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Subject-matter didactics in German traditions

Early historical developments

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Abstract Subject-related approaches have played an important role in the development of the didactics of mathematics as a professional field in Germany. In the 1980s, views of the nature of learning as well as of the objects and methods of research in mathematics education changed and the perspective was widened and opened toward new directions: Mathematics teaching was also considered a social reality and the process of teaching and learning was analyzed not only by psychological means but also under sociological, epistemological, and historical perspectives. This shift of view issued new challenges to subject-related considerations that were enhanced by the recent discussion about professional mathematical knowledge for teaching. The article offers a perspective on these early developments.

Keywords Subject-matter didactics · Professional mathematical knowledge for teaching

MESC Codes A30 · A40 · D30

Didaktik von Lerninhalten in der deutschen Tradition

Frühe geschichtliche Entwicklungen

Zusammenfassung In der Entwicklung der Mathematikdidaktik zu einer wissenschaftlichen Disziplin spielten in Deutschland auf das Fach Mathematik bezogene Zugänge eine wichtige Rolle. In den 1980er-Jahren änderten sich die Sichtweisen auf die Natur des Lernens wie auch die Gegenstände und Methoden der mathematikdidaktischen Forschung, und die Perspektive wurde ausgeweitet und in neue Richtungen geöffnet: Mathematikunterricht wurde auch als soziale Realität betrach-

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tet, und so wurde der Prozess des Lehrens und Lernens nicht nur mit psychologischen Mitteln analysiert, sondern auch unter soziologischen, epistemologischen und historischen Gesichtspunkten betrachtet. Aus diesen Sichtverschiebungen ergaben sich neue Herausforderungen an fachbezogene Überlegungen, die durch die neueren Diskussionen zum fachbezogenen Lehrerprofessionswissen zusätzlichen Auftrieb erhielten. Der folgende Artikel bietet eine Perspektive auf diese Entwicklungen.

1 Introduction

In the development of the didactics of mathematics as a professional field in Germany, subject-related approaches have played an important role from the very beginning. However, perspectives and functions have varied under the influence of changing circumstances.

Historical studies show that mathematics teaching "is intimately tied to the social and political context in the respective country and to dominant epistemological conceptions" (Schubring 2012, p. 525). This context determines the way mathematics is institutionalized in the educational system, and it has an important impact on issues like the weekly number of teaching hours, the choice and extensiveness of teaching subjects, the range of autonomy allowed to the teachers, and the prevailing teacher education programs. Under the influence of these changing contexts, subject-matter didactics provided varying foundations and guidelines for mathematics instruction.

In addition, views of the nature of learning as well as of the focus and methods of research in mathematics education changed. The theoretical and empirical perspectives were widened and opened toward new directions: Mathematics teaching started being considered a social reality and the process of teaching and learning was analyzed not only in psychological, but in sociological, epistemological, and historical perspectives. These widened perspectives created new challenges to subject-related considerations that were enhanced by the recent discussion about professional mathematical knowledge for teaching (e. g. Ball et al. 2008).

This article offers a perspective on these early developments.

2 The period before World War I: Origins in the 19th century

In the period before World War I, mathematics became a major teaching discipline and the necessity for didactical considerations increased.

2.1 The social and political context and the role of mathematics as a teaching discipline

In the next section we will mainly follow the instructive investigations of Schubring about the history of mathematics education in Germany (Schubring 2007, 2012, 2014).

An important impact of the French Revolution was the establishment of a system of mandatory public schooling in France with mathematics as a major teaching subject. Most other European countries followed France in establishing such a system.

After the definite defeat of Napoleon and the reorganization of the European political map in the course of the Vienna Congress of 1815, Germany became a loosely connected collection of 39 independent states with very different school systems and two entirely different conceptions of mathematics instruction: the neo-humanist in Prussia and the classical humanist in the majority of the other states. Within the classical humanist conception, the main emphasis was on classical languages, especially Latin, while the neo-humanist conception considered three key components to be substantial for cognitive development: classical languages, history and geography, and mathematics and sciences.

Prussia played a special role in the development of mathematics as a major teaching discipline. In response to the devastating defeat of 1806 in the war against the French armies, the state initiated profound social, political, and educational reforms, following a novel concept of *Allgemeinbildung* (general education) wherein mathematics had a key function.

> The change of mathematics from a marginal to a major teaching subject was a part of a systematic reform of the entire educational system, from primary schooling to higher education, issued in 1810. The essential elements were the reconstruction of the secondary schools and the universities and of the relation between these two subsystems. Regarding secondary schools, their traditional form of *Gelehrtenschulen* – "scholarly" schools, so called since they should prepare students for university studies – as practiced from the sixteenth to the eighteenth centuries, understood this schooling as forming the mind by the study of classical languages. "Latinität" [...] was the predominant ideal for learning. The neo-humanist conception of learning replaced this almost monopoly by an entirely novel understanding of cognitive development, which should be instigated by schools: envisaging human knowledge as an organic unity, its key components should interact for developing the cognitive capacities. In this conception there were three key components: classical languages, history and geography, and mathematics and the sciences. The curriculum for the new Gymnasia, hence, was tripartite in all the nine grades, even stipulating a weekly share of 6 h for mathematics in each grade. Likewise, there should be three main types of teachers, for these three disciplines. (Schubring 2012, p. 527)

The state also established teacher education for the three types of Gymnasium teachers by reconstructing the Prussian universities. The philosophical faculties, which traditionally had to prepare students to study at one of the paramount professional faculties (theology, law, or medicine), were given an independent function. They had to provide teacher education for the Gymnasia and thus to prepare trainee teachers for the final exam organized by the state. On the whole, the reform entailed two connected effects: the establishment of a new teaching profession by way of an enhancement of the philosophical faculties, and a firm development of mathematics as a scientific discipline within the reformed universities.

> Even in the long run, throughout the nineteenth century, mathematics was able to maintain its strong role as a major discipline in the Gymnasia, despite attacks by philology teachers aspiring to reinforce again Latinity and despite the rise of new types of secondary schools – *Realschulen* and *Realgymnasien* – with more realist minded teaching of mathematics and the sciences, reducing classical studies. (Schubring ibid.)

In 1871, the system established by the Vienna Congress was replaced by the confederative state *Deutsches Reich* ("German Empire"), which now comprised 25 states, and no longer included Austria. The member states continued to be independent, in particular with regard to their educational systems, which entailed the necessity of a procedure for mutual acknowledgement of diplomas, in particular of the *Abitur*, the university entrance degree, and the teacher examinations. To address this issue, a certain coordination and alignment of the respective curricula was brought about. However, a considerable variety with regard to teaching hours and extension of teaching subjects for mathematics and the sciences persisted, and consequently the "teaching of mathematics and the sciences continued to be only formally compatible, and not actually with regard to the conceptions" (Schubring 2012, p. 530). It was a harmonization at a low common denominator.

