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"Grundvorstellungen" as a Category of Subject-Matter Didactics

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Abstract One of the central issues that has long captivated research efforts in mathematical education concerns the question of what mental representations people have of mathematical content. That is, what meaning do people associate with mathematical content, and what role do these mental representations play in teaching and learning mathematics? To this end, a number of different categories concerning mental representations have been developed, including, e.g. "intuition", "use meaning" or "concept image". One approach, stemming from the German *Stoffdidaktik* (subject-matter didactics) tradition, is constituted by the didactic category of *Grundvorstellung(en)*. In this contribution we begin by investigating the origins and the development of the concept of *Grundvorstellung* in the German-speaking research literature on mathematics education. Then, we summarize the main ideas of the concept, particularly concerning its normative, descriptive, and constructive use as well as the distinction between primary and secondary *Grundvorstellungen*. Following this, a number of typical areas of application will be considered. We will then discuss the use of *Grundvorstellungen* in the construction and classification of items for quantitative assessment, as well as the role of *Grundvorstellungen* as a descriptive and explanatory category in empirical classroom research.

Keywords Subject-matter didactics (German: Stoffdidaktik) · Mental mathematical representations (German: Grundvorstellungen) · Didactically-oriented mathematical analysis

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Grundvorstellungen als stoffdidaktische Kategorie

Zusammenfassung Zu den wichtigen Forschungsinteressen der Didaktik der Mathematik gehört seit langem die Frage, was Menschen sich unter mathematischen Inhalten vorstellen, welche inhaltliche Bedeutung sie damit verbinden und welche Rolle diese Vorstellungen beim Lehren und Lernen von Mathematik spielen. Hierzu haben sich unterschiedliche Konzepte über mentale Repräsentationen entwickelt, z. B. unter den Begriffen „intuition", „use meaning" oder „concept image". Ein Ansatz, der sich im Zusammenhang mit der deutschen Stoffdidaktik entwickelt hat, ist das Grundvorstellungskonzept. Im Beitrag gehen wir zunächst auf Entstehung und Entwicklung dieses Konzepts in der deutschsprachigen Mathematikdidaktik ein. Danach werden Kernpunkte dieses Konzepts dargestellt wie die Unterscheidung zwischen normativer, deskriptiver und konstruktiver Verwendung sowie zwischen primären und sekundären Grundvorstellungen. Weiterhin werden exemplarisch wichtige Anwendungsfelder dargestellt. Hierbei gehen wir auch auf die Verwendung von Grundvorstellungen bei der Konstruktion und Klassifizierung von Items für quantitative Studien ein, ebenso wie auf die Rolle von Grundvorstellungen als Beschreibungs- und Erklärungskategorie in der empirischen Unterrichtsforschung.

Schlüsselwörter Stoffdidaktik · Grundvorstellungen · Didaktisch orientierte Sachanalyse

1 The Origins and Characteristics of the Concept of *Grundvorstellungen*

1.1 Grundvorstellungen[1] as Subject-Matter Didactic Concept

Subject-matter didactics (German: *Stoffdidaktik*) was the most important school of thought in German-speaking mathematical didactics research until the 1980's. It was developed during the heyday of the "New Math", where "set theory" was forced into elementary schools, and particularly rigorous *academic* curricula were commonly present in lesson plans, textbooks, and in the classrooms of secondary school education. During this time, for instance, fractions were no longer introduced to students using pies and pizzas, but rather with equivalence classes and paired numbers, things more commonly seen in algebra lectures. Similarly, in calculus, epsilon-delta definitions became the standard. However, it soon became apparent that simply importing and adapting academic mathematics from universities for

[1] Though the term *Grundvorstellung* (plural: *Grundvorstellungen)* roughly translates to 'basic notion' or 'basic idea', we have decided to keep the German term here due to the ability of the term's root word, *Vorstellung* (roughly 'idea' or in the context of *Grundvorstellung* 'conception' or 'notion'; the prefix *Grund-* meaning 'basic'). This is meant to articulate more adequately our emphasis on *mental states*, i. e. mental representations, as a basis for thinking about the learning and teaching of mathematics. In particular, this goes beyond, e. g. the everyday language meaning of 'having a basic idea', or 'the basic notion', of a particular term or procedure in math. For further semantic clarification cf. Bender (1998, p. 110 ff).

use in pre-university schools did not result in an increased comprehension of the mathematical material by the students. As a consequence, many students could recite epsilon-delta definitions but were baffled as to the meaning of the various cryptic symbols scribbled on the blackboard, or even their relevance to everyday life.

It was during this time that early subject-matter didactics was developed (cf. Griesel 1968, for fractions; Kirsch 1969, for everyday arithmetic; or Blum 1979, for calculus). The goal of this new approach was not to simply continue to import more precise academic mathematics into schools. Rather, the goal was to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, while simultaneously simplifying mathematical material without distorting it from its original form, with the aim of making it accessible for learners. Proponents of subject-matter didactics developed several important principles in order to achieve these goals, and provided suggestions for applying these principles in teaching. Key among these was that the simplifications introduced into mathematical material should be "intellectually honest" and "upwardly compatible" (Kirsch 1987). That is, concepts and explanations should be taught to students with sufficient mathematical rigor in a manner that connects with and expands their knowledge of the subject. This approach was meant to avoid the mentality of students "forgetting what you have learned earlier" as mathematical material will be revised at a later date. For instance, "a null sequence is a sequence in which the members of that sequence become smaller and smaller" is an example of just such a (mental) representation that is not helpful in a school setting, since then the descending sequence of negative integers would also qualify as a null sequence. In contrast to this, "a null sequence is a sequence in which all except for finitely many members of that sequence lie within each vicinity of zero" is an explanation that does not need any formalism, is easily made accessible by students, and adequately depicts the mathematical content of the subject.

Subject-matter didactics, in other words, resisted the tendency of formalizing concepts and practices exhibited by the New Math, and instead placed more value on constructing viable and robust mental representations – i. e. *Grundvorstellungen*[2] – with which to capture mathematical concepts and procedures as they are represented in the mental realm. GVs should be able to, on the one hand, accurately fit to the cognitive qualifications of students and, on the other hand, also capture the substance of the mathematical content at hand.

The concept of a *Grundvorstellungen* in subject-matter didactics borrows from two different traditions in German-speaking mathematical didactics, namely the college-preparatory mathematical didactics of Felix Klein, and the so-called *Rechendidaktik* that has been present in historical German compulsory schools (i. e. *Volksschulen*) since Herbart and Diesterweg.

[2] Hereafter often abbreviated as GVs. (cf. also Footnote 1.).

1.2 Origins in College-preparatory Mathematical Didactics

In addition to the schools of thought pushing for more university-level mathematics in schools, there have always been other voices that interpret learning mathematics as an epistemological-genetic process. These voices warned against a one-sided orientation of math that leans toward structural mathematics. Moreover, these voices were often associated with calls for more transparent and intuitively accessible introductions to new material, more orientation towards application, and the construction of viable and robust approaches with which to teach math. Such approaches were already present in the work of Felix Klein, one of the most influential thinkers in mathematical didactics in the German-speaking area. Klein's approach emphasized the importance of building on familiar actions and experience when introducing new mathematical material, for instance by drawing, cutting out, and measuring surfaces to be calculated on a piece of paper before moving on to a systematic treatment of an integral (Klein 1928, p. 30). A similar approach can be found in the proposals offered by Lietzmann, who, in the tradition of Klein, argued against unnecessarily early formalization of mathematics, and instead argued for descriptive introductions to new material and for the construction of contextualized "*Anschauungen",* i. e. ways of looking at something, that would support such an approach (Lietzmann 1916, p. 3 ff). Lietzmann viewed this activity of (mentally) forming concepts as strongly linked to the ability to apply them, suggesting that the real issue lies in "lifting a mathematical problem from reality and then plugging the mathematical solution back into reality"[3] (ibid., p. 323). Courant and Robbins also deserve mention here, as they not only supported clarity as central to teaching math, but also argued for both the construction of viable and robust ideas and intuitions, as well as a reduced significance of structural mathematics. As they put it:

> Although axiomatic form indeed represents an ideal to be striven towards, it would be a mistake to believe that axioms designate *the* essence of mathematics. The constructive viewpoint of the mathematician is a non-deductive, non-rational element, which is just as vital to mathematics as the creative imagination is to visual arts and music. (Courant and Robbins 1962, p. 165)

In this tradition one can already see several key elements of GVs, particularly the requirement of an illustrative introduction of new concepts by linking familiar contexts of action and experience to related viable and robust ideas. Nonetheless, these ideas remained rather general and imprecise in college-prep didactics, and were rarely clarified further or distinguished from one another. Details pertaining to individual concept formation or other links to psychology were rarely or only tentatively considered, though admittedly these were not the focus of this tradition, which was focused more on constructive improvement of courses for mathematics in college-preparatory and higher education.

