = \{r\}$ and $T = V \setminus \{r\}$. Then, it is easy to see that a minimal S - T bibranching is an r -arborescence.

A typical optimization problem concerning edge covers and arborescences is to find those of minimum weight, when a non-negative edge/arc weight is given. So, our main interest is in the *shortest S - T bibranching problem*, the objective of which is to find an S - T bibranching B minimizing $w(B) = \sum_{e \in B} w(e)$. Generally, for a finite set V and a vector $x \in \mathbf{R}^V$, we denote $x(X) = \sum_{v \in X} x(v)$ for $X \subseteq V$.

A min-max relation for the minimum size of an S - T bibranching follows from the König-Rado edge cover theorem. Furthermore, Schrijver [17] presented a totally dual integral linear system determining the S - T bibranching polytope (see also Schrijver [19]). The shortest S - T bibranching problem can be solved in polynomial time by the ellipsoid method [17], and Keijsper and Pendavingh [9] designed a combinatorial primal-dual algorithm with running time $O(n'(m + n \log n))$, where $n' = \min\{|S|, |T|\}$, $n = |S \cup T|$ and $m = |A|$. This time complexity is almost best possible, since the current best time complexity for the minimum-weight edge cover problem is $O(n'(m + n \log n))$ [3, 24] and that for the minimum-weight arborescence problem is $O(m + n \log n)$ [7]. Therefore, it could be understood that the shortest S - T bibranching problem is as tractable as its special cases of the minimum-weight edge cover and minimum-weight arborescence problems, and that the tractability derives from the total dual integrality of the linear system determining the S - T bibranching polytope.

In this paper, we exhibit a new feature of the shortest S - T bibranching problem which explains its tractability: the shortest S - T bibranching problem is a special case of the *valuated matroid intersection problem*, introduced by Murota [12, 13]. For a function $\omega : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$, let $\text{dom } \omega$ denote the *effective domain* of ω , which is defined

by $\text{dom } \omega = \{X \mid X \subseteq V, \omega(X) > -\infty\}$. A function $\omega : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$ with $\text{dom } \omega \neq \emptyset$ is called a *valuated matroid* [1, 2] if it satisfies the following exchange property:

(VM) for each $X, Y \in 2^V$ and $u \in X \setminus Y$, there exists $v \in Y \setminus X$ such that

$$\omega((X \setminus \{u\}) \cup \{v\}) + \omega((Y \setminus \{v\}) \cup \{u\}) \geq \omega(X) + \omega(Y).$$

It is easy to see that $\text{dom } \omega$ forms the base family of a matroid of ground set V . The valuated matroid intersection problem is described as follows.

The valuated matroid intersection problem

Instance: – a ground set V ,
 – a weight vector $w \in \mathbf{R}^V$, and
 – two valuated matroids $\omega^+ : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$ and $\omega^- : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$.

Objective: maximize $w(X) + \omega^+(X) + \omega^-(X)$.

The valuated matroid intersection problem is a special case of the *valuated independent assignment problem*, which is a weighted generalization of the independent matching problem, and includes other important problems such as the independent assignment problem [8] and the weighted matroid intersection problem. The valuated independent assignment problem can be solved efficiently with polynomially many value oracles: Murota [13] designed combinatorial algorithms by naturally extending the classical weighted matroid intersection algorithms [4, 8, 10]. For more information on the valuated matroid intersection problem, readers are referred to Murota [15].

Our main result shows a relation between the shortest bibranching and valuated matroid intersection problems.

Theorem 1 *The shortest S - T bibranching problem is polynomially reducible to the valuated matroid intersection problem.*

Theorem 1 reveals a new feature of bibranchings underlying the tractability of the shortest S - T bibranching problem. Furthermore, by combining this reduction and the valuated independent assignment algorithms in [13], we obtain new combinatorial algorithms for the shortest S - T bibranching problem which is based on the weighted matroid intersection algorithms.

In proving Theorem 1, we make use of theory on a more general framework of discrete convex analysis [14], in particular theory on M -convex and M^\natural -convex functions [11, 16] (see also [6]). Our argument is an example where theory of a generalized framework gives a new insight of a certain special case.