The primary level was characterized by a social segregation between an elementary school system for the lower social classes (*Volksschule*) and the secondary school system with separate preparatory schooling for the upper social classes. Teachers for the elementary schools attended special teacher-preparing seminaries to attain some knowledge of content and practical teaching skills. The topics related to mathematics consisted mainly in arithmetic calculations with extended everyday applications. The role of geometry was controversial for a long time; toward the end of the nineteenth century, however, it became an integral part of the syllabi for the upper grades of elementary school.

2.2 The Meran reform movement initiated by Felix Klein

Since the formation of the Deutsches Reich, the economy and industry had experienced an enormous upswing, which entailed the need for an adapted modern education. Nevertheless:

> [...] mathematics instruction everywhere was dominated by elementary teaching goals, focusing on classical, Euclidean geometry and enhancing the formation of logical thinking as a key function. The teaching of variables was banned as not being elementary in that sense, and therefore the teaching of functions was also banned. (Schubring 2014, p. 247)

In addition, "the traditional tension and conflict between orientation towards classical humanities and orientation towards modernity had effected eventually the establishment of three different types of secondary schools" (ibid. p. 248), namely, the classical school type, now called *humanistisches Gymnasium* (with Greek and Latin) and two new types, *Realgymnasium* (with Latin only) and *Oberrealschule* (without ancient languages), which had an extended teaching of mathematics and the sciences and thus claimed to have a more practical orientation and preparation for life in the present world. In 1900, the existing conflicts between these competing partial systems were resolved for the entire Confederation in a far-reaching compromise.

> All three types of secondary schools were granted the right to hold the *Abitur* exam and thus give access to higher education. [...] An essential feature of this compromise was that mathematics constituted a major teaching subject in any of these types, albeit according to different views of mathematics. (ibid. p. 248)

An analogous compromise was found at the level of higher education where also two competing institutions existed since the universities had been complemented by the new technical colleges (*Technische Hochschulen*). These colleges were given full academic status and the graduates of the three school types were granted free access to the two types of higher education. However, the different views of mathematics existing on both levels of education caused a new structural problem.

> Thus, at stake for mathematics was a problem of transition from secondary schooling to higher education. The problem was all the more acute as the technical colleges, due to their origin as polytechnic schools, provided a large portion of basically elementary mathematics. When young mathematics professors, formed in the spirit of the new Weierstrassian rigor in analysis, used them to present rigorous foundations of mathematics, this not only annoyed their students, but even provoked the emergence of an anti-mathematical movement among engineers. (ibid. p. 248)

Felix Klein (1849–1925), professor at the University of Erlangen, the Technical College in Munich, the University of Leipzig, and finally at the University of Göttingen, had gained experience with the competitive situation between the two types of institutions and the growing anti-mathematical movement. He had always been engaged in educational aspects of mathematics and took up the challenge of the situation to initiate a profound reform of mathematics teaching, forging a broad alliance of teachers, scientists, and engineers to support his ambitious plans.

The slogan for his Meran reform program was the famous notion of *functional reasoning*, namely, the idea that the function concept should pervade all parts of the mathematics curriculum "like an enzyme" (Klein 1904). As Krüger (1999) pointed out, this notion has to be considered as an overarching curricular guideline and orientation, which refers to a renewal of content and teaching methods of the mathematics curriculum at once.

With respect to content, the aim of the mathematics curriculum in the Meran reform program was to gradually create in the pupil's mind a consciousness for the variability of quantities – in arithmetic as well as geometric contexts – for their mutual interdependence and relationships, and thus to raise a habit of thinking and to prepare an access to analysis and differential and integral calculus and to bridge the gap between secondary and higher education. With respect to didactic approaches and concrete pedagogies, the aim was to reject systematic-deductive approaches and to turn to heuristic and genetic approaches as well as to change from rather static logical conclusions to more flexible ways of thinking.

> A committee, established 1904 in Breslau, reflecting this broad movement in its composition [...] presented at the annual meeting of the association of German mathematicians a year later, in 1905 at Meran, a profoundly revised syllabus for a modernized course, based in fact on that idea of functional reasoning and ending with the elements of calculus. This later became the famous Meran program. The Meran text contained but one shadow: because of the resistance of some functionaries [...] calculus was recommended for both realist school types, but was optional for the *humanistisches Gymnasium.* Klein's conception of free transition should likewise apply to the realist and to the classical school types and, hence, contribute to overcoming – at least for mathematics – the split along contrasting views of culture or cultures. (Schubring 2014, p. 249)

Demands for modernizing mathematics teaching emerged in other European countries as well, in particular in France and in Great Britain. Klein vitally supported the internationalization of the movement within the *Internationale Mathematische Unterrichtskommission* (IMUK) or *Commission Internationale de l'Enseignement Mathématique* (CIEM), which was founded at the fourth International Congress of Mathematicians in Rome (1908). IMUK or CIEM today is better known as the International Commission for Mathematics Instruction (ICMI).

Thus the reform movement determined the course for a profound modernization. Nonetheless, the teaching practice often missed its spirit, owing to an obvious tendency to deal with rigid subjects of instruction instead of developing habits of flexible thinking (Krüger1999, p. 304).

2.3 The Role of psychology in the Meran reform movement

Since the turn of the nineteenth to the twentieth century, psychological research intensified and interest in learning processes emerged. Schubring (2013) points out that mathematics has always been a privileged subject in this context. For example, Paul Ranschburg (1870–1945), who later became an authority in dyscalculia, studied learning difficulties in basic arithmetic and even progressed beyond the merely descriptive approach and "discussed difficulties in learning determined by mathematical concepts themselves, namely, by various operations with numbers" (ibid. p. 6).

An effect of the efforts of IMUK was that research on the psychology of mathematics teaching entered the focus of mathematics education itself. Felix Klein himself was aware of the relevance of this dimension and commissioned David Katz (1888–1953) to write the first monograph on the psychology of mathematics education (Katz 1913), which appeared in 1913 as *Psychology of Mathematics Instruction*. The author had studied mathematics and science at the University of Göttingen and then changed to psychology. His monograph aimed at providing insights into the process of learning mathematics based on experimental psychology, comprising the entire individual development from the "pre-conceptual stage" throughout all levels of schooling and regarding not only arithmetic, but also the development of spatial notions (Schubring 2013, p. 7).

