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A view on subject matter didactics from the left side of the Rhine

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Abstract Although subject matter didactics did not prevail in France as it did in Western Germany in the 1960s and 1970s, the design of the mathematical content to be taught and the preparation of its teaching gave rise to numerous local and national research projects in France. Subsequently, subject matter didactics became part of the didactical engineering research method that was forged in the 1980s. After considering subject matter didactics from a praxeological point of view, this article aims at unfolding how didactical engineering emerged from a subject-focused approach and developed over time. It analyzes some of the salient features of didactical engineering by means of examples.

Keywords Comparison · Subject matter didactics · Didactical engineering · Teaching sequence · France

Über Stoffdidaktik aus der Sicht eines Nachbarn von der anderen Rheinseite

Zusammenfassung In den 60er und 70er Jahren war die Stoffdidaktik in Frankreich in der Mathematikdidaktik nicht so verbreitet wie in Westdeutschland. Dennoch wurde die Entwicklung mathematischer Themen und die Vorbereitung der Themen für den Unterricht immer wieder und lokal wie national bearbeitet. Diese Arbeit wurde dann in den 80er Jahren Teil einer Forschungsmethode, dem ‚didactical engineering'. Nach Betrachtung von ‚subject matter didactics' unter einem praxeologischen Blickwinkel möchte der Text herausarbeiten, wie aus einer

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Fokussierung auf den Gegenstand ‚didactical engineering' entstand und sich mit der Zeit entwickelte. Mit Beispielen werden die zentralen Eigenschaften des ‚didactical engineering' herausgearbeitet.

Schlüsselwörter Vergleich · Stoffdidaktik · Didactical engineering · Unterrichtseinheit · Frankreich

MESC-Codes D20 · C70 · U10

At first glance, the term "subject matter didactics" may seem to be a truism for the French research community on the didactics of mathematics. At the end of the 1970s, when starting to establish an institutional community within universities, French researchers stressed the fact that the focus of the didactics of mathematics was to study the teaching and learning phenomena specific to the mathematical content, as opposed to pedagogy studying the general phenomena of teaching/learning independent of the content to be taught.

The French community learnt more about German subject matter didactics mainly with the lecture of Rudolf Sträßer (1994) at the conference celebrating 20 years of French didactics of mathematics in 1993. This does not mean that there were no exchanges between the French and (Western) German communities before 1993. International groups, conferences, and congresses as well as bilateral exchanges were places where German and French researchers could communicate. The term was known but what was underlying subject matter didactics remained somewhat of a mystery for the French community until Sträßer's lecture. Since then, international conferences gave rise to the presentation of German didactics for an international audience, for example, a conference of the International Group for Psychology of Mathematics Education (Bruder et al. 2013).

This text is written 20 years after Sträßer's lecture and is based on a point of view external to Germany, written from the other side of the Rhine River. It will revisit subject matter didactics in light of the French approaches for choosing and preparing the content to be taught by focusing on the method of didactical engineering, which was developed in France in 1980. As the term subject matter didactics has been used in an ambiguous way (vom Hofe, 1995, p. 329, footnote 4) and also been subject to changes in the last quarter of the twentieth century (Sträßer, 2013), we will retain only some prominent points that are presented in the next section. The "didactical engineering" approach developed later in France and still prevailing shares some common aims with the traditional German subject matter didactics.

1 Place of subject matter didactics in mathematics education

The first main characteristic of subject matter didactics lies in its goal of defining the mathematical content to be taught and making them accessible and understandable to learners, mainly by concentrating on their mathematical aspects and on their place in mathematics, as described briefly by Törner and Sriraman (2006): "The focus of research was on analysing specific content and use this as a basis to elaborate on instructional design.” Subject matter didactics “is an approach to […] mathematics education and research on teaching and learning mathematics (i.e. didactics of mathematics), which concentrates on the mathematical contents of the subject matter to be taught, attempting to be as close as possible to disciplinary mathematics. A major aim is to make mathematics accessible and understandable to the learner” (Barzel and Sträßer in Bruder et al. 2013).

A second feature is related to the method used to prepare the content for the students. In terms of Klein’s tradition, the word “elementarize” is used to describe the process of designing the content to be taught and learnt. This design may be based on mathematical methods. “Subject matter didactics proceeds to prepare the pre-given mathematical disciplinary knowledge for instruction as a mathematical content, to elementarize it and to arrange it methodically” (Steinbring 2011, p. 45). This didactically oriented content analysis, which follows mathematical methods, is to give a better foundation for the formulation of content-related learning goals and for the development, definition, and use of a differentiated methodical set of instruments. (Griesel 1974, p. 118, translation by H. Steinbring 2011, p. 45).

An example of the preparation of the foundations of a geometry curriculum based only on a mathematical method is provided by the New Math reform in France. Geometry was conceived as an example of the construction of mathematical theory based on axioms, appropriate for school because elementary geometry was a closed theory from a mathematical point of view and a perfect illustration of linear phenomena. The meaning of taught geometry was not to be found in its modeling power of spatial situations but in the internal steps of the construction of a mathematical theory.