The organization of this paper is as follows. In Sect. 2, we review bibranchings, valuated matroids, and M -convex and M^\natural -convex functions. Section 3 is devoted to proving Theorem 1. In Sect. 4, we conclude this paper with a few remarks.

2 Preliminaries

2.1 Digraphs

Let $D = (V, A)$ be a digraph. An arc from u to v is denoted by uv . For an arc a , the initial vertex is denoted by ∂^+a and the terminal vertex is denoted by ∂^-a . For a vertex $v \in V$, the set of arcs with tail v is denoted by δ^+v , and those with head v is denoted by δ^-v . For $X \subseteq V$, let $D[X] = (U, A[X])$ denote the *subgraph induced by X* , i.e., $A[X] = \{a \mid a \in A, \partial^+a \in X, \partial^-a \in X\}$. For $X, Y \subseteq V$, let $A[X, Y] = \{a \mid a \in A, \partial^+a \in X, \partial^-a \in Y\}$.

A *path* in a digraph is a sequence $(v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k)$, where $a_i = v_{i-1}v_i$ for every $i = 1, \dots, k$ and v_0, v_1, \dots, v_k are distinct. A *cycle* is a sequence $(v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k)$, where $a_i = v_{i-1}v_i$ ($i = 1, \dots, k$), v_0, v_1, \dots, v_{k-1} are distinct, and $v_0 = v_k$. For $v \in V$ and $U \subseteq V$, we say that v *reaches* U if there exists a path starting from v and ending in some vertex in U , and that v is *reachable from* U if there exists a path starting from some vertex in U and ending in v .

For an arc set $F \subseteq A$, its *reversal* is an arc set obtained by reversing every arc in F , that is, $\{vu \mid uv \in F\}$. For a digraph $D = (V, A)$, an arc subset $B \subseteq A$ is a *branching* if $|B \cap \delta^-v| \leq 1$ for every $v \in V$ and B contains no cycle. A *cobranching* is a set of edges whose reversal is a branching.

For an arc subset $B \subseteq A$, define $\partial^+B = \bigcup_{a \in B} \partial^+a$ and $\partial^-B = \bigcup_{a \in B} \partial^-a$. If B is a branching, a vertex in $V \setminus \partial^-B$ is called a *root* of B and we denote the set of roots by $R(B)$, i.e., $R(B) = V \setminus \partial^-B$. Similarly, for a cobranching B' , we define $R^*(B') = V \setminus \partial^+B'$. Recall that, for a vertex $r \in V$, an *r -arborescence* is a branching B with $R(B) = \{r\}$.

Let $D = (V, A)$ be a digraph and let $\{S, T\}$ be a partition of V . That is, $S \cap T = \emptyset$ and $S \cup T = V$. Recall that an arc set $B \subseteq A$ is an *S - T bibranching* if, in the subgraph (V, B) , every vertex in T is reachable from S and every vertex in S reaches T . A set of arcs $C \subseteq A$ is called an *S - T bicut* if $C = A[V \setminus U, U] = \{a \mid \partial^+a \in V \setminus U, \partial^-a \in U\}$ for some nonempty proper subset U of V such that $U \subseteq T$ or $U \supseteq T$.

The following linear system describes the S - T bibranching polytope:

$$0 \leq x(a) \leq 1 \quad \text{for each } a \in A, \tag{1}$$

$$x(C) \geq 1 \quad \text{for each } S\text{-}T \text{ bicut } C. \tag{2}$$

Theorem 2 (Schrijver [17]) *The linear system consisting of Eqs. (1) and (2) is totally dual integral.*

2.2 Valuated matroids, M -convex functions and M^\natural -convex functions

In this subsection, beginning with valuated matroids, we exhibit several basic facts on *M -convex functions* [11] and *M^\natural -convex functions* [16], which provide broader frameworks than that of valuated matroids, and also their relation to minimum-weight

branchings. For more detailed information on these discrete convex functions, readers are referred to Murota [14] and Fujishige [6].