It is remarkable that awareness of psychological needs in mathematics teaching is already articulated in the Meran syllabus. In the preamble, the following demands are formulated: to adapt the course better to the natural development of mind, to take into account students' preliminary ideas and concepts, and to establish organic connections between existing knowledge and new insights (Krüger1999, p. 168).

2.4 Subject-matter didactics during the Meran reform movement

The Meran reform movement entailed extensive publication activities in subject-specific journals and monographs, for instance, reports of experiences with the new subjects, proposals for the structure of the syllabus and for the design of teaching with regard to content and methods, and finally complete textbooks and task collections (Krüger1999, p. 149).

Another contribution to the development of the didactics of mathematics was made by efforts to elaborate an encyclopedia of elementary mathematics containing the entire inner network of paths by which the components are connected. The first volume of the *Enzyklopädie der Elementarmathematik* (Encyclopedia of Elementary Mathematics; Weber and Wellstein 1903) appeared and became an influential source.

> This is a kind of anticipation of mathematical content analysis for the sake of didactical applications in which the flexibility of the existing knowledge and different possibilities of elementarization are exhibited as it became an important component in research in didactics of mathematics in Germany in the 1960s. (Griesel and Steiner 1992, p. 290)

Klein himself created an example of continued relevance by his lessons on *Elementary Mathematics from a Higher Standpoint*, which later appeared in a series of books in several editions (Klein 1968). He intended to give an overview over the academic discipline of mathematics as a whole including the historical development and thus to convey a general education (*Allgemeinbildung*) to the teachers. The presentation in detail considers essential didactical categories as the following:

- The relation between heuristic approaches and strict logical reasoning in a process of discovery.
- The relation between provable conclusions and reasonable determinations.
- Different stages in the development of a concept; for example, the concept of natural numbers and operations with natural numbers can be based on practical experience with concrete objects, whereas the transition to negative numbers deserves the transition to a formal stage of reasoning.
- The role of concept images at any stage and the distinction between initial images, based on concrete experiences, and more abstract images.

O. Toeplitz (1881–1940), to whom Klein was an inspiring example (Behnke and Köthe 1963) and who was also interested in issues of mathematics teaching, later explained where he saw the specific didactical value of Klein's approach (Toeplitz 1928, 1932). In the first instance it was his individuality, his universal overview, presented with an impressive narrative talent beyond the usual systematic structure of university lessons. Furthermore, it was the systematic interpretation of knowledge of elementary mathematics within the context of advanced mathematics and finally the potential thereby to convey a deeper insight into the "gear system" (*Getriebe*) of a mature mathematical theory.

3 The interwar period (1918–1939)

After World War I, Germany became a federal republic, still named *Deutsches Reich* (German Empire) with 18 member states, which continued to be autonomous about education. Nonetheless, a decisive structural reform was enacted.

> Despite the competence of each state for educational policy, a decisive law for the entire Reich had been enacted: The Reichsschulgesetz of 1920 abolished the social segregation between a primary school system for the lower classes and a secondary school system with separate preparatory schooling for higher classes and established a consecutive system of primary school for all, followed by a streaming into diverse types of secondary schools. Moreover, the formation of teachers for these new primary schools became attributed to institutions belonging to higher education – the pedagogical Academies, admitting only students provided with an *Abitur* – whereas the earlier seminaries had as students graduates of the *Volksschulen* so that the formation of these instructors and the *Volksschulen* had constituted a closed system. The professorships established there for the methodology [pedagogy, insert LHH] of teaching reckoning and geometry (*Raumlehre*) paved the way for seriously studying pedagogical conditions for learning – and teaching – (elementary) mathematical knowledge, the first step towards what would later become didactics of mathematics in West Germany. (Schubring 2014, p. 250)

3.1 Primary education

In the beginning of the twentieth century, new educational methods (such as the so-called *Reformpädagogik*) gained increasing influence. They aspired to replace the old "learn and drill school" by an education toward self-acting and consequently self-reliance. On this basis. J. Kühnel (1869–1928) developed his reality-oriented didactics of arithmetic (Kühnel 1916), which claimed that education and teaching should be oriented toward the natural development of mind ("start from the child"). To reveal traces of this development is considered to be the task of scientific research. In addition to his fundamental principle, Kühnel formulated didactical guidelines, wherein he claimed (Schmidt 1978):

- To release mathematical concepts from their relationships to the environment
- To have pupils work on their own on mathematical problems and find different solutions
- To isolate difficulties moderately and afterward to integrate the different aspects of the subject in a comprehensive overview as well as to work through the subject in an operative manner

On the whole, the teaching of arithmetic should aim at independent action ability and give the pupils an active role. On this basis, Kühnel designed lesson plans, which he tried out himself as far as possible and used as teachers' instructions. The underlying number concept was focused on counting but not elaborated in detail (e. g. with respect to the connection between numbers and quantities), because Kühnel did not regard himself as a mathematician (Schmidt 1978).

In the 1920s and 1930s, J. Wittmann (1885–1960), who was a psychologist, established his "holistic didactics of reckoning," which was considered as an application of Gestalt psychology (Wittmann 1939). Therefore his concept of arithmetic started with doodle patterns, which children had to arrange, to rearrange, to compose, to decompose, and to compare. Thus they were trained to discover relationships, which served as an illustrative base for the subsequent systematic consideration of numbers. Many of his figurative patterns are in use till today.

These approaches marked a rejection of formalist teaching methods, which mainly consisted in training procedures and memorizing rules. Mathematics was seen as offering an opportunity for developing cognitive abilities and forming the mind even at an early age.

3.2 Secondary education

After World War I, the political, social, cultural, and economic situation had changed, which entailed a loss of esteem for mathematics and sciences:

> A cultural crisis of mathematics and the sciences arose. Subjects now valued in the school context were of a different, nationalist character: *kulturkundliche* subjects [subjects concerned with culture, insert LHH], i. e., German language and literature, geography and history, were favoured, to the disadvantage of mathematics and the sciences. Weekly hours for the latter subjects were reduced in all types of secondary schools. Nevertheless, a considerable progress was achieved. In the new curriculum for all secondary schools in Prussia of 1925, the so-called *Richertsche Richtlinien* now officially enacted what had for a long time been practiced by mathematics teachers: the Klein program with the elements of calculus in all three types of secondary schools. (Schubring 2014, p. 250)

W. Lietzmann (1880–1959), who was directly involved in the Merano reform, created a classic with his "teaching methods for mathematics teaching" (Lietzmann 1923). Working in the tradition of Felix Klein, his purpose was to provide practicing teachers with a detailed insight into the subject matter, to propose appropriate formulations and illustrations such as possible learning paths, and to indicate obvious difficulties.