[3] Original quote: "aus der Wirklichkeit das mathematische Problem herauszuschälen und das mathematische Ergebnis wieder in die Wirklichkeit zu übertragen".

1.3 Origins in Rechendidaktik in German Volksschulen

A more precise expression of GVs is found in another historical tradition of German didactics, namely that of *Rechendidaktik*.[4] The proponents of *Rechendidaktik*, in contrast to the college-preparatory didactic tradition, were more closely associated with concepts and ideas based in the social sciences and psychology. Here a number of different, psychology-based concepts dealing with the generation of basic mathematical concepts can be traced back to the beginning of the 19th century. Herbart (1887) and Diesterweg (1850) interpreted the development of mental representations of ideas and concepts in the context of the psychological school of associationism. Mental representations, according to this tradition, were seen as a kind of mapping process of objects in the real world into the mental world of a given learner. A similar receptive interpretation of perception is also present in the work of Kühnel (1916), whose approach built on the apperceptive psychology of Wundt (1907). Around this time, the increasing influence of 20th century Gestalt psychology became more noticeable in the way that the processes of constructing GVs were understood. This can be seen in Johannes Wittmann (1929), who interpreted this process in a less static, and more constructivist, manner. The decisive turn to a true constructivist interpretation of the concept of GV came soon after as a result of Piaget's (1947) work in psychology, which exerted a significant influence on German didactics. This became apparent in the GV concept of Oehl (1962), whose work forms the contemporary basis for the concept of GVs in subject-matter didactics. Oehl picked up the above mentioned approaches and formed them into an integrated concept. He was the first to use the term *Grundvorstellungen* in an explicit and systematic way. Taking up the principles of visual instruction (Pestalozzi) and self doing and insight (Kühnel), he merged knowledge of traditional lesson practice, didactic analyses of mathematical subject-matter, and knowledge gained from psychological approaches, especially the cognitive psychology of Piaget. Oehl uses the term *Grundvorstellung* – like his predecessors – in a twofold way, characterizing psychological mental models which students are supposed to generate, on the one hand, and describing concrete or graphic models or schemes of action which are intended to initiate and support the corresponding learning process, on the other hand. The second meaning was put into concrete form in his courses for mathematics instruction, which had a wide circulation in Germany. Here, GVs were used as prescriptive notions which are supposed to characterize an adequate meaning of mathematical content by giving verbal or graphic descriptions. To choose a simple example: GVs of dividing are, in particular, the ideas of "splitting up" or "sharing out". Under the influence of Piaget, Oehl interpreted the generation of GVs in a dynamic way, assuming learning and understanding as a process of change, modification and reinterpretation. In this (psychological) connection, GVs are not regarded as isolated fixed entities, but as variable parts of a growing and changing system (cf. vom Hofe 1995, for more details).

[4] "Rechendidaktik" names a tradition in German mathematics education, which occurred from primary school didactics at the beginning of the 19^{th} century and dominated math education in elementary schools until the era of new math.

1.4 Formation of the Concept of Grundvorstellungen

In German subject-matter didactics, a connection was made between the notion of intuitive accessibility, which stemmed from college-preparatory mathematical didactics, and the notion of the mentally represented idea (i. e. the *Vorstellung*) stemming from the *Rechendidaktik* tradition. This is in itself significant, since both of these other traditions up until that time – and largely unnoticed by the other – operated in parallel with one another. Interestingly, the New Math was at least partially responsible for the convergence of these two traditions, even though the roots of both traditions opposed the basic principles of the New Math. This was a result of the widespread mathematization of school education brought on by the New Math in general, a trend which was not limited to college-preparatory education. This development was also evident through the emergence of new textbooks in primary school education, specifically through the creation of proper *textbooks for mathematics*, as up until that point *arithmetic books* (German: *Rechenbuch*) comprised the sole texts for primary mathematics education. Moreover, the previously separate cultures of primary and secondary education came to be seen from a more overarching perspective concerning mathematical education, which opened the possibility of a general early mathematical education for primary and secondary school educators, respectively. The combination of these different viewpoints subsequently had a productive effect on the concept of *Grundvorstellungen* particularly evident through the take-up, and further development, of the concept of mental representations (i. e. *Vorstellungen*) from the *Rechendidatik* approach by proponents of college-preparatory didactics. In particular, the work of Oehl was subsequently developed in different directions, and applied to mathematical content pertaining to secondary education (cf. e. g. Griesel 1973; Blum 1979; Kirsch 1979; vom Hofe 1998; Bender 1998).

In summary, the concept of a *Grundvorstellung* can be summarized using the following main features (vom Hofe 1995): This concept describes the relationships between mathematical content and the phenomenon of individual concept formation. The numerous treatments that the GV concept has received over time nonetheless focus on three particular aspects of this phenomenon, albeit with different emphases among these treatments:

- The *constitution of meaning* of a mathematical concept by linking it back to a familiar knowledge or experiences, or back to (mentally) represented actions,
- The *generation of a corresponding mental representation* of that concept; that is, an "internalization", which (following Piaget) enables operative action at the level of thought,
- The *ability to apply* a concept *to* real-life situations by recognizing a corresponding structure in subject-related contexts or by modelling a subject-related problem with the aid of mathematical structures.

In general, GVs characterize mathematical concepts or procedures and their possible interpretations in real-life situations. Thus, familiar knowledge and application contexts play an important role for the above-mentioned process of the constitution of meaning. Furthermore, familiar intra-mathematical contexts can provide basic structures for the constitution of meaning. This is in particular the case as far as advanced mathematical concepts or procedures are concerned (cf. Sects. 2.2 and 3.2).

So, *Grundvorstellungen* can be construed as mediating elements or as objects of transition between the world of mathematics and the individual conceptual world of the learner. GVs thus describe relationships between mathematical structures, individual-psychological processes, and subject-related contexts, or, in short: the relationships between mathematics, the individual, and reality.

Examples of GVs relevant for the primary school level include the addition of natural numbers ("putting together", "subjoining", "change"), the concept of quantity (cf. Sect. 3.1) or the fraction concept ("part-whole", "operator", "ratio"). For the secondary level we mention GVs of the angle concept ("pair of rays", "field", "rotation"), the concept of percentage (cf. Sect. 3.4), the function concept (cf. also Sect. 3.4), and the concepts of derivative and integral (cf. Sect. 3.2). An extensive list of GVs for arithmetic, algebra, geometry and probability can be found in Jordan (2006, p. 148–154). In all cases, the *method* to find the GVs, in a normative sense, of a given mathematical entity (a concept or a procedure) is a careful and detailed analysis of this entity, including its links to other mathematical entities as well as to extra-mathematical situations, in short: a subject matter analysis; cf. Sect. 3.1 for an example of such an analysis. The aim of these analyses is to identify possible approaches to the mathematical object at hand in order to make it *accessible for learners* in a natural way (cf. Kirsch 1977, for general aspects and concrete examples of making mathematical content accessible).

2 Central Aspects of Grundvorstellungen

2.1 Normative, Descriptive, and Constructive Aspects

The concept of GVs was initially developed in the *Volksschule* tradition (cf. Sect. 1), and later found its way into many areas of didactical and practical teaching literature. The scope of the concept was, however, constrained in several decisive ways: For one thing, until the 1980's the concept was consistently applied in a normative fashion, and GVs were used as didactic categories expressing how students *should* be taught in order to possess adequate conceptual understanding of different subjects. The question of what mental representations students *actually* form, and thereby also questions regarding what kinds of misrepresentations they have, were rarely addressed.

The growing influence of descriptive studies in mathematical didactics along with the increasing interest in gaining insight into concrete manifestations of student learning processes pushed research efforts, particularly in mathematical didactics, towards reconstructing individual learning processes (e. g. Bauersfeld 1982; Voigt 1984).