Let us begin with matroids. Let V be a finite set and $\mathcal{B} \subseteq 2^V$ be its subset family with $\mathcal{B} \neq \emptyset$. Then, we say that (V, \mathcal{B}) is a *matroid* if, for each $X, Y \in \mathcal{B}$ and $u \in X \setminus Y$, there exists $v \in Y \setminus X$ such that $(X \setminus \{u\}) \cup \{v\} \in \mathcal{B}$.

Actually, this exchange axiom is equivalent to the following simultaneous exchange property:

(M) for each $X, Y \in \mathcal{B}$ and $u \in X \setminus Y$, there exists $v \in Y \setminus X$ such that

$$(X \setminus \{u\}) \cup \{v\} \in \mathcal{B} \quad \text{and} \quad (Y \cup \{u\}) \setminus \{v\} \in \mathcal{B}.$$

By considering this simultaneous exchange property, we can define a quantitative generalization (VM) in Sect. 1, which defines valuated matroids.

Matroids are defined as a pair of a ground set V and the base family $\mathcal{B} \subseteq 2^V$. By identifying a subset of V as a vector in $\{0, 1\}^V$, \mathcal{B} could be seen as a subfamily of $\{0, 1\}^V$. *M-convex sets* offer a generalization of matroids which is defined as a subfamily of \mathbf{Z}^V .

For $X \subseteq V$, the *characteristic vector* of X is a vector $\chi_X \in \mathbf{Z}^V$ defined by $\chi_X(v) = 1$ for $v \in X$ and $\chi_X(v) = 0$ for $v \in V \setminus X$. For an element $u \in V$, $\chi_{\{u\}}$ is abbreviated as χ_u . For a vector $x \in \mathbf{Z}^V$, define $\text{supp}^+(x) = \{v \mid v \in V, x(v) > 0\}$ and $\text{supp}^-(x) = \{v \mid v \in V, x(v) < 0\}$

A set of integer points $B \subseteq \mathbf{Z}^V$ is an *M-convex set* if it satisfies the following exchange axiom:

(B- EXC) for each $x, y \in B$ and $u \in \text{supp}^+(x - y)$, there exists $v \in \text{supp}^-(x - y)$ such that

$$x - \chi_u + \chi_v \in B \quad \text{and} \quad y + \chi_u - \chi_v \in B.$$

Note that (B- EXC) generalizes the axiom (M). Now, we can define M-convex functions by generalizing (DM). For a function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$, let $\text{dom } f = \{x \mid f(x) < +\infty\}$. A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$ is called an *M-convex function* if it satisfies the following property:

(M- EXC) for each $x, y \in \mathbf{Z}^V$ and $u \in \text{supp}^+(x - y)$, there exists $v \in \text{supp}^-(x - y)$ such that

$$f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \leq f(x) + f(y).$$

A function g is said to be *M-concave* if $-g$ is M-convex. It is straightforward to see that a valuated matroid is exactly an M-concave function g with $\text{dom } g \subseteq \{0, 1\}^V$.

We can also consider slightly weaker exchange properties. A set $Q \subseteq \mathbf{Z}^V$ is an *M[♯]-convex set* if it satisfies the following exchange axiom:

(B[♯]- EXC) for each $x, y \in Q$ and $u \in \text{supp}^+(x - y)$, it holds that

$$x - \chi_u \in Q \quad \text{and} \quad y + \chi_u \in Q,$$

or there exists $v \in \text{supp}^-(x - y)$ such that

$$x - \chi_u + \chi_v \in Q \quad \text{and} \quad y + \chi_u - \chi_v \in Q.$$

It is known that an M^{\natural} -convex set is the set of integer vectors in an integral *generalized polymatroid* [5,23] (see also [6,14]). Thus, we often refer to a pair (V, \mathcal{F}) , where $\mathcal{F} \subseteq 2^V$, as a *generalized matroid* if $(M^{\natural}\text{-EXC})$ holds for $Q = \{\chi_X \mid X \in \mathcal{F}\} \subseteq \{0, 1\}^V$.

A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$ is called an M^{\natural} -convex function if it satisfies the following property:

$(M^{\natural}\text{-EXC})$ for each $x, y \in \mathbf{Z}^V$ and $u \in \text{supp}^+(x - y)$, it holds that

$$f(x - \chi_u) + f(y + \chi_u) \leq f(x) + f(y),$$

or there exists $v \in \text{supp}^-(x - y)$ such that

$$f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \leq f(x) + f(y).$$

A function g is said to be M^{\natural} -concave if $-g$ is M^{\natural} -convex.