Nevertheless, mathematics teaching in secondary schools did not attain a unique character in the sense of the Meran movement. According to the fundamental criticism of Lenné (1975), there were too many tensions between traditional habits and new endeavors that could not be reconciled in practice. For example, the implementation of the concept of mapping in geometry was stagnating because first an appropriate system had to be found (ibid. p. 47). A severe problem was the predominance of a traditional organizing principle, which Lenné calls *Aufgabendidaktik* (didactic of tasks). According to this principle, a mathematical domain was divided into subdomains, which were characterized by special types of exercises such as rule of three calculations or constructions of triangles. These exercises were trained systematically in steps leading from the simple to the more complex forms. In the end, mathematics appeared as a collection of more or less isolated exercises and not as a comprehensive body of knowledge guided by overarching ideas such as "functional thinking" (ibid. pp. 34–35).

3.3 Summary: traditions from different ends of the field

In the nineteenth and the beginning of the twentieth century the development of the didactics of mathematics as a professional field originated in different traditions that partially started from opposite ends of the field. The secondary education tradition was clearly orientated toward the subject as a scientific domain with a more or less pronounced awareness of psychological needs. The new developments in primary education were mainly oriented toward contemporary psychology, whereas hardly any connections to mathematics as a field of science and research were established (Müller and Wittmann 1984, p. 147). Some decades later, the development led to syntheses of the approaches, which also resulted in a more differentiated view of the subject matter.

4 From 1945 to the 1990s: Intensified integration of psychological, epistemological, and sociocultural aspects

In the period after World War II, subject-related didactics obtained high importance in Germany. Especially with respect to secondary education, it was the predominant approach, following the paths that were laid in the Kleinian era.

4.1 Political developments from 1945 to 1990

After World War II, Germany was split into two states in 1949: the Federal Republic of Germany (FRG, West Germany) and the German Democratic Republic (GDR, East Germany).

In the West German society, Schubring (2014) observed initially a conservative stabilization, which directly affected the teaching of mathematics and sciences.

The universities continued to be in charge of secondary teacher education. Prospective teachers studied two or three subjects (e. g. mathematics and physics) and attained basic knowledge in philosophy and/or general education (*Philosophikum*). Additional studies with respect to school practice and mathematics education were voluntary and suitable offers were rare. However, individual mathematicians at university took their responsibility for the quality of mathematics teaching at Gymnasium seriously. H. Behnke (1898–1979) in particular was engaged in this area. He gave support to the communication among teachers and took care of newly established or continued journals. The domains of interest were subject-oriented proposals for teaching and special pedagogies (teaching methods). Behnke also initiated a new compendium with respect to the scientific foundation of school mathematics (Behnke et al. 1958).

As a reaction to the Sputnik shock, fundamental reforms of teacher education for elementary and middle schools were initiated:

> The second half of the 1960s brought a profound turnaround: the educational system became massively expanded [...] and the institutional system was improved and upgraded. For instance, the Pedagogical Academies that had been inherited from the Weimar period as institutions for primary and middle school types were "upgraded" to Pedagogical Colleges (*Pädagogische Hochschulen*), raising former lectureship for the methodology of teaching subjects to the level of professorships for the "Didaktik" of these subjects. This thus gave scientific status to the discipline of mathematics teacher training. As a result of many innovations enacted now, the world of schooling for lower classes became definitely united with the world of schooling for middle and upper classes. (Schubring 2014, p. 252)

The general aim of the reform was to provide a scientific foundation of teacher education. As a consequence, in the training of prospective elementary teachers, the pedagogical part was complemented by subject-oriented courses. The objective was to integrate didactical basic knowledge and didactically oriented content knowledge. In the domain of higher education, however, compulsory parts in didactics were established only from the 1980s onward (Burscheid 2003).

Parallel to the extension of the educational system, a profound curricular reform was decreed, which was to be enacted from 1972 on. It followed the "new math movement" of the epoch with the desire to bring school mathematics closer to the academic mathematics of the twentieth century and to adopt the structuralist view of the Bourbaki group together with the set theoretic language (Kilpatrick 2012; Schubring 2014).

> Primary and secondary schooling were now seen as a unity, subject to a common curriculum developing the key thematic issues of mathematics over the school years, from the first grade on. (Schubring 2014, p. 252)

However, teachers and educators were overtaken by the reform entirely unprepared. The consequence was a growing public resistance especially against "the alleged set theoretical nonsense" (Schubring, ibid.) and the exaggerated mathematical rigor. The result was a modification of the reform in 1975 with a moderate and less demanding version of the curriculum, again free of set theory.

> Rather, the main effect of a common curricular structure of the entire school mathematics, developing the fundamental concepts of mathematics, was maintained. As a result, a consensus emerged on all syllabi of the federal states, which established a few conceptual fields as constituting school mathematics for primary and lower secondary grades: number, figure and form, magnitudes, functions and data. (Schubring 2014, p. 253)

Under the influence of H. Winter (1975), there arose a general consciousness that mathematics teaching should fulfill different requirements: the standards of mathematics as a scientific discipline, the demands of society, the dispositions of the learners and their right to free self-realization. In this sense, primary teaching should be considered as an organic part within an overarching concept of mathematics teaching as a whole, wherein specific activities such as mathematizing, exploring, reasoning, and communicating play an essential part.

The GDR in East Germany abolished the federal structures and established a centralized system of education with an obligatory 10-year school for all, an ensuing differentiation into professional and university-preparation schooling, and an increasing orientation toward Soviet psychology. Mathematics and sciences were greatly valued teaching subjects with high curricular standards. Steady revisions of the mathematics curriculum and systematic evaluation of the teaching practice were constitutive as well (Schubring 2014).

4.2 The role of psychology from 1945 onward

Piaget's genetic psychology (Piaget 1948) gained great influence upon the development of the didactics of mathematics, in the first instance upon primary teaching. Piaget taught that human intelligence developed in a process of active adaptation to the necessities of the environment and that logical thinking was characterized by structures that were, at least, amazingly similar to mathematical structures. Following the concepts of Piaget and his disciple Aebli, A. Fricke (1913–1986) elaborated his "operative didactics" (Fricke 1968).