In the course of this development, the *Grundvorstellung* concept was – in addition to its normative application – extended to the description, analysis, and interpretation of teaching situations and for the constructive correction of errors and misconceptions (vom Hofe 1995). This led to a systematic distinction between descriptive and normative aspects in the use of the GV concept:

- The *normative aspect* works as an *educational guideline*, following a particular educational goal, and describes *GVs as adequate interpretations* of the use of mathematical concepts in a subject-related context, or rather their mathematical core.
- The *descriptive aspect* allows mental representations of students to be characterized, which provide information about the *individual images and explanatory models* of each student. These individual images usually deviate more or less from the GVs that are intended as normative guidelines.

So the adjectives *normative* and *descriptive* do not refer to different kinds of GVs but to different methods of working with the GV-concept. We use the term *Grundvorstellung* throughout to describe *appropriate explanation models*. GVs as a normative notion can be investigated by analysing the mathematical content and corresponding areas of application. Descriptive working methods are required to determine to what extent the actual explanation models of the students correspond to the intended GVs. Examples of normative and descriptive working methods are presented in Sect. 3.

The basic assumption is that in the treatment of a mathematical concept in a teaching situation, corresponding individual images and explanatory models can be constructed. Each of these share a common core across all subjective interpretations, in other words, GVs can be generated – the *constructive aspect* of GVs. This process is depicted in Fig. 1 (cf. vom Hofe 1995).

In this figure, didactic decisions are depicted on the left side, while the learners' activities that can be supported by appropriate didactic measures are depicted on the right side. The starting point of didactic decisions the teacher has to make, is reflecting on the respective mathematical content or procedure in order to *specify*

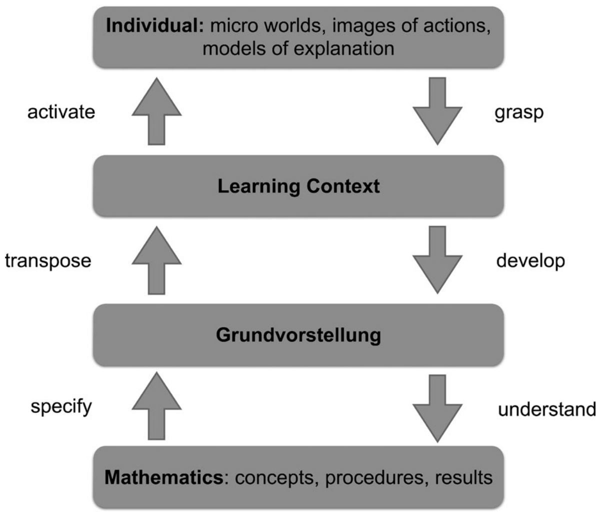


Fig. 1 Generation of Grundvorstellungen

an adequate normative GV. Just analyzing the mathematical structure is not sufficient, as the pre-knowledge of the learners has to be considered. In particular, this step requires reflecting on the meaning that the mathematical content could have, considering the individual experience of the students. This, hence, represents a fundamental didactic decision. The *transposition* of the given content, which involves the formation, or rather identification, of a relevant context and a corresponding learning environment, should represent the structural core of a particular concept to be formed. The learning context should be suitable for *activating* appropriate individual images and explanatory models for the student. Under ideal conditions she or he will then be able to *grasp* the learning context from the angle of her/his mental models and experiences. This can lead in the long run to the *development* of the intended GV by the student who integrates this new idea into her/his own system of explanatory models. In this way s/he can finally – accordingly to her/his individual conditions – participate in the core of the mathematical concept, which means to *understand* it.

The foregoing described processes can, firstly, aid in the construction of class lessons, and can, secondly, be used as a model that contributes to the detection of, and constructive resolution for, conflicts between formal and intuitive approaches. This may even raise awareness of fundamental didactic-related decision fields, and thereby offer sharper insight into tracking and identifying learning problems. Detecting potential conflicts between normative and descriptive aspects, that is between intended and actual, observable GVs can provide starting points for a constructive resolution to particular problems in comprehension.

2.2 Primary and Secondary GVs

An important didactical purpose in the formation of GVs is the ability to represent an abstract concept on a concrete or otherwise vividly accessible level, i. e., to make it accessible or to provide insight into the concept for the learner. Looking more closely at this kind of concretization or illustration with respect to basic versus advanced mathematical content, there are clear differences in typical and didactically meaningful representations of a given concept. For instance, representations of the basic meaning of simple arithmetic operations such as adding, subtracting, multiplying, and dividing, are not only able to be figuratively represented, but also objectively realizable. In this way, one can speak of a "representative level" and a corresponding underlying "realizing level". Numbers are realized as concrete sets of things, and the compositions between them are realized by feasible actions. These mental representations underlie a variety of action experiences that are acquired long before any systematic acquaintance with mathematics, e. g. by a child playing.

The mental representation of addition as "rising up" along a number line, corresponding to an interpretation of "state – change – state", already exhibits a less objective character than the unification of sets of things. This was already emphasized by Kühnel (1916) and even more significantly by Oehl (1962). After all, a number line is not an empirical object encountered in everyday life, but rather a visual representation. Nonetheless, objects of everyday experience can still play a certain role when construing addition as "moving" along a number line, such as when associating it with ascending and descending a ladder or stairs, or jumping around the playground during a game of hopscotch. However, what is characteristic in the case of the number line is the disentanglement of real movement processes and the ability to mentally structure this representational form. In this case, dealing with fully mental representations of the movement along the number line does not comprise a true realization of an actual experience of movement, but rather the abstract representation of movement that is bound to a visual means of depiction. The detachment from the level of concrete objects becomes especially clear when dealing with arrows, particularly in mental representations of displacement, stretching, shrinking, and reflection, such as used for the introduction of negative numbers (cf. vom Hofe & Fast 2015). This becomes even clearer in representations of functions and functional concept formation, which comprise typical representational forms for not only typical application contexts, but also symbolic objects such as terms, equations and graphs. In terms of elementary versus advanced concept formation, the following kinds of GVs can thus be distinguished:

- *Primary GVs* are based in concrete actions with real objects. The corresponding concepts can be, so to speak, semi-isomorphically represented by real objects and actions, for instance by joining or dividing real sets of things. Primary GVs for this reason possess a *representational character*.
- *Secondary GVs* are based on mathematical operations with symbolic objects. Constituents of the corresponding mathematical structures are not real actions, but rather imagined actions dealing with mathematical objects and means of representing these objects, such as number lines, terms, and function graphs. Secondary GVs for this reason are said to have a *symbolic character*.

The subject-matter didactics-based distinction between primary and secondary GVs (vom Hofe and Jordan 2009) largely corresponds to the distinction given by Fischbein between "primary" and "secondary intuitions" (Fischbein 1987; for more details, cf. Sect. 2.4).

The research on secondary GVs is occasionally combined with other theories of the development of advanced mathematical thinking. This is especially the case when concepts, e. g. function or derivative, are investigated. Here it has proven to be useful to assign GVs to special levels of concept development, for example to the levels of function concepts referring to Dubinsky and Harel (1992) or to the steps of understanding referring to Vollrath (1989). Furthermore, conjunctions of the GV concept and the semiotic register theory of Duval have proven to be useful when describing mental translations between different representation levels, e. g. between algebra and geometry. Approaches in this direction have been carried out in Stoelting (2008), vom Hofe and Jordan (2009), and vom Hofe et al. (2015).

2.3 Grundvorstellungen and Modelling

Primary and secondary GVs play a key role in the process of modelling, that is, in the translation between mathematics and the "rest of the world" (Pollak 1979). Central mental activities work when an adequate mathematization for a real world problem situation has to be found or when a mathematical result has to be interpreted

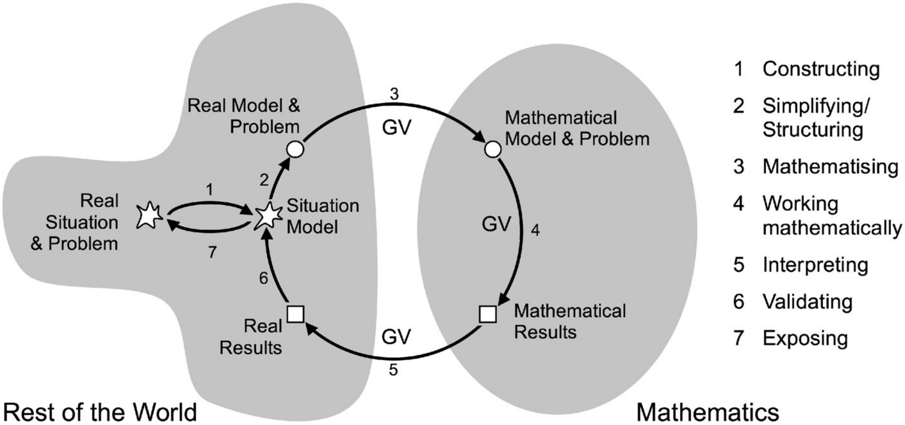


Fig. 2 GVs and Modelling (according to Blum and Leiß 2007)

in relation to the problem situation. For this, one needs mental representations of which mathematical content or method can fit a particular problem situation, or, vice versa, which problem situations can be modeled with specific mathematical content (cf. Fig. 2: Mathematizing and Interpreting). Furthermore, modelling processes often require translation between different levels of representation within mathematics, e. g. between geometry and algebra. Again, GVs are required here to assign, for example, geometric representations of concepts such as slope or monotonicity to their corresponding algebraic representations (cf. Fig. 2, step 4: Working mathematically). These are typically accomplished with secondary GVs.