The rationale for the focus on subject matter didactics lies in the fact that mathematics taught or used in different institutions may differ even if related to the same mathematical notions. More than 20 years ago, Chevallard analyzed in depth the process of didactical transposition (Chevallard 1991) comprising two steps:

- The first step external to school, the move from reference knowledge (scholarly knowledge later expanded to social reference practices and knowledge) to the content to be taught as presented in the programs of studies or developed in curricula
- The second step internal to school, in which the content to be taught is actually taught in the real classroom and transformed according to the constraints of real teaching (time, students, examinations, specific school culture and tradition, etc.)

In the aforementioned definitions, subject matter didactics seems to deal mainly with the delimitation and justification of the choice and organization of mathematical content to be taught, that is, with the first step of the didactical transposition. A book like *Geometrie für Leher und Studenten* (Geometry for teachers and students) by Holland (1974/1977) is a typical example of subject matter didactics (Sträßer 1994) presenting a background theory of plane Euclidean geometry for teaching in classrooms of secondary school. “The book offers an axiomatic structure of Euclidean geometry that through the system of concepts, the choice and organisation of the contents is adaptable to the teaching of geometry in schools, to the textbooks, the programme of studies, and the teaching objectives of German 'Länder' (states)" (foreword by Holland quoted by Sträßer 1994).

The field of action of subject matter didactics may go beyond the mere choice and organization of the content to be taught. It may also encompass the development of teaching material, including teaching methods, thereby covering completely the first step of didactical transposition. Later, German researchers requested that results from empirical investigations be included in subject matter analysis (Sträßer 1994, p. 168; second step of the didactic transposition). Subsequently, several subject matter analyses added evaluation elements about student difficulties related to specific aspects of the content. For example, the review done by Vollrath (1992) of the structural analyses of practical arithmetic taught in Volksschulen (especially of problems involving proportionality) was complemented by the results of empirical investigations on the procedures used by students to solve these problems.

The part of subject matter analyses devoted to preparing the content to be taught can be interpreted in terms of the French "anthropological theory of didactics" developed by Chevallard (1999). A praxeology is described by Chevallard as a four-tuple consisting of the set of tasks defining the field of human activity in question, of techniques developed to solve the tasks, of technologies justifying the techniques, and of theories founding the technologies. The evolution and changes of the teaching of a mathematical notion over time are made visible through the description of the various praxeologies in use.

A typical case of changes in praxeologies in curricula is the New Math reform in France, which saw the move from teaching the three cases of congruence of triangles to teaching geometrical transformations. Techniques used in tasks for proving congruence of segments or of angles moved from the use of congruence of triangles to the use, sometimes very sophisticated, of transformations such as reflections, rotations, and shifts. In 2000, the triangle congruence cases returned to teaching but coexisted with transformations that in the present curricula have partly disappeared from the teaching of geometry.

Chevallard distinguishes between two kinds of praxeologies or organizations: mathematical and didactical. Whereas the mathematical praxeologies deal with mathematics, the latter answer the question of how to organize the study (*l'étude*) of a given mathematical praxeology by identifying the various steps and the forms of the encounter of the students with the elements of the respective mathematical praxeology. Viewed from the perspective of praxeologies, subject matter analyses offer "mathematical praxeologies" of the content to be taught. They develop, within mathematics, technologies and theories justifying the mathematical tasks to be given to students and the techniques to solve these tasks. The relative importance of the practice part (describing the tasks given to students and the associated techniques) and of the knowledge part (formulation of technologies and theory) may vary according to the authors and their role in the teaching system. Authors of curricula and of textbooks focus on the practice part, whereas individuals in charge of changes in study programs will mainly offer suggestions in the knowledge part.

This process of going from a delimitation of the content to be taught governed by essentially mathematical methods to their real teaching took place in France and gave birth to the method of "didactical engineering." Its roots can be found in the reforms of the teaching of mathematics from the very beginning of the twentieth century.

2 The origins: changes in the French program of studies

France shares with Germany the fact that the teaching of mathematics and the design of study programs in mathematics have drawn the attention of many members of the "noosphere", "the sphere of those who think about teaching […] who share an interest in the teaching system, and who 'act out' their impulses in some way or another" (Chevallard, 1991). These members come from various places: university mathematicians, leaders of the association of mathematics teachers (APMEP), persons in charge of controlling the content to be taught, e.g., state inspectors of mathematics teaching, and after 1969 members of the IREMs (*Instituts de Recherche sur l'Enseignement des Mathématiques*), institutes created after 1968 in particular at the strong request of the mathematics teacher association.

Among these members of the noosphere, mathematicians played a decisive role. Artigue (1995) gives two examples of reforms made in France at the initiative of mathematicians: the reform of 1902 and the New Math reform that affected also a wide range of countries across the world, including the Federal Republic of Germany.