Later, constructivist learning theories in a broader sense gained acceptance. They postulate that learning is an active, constructive process, where the learner actively constructs or creates his or her own subjective representations of reality. A protagonist in this field was E. von Glasersfeld (1995), whose radical constructivism has been highly influential in the fields of mathematics and science education. His epistemological position was taken up, for example, by P. A. Cobb, who directed his research interest on the implications of constructivism for supporting the improvement of mathematics teaching (Cobb 2011).

Constructivist views date back to Dewey, Vygotski, Piaget, and Bruner. They directed research interests toward the mental state of the learners, their subjective representations of the subject under consideration, and in particular typical misconceptions, which were considered as natural concomitant phenomena within the learning process. In this way, the early ideas of the Meran reform about systematically taking into account student thinking became more mainstream.

4.3 Facets of subject-oriented didactics in the secondary school tradition from 1945 to 1990

The development of new structures within the educational system, the introduction of new curricular contents, and a refined consciousness of the facets of subjective representations of knowledge created the need for appropriate literature on subject-oriented didactics as well as discussions about the nature and place of such contributions.

4.3.1 Subject-related didactics according to Drenckhahn

F. Drenckhahn (1894–1977), who taught at the Pedagogical Institute in Rostock, wrote a clarifying contribution wherein he defined the didactics of mathematics to be "the presentation of the subject matter with respect to teaching" (Drenckhahn 1952/1953, p. 205). As he pointed out, the difference between mathematics as a scientific discipline and the didactics of mathematics exists not so much in the content, but in the aspects guiding the presentation, which are linked to different aims:

- Mathematics as a scientific discipline strives for a tight systematization and logical compression. The presentations reproduce the final stage of mathematical insight (according to the latest scientific findings), where the tracks of the thought process are covered up.
- In didactics of mathematics not only formal logic, but also the inner logic of the subject plays an important part. First of all, the didactics of mathematics has the task to reveal (uncover) the images, notions, ideas, concepts, judgments, and conclusions, but also the impulses and working methods, which originally result from the subject matter with logical necessity. This results in a new subject-related architecture of different layers of mathematics with distinct subject-related logics. In this sense, for example, the rules for the multiplication of fractions or negative numbers follow a different subject-related logic than the empirically based rules for calculation with natural numbers.

Thus, the didactics of mathematics is oriented toward the origins and takes into consideration "the natural tension between problems and solutions such as the factual utility of tools, auxiliary lines and so on" (ibid., translated by the author).

With these clarifications, Drenckhahn took up the secondary school tradition and influenced the main orientation up to the 1970s. Mathematics remained the academic discipline to be referred to by the didactics of mathematics (Burscheid 2003).

4.3.2 Didactically oriented content analysis

Whereas Kühnel placed his focus on psychological issues for the elementary school, W. Oehl (1904–1991), who taught at the Pedagogical Institute in Dortmund, accentuated the importance of the essential structures of the subject and claimed that subject-related considerations and domain-specific ways of thinking should play a central role in mathematics education from the very beginning (Oehl 1962, 1965).

On the basis of the approaches of Drenckhahn and Oehl, "didactically oriented content analysis" was developed as a tool for research in the didactics of mathematics, resulting from the ambition for solid foundations and conducted with the aim to present the contents in a way that is compatible with the standards of the field and at the same time appropriate to the learners and the requirements of teaching (Griesel 1974).

First, the main emphasis was placed on the lower secondary school level, especially in the domain of algebra and arithmetic, complemented by an analysis and the concept of function (Vollrath 1974). For example, different concepts of fractions were discussed (Griesel 1978); Kirsch presented far-reaching analyses of the foundations of proportional reasoning such as linear and exponential growth (Kirsch 1969, 1976a). A special branch was the attempt to find exact foundations for using the relations between numbers and quantities in primary and secondary teaching (Kirsch 1970; Griesel 1997). The approach of a formal reconstruction of didactical knowledge is continued by Burscheid and Struve (2009). The focus in geometry was determined by the ambition to organize geometry by a unifying concept corresponding to the concept of function in arithmetic and algebra (Struve 2015). Thus didactically oriented content analysis in geometry was mostly centered on transformation geometry (see, e. g. Holland 1974/1977, Schupp 1967). Overviews of the discussion were given by Bender (1982) and Struve (1984).

Finally, the didactically oriented content analyses were expanded to domains of upper secondary school teaching. Here the contents already had a solid scientific foundation and the problems were mostly the opposite of those of the lower stages. The question was how mathematical theories and concepts can be simplified and elementarized without falsifying the central mathematical content. Blum and Kirsch suggested more intuitive approaches (at least for basic courses) with the original naïve ideas of function and limit and sequential steps of exactitude, which could be achieved according to the capacity of the learners (Blum and Kirsch 1979; Kirsch 1976b).

A general goal was to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, while simultaneously simplifying mathematical material without distorting it from its original form, with the aim of making it accessible for learners (Kirsch 1977). The simplifications introduced into mathematical material should be “intellectually honest” and “upwardly compatible” (Kirsch 1987). That is, concepts and explanations should be taught to students with sufficient mathematical rigor in a manner that connects with and expands their knowledge of the subject.

4.3.3 Guiding orientations

The goal to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, taking into account processes of natural development, also led to a search for guiding orientations in a local and global sense, and produced paramount constructs of subject-matter didactics.

The concept of *Grundvorstellungen* (Oehl 1962; vom Hofe 1995), which could be roughly translated as “basic mental models”, describes the relationships between mathematical content and the phenomenon of individual concept formation. For example, the actions of distributing and measuring provide basic mental models for the operation of division within the domain of natural numbers (partitive and quotitive basic model). The numerous treatments that the basic mental model concept has received over time, nonetheless, focus on three particular aspects of this phenomenon, albeit with different emphases among these treatments:

- The *constitution of meaning* of a mathematical concept by linking it back to a familiar knowledge or experiences, or back to (mentally) represented actions
- The *generation of a corresponding mental representation* of that concept; that is, an "internalization," which (following Piaget) enables operative action at the level of thought
- The *ability to apply* a concept *to* real-life situations by recognizing a corresponding structure in subject-related contexts or by modeling a subject-related problem with the aid of mathematical structures

In general, basic mental models characterize mathematical concepts or procedures and their possible interpretations in real-life situations. Thus, familiar knowledge and application contexts play a primary role in the aforementioned process of the constitution of meaning (see Blum and vom Hofe in this issue).

The concept of *fundamental ideas* (e. g. Schweiger 1992) tries to describe the underlying principle or the essence of a subject domain, for example, the idea of symmetry. However, this term is relatively vague and used with different meanings (for a detailed discussion, see Vohns 2007).