Reversing this modelling cycle and starting with a mathematical statement that has to be proved leads to the concept of "*reality-related proof*" (Blum 1998) which is a special case of pre-formal proofs (Blum and Kirsch 1991). The first step is an interpretation of the statement and the involved objects in a certain real world context. The second step consists of arguments and actions (actually carried out or only imagined) within this context, accompanied by reflections upon their validity. In the third step, the results achieved by this are mathematised, leading to mathematical results that prove the initial statement. If formalized, the arguments and actions in the real context have to correspond to correct formal-mathematical arguments. However, it is not necessary for such a formalization to be actually effected. An example is a proof of certain combinatorial identities [such as $\binom{n}{k} = \binom{n}{n-k}$] by interpreting the objects (in this case binomial coefficients) as certain numbers in real world settings (in this case number of possibilities for committees). Indispensable for the required translation processes (steps 1 and 3) in all reality-related proofs are GVs of the given mathematical objects, and again careful subject matter analyses are the methodological basis for creating such proofs (cf. Blum 1998, for more details of this concept).

These relationships demonstrate the important role of GVs for the development of mathematical competencies. Ideally, this development will be accompanied by the formation of both primary GVs and, with progressive learning, increasingly also secondary GVs, into a growing and networked system. In particular, the ability to apply mathematical skills is based, according to this view, on the quality of development and the degree of cross-linking of GVs, as well as on the ability to activate and coordinate GVs.

2.4 Relations Between the Concept of Grundvorstellungen and Other Concepts of Mental Representations

The formation of the GV concept as well as its interrelated basic assumptions can be considered in the context of other concepts dealing with mental mathematical representations. We focus on three important approaches: The research on *intuition in mathematical thinking*, the work on *concept image and concept definition*, and the theory of *conceptual change*.

According to Fischbein et al. (1990) the process of mathematical problem solving is, even at higher levels, always combined with intuitive images and assumptions which affect the way of problem solving more or less unconsciously (cf. Fischbein 1987, 1989). In this respect, there is no thinking without mental images. At best, these images can affect mathematical thinking positively. But they can also lead us astray – also shown by Fischbein – if inadequate images become fixed as unconsciously effective "tacit models". Fischbein's research focuses on the contrast between the *intuitive level* and the *formal level* and the finding that many errors in students' mathematical thinking and acting are based on intuitive tendencies which interfere with correct reasoning. According to Fischbein (1987) intuitions can be characterized by special properties that form the nature of intuitive thinking. Particularly important for understanding intuitions are the characteristics that the connected insight is experienced as certainly true (intrinsic certainty), that there is no feeling for the need of any justification (self-evidence), that the idea comes quickly to mind (immediacy), and that intuitive judgments are resistant to differing interpretations (perseverance). Concerning the origin of intuitions, Fischbein distinguishes between *primary* and *secondary intuitions*. While primary intuitions are formed by personal experience, mainly before school, secondary intuitions develop through systematic mathematical instruction.

Fischbein's insight that serious problems of understanding and communication found in mathematics lessons are based on conflicts concerning the intuitive level, corresponds clearly to the findings of the early advocators of subject-matter didactics. These findings suggest that serious learning problems are based on the fact that concepts and symbols used in the teaching of mathematics are often understood by students in a completely different way from what would be considered adequate from the teacher's point of view. This was especially realized in the area of New Math and led, in Germany, to the concept of Grundvorstellungen (com. Sect. 1.1). In addition, the difference between primary and secondary intuitions corresponds broadly to the distinction between primary and secondary GVs. A major difference between these concepts is that, in the work of Fischbein, intuitions are used exclusively as a descriptive notion while the GV concept uses Grundvorstellungen primarily as a normative notion. The extension of the early GV concept by the inclusion of descriptive methods which aim at the identification of actual mental images of students was influenced significantly by the ideas of Fischbein and led to the distinction of the normative and the descriptive aspect of GVs (cf. above).

Building on Fischbeins work, further concepts have been developed which can be related to aspects of the GV concept. We only mention briefly the *intuitive rules* theory (Triosh and Stavy 1999) and the theory of *inhibitory control* (Van Dooren and Inglis 2015). The notion of misleading *intuitive rules* can be related to unconsciously working tacit models (Fischbein 1989) or, from the perspective of the GV concept, as GVs which have been developed in a specific area (e. g. natural numbers) and are incorrectly used in a new area (e. g. rational numbers) in which they are no longer valid (e. g. the assumption that multiplication always makes numbers bigger). *Inhibitory control* can basically be seen as the mental capability to control unconsciously working tacit models through reflecting and monitoring mental activities.

Another important concept in this area is the theory of *concept image* and *concept definition* by Tall and Vinner (1981). The term *concept image* describes "the total cognitive structure that is associated with the concept, including mental pictures and associated properties and processes. It is built up over the years through experience of all kinds, changing as the individual meets new stimuli and matures" (Tall and Vinner 1981, p. 2). It can be more or less in accordance with, or in contrast to, the concept definition. While doing mathematics, activated elements of the concept image play an important role, in many cases a more deciding one than the knowledge of the concept definition. In this sense, certain problems in mathematical thinking and learning can be interpreted as conflicts between concept image and concept definition. The focus of this approach is on descriptive studies concerning the concept image and its relations to the concept definition.

These ideas developed similarly to the GV approach during the area of New Math and provide the insight that teaching mathematics on a formal level does not lead without further ado to appropriate understanding of the students. The contrast of concept image and concept definition may seem parallel to the distinction between the descriptive and the normative aspect of the GV concept, but this would be a misunderstanding: Both, the descriptive and the normative aspect of the GV concept refer to the field of concept image and describe actual detectable individual images (descriptive aspect) and didactically intended GVs (normative aspect), and the main focus of the early GV concept lay on deriving normative GVs by subject-matter analyses. In this aspect, there is a significant difference between these concepts.

Finally, we take a look at the theory of *conceptual change*. A key assumption of this theory is that individuals generate early and robust concepts by interpreting their daily lives, which may become inadequate in the course of time when facing new information. Therefore, processes of conceptual change are needed to extend the existing mental structures and to adapt them to the new requirements. This concept has roots in Piaget's theory of assimilation and accommodation and was developed especially in science education (Posner et al. 1982). Furthermore, this concept has also been applied to the learning of mathematics, particularly concerning the advancement from natural to rational numbers (Vonsiadou and Verschaffel 2004; Kleine et al. 2005a; Prediger 2008).

Considering that the generation of Grundvorstellungen in the long run is supposed to be a dynamic process with changes, reinterpretations and substantial modifications (vom Hofe 1998), affinities between the idea of conceptual change and the generation of GVs are obvious. Early advocators of the GV concept have already postulated that the individual scope of mathematical content and procedures has its roots in the development of a network of many figurative and symbolic basic ideas (Kühnel 1916). In particular, if the individual is dealing with new mathematical subjects, she or he will have to modify and extend her or his system of mental models, otherwise basic ideas, which have been successful for so long, could become misleading concepts when dealing with new mathematical subjects. These mental procedures can also be interpreted from the perspective of the conceptual change theory (for more details cf. Kleine et al. 2005a, or Prediger 2008).