The class of M^{\natural} -convex functions is indeed a proper generalization of M -convex functions. On the other hand, in the following sense these two are essentially equivalent. Let v_0 be an element distinct from V and denote $\tilde{V} = \{v_0\} \cup V$. Let α be an integer. For a function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$, define $\tilde{f} : \mathbf{Z}^{\tilde{V}} \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x(\tilde{V}) = \alpha, \\ +\infty & \text{otherwise} \end{cases} \quad (x_0 \in \mathbf{Z}, x \in \mathbf{Z}^V). \tag{3}$$

Murota and Shioura [16] proved the following relation between M -convexity and M^{\natural} -convexity.

Theorem 3 [16] *A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ is an M^{\natural} -convex function if and only if \tilde{f} defined by (3) is an M -convex function.*

An important operation on M -convex (M^{\natural} -convex) functions is *transformation through a network*. Let (N, E) be a digraph with entrance set $N_1 \subseteq N$ and exit set $N_2 \subseteq N$, where N_1 and N_2 are disjoint. Let $\underline{c} : E \rightarrow \mathbf{Z} \cup \{-\infty\}$ and $\bar{c} : E \rightarrow \mathbf{Z} \cup \{+\infty\}$ denote the lower and upper capacity functions, respectively. For each arc $e \in E$, a cost function $\gamma_e : \mathbf{Z} \rightarrow \mathbf{R} \cup \{+\infty\}$ is attached. We assume that γ_e is a univariate convex function for every $e \in E$. For a flow $\xi \in \mathbf{Z}^E$, define its boundary $\partial\xi \in \mathbf{Z}^N$ by $\partial\xi(v) = \xi(\delta^+v) - \xi(\delta^-v)$.

Given a function $f_1 : \mathbf{Z}^{N_1} \rightarrow \mathbf{R} \cup \{+\infty\}$, define a function $f_2 : \mathbf{Z}^{N_2} \rightarrow \mathbf{R} \cup \{\pm\infty\}$ by

$$f_2(y) = \inf_{x, \xi} \left\{ f_1(x) + \sum_{e \in E} \gamma_e(\xi(e)) \mid \xi \in \mathbf{Z}^E, \underline{c}(e) \leq \xi(e) \leq \bar{c}(e) \forall e \in E, \right. \\ \left. \partial \xi = (x, -y, 0), (x, -y, 0) \in \mathbf{Z}^{N_1} \times \mathbf{Z}^{N_2} \times \mathbf{Z}^{N \setminus (N_1 \cup N_2)} \right\}. \tag{4}$$

We have that M-convexity and M^\natural -convexity are maintained in this transformation.

Theorem 4 [11,20] *Assume that f_2 defined in (4) satisfies $f_2 > -\infty$. It holds that*

- if f_1 is an M-convex function, then f_2 is also an M-convex function; and
- if f_1 is an M^\natural -convex function, then f_2 is also an M^\natural -convex function.

If an M^\natural -convex function f satisfies that $\text{dom } f \subseteq \{0, 1\}^V$, then f could be seen as a valuation on a generalized matroid by corresponding a subset $X \subseteq V$ to a vector $\chi_X \in \{0, 1\}^V$. That is, we can define a function $g : 2^V \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$g(X) = f(\chi_X) \quad (X \subseteq V).$$

We refer to such a set function g obtained from an M^\natural -convex function f with $\text{dom } f \subseteq \{0, 1\}^V$ as an M^\natural -convex function, as well.

An example of such an M^\natural -convex function arises from branchings. For a weighted digraph (D, w) of $D = (V, A)$ and $w \in \mathbf{R}^A$, define $\mathcal{F} \subseteq 2^V$ by

$$\mathcal{F} = \{R(B) \mid B \subseteq A \text{ is a branching in } G\}. \tag{5}$$

Further, define a function $f : 2^V \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$f(X) = \begin{cases} \min\{w(B) \mid B \text{ is a branching, } R(B) = X\} & \text{if } X \in \mathcal{F}, \\ +\infty & \text{if } X \notin \mathcal{F}. \end{cases} \tag{6}$$

In [22], it is declared that the following theorems follow from arguments in Schrijver [18].