The concept of *stages* of exactness takes into account that mathematical concepts developed step by step with respect to abstractness, sophistication, and exactness. This notion appears in the concept of Drenckhahn and was elaborated by Freudenthal within his theory that mathematics is developed in a stepwise process of the mental organization of domains of experience (Freudenthal 1973, 1983).

A special curriculum project in grammar schools for grades 7–10, entitled "Mathematics as a Tool for Representation of Knowledge," was conducted in Osnabrück (Cohors-Fresenborg and Kaune 1993). In this project the construction of a cognitive mathematical operating system in the pupils' heads is put in the center of the conceptual work, taking into account individual differences in cognitive structures – predicative versus functional (Schwank 1993). The special quality of the developed concept and the intended teaching culture made it necessary to develop completely new textbooks for the pupils and extensive handbooks for the teachers.

4.4 Content and process

Within a changed view of the nature of learning (see Sect. 4.2), learners were no longer considered as recipients of instruction but assigned an active part in the learning process. Consequently, the focus of teaching had to be shifted from the conveyance of knowledge to the inspiration and organization of learning processes. In this setting, approaches to mathematical content knowledge, which were based on instruction in small steps along a predetermined development of the subject matter, appeared restricted and too narrow.

The books of Freudenthal (1973, 1983) had a stimulating effect also on the German didactic community. Basic pillars of his didactical philosophy offered an active access to mathematics ("guided reinvention") and processes of sense making, where connections to reality are very important.

Winter (1975) influenced the ongoing discussion with his catalogue of general objectives where he stresses:

- The ability to argue objectively and to the point
- The ability to cognitively structure situations of everyday experience, to detect relationships and describe them in mathematical terms, or to develop mathematical tools and concepts with this in mind
- Creativity; that is, to acquire and use heuristic strategies to cope with unknown problems, especially strategies for developing and examining hypotheses

From this point of view, Winter developed a pedagogically oriented philosophy of didactics, which combined contents and processes into a synthesis. This philosophy greatly influenced the conceptions of curricula in Germany. Teachers should be prepared for this task by three domains of knowledge, which are usually not sufficiently represented in university mathematics courses (Winter 1994):

- Knowledge about the genesis and the cultural context of mathematical ideas
- Knowledge about heuristic approaches to mathematical problem solving and about ways to organize mathematical contents for being remembered
- Knowledge about applications, embodiments, and realizations of mathematical concepts and theories including knowledge about the importance of mathematics in science and society

These domains are exemplarily unfolded in the book about discovery learning in mathematics (Winter 1989).

Steinbring (1994) pointed out the particular epistemological nature of mathematical knowledge, which is characterized essentially as theoretical knowledge. This assumption influences the possibilities of how to analyze and interpret mathematical communication and shows that mathematical knowledge cannot sufficiently be determined by a fixed pre-given description.

> Mathematical concepts are constructed in interaction processes as symbolic relational structures and are coded by means of signs and symbols that can be combined logically in mathematical operations. This interpretation does not require a fixed, pre-given description for the mathematical knowledge (the symbolic relations have to be actively constructed and controlled by the subject in interactions). Further, certain epistemological characteristics of this knowledge are required and explicitly used in the analysis process (i. e., mathematical knowledge is characterized in a consistent way as a structure of relations between (new) symbols and reference contexts). (Steinbring 2011, p. 54)

These different aspects were taken into account in the project "mathe 2000" (Müller et al. 1997), which considers the didactics of mathematics as a "design science" with a focus on "mathematics education *emerging from the subject*" (Wittmann 2012). In this concept, mathematics curricula are organized around "*substantial learning environments,*" where children can gain mathematical experience, recognize patterns, and solve problems. To construct such environments requires a "structure-genetic empirical analysis" (ibid), which comprises content-related analyses of the traditional type, but in addition the analysis of the cognitive preconditions of the learners, mathematical practices of exploring patterns, and the objectives of teaching. With this philosophy, Wittmann and Müller came back to ideas that were already articulated by Klein and Dewey and converted them into rich curriculum materials and textbooks, which are now used in many German elementary schools.

4.5 Discourses about limitations of *Stoffdidaktik* since the 1980s

The didactically oriented content analysis as a research method was strongly related to the teaching methods of the 1970s and 1980s, which were primarily based upon instruction and supported by the hidden belief that, in an appropriate ready-made setting, knowledge could be handed over or transmitted from the teacher to the learners – an attitude that was termed by its critics as "broadcast metaphor":

> The crisis of the broadcast metaphor shows that the belief can no longer be held that knowledge is "sent" by the teacher and "received" by the students, or in other words that knowledge is something like property that can be handed over or transmitted. Rather, it is becoming increasingly important to understand that in the process of teaching as well as in the process of learning, knowledge is subjected to changes and transformations that fundamentally determine learning and what is learned. (Seeger and Steinbring 1992, p. 280)

Qualitative analysis of classroom interaction had undoubtedly revealed "hidden dimensions" of the mathematics classroom (Bauersfeld 1980) and shown that the processes could not be determined in detail and that the negotiation of meaning is fragile (Bauersfeld et al. 1988; Steinbring 1994).

These findings initially caused heated debates, wherein subject-related positions were defended with arguments of the type: "If the teacher in this situation would have reacted in this way, then the pupils would ..." In this situation the expression *Stoffdidaktik* came into use as a critical label for an approach together with a restricted attitude:

> ... "stoffdidaktik" is dominated by too simple a model to solve didactical questions and research problems. [It] acts on the assumption that mathematical knowledge – as researched and developed in the academic discipline – is essentially unchanged and absolute ... Though "stoffdidaktik" in the meantime notices the problems of understanding that students have in learning, and accordingly it specifically proceeds to prepare the pre-given mathematical disciplinary knowledge for instruction as a mathematical content, to elementarise it and to arrange it methodically; yet the principle remains unchallenged that mathematical knowledge represents a finished product, and that the teaching-learning-process can be organised linearly, emanating from the content, over the teacher, into the students' heads, and can ultimately be controlled and influenced at every step by mathematics educators. (Steinbring 2011, p. 44; Steinbring 1997, p. 67)

In addition Sträßer states:

> The practice of "content-oriented analysis" up to the 1960s suggests that implicitly Stoffdidaktik starts from the assumption that after a decent mathematical analysis, one will find one and only one best way to teach a certain content matter, which then should be incorporated into mathematics textbooks. (Sträßer 2014, p. 568; cf. also Sträßer 1996)

5 New developments

Since then, the expression *Stoffdidaktik* has been replaced by wider programs of "mathematics education *emerging from the subject*" (Wittmann 2012), which relate the analysis of mathematical content knowledge to the learning process of students and include epistemological aspects, such as the project "mathe 2000" (see previous section).