In summary, we can state that certain aspects of the GV concept can also be interpreted from the perspective of the above-mentioned theories. In particular Fischbein's theory of intuition had a significant influence on the GV concept, specifically on its further development from a purely normative approach to a descriptive approach as well. However, a substantial difference exists concerning the main foci of these concepts: Whereas research concerning the above mentioned concepts has mainly or exclusively a descriptive emphasis, the approach of the GV concept combines normative and descriptive methods with a constructive aim. In this sense, analyses of students' work based on the GV concept typically do not remain at the descriptive level but lead to indications of a constructive "repairing" of the analyzed problems. This will be explicated in the following examples.

3 Examples of the Use of GVs

We will now consider some characteristic examples of the different uses of GVs, in order to explicate this didactic category and its characteristic features in more detail. We will begin with the description of basic normative GVs given by Griesel, which was of particular importance for the subject-matter didactic concretization of the GV concept. Then we will consider the normative use of GVs in differential and integral calculus by Blum and Kirsch. Following this, we will present examples of a descriptive use of the concept of GVs using a sequence of interviews from the PALMA study. Finally, we will offer some insight into the use of GVs as categories of analysis and explanation in quantitative studies, using the data compiled in the PISA 2000 comparative study. So, the following examples ought to provide insight into the normative use of primary (3.1) and secondary GVs (3.2) as well as in the descriptive use of the GV concept concerning primary (3.3) and secondary GVs (3.4).

3.1 GVs of Quantities

The means of developing normative GVs is a didactically-oriented mathematical analysis of a given subject matter, in short: *subject matter analysis*. This type of analysis was developed in the early years of subject-matter didactics and forms the main basis of work in subject-matter didactics. According to Griesel, the goal of this analysis is to

> analyze what might be called a kind of natural access [to mathematics], whereby one is oriented toward the basic needs and applications of numbers in everyday life, including the measurement of quantities. In a manner of speaking, here one has to watch at a distance a man on the street, a craftsman, or a housewife, and observe all situations where numbers are used in order to find aspects for a mathematical analysis. (Griesel, 1972, p. 80)

Griesel's early work was guided by the intention to elucidate, and explain, the obvious discrepancies between the conceptualization of academic mathematics and intuitively accessible introductions found in school mathematics. For example, concerning calculating fractions, intra-mathematical constructions of positive rational numbers as classes of pairs of natural numbers stood apart from, and apparently unrelated to, everyday dealings with pies and pancakes, which were used in schools. The aim of his analysis was hence, among other things, to provide a clarification of the relationship between applications, illustrations, and corresponding mathematical structures. This led to a re-description of the relations between the real world and mathematics previously analyzed by Oehl, now seen from the perspective of its subject matter logic. In contrast to traditional approaches of Rechendidaktik, Griesel differentiates systematically between quantities and representatives of quantities, and analyzes their meaning for the formation of the quantity concept and the corresponding arithmetical operations. This proceeds, according to Griesel, on the level of representatives:

> A quantity such as 3/4 m cannot be mentally represented as such, but rather by their representatives, such as, in this case, line segments with length of 3/4 m. For use-orientated thinking about, and for intuitive reasoning for, mathematical relationships, such mental representatives are essential. (Griesel 1981)

So, in order to generate the concept of length, a student must interact with a plethora of different representatives, e. g. line segments, edges, and sticks, and then compare them. The length of a line segment will then appear as a common property of all line segments, or rather length representatives, which are of equal length. The relationships between quantities and quantity representatives can be – in analogy to the relationship between sets and cardinal numbers – depicted as a hierarchy with transitions via representation and abstraction (cf. Fig. 3).

The basis of this abstraction process is an equivalence relation; in the above example "*is the same length as*", which is settled through the representatives and leads to the concept of quantity. The student's ability to produce or recognize appropriate and diverse representatives determines the individual scope of the concept; the

Fig. 3 Quantities and Representatives (according to Griesel 1973)

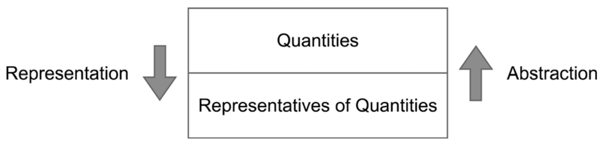


Table 1 GVs of the Concept of Quantity (Griesel 1973, p. 71)

Operation	Accompanying GV
Addition of two quantities	Aggregation (set union) of (semi-disjunctive) representatives
Subtraction of two quantities	Separation of one representative from another
Multiplication of one quantity with a natural number *n*	Aggregation (set union) of *n* equivalent (pairwise disjunctive) representatives of a quantity
Division of a quantity by a natural number *n*	Breaking down of a quantity representative into *n* equivalent pairwise disjunctive sub-representatives
Division of a quantity *a* by another quantity *b*	Assembling (or decomposing) of a representative of quantity *a* out of representatives of quantity *b*. (Repeated separation of representatives of quantity *b* from a representative of quantity *a*)

ability to apply the concept then rests on the ability to recognize instances of the appropriate representatives that are hidden in a mathematical task or context.

Using these resources, Griesel describes the formation of the concept of quantity and its links from environmental relationships by means of GVs. They are related to objectively interpretable situations and/or actions at the level of representatives (cf. Table 1).

These GVs can be construed in several ways as didactical guidelines:

- They describe foundational intuitive interpretations for an adequate conceptual understanding by the student,
- they describe a framework for the construction of corresponding teaching or game situations,
- they give indications for adequate and intuitively accessible depictions or situation sketches of subject-related issues and/or their mathematical core.

Griesel's work resulted in a subject-matter didactical specification of GVs, thereby offering for the first time a systematic analysis of relationships between quantities and their representatives. Herby GVs are specified on the basis of adequate constellations of representatives. Griesel's GVs form normative categories, with the aim of developing guidelines for constructing and organizing practical curricula.

3.2 GVs of Derivatives and Integrals

The concept of secondary *Grundvorstellungen* can also be applied to advanced mathematical content. As an example we take the central infinitesimal concepts from upper-level secondary mathematics: derivatives and integrals. Both concepts have a geometric interpretation, each of which serves as universal illustration: derivatives as tangent or graph slopes, and integrals as (oriented) areas. Both of these are the dominant meanings attributed to these concepts in schools. The actual meaning of these terms, however, lies in their related GVs, i. e., in the anchoring of these concepts within real-life situations (cf. Sect. 1). This leads to derivatives being seen as local rates of change and to integrals being seen as generalized products or overall effects.

Derivative. As a generic example of a derivative one can consider a functional distance-time relationship $f\colon t \to s$ given on a time interval $[a, b]$. The question is how much the distance will change in a given amount of time. More specifically, of interest is the relative change in distance for a given change in time, expressed in quotients

$$\frac{\Delta s}{\Delta t} = \frac{f(t+h) - f(t)}{h}$$

on certain sub-intervals (where $h \neq 0$). These are the average velocities in the considered sub-intervals, or, more generally, the average rates of change of distance for a given amount of time. For a fixed $t \in [a, b]$, one can now imagine making the length $|h|$ of the time interval smaller, i. e. $h \to 0$, and observe what happens. The mean velocities will then – provided that the function f is sufficiently "benign" (that means differentiable) – approach a fixed value that can be interpreted as the instantaneous velocity at time t. In more general terms, this constitutes the first GV for the concept of derivative.

The resulting number is called the derivative $f'(t)$ of f at point t, interpreted as a *local* (in time-related contexts: *instantaneous*) *rate of change* of f in relation to t.

The corresponding formal definition of the concept of a derivative is as a limit value of the difference quotients. Questions of existence and uniqueness are not of interest in such content-based approaches used in schools (cf. Blum and Törner 1983; Tietze et al. 1997). Other examples of local rates of change can be found in Table 2 (cf. Blum and Kirsch 1996). The corresponding key learning objective for school is this: the ability to specify, for any given meaning of x and $f(x)$ as quantities, the meaning of $f'(x)$ as a local rate of change.

Another meaning of the derivative can be understood in the sense of an *optimal linear approximation*, that is, the derivative $f'(x)$ as the slope of the (uniquely determined) linear function that best approximates the given function f around x among all linear functions (for particulars, cf. Danckwerts and Vogel 2006). The corresponding definition of the concept of a derivative is as a (uniquely determined) number m in the equation

$$f(x+h) = f(x) + h \cdot m + r(h) \text{ with } \frac{r(h)}{h} \to 0, \text{ where } h \to 0$$

This GV is relevant only at a higher level, especially for functions of several variables. We will not discuss this in detail here.