Theorem 5 [22] *For a digraph $D = (V, A)$, the pair (V, \mathcal{F}) defined by (5) is a generalized matroid.*

Theorem 6 [22] *For a weighted digraph (D, w) of $D = (V, A)$ and $w \in \mathbf{R}^A$, the function f defined by (6) is an M^\natural -convex function.*

Note that \mathcal{F} is not an M-convex set, and f is not an M-convex function.

Their cobranching counterparts $\mathcal{F}^* \subseteq 2^V$ and $f^* : 2^V \rightarrow \mathbf{R} \cup \{+\infty\}$ are defined as follows:

$$\mathcal{F}^* = \{R^*(B) \mid B \subseteq A \text{ is a cobranching in } G\}, \\ f^*(X) = \begin{cases} \min\{w(B) \mid B \text{ is a cobranching, } R^*(B) = X\} & \text{if } X \in \mathcal{F}^*, \\ +\infty & \text{if } X \notin \mathcal{F}^*. \end{cases} \tag{7}$$

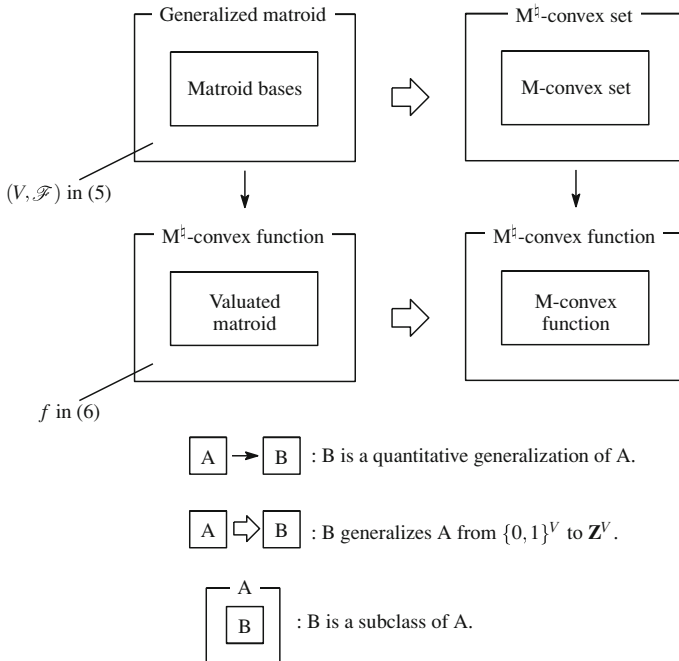


Fig. 1 Relation among the notions

It is straightforward to see that (V, \mathcal{F}^*) is a generalized matroid and f^* is an M^\sharp -convex function.

We summarize the relation among the notions mentioned in this subsection in Fig. 1.

3 Reduction of shortest bibranching to valuated matroid intersection

In this section, we prove Theorem 1 by exhibiting a polynomial reduction of the shortest S - T bibranching problem to the valuated matroid intersection problem.

3.1 A matroidal formulation of the shortest bibranching problem

Let (D, w) be a weighted digraph with $D = (V, A)$ and $w \in \mathbf{R}_+^A$, and let $\{S, T\}$ be a partition of V . As stated in Sect. 2.1, the shortest S - T bibranching problem is exactly finding an integer vector $x \in \mathbf{R}^A$ minimizing $\sum_{a \in A} w(a)x(a)$ under the constraints (1) and (2), which is tractable by the fact that the system defined by (1) and (2) is totally dual integral (Theorem 2).