Hußmann and Prediger (in this issue) present a four-level approach for specifying and structuring mathematical learning contents developed within the research program of topic-specific design research. They also understand this approach as an extension of classical "didactical analysis of subject matters," following the tradition of *Stoffdidaktik* and extend it by a combination with an empirical component.

6 Conclusion

The sketched lines of historical traditions have shown that within the last 200 years, subject-matter didactics has developed substantially, but not linearly. Often, early ideas needed time to be realized in curriculum material. The driving forces for the developments were the changing external contexts influencing schools, mathematics classrooms, and mathematics teacher education on the one hand, and the integration of previously antagonist approaches on the other hand.

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Literature

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of mathematics classroom. *Educational Studies of Mathematics*, *11*, 23–29.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner, & A. Vermandel (Eds.), *Foundations and methodology of the discipline mathematics education* (pp. 174–188). Antwerpen: University of Antwerp.

Bender, P. (1982). Abbildungsgeometrie in der didaktischen Diskussion. *Zentralblatt für Didaktik der Mathematik*, *14*(1), 9–24.

Behnke, H., Fladt, K., & Süss, W. (Eds.) (1958). *Grundzüge der Mathematik*. Göttingen: Vandenhoeck & Ruprecht.

Behnke, H., & Koethe, G. (1963). Otto Toeplitz zum Gedächtnis. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *66*, 1–16.

Blum, W., & Kirsch, A. (1979). Zur Konzeption des Analysisunterrichts in Grundkursen. *Der Mathematikunterricht*, *25*(3), 6–24.

Burscheid, H.J. (2003). Entwicklung der Disziplin Mathematikdidaktik. *Zentralblatt für Didaktik der Mathematik*, *35*(4), 146–152.

Burscheid, H.J., & Struve, H. (2009). *Mathematikdidaktik in Rekonstruktionen. Ein Beitrag zu ihrer Grundlegung*. Hildesheim Berlin: Franzbecker.

Cobb, P.A. (2011). Implications of Ernst von Glaserfeld's constructivism for supporting the improvement of teaching on a large scale. *Constructivist Foundations*, *6*(2), 157–161.

Cohors-Freseborg, E., & Kaune, C. (1993). Zur Konzeption eines gymnasialen mathematischen Anfangsunterrichts unter kognitionstheoretischem Aspekt. *Der Mathematikunterricht*, *39*(3), 4–11.

Drenckhahn, F. (1952/1953). Zur Didaktik der Mathematik und ihrer Wissenschaftsmethodik. *Der mathematische und naturwissenschaftliche Unterricht 5*, 205–211. Reprinted in H.-G. Steiner (Ed.) (1978): Didaktik der Mathematik (pp. 3–18). Darmstadt: Wissenschaftliche Buchgesellschaft.

Freudenthal, H. (1973). *Mathematik als pädagogische Aufgabe* (vol. 2). Stuttgart: Klett.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Kluwer.

Fricke, A. (1968). Operative Lernprinzipien. In A. Fricke, & H. Besuden (Eds.), *Mathematik. Elemente einer Didaktik und Methodik*. Stuttgart: Klett.

Glasersfeld, E. von (1995). *Radical constructivism, a way of knowing and learning*. London: Falmer.

Griesel, H. (1974). Überlegungen zur Didaktik der Mathematik als Wissenschaft. *Zentralblatt für Didaktik der Mathematik*, *6*(3), 115–119.

Griesel, H. (1978). Eine Analyse und Neubegründung der Bruchrechnung. *Mathematisch-Physikalische Semesterberichte*, *15*(1), 48–68.

Griesel, H., & Steiner, H.-G. (1992). The Organisation of didactics of mathematics as a professional field. *Zentralblatt für Didaktik der Mathematik*, *24*(7), 287–295.

Griesel, H. (1997). Zur didaktisch orientierten Sachanalyse des Begriffs Größe. *Journal für Mathematik-Didaktik*, *18*(4), 259–284.

Hofe, R. vom (1995). *Grundvorstellungen mathematischer Inhalte*. Heidelberg Berlin Oxford: Spektrum.

Holland, G. (1974/1977). *Geometrie für Lehrer und Studenten* (vol 2.). Hannover: Schroedel.

Katz, D. (1913). *Psychologie und mathematischer Unterricht*. Leipzig: Teubner.

Kilpatrick, J. (2012). The new math as an international phenomenon. *ZDM – The International Journal on Mathematics Education*, *44*(4), 563–571.

Kirsch, A. (1969). Eine Analyse der sogenannten Schlussrechnung. *Mathematisch-Physikalische Semesterberichte*, *16*(1), 41–55.

Kirsch, A. (1970). *Elementare Zahlen- und Größenbereiche*. Göttingen: Vandenhoeck & Ruprecht.

Kirsch, A. (1976a). Vorschläge zur Behandlung von Wachstumsprozessen und Exponentialfunktionen im Mittelstufenunterricht. *Didaktik der Mathematik*, *4*(4), 257–284.

Kirsch, A. (1976b). Eine „intellektuell ehrliche“ Einführung des Integralbegriffs in Grundkursen. *Didaktik der Mathematik*, *4*(2), 87–105.

Kirsch, A. (1977). Aspects of simplification in mathematics teaching. In H. Athen, & H. Kunle (Eds.), *Proceedings of the third international congress on mathematical education* (pp. 98–120). Karlsruhe: Zentralblatt für Didaktik der Mathematik.

Kirsch, A. (1987). *Mathematik wirklich verstehen*. Köln: Aulis Deubner.

Klein, F. (1904). Über eine zeitgemäße Umgestaltung des Mathematikunterrichts an höheren Schulen. In F. Klein, & E. Riecke (Eds.), *Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts an höheren Schulen* (pp. 1–32). Leipzig Berlin: Teubner.

Klein, F. (1968). *Elementarmathematik vom höheren Standpunkte aus: 3rd Edition, 1st Volume: Arithmetik, Algebra, Analysis 1933; 2nd Volume: Geometrie 1925; 3rd Volume: Präzisions- und Approximationsmathematik 1928*. Berlin: Springer.

Krüger, K. (1999). *Erziehung zum funktionalen Denken – Zur Begriffsgeschichte eines didaktischen Prinzips*. Berlin: Logos.

Kühnel, J. (1916). *Neubau des Rechenunterrichts – Ein Handbuch der Pädagogik für ein Sondergebiet*. Leipzig: Klinkhardt.