Integral. As a generic example another kinematic relation will be used, this time with a given function $f\colon t \to v$ on a time interval $[a, b]$, where t again designates

Table 2 Examples of local rates of change

Meaning of x as quantity	Meaning of $f(x)$ as quantity	Meaning of $f'(x)$ as local rate of change
Time	Distance	Instantaneous velocity
Time	Instantaneous velocity	Instantaneous acceleration
Time	Population size	Growth velocity
Income	Income tax	Marginal tax rate
Circle radius	Circle area	Circle circumference
Sphere radius	Sphere volume	Sphere surface area

time and v is the instantaneous velocity at time t, i. e., the local (instantaneous) rate of change of distance traveled in relation to time. The question is whether and how one can reconstruct the traveled distance in $[a, b]$. The basic idea is to approximate the function f by a piecewise constant function on small time intervals, since at constant speed the distance traveled is calculated simply as velocity multiplied by time difference. One hence decomposes the given interval $[a, b]$ into smaller parts of length Δ_i where the velocity v_i is constant. The distance traveled in these parts of the interval is: $s_i = v_i \cdot \Delta_i$. Hence, the total distance traveled is $\Sigma\ s_i$. If one imagines the subintervals to be increasingly smaller, and allows their number to grow without limit, one obtains the distance traveled in $[a, b]$ arbitrarily precisely. The resulting number is called the *integral* of f over $[a, b]$, interpreted as the *overall effect* of the rate of change function f in this interval, in other words (since during this process products are formed from the f-values and small t-differences) the *generalized product* of f-values (here: instantaneous velocities) and t-differences (here: pieces of time). The corresponding formal definition of the concept of an integral is that the integral $\int_a^b f(x)\,dx$ is a limit value of summed products. Like above, questions of existence and uniqueness are not of interest in such content-based approaches used in school (cf. Blum and Törner 1983; Tietze et al. 1997). Further examples of generalized products can be found in the following Table 3 (cf. Blum and Kirsch 1996), where "growth" can also be negative, depending on the behavior of f. The corresponding key learning objective for schools is: the ability to specify, for any given meaning of x and $f(x)$ as quantities, the meaning of $\int_a^b f(x)\,dx$ as a generalized product.

In Table 3 one can see that the second and third columns are switched from their positions in Table 2. This simply reflects nothing more than the content of the two fundamental theorems of differential and integral calculus (cf. Kirsch 1996; Blum and Kirsch 1996) as they are concretely given in the examples: differentiating and integrating are in a way reverse operations of one another. The considerations that led to the fact that the generalized product of instantaneous velocity and time is the distance are a substantive derivation of the second fundamental theorem: The generalized product of the local rate of change of a "stock function" and of the abscissa is the increase of this stock function in the given interval, or mathematically: The integral of the derivative of a function is the function growth in this interval. From this, one can easily deduce the first fundamental theorem, or one can also derive it related to a context, independently from the other. Thus it has been shown

Table 3 Examples of generalized products (total effects)

Meaning of x as a quantity	Meaning of $f(x)$ as a quantity	Meaning of $\int_a^b f(x)\,dx$ as generalized product
Time	Instantaneous velocity	Distance covered between a and b
Time	Instantaneous acceleration	Increase in instantaneous velocity between a and b
Time	Growth velocity	Growth of population size between a and b
Income	Marginal tax rate	Growth of income tax between a and b
Circle radius	Circle circumference	Area of annulus between a and b
Sphere radius	Sphere surface area	Volume of spherical shell between a and b

here that adequate GVs of the basic concepts of calculus naturally lead to the two fundamental theorems in a straightforward and obvious way, while the same laws, seen from a purely geometric point of view, appear to be barely-understandable mysteries. One can address the necessary assumptions for these two theorems concerning the given function f (mathematically: the continuity of f) afterwards by retrospectively going through these considerations.

There are several reasons why first derivatives and then integrals are treated in calculus, though historically this has been different. Among other reasons, the concept of derivative appears cognitively less complex, and calculating derivatives of functions is usually done more easily than calculating integrals (cf. Blum and Törner 1983). When derivatives come first, the reconstruction of the stock from the rate of change is a natural introduction to integral calculus. The second fundamental theorem is, as just mentioned, the primary theorem now. This corresponds to its importance, which, in addition to reconstructing, lies in simple integral calculations: Integration is now simply differentiating in reverse ("antidifferentiating").

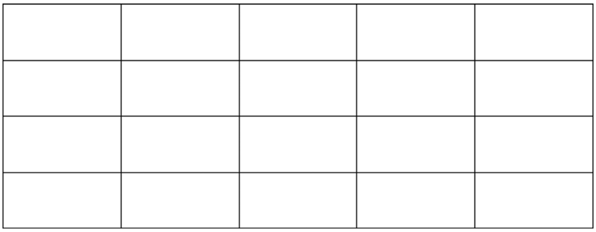
3.3 Assessment of GVs

While in Sects. 3.1 and 3.2 examples for the normative and constructive aspect were demonstrated, the following Sects. 3.3 and 3.4 are dealing with the inclusion of descriptive methods and the interplay of the normative and descriptive aspects of the GV category. The assessment of GVs initially took place in a number of individual case studies (vom Hofe 1995), and later more systematically as part of larger research projects such as interview studies. In these instances the descriptive studies in question were often preceded by a normative task analysis in which the normative GVs operating within the used tasks were identified.

We will consider as an example an excerpt from an interview study by the PALMA Project conducted from 2000–2007. PALMA encompassed a longitudinal survey using quantitative and qualitative methods, and investigated the development of mathematical competence in Bavarian schools using a sample of about 2000 students (Pekrun et al. 2007). The *Grundvorstellung* category was used as one of the design principles in the development of the PALMA performance test for math. Specifically, items were classified according to the number and complexity of GVs that are necessary to solve them (vom Hofe et al. 2002). The test was subdivided into subscales by using the GV concept, in order to provide a quantitative measure that differentiates between mathematical performance development with respect to item groups that require GVs, and to other groups that require only pure calculations. The results suggest a significant deficit in GVs in some phases of the measurement period. However, due to the quantitative data used by the PALMA main study, no statements can be made concerning the nature and cause of these deficits. The main quantitative study was for this reason supplemented by interview studies on specific topics in order to obtain information about the nature of these deficits.

In what follows, excerpts of transcripts will be presented to provide insight into the method of identifying and describing GVs recognizable in students' work. It originates from a series of interviews with 36 sixth-grade students from three Bavarian college-preparatory schools, which were conducted in June 2003 (cf. vom Hofe

Fig. 4 Task: The Chocolate Bar



and Wartha 2005). We will begin with a description of a task involved in the interviews and with a brief analysis of the obvious means of solving as well as of the expected GVs from a normative perspective.

The Chocolate Bar Task: Lilly breaks off half of the chocolate bar illustrated in Fig. 4. Then she eats 3/5 of the half bar. How many pieces of chocolate did she eat?

This task is expected to be solved in two steps; (1) first halving the chocolate (2), then, for one half (10 pieces), determining the correct proportion. There are several easily accessible means of ascertaining the proportion, for instance:

- Calculating the result with fraction operators: 10 pieces * 3/5 = 6 pieces. Here it is necessary to translate the formation "taking 3/5 of 10" to a multiplication "3/5 * 10".
- Calculating the result by inferring the proportionality by which, e. g. half of the chocolate (10 pieces) is taken to be the whole, using reasoning such as: 5/5 ↔ 10 pcs, 1/5 ↔ 2 pcs., 3/5 ↔ 6 pcs. Here a GV for the concept of fractions is required, for instance an idea of a proportion (3/5 as three times one-fifth of the whole; cf. Sect. 2). Moreover, mental attributions, and some idea of proportionality, are also necessary, e. g.; "if one doubles (halves) the output quantity, one doubles (halves) the attributed quantity".