Here, we present another formulation of the shortest S - T bibranching problem focusing on the matroidal structure of branchings and cobbranchings (Theorems 5 and

6). In $D[S]$, define $\mathcal{F}_S^* \subseteq 2^S$ and $f_S^* : 2^S \rightarrow \mathbf{R} \cup \{+\infty\}$ as follows:

$$\mathcal{F}_S^* = \{R^*(B) \mid B \text{ is a cobranching}\},$$

$$f_S^*(X) = \begin{cases} \min\{w(B) \mid B \text{ is a cobranching, } R^*(B) = X\} & \text{if } X \in \mathcal{F}_S^*, \\ +\infty & \text{if } X \notin \mathcal{F}_S^*. \end{cases}$$

Similarly, in $D[T]$, define $\mathcal{F}_T \subseteq 2^T$ and $f_T : 2^T \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$\mathcal{F}_T = \{R(B) \cap T \mid B \text{ is a branching}\},$$

$$f_T(X) = \begin{cases} \min\{w(B) \mid B \text{ is a branching, } R(B) = X\} & \text{if } X \in \mathcal{F}_T, \\ +\infty & \text{if } X \notin \mathcal{F}_T. \end{cases}$$

By Theorem 6, both f_S^* and f_T are M^\square -convex functions. Furthermore, the following proposition immediately follows from nonnegativity of w .

Proposition 1 *The functions f_S^* and f_T are monotone non-increasing, that is, $f_S^*(X') \geq f_S^*(X)$ if $X' \subseteq X$ and $f_T(X') \geq f_T(X)$ if $X' \subseteq X$.*

Proof Consider f_T . Let $X' \subseteq X \subseteq V$ and suppose $X' \in \text{dom } f_T$. Let $B' \subseteq A[T]$ be a branching in $D[T]$ attaining $f_T(X')$, that is, $R(B') = X'$ and $w(B') = f_T(X')$. By removing arcs in $B' \cap \bigcup_{v \in X \setminus X'} \delta^- v$ from B' , we obtain another branching B with $R(B) = X$. By nonnegativity of w , it holds that $w(B) \leq w(B')$. We consequently obtain that $f_T(X) \leq w(B) \leq w(B') = f_T(X')$.

A similar argument holds for f_S^* . □

Now, the shortest S - T bibranching problem is described as the following optimization problem:

$$\begin{aligned} \text{(SBP)} \quad & \text{minimize} && w(F) + f_S^*(\partial^+ F) + f_T(\partial^- F) \\ & \text{subject to} && F \subseteq A[S, T]. \end{aligned}$$

In order to obtain a shortest S - T bibranching from an optimal solution F_0 for (SBP), it suffices to find a minimum-weight cobranching B_S in $D[S]$ with $R^*(B_S) = \partial^- F_0$ and a minimum-weight branching B_T in $D[T]$ with $R(B_T) = \partial^+ F_0$, and then return $F_0 \cup B_S \cup B_T$.

In the next subsection, we show a polynomial reduction of (SBP) to the valuated matroid intersection problem.

3.2 Reduction to valuated matroid intersection

We consider to transform f_S^* and f_T to valuated matroids whose ground sets correspond to $A[S, T]$. First, we extend the domain of f_S^* to \mathbf{Z}^S . Define $g_S : \mathbf{Z}^S \rightarrow$

$\mathbf{R} \cup \{+\infty\}$ by

$$g_S(x) = \begin{cases} f_S^*(\text{supp}^+(x)) & \text{if } x \in \mathbf{Z}_+^S, \\ +\infty & \text{if } x(v) < 0 \text{ for some } v \in S. \end{cases}$$

Proposition 2 *The function g_S is an M^\sharp -convex function.*

Proof For brevity, let us abbreviate f_S^* and g_S as f and g , respectively. We prove that g satisfies $(M^\sharp\text{-EXC})$.

Let $x, y \in \text{dom } g$ and $u \in \text{supp}^+(x - y)$.

Case 1 ($y(u) \geq 1$). Note that $x(u) \geq 2$ holds. It is easy to observe that

$$g(x - \chi_u) = g(x) = f(\text{supp}^+(x)), \quad g(y + \chi_u) = g(y) = f(\text{supp}^+(y)).$$

Thus, $g(x - \chi_u) + g(y + \chi_u) = g(x) + g(y)$ follows.

Case 2 ($x(u) \geq 2, y(u) = 0$). In this case, we have that $g(x - \chi_u) = g(x)$, and $g(y + \chi_u) \leq g(y)$ by Proposition 1. Thus, $g(x - \chi_u) + g(y + \chi_u) \leq g(x) + g(y)$ follows.