Lenné, H. (1975). *Analyse der Mathematikdidaktik in Deutschland* (2nd edn.). Stuttgart: Klett.

Lietzmann, W. (1923). *Methodik des mathematischen Unterrichts. 2. Teil: Didaktik der einzelnen Gebiete des mathematischen Unterrichts* (2nd edn.). Leipzig: Quelle & Meyer.

Müller, G., & Wittmann, E.C. (1984). *Der Mathematikunterricht in der Primarstufe* (3rd edn.). Braunschweig Wiesbaden: Vieweg.
Müller, G.N., Steinbring, H., & Wittmann, E.Ch. (Eds.) (1997). *10 Jahre "mathe 2000" – Bilanz und Perspektive*. Leipzig: Ernst Klett Grundschulverlag.
Oehl, W. (1962). *Der Rechenunterricht in der Grundschule*. Hannover: Schroedel.
Oehl, W. (1965). *Der Rechenunterricht in der Hauptschule*. Hannover: Schroedel.
Piaget, J. (1948). *Psychologie der Intelligenz*. Zürich: Rascher.
Schmidt, S. (1978). *Die Rechendidaktik von Johnnes Kühnel (1869–1928) – Wissenschaftsverständnis, deskriptive und normative Grundlagen sowie deren Bedeutung für die Vorschläge zur Gestaltung des elementaren arithmetischen Unterrichts*. Köln: Dissertation PH Rheinland.
Schubring, G. (2007). Der Aufbruch zum ‚funktionalen Denken': Geschichte des Mathematikunterrichts im Kaiserreich. *NTM Zeitschrift für Geschichte der Wissenschaften, Technik und Medizin*, *15*, 1–17.
Schubring, G. (2012). Antagonism between German states regarding the status of mathematics teaching during the 19th century: processes of reconciling them. *ZDM – The International Journal on Mathematics Education*, *44*(4), 525–535.
Schubring, G. (2013). The history of empirical research into mathematics learning in Germany in the 20th century. In A.M. Lindmeier, & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 237–241). Kiel: PME.
Schubring, G. (2014). Mathematics Education in Germany (Modern Times). In A. Karp, & G. Schubring (Eds.), *Handbook on the History of Mathematics Education* (pp. 241–255). New York: Springer.
Schupp, H. (1967). *Abbildungsgeometrie*. Weinheim Basel: Beltz.
Schwank, I. (1993). Verschiedene Repräsentationen algorithmischer Begriffe und der Aufbau mentaler Modelle. *Der Mathematikunterricht*, *39*(3), 12–26.
Schweiger, F. (1992). Fundamentale Ideen. Eine geistesgeschichtliche Studie zur Mathematikdidaktik. *Journal für Mathematikdidaktik*, *13*(2), 199–214.
Seeger, F., & Steinbring, H. (1992). The practical phase in teacher training: preparing for professional practice under changing conditions. *Zentralblatt für Didaktik der Mathematik*, *24*(7), 280–286.
Steinbring, H. (1994). Frosch, Känguruh und Zehnerübergang – Epistemologische Probleme beim Verstehen von Rechenstrategien im Mathematikunterricht der Grundschule. In H. Maier (Ed.), *Verstehen und Verständigung* (pp. 192–217). Köln: Aulis.
Steinbring, H. (1997). Voraussetzungen und Perspektiven der Erforschung mathematischer Kommunikationsprozesse. In G.N. Müller, H. Steinbring, & E.Ch. Wittmann (Eds.), *10 Jahre "mathe 2000" – Bilanz und Perspektiven* (pp. 66–75). Leipzig: Ernst Klett Grundschulverlag.
Steinbring, H. (2011). Changed views on mathematical knowledge in the course of didactical theory development: independent corpus of scientific knowledge or result of social constructions? In T. Rowland, & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (pp. 43–64). Berlin New York: Springer.
Straesser, R. (1996). Stoffdidaktik und Ingénierie didactique – ein Vergleich ['Stoffdidactic' and 'Ingénierie didactique' a comparison. In G. Kadunz et al. (Ed.), *Trends und Perspektiven,* Schriftenreihe Didaktik der Mathematik (vol. 23, pp. 369–376). Vienna: Hölder-Pichler-Tempsky.
Straesser, R. (2014). Stoffdidaktik in Mathematics Education. In St. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 566–570). New York: Springer.
Struve, H. (1984). Zur Diskussion um die Abbildungsgeometrie. *Zentralblatt für Didaktik der Mathematik*, *16*(1), 69–74.
Struve, H. (2015). Zur geschichtlichen Entwicklung der Mathematikdidaktik als wissenschaftlicher Disziplin. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* (pp. 539–566). Berlin Heidelberg: Springer.
Toeplitz, O. (1928). Die Spannungen zwischen den Aufgaben und Zielen der Mathematik an der Hochschule und an der höheren Schule. *Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht*, *11*(10), 1–16.
Toeplitz, O. (1932). Das Problem der „Elementarmathematik vom höheren Standpunkt aus". *Semesterberichte*, *I*, 1–15.
Vohns, A. (2007). *Grundlegende Ideen und Mathematikunterricht – Entwicklung und Perspektiven einer fachdidaktischen Kategorie*. Norderstedt: BoD.
Vollrath, H.-J. (1974). *Didaktik der Algebra*. Stuttgart: Klett.
Weber, H., & Wellstein, J. (Eds.). (1903). *Encyklopädie der Elementar-Mathematik. Ein Handbuch für Studierende* (5th edn.). Leipzig: Teubner.

Winter, H. (1975). Allgemeine Lernziele für den Mathematikunterricht? *Zentralblatt für Didaktik der Mathematik*, *7*(3), 106–116.

Winter, H. (1989). *Entdeckendes Lernen im Mathematikunterricht. Einblicke in die Ideengeschichte und ihre Bedeutung für die Pädagogik*. Braunschweig: Vieweg.

Winter, H. (1994). Wie lässt sich Mathematikdidaktik als Hochschuldisziplin legitimieren? *Mitteilungen der Deutschen Mathematiker-Vereinigung*, *94*(2), 14–18.

Wittmann, E. Ch. (2012). Das Projekt "mathe 2000": Wissenschaft für die Praxis – eine Bilanz aus 25 Jahren didaktischer Entwicklungsforschung. In G. N. Müller, Ch. Selter, & E. Ch. Wittmann (Eds.), *Zahlen, Muster und Strukturen. Spielräume für aktives Lernen und Üben* (pp. 265–279). Stuttgart: Klett.

Wittmann, J. (1939). *Ganzheitliches Rechnen*. Dortmund: Crüwell.