In addition to these ways, numerous other approaches are also conceivable, but all require a mental representation of partitioning of a fraction as being a part of a whole. Consider now an example:

3.3.1 First Scene

```
S:    ...one-fifth is zero-point-two. So two-fifths are zero-point-
      four and three-fifths, um, zero-point-six.
      And, um, half, I mean one, one-half has to be...
      (S thinks) (9 seconds)
S:    Well, since that is the whole, then that's two-halves.
      It's equal to one. And, um...that's zero-point-five, or one-
      half. And, um, then three-point-five...
      (meaning obviously the fraction three-fifths)...that makes zero·
      point-six, I think. And there one has to, uhh, ten...
      (S thinks) (6 seconds)
S:    ...mmm. (S begins to calculate 10 : 0,6)
```

Sophia firstly converts the given fraction into a decimal fraction. Here she apparently uses her knowledge that one-fifth is the same as 0.2, and draws inferences using proportionality: "... two-fifths are 0.4, and three-fifths are 0.6" (18 f). Next, she forms half of the chocolate bar, which she expresses using the decimal fraction 0.5 (23 f). She then repeats the relationship already established with the fraction 3/5, which she incorrectly designates here as "three-point-five" (24 f), and 0.6. Then she seems to try and find a matching arithmetic operation to connect the two focused quantities (half or, respectively, 0.5 and 3/5, or 0.6). She decides not to express half as 0.5, but rather with 10 pieces of chocolate (26). Finally, she connects the two quantities of 10 pieces and 0.6 with a division sign and starts calculating by considering 10 : 0.6 (28). To this the interviewer then asks:

3.3.2 Second Scene

```
I:   Why did you divide by zero-point-six just now?
S:   Yeah, uh, um, because, uh, ten is zero-point-five and that is
     half of the whole. And the zero-point-six is three-fifths. And
     you need that from the one-half because it's only half of it,
     and not the whole.
I:   My question is about the operating sign.
     Why is it divided by? And not times, or plus or minus?
S:   Yeah, because, um, then it would be more and it has to be less
     and less, because she doesn't eat more than chocolate is there,
     but she eats less than chocolate is there.
S    performs the division 10 : 0,6.
     (26 seconds)
I:   Okay, what do you get as a result?
S:   Well, sixteen, um, six six six period. And then I have to, uh,
     because that is more than half,....I have to , um, since
     I....since there is a comma here, it's one-point-six-six-six
     period.
     So, six, we can leave the last one out.
     And then, I think, um, if you shorten it, then it would be, so
     then, um, change it, because starting at five it's always
     rounded up and it would be two. Then she eats about two pieces.
     About, yes.
```

Sophia came to the result 16.666 by using the calculation illustrated in Fig. 5. She obviously recognizes that this result cannot be a meaningful solution to the given task. Without hesitating, the result was altered in two steps: First she shifts the comma left one place. She explains this by reasoning that 16.666 is "more than half" (55 f), and the chocolate eaten has to be less than half. In the second step she rounds 1.666 up to 2, since "starting at five it's always rounded up" (60 f). The inaccuracies in the calculation, caused by this rounding up, are then interpreted in terms of an approximate result: "Then she eats about two pieces" (61).

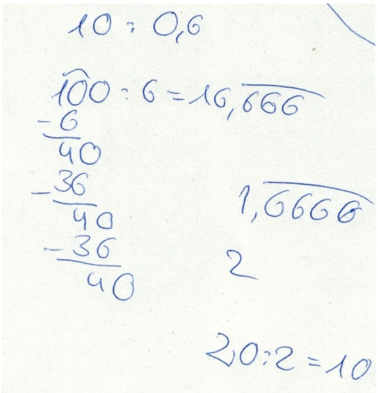
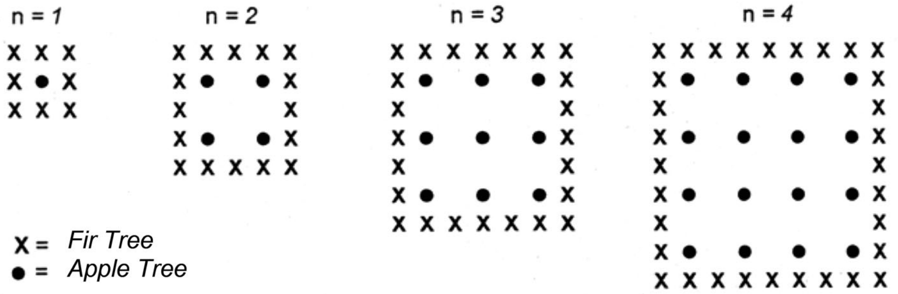


Fig. 5 Sophia's writing

In this attempt to solve the task, several characteristic strategies, and typical sources of error, become clear: It is striking that Sophia converts the given fractions into decimal fractions even though the corresponding constructed proportion could be more clearly and easily calculated with ordinary fractions. This suggests that the corresponding mental representations she entertains are not quite consolidated such that her dealing with fractions does not exhibit the same kind of familiarity as with decimal numbers. The decisive point in her (mistaken) strategy is the identification of proportion formation with division. The intuitive assumption that hinders the correct operation becomes clear when she explains her reasoning: "then it would be more and it has to be less and less" (48 f). It is interesting to note her adaptation of the unrealistic result to the given task situation. Here one can see techniques learned in math class (moving commas, rounding) being used without regard for their objective meaning, instead they seem to be merely following the subjective goal of achieving a plausible result from the task situation.

Though Sophia demonstrates that she can deal formally with fractions, as shown in her correct calculation of decimal fractions and ratios, she does not possess an extended GV for dealing with fractions. In this regard, it is interesting that neither Sophia nor any of the other students used the given illustration to find a solution. The "old" intuitive assumptions, from the realm of natural numbers (multiplication enlarges, division reduces), continue to be operative. This, in turn, becomes a substantial factor in the mistake committed in this approach to constructing the solution, which then results in the "rescue techniques" seen afterward. The means of constructing the solution observed in this case represents a typical example of a proven error-generating transference of intuitive ideas from the realm of rational numbers. Of the 36 students interviewed, a quarter of them failed to solve the chocolate task. Approximately two-thirds of the false solutions resulted from these kinds of representation-related mistakes (cf. Wartha 2007 for more details).

The mistaken mental representations analyzed in the PALMA interview study were also seen in several other descriptive studies (e. g. Padberg 2009). The empirical analysis of GVs hence confirms numerous investigations done in other contexts. An advantage of the category of GVs over purely descriptive or more psychology-oriented approaches (cf. Sect. 2.4) presents itself particularly when juxtaposed with



n	*Number of Apple Trees*	*Number of Fir Trees*
1	*1*	*8*
2	*4*	
3		
4		
5		

Fig. 6 Task Apples. (PISA 2000)

normative analyses or descriptive findings. Here considerations can be derived regarding the points at which the actual mental representations of the students fail to correspond to intended normative guidelines. From this, starting points for a constructive treatment of the respective deficits can be further derived. In the example above, didactic support for the extension of the student's mental representation of multiplication would certainly represent useful help for the student.

Overall, an empirical analysis of students' GVs results in helpful considerations for the constructive improvement of the generation of GVs, and offers means to reveal previously unconscious misconceptions and to reflect on these (e. g. Wittmann 2006; Prediger 2008, 2009; Hafner 2012; Hattermann 2015).

3.4 GVs as Instrument of Analysis in Quantitative Studies

A fundamental problem in realizing the role of GVs in quantitative kinds of tests is the fact that a mathematical task typically requires both knowledge and skills, as well as appropriate mental representations (GVs) and competencies, in order to solve them. That is, mental representations do not appear isolated from other factors and thus cannot be measured separately from them. Nonetheless, the extent to which the nature and scope of the GVs that are necessary for solving a given problem or task is a relevant measure of the difficulty of that task, remains an interesting empirical question. We will now summarize a study based on the results gathered by PISA 2000, which pursued exactly this question (Blum et al. 2004). This study:

- showed which GVs (normatively identified) must be available, and which conceptions must be activated in order to successfully work through the tasks given by PISA,
- defined a variable for "GV-intensity" to (one-dimensionally) classify given tasks according to the nature and extent of the required GVs,
- tested empirically how well this variable – in conjunction with other variables – can explain the difficulty of the PISA tasks.

3.4.1 Examples of PISA Tasks

First, we will analyse the GVs needed to solve three selected PISA tasks, following Blum et al. (2004). This analysis deals with idealized types of processing: By means of a subject-matter analysis we investigated which GVs and which knowledge, skills and competencies are required for the solution process from a normative point of view. We will start with an example from algebra ("Apples" a and c; Source: PISA 2000 international; cf. Baumert et al. 2001):

> ***Apples***. *A farmer plants several apple trees, which are ordered into a square pattern. In order to protect these trees from wind, the farmer plants fir trees around the fruit garden. The* Fig. 6 *illustrates this pattern of apple trees and fir trees, which are planted for an arbitrary number (n) of rows of apple trees.*
>
> Apple a) Complete the table.
>
> Apple c) Assume that the Farmer wants to plant a much larger fruit garden with many rows of trees. Which will increase in number more quickly: the number of apple trees or the number of fir trees? Explain how you arrived at your answer.