Case 3 ($x(u) = 1, y(u) = 0$). Consider Property $(M^\sharp\text{-EXC})$ of f . Let $X = \text{supp}^+(x)$, $Y = \text{supp}^+(y)$. If $f(X \setminus \{u\}) + f(Y \cup \{u\}) \leq f(X) + f(Y)$ holds, then $g(x - \chi_u) + g(y + \chi_u) \leq g(x) + g(y)$ follows immediately.

Suppose $f((X \setminus \{u\}) \cup \{v\}) + f((Y \cup \{u\}) \setminus \{v\}) \leq f(X) + f(Y)$ for some $v \in Y \setminus X$. Then, it follows that

$$g(x - \chi_u + \chi_v) = f(\text{supp}^+(x - \chi_u + \chi_v)) = f((X \setminus \{u\}) \cup \{v\}),$$

and, from Proposition 1,

$$g(y + \chi_u - \chi_v) = f(\text{supp}^+(y + \chi_u - \chi_v)) \leq f((Y \cup \{u\}) \setminus \{v\}).$$

Thus, we have that $g(x - \chi_u + \chi_v) + g(y + \chi_u - \chi_v) \leq g(x) + g(y)$. \square

We then transform g_S so that its domain could be associated to $A[S, T]$. Define $h_S : \mathbf{Z}^{A[S, T]} \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$h_S(x) = \begin{cases} g_S(x') & \text{if } x \in \{0, 1\}^{A[S, T]}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $x' \in \mathbf{Z}^S$ is a vector defined by

$$x'(u) = \sum_{a \in \delta^+ u} x(a) \quad \text{for each } u \in S.$$

It is straightforward to see that h_S is obtained from g_S by transformation through the following network:

- The digraph is (N, E) , where $N = S \cup A[S, T]$ and $E = \{ua \mid u \in S, a \in A[S, T] \cap \delta^+u\}$. The entrance set is S and the exit set is $A[S, T]$.
- $\underline{c}(e) = 0, \bar{c}(e) = 1$, and $\gamma_e \equiv 0$ for every $e \in E$.

By Theorem 4, we have that h_S is an M^{\square} -convex function.

We further transform the M^{\square} -convex function h_S to an M -convex function. Let a_0 be an element distinct from $A[S, T]$, and let $\tilde{A} = \{a_0\} \cup A[S, T]$. Let $\alpha \in \mathbf{Z}_+$ be a sufficiently large integer¹. Define $\tilde{h}_S : \mathbf{Z}^{\tilde{A}} \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{h}_S(x_0, x) = \begin{cases} h'_S(x) & \text{if } x_0 = \alpha - x(A[S, T]), \\ +\infty & \text{otherwise} \end{cases} \quad (x_0 \in \mathbf{Z}, x \in \mathbf{Z}^{A[S, T]}).$$

By Theorem 3, \tilde{h}_S is an M -convex function.

Let U be a set of elements disjoint from $A[S, T]$ with size α and denote $W = U \cup A_W$, where A_W is a copy of $A[S, T]$. An element in A_W corresponding to $a \in A[S, T]$ is denoted by a_w . Define $h^+ : 2^W \rightarrow \mathbf{R} \cup \{+\infty\}$ by

$$h^+(X) = \begin{cases} \tilde{h}_S(|X \cap U|, \chi_{X \cap A_W}) & \text{if } |X| = \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to observe that h^+ is obtained from \tilde{h}_S by transformation through the following network and identifying $X \subseteq V$ as $\chi_X \in \{0, 1\}^V$:

- The digraph is (N, E) , where $N = \tilde{A} \cup W$ and $E = \{a_0u \mid u \in U\} \cup \{aa_w \mid a \in A[S, T]\}$. The entrance set is \tilde{A} and the exit set is W .
- $\underline{c}(e) = 0, \bar{c}(e) = 1$, and $\gamma_e \equiv 0$ for every $e \in E$.

Thus, by Theorem 4, $\omega^+ = -h^+$ is a valuated matroid. We also obtain a valuated matroid $\omega^- : 2^W \rightarrow \mathbf{R} \cup \{-\infty\}$ from f_T by similar transformation.