In the reform of France's high school curriculum in 1902, famous mathematicians wanted to introduce the idea of "scientific humanities" as opposed to the prevailing classical literary culture. Henri Poincaré and Henri Borel gave two public lectures entitled "The general definitions in mathematics" (Les définitions générales en mathématiques) and "The practical exercises of mathematics in secondary school" (Les exercices pratiques de mathématiques dans l'enseignement secondaire).

"What is a good definition?" asked Poincaré at the beginning of his lecture. He added that his question did not deal with a good definition for a philosopher or a scholar but with a good definition for teaching. His answer was very simple. A good definition is the one that is understood by students. ("C'est celle qui est comprise par les élèves"). Then he posed another question: "Why do the majority of individuals not understand what is so clear in mathematics?" Starting from this second question, he introduced the distinction between a logical explanation and an intuitive explanation. Poincaré referred to the history of mathematics and showed that often in the construction of a mathematical notion, such as a fraction or function, an intuitive image is first composed and only later is this image transformed into a logical definition. He then argued that teaching should organize this process from an intuitive encounter with the concept leading only in a final step to its definition (cf. Gispert 2013).

As is very clear in the quoted excerpt of Poincaré's presentation, these lectures dealt with not only the content but also with the appropriate methods for its teaching. The mathematical reputation of these mathematicians was a warrant of their expertise on teaching methods (Gispert 2011). However, this expertise was not necessarily accepted by members of the French teaching system. In 1904, a representative of the teachers of the lycées, Emile Blutel, one of the founders of the association of mathematics teachers, complained that the activity of teachers was more ruled than solicited ("L'activité des enseignants du secondaire a été plus réglementée que sollicitée," Gispert, ibid.).

The second example is the well-known New Math reform of the 1960s. The mathematics study program for middle and secondary schools (enseignement secondaire, 11- to 18-year-old students) appeared as obsolete and inadequate with respect to the scientific and technical progress of society, according to various members of the noosphere. In particular the CIEAEM (Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques) managed to mobilize the psychologist Piaget, the philosopher Gonseth, and the mathematicians Dieudonné, Lichnerowicz, and Choquet (Gispert, ibid.). The mathematicians involved in the reform were probably also influenced by the prominent place of structuralism in various domains of scholarly knowledge, such as psychology, linguistics, and of course mathematics, deeply marked by the French Bourbaki movement. The same strong hypothesis about learning prevailed in Germany and in France: The content of teaching did not give a chance to students to truly understand it because it was mathematically unfounded.

The mathematical content to be taught was extensively changed under the umbrella of structuralism without taking into consideration its impact on the actual everyday teaching in classroom and on learning processes. In the case of geometry, a new organization was proposed especially by the mathematician Choquet (1964), aiming at finding the best set of axioms for a presentation providing a logic sequencing of the content appropriate for secondary mathematics. In particular, a choice had to be made between a light system of axioms and a heavier set. A heavier set avoids the long and tedious path to theorems, whereas a light system minimized what had to be accepted. Such a proposal for the reorganization of geometry was based primarily only on the block of knowledge from a praxeological approach. It was proposed as an example for future curricula and certainly affected the design of the New Math program of studies. One can recognize in this enterprise a "radical" subject matter didactic approach, the concentration on the pure mathematical content, and the wish to give a better foundation to the mathematical content before embedding it into a curriculum.

The same concern for a good axiomatic system for the teaching of geometry took also place in Germany during the same period (Steiner and Winkelmann 1981). For example, the book by Holland, mentioned earlier, seems to have the same goal of establishing a complete mathematical structure for theorizing the organization of geometry to be taught. In Germany, subject matter didactics provided background theories for other mathematical domains with the aim of mathematically justifying new ways of using them in school in order to help students truly understand the concepts. For example, Griesel (1997) described exactly in these terms the work of Lugowski (1962) on an axiomatic foundation of the demonstrative-genetic construction of arithmetic in school mathematics.

Geometry was a part of the French curricula strongly affected by the New Math reform. This reform was indeed presented by its promoters as a way of ordering the chaos of old geometry, which consisted of several local facts and required more erudition than understanding. This reform was guided in France by the underlying idea of the universal power of mathematics conceived as constructive, axiomatic, and structural. It was unnecessary in the contemporary world to teach numerous isolated facts (especially in geometry). Only the main and fundamental topics had to be taught and all the problems in various fields could be solved. The emphasis placed on the structures was substantiated by an important work of didactical transposition about an axiomatic presentation of school geometry. As a result, all textbooks of that time shared some common points:

- Expression of a system of axioms
- Description of geometrical objects in terms of set theory (see Sect. 2); geometry was only an example of the use of mathematical structures
- Importance of the linear aspects of geometry (transformations) and of vectors
- Clear-cut distinction between the vectorial, affine, and metric properties imposed by the national curricula; the affine structure was taught before the metric structure

The reform of 1969 in France gave a crucial place to geometrical transformations that were considered as the core of geometry. An emphasis