Both of these tasks require the activation of attribution processes in the context of a functional relation. According to Vollrath (1989), three GVs can be distinguished for the concept of function:

- The *Mapping GV* describes the unique mapping of numbers or quantities to other numbers and quantities,
- the *Covariation GV* captures the characteristic change of a quantity when another quantity changes,
- the *Object GV* relates to functions as objects, with which functional relations as a whole can be described.

To successfully complete task a), in addition to the ability to understanding the text, it is sufficient to conduct simple translations between reality and mathematics whereby a pre-structuring follows from the diagram: Information must be extracted from the given pattern and then transferred and made suitable for the table to be filled in. Additionally, basic skills in calculating with natural numbers are also required. Task c) is more demanding because (discrete) linear, and quadratic growth must be compared asymptotically. For this, an activation of mental representations of covariation is required, one GV of the concept of function. Consider here a second, simpler example (glass factory 1; Source: PISA 2000 national):

> ***Glass factory***. *1) A glass factory produces 8000 bottles each day. 2 % of the bottles contain flaws. How many bottles contain flaws?*

To solve this task a GV of the concept of percentage has to be activated with which one can grasp the meaning of "2 %" in a given context. Following Oehl (1970), one can normatively distinguish here between three different GVs, which have since been also empirically confirmed (Hafner 2012):

- *GV of Out-of-One-Hundred:* The statement $p\%$ is transferred to a (fictitious) situation with 100 units: Out of 100 units, p units result, or, with 100 units it would be p units.
- *GV of Hundredths (or Operator-GV):* The specification $p\,\%$ is understood as the fraction $p/100$, and "$p\,\%$ of" as "$p/100$ times".
- *GV of Units-per-Demand:* The whole corresponds to 100 %, and one-hundredth to 1 %, so $p\,\%$ of the whole here means p-times the hundredth part of the whole.

Elementary mental representations like those introduced in the foregoing examples form the foundation of a sense making mathematical work. These include GVs of arithmetical operations such as adding (merging, appending, combining) or subtracting (taking away, supplementing, comparing). If one combines such elementary GVs, then the task becomes more cognitively complex, as the following example shows (Saving money; Source: PISA 2000 national, Baumert et al. 2001):

> **Saving money**. *Karina earned 1000 € at her summer job. Her mother suggests placing her money into a savings account at the bank for two years (interest on interest!). Here she has two options:*
>
> *a) "Plus"-Savings: In the first year 3 % interest, in the second year 5 % interest.*
>
> *b) "Extra"-Savings: Both in the first and in the second year 4 % interest.*
>
> *Karina answers: "Both offers are equally good." What do you think of this? Explain your answer!*

Aside from understanding the text and an elementary knowledge of interest, this example requires the creation of an *increased basic value* by combining GVs of *percentage* and *addition* (perhaps in a multiplicative way).

3.4.2 GV-intensity as an Indicator of Difficulty in PISA-2000

In the preceding examples, GVs appear both singly and in combination with others. The cognitive demands in these different instances vary both qualitatively and quantitatively. In order to make this more precise, Blum et al. (2004) distinguish between three kinds of *Grundvorstellungen*: (1) Elementary GVs, (2) Extended GVs, and (3) Complex GVs. This distinction forms the basis for a classification scheme, by which it is possible to categorize tasks based on the respective yields of required

Table 4 Explanation of task difficulty using GV-Intensity

Item characteristic	Extra-Mathematical			Intra-Mathematical		
	β	Level of significance	R²	β	Level of significance	R²
Calculational Items	0.686	0.000	0.457	–[a]	–[a]	–[a]
Conceptual Items	0.616	0.001	0.353	0.543	0.013	0.255

[a]Number of items too small for a multiple regression analysis

GVs by means of a "GV-intensity variable". Hereby the continuum of levels of requirement in the activation of GVs can be divided into four domains: Level 0 (no GV required), Level 1 (one elementary GV required), Level 2 (one extended GV or several elementary GVs required), and Level 3 (more, e. g. one complex GV required) (cf. Blum et al. 2004, for details of the classification scheme). The basis for this classification was a catalogue of all relevant GVs similar to the list in Jordan (2006, mentioned in Sect. 1.4); concerning details of the classification method outlined in Kleine et al. (2005b).

Blum et al. (2004) investigated the correlation between the "GV-intensity" variable and the empirical difficulty of the items in the PISA-2000 study. Here, the 31 international and 86 national items of the PISA-2000 test were classified according to context into the dichotomous terms of being "extra-mathematical" (in the sense of some real-life stimulus) or "intra-mathematical" (i. e. without direct orientation to reality). In order to investigate the relationship between the "GV-intensity" variable and the difficulty of the items used in the respective task, all of the 117 PISA items were classified according to their GV-intensity. Of these, 23 dealt with technical items that only require the use of technical calculations and, by definition, possess a GV-intensity of 0.

To investigate the variance of difficulty in PISA, multiple regression analyses with the GV-intensity variable were calculated (cf. Tabachnik and Fildell 1996). Thresholds for the dependent difficulty parameter were derived from the Rasch model Table 4. shows that the difficulty of these items depends not only theoretically, but also in an empirically significant way, on which GV-level the demand of the task in a question is placed. For calculation tasks in extra-mathematical contexts, the explained variation rests at 0.457; a regression analysis of the corresponding tasks in intra-mathematical contexts could not be performed due to the small number of items (6). The variation in extra-mathematical conceptual items of 0.353 can be also explained quite well; concerning intra-mathematical contexts the correlation is somewhat smaller, with a value of 0.255. All of these explained variations are significant at least at a 5 %-level.

The lower correlation values for the conceptual PISA items are plausible since in many of these cases the essential cognitive demands arise from lower level GV-intensive steps such as multi-step reasoning. Hence, these analyses demonstrate that the difficulties of the tasks are significantly elucidated by means of a singular content-specific feature – i. e. GV-intensity.

These results suggest the following hypothesis: The revelation by PISA of the serious problems encountered by German students in dealing with more demanding mathematical tasks points to an overemphasis in German mathematics teaching on acquiring calculational proficiencies and on learning standard methods and procedures. Indeed, the results of German 15-year-olds have increased in the following PISA-cycles, which may be due to the introduction of competency-oriented education standards in Germany as a consequence of the first PISA cycle. However, from a normative standpoint the results are still unsatisfactory.

4 Summary and Conclusion

Building on earlier German-speaking traditions in mathematical education, the term *Grundvorstellung* was actively developed in the 1970's and 1980's into a subject-matter didactics category poised between mathematics and psychology. The use of GVs was initially normative, and served to describe elementary mathematical concepts and procedures for primary schools. The GV concept was then later extended to more advanced material in secondary education. As a result, the distinction was established between "primary" GVs, which have a representational character and are rooted in concrete actions, and "secondary" GVs, which possess a symbolic character and can be traced back to mathematical instruction. In turn both of these forms of GVs became the focus of further investigation.

Since the 1990's the concept of GVs has been further developed for use in descriptive studies. Here, the subject-matter didactics' principle of didactic lesson plan assessment merged with the methods of interpretive educational research. Descriptive analyses of GVs have since found application in interpretive case studies as well as field studies, where not only quantitative data but also qualitative assessment of specific student strategies are taken into account. Furthermore, in empirical comparative studies the category of *Grundvorstellung* has been used as a design principle for test development as well as an analytic tool for data analysis.

Besides these advances there are still open questions for further research concerning GVs. The first concerns the field of subject-matter: while numerous GV descriptions have been developed for numbers and quantities as well as for functions and calculus, less work on this topic has been produced for geometry and stochastics so far. Initial attempts at incorporating GVs into geometry have been offered by Filler et al. (2015). Another potential field of application for GVs is the constructive use in diagnostic analysis in schools and individual support of students. Promising approaches for this can be found, for instance, in Wartha and Schulz (2011). Further research and clarification is also needed on the question of what relationship the subject-matter didactic category of GV possesses to other internationally established categories of mental representations in mathematics education and psychology, and how this can contribute to establish connections for mutual development. Here, initial attempts have been offered (cf. Sect. 2.4); a comprehensive reappraisal of this issue is still pending.

A treatment of this complex topic may therefore be particularly promising, since the didactic use of *Grundvorstellungen* has proven to be an approach which continues traditional principles of subject-matters didactics, while simultaneously remaining open to new methods, theories, and research directions.

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