We are now ready to define an instance of the valuated matroid intersection problem. The ground set is $W = U \cup A[S, T]$ and define a weight vector $\tilde{w} \in \mathbf{R}^W$ by

$$\tilde{w}(v) = \begin{cases} 0 & \text{if } v \in U, \\ -w(a) & \text{if } v \in A[S, T]. \end{cases}$$

Now, we have constructed an instance $(W, \tilde{w}, \omega^+, \omega^-)$ of the valuated matroid intersection problem. It immediately follows that, for an optimal solution X for $(W, \tilde{w}, \omega^+, \omega^-)$, the set of arcs in $A[S, T]$ corresponding to $X \cap A_W[S, T]$ is an optimal solution for (SBP).

4 Concluding remarks

We have shown that the shortest S - T bibranching problem falls in the framework of the valuated matroid intersection problem. Our argument begins with the fact that

¹ It suffices that $\alpha \geq |A[S, T]| - \min\{\min\{|X| \mid X \in \text{dom } f_S^*\}, \min\{|X| \mid X \in \text{dom } f_T\}\}$.

weighted branchings determine a valuation on a generalized matroid (Theorems 5 and 6), and then make use of theory of a more general framework of M -convex functions. That is, we transformed the valuation to an M -convex function, and then applied the operation of transformation through a network, in order to make the two valuations on $\{0, 1\}^S$ and $\{0, 1\}^T$ have the same ground set.

Let us mention the shortest S - T bibranching algorithms obtained from this reduction. By applying algorithms for the valuated matroid intersection problem [13] to the instance $(W, \tilde{w}, \omega^+, \omega^-)$ constructed in Sect. 3.2, we obtain new combinatorial algorithms for the shortest S - T bibranching problem. The algorithms in [13] essentially resemble to classical combinatorial matroid intersection algorithms, and we need a value oracle for ω^+ or ω^- in constructing every exchangeability arc in auxiliary graphs. In our case, this value oracle is exactly computing $f(X)$ in (6) or $f^*(X)$ in (7), which can be performed efficiently by calling a minimum-weight arborescence algorithm.

We conclude this paper by mentioning one more application of Theorem 6. In a recent paper of Shioura [21], the following problem is considered.

The k -budgeted M^\natural -concave maximization problem

Input: – an M^\natural -concave function $f : 2^V \rightarrow \mathbf{R} \cup \{-\infty\}$,
 – cost vectors $\gamma_i : V \rightarrow \mathbf{R}_+$ and budgets $b_i \in \mathbf{R}_+$ ($i = 1, \dots, k$).

Objective: maximize $f(X)$ subject to $\gamma_i(X) \leq b_i$ for each $i = 1, \dots, k$.

For the k -budgeted M^\natural -concave maximization problem, Shioura [21] presented a PTAS. We can consider one kind of the network design problem which falls in the framework of the k -budgeted M^\natural -concave maximization problem.

Suppose that we are given a weighted digraph (D, w) of $D = (V, A)$ and $w \in \mathbf{R}_+^A$. We should determine $X \subseteq V$, where to open facilities, and construct a subgraph of D in which every vertex in $V \setminus X$ reaches some facility in X .

In opening facilities, we have k budget-constraints, which are represented as linear constraints $\gamma_i(X) \leq b_i$ ($i = 1, \dots, k$). Connecting vertices by an arc $a \in A$ costs $w(a)$, and thus construction of a subgraph (V, B) costs $w(B)$. Our objective is to find $X \subseteq V$ and $B \subseteq A$ which satisfy the budget constraints and achieve the minimum connection cost.

If X is given, a minimal subgraph (V, B) achieving the connection from $V \setminus X$ to X is exactly a cobranching B with $R^*(B) = X$, and thus the connection cost is represented by f^* in (7). By Theorem 6, $-f^*$ is an M^\natural -concave function, and thus minimizing $f^*(X)$ subject to $\gamma_i(X) \leq b_i$ ($i = 1, \dots, k$) is a special case of the k -budgeted M^\natural -concave maximization problem. In this special case, again the value oracle is computing $f^*(X)$, which can be done efficiently, and thus the PTAS in [21] can be applied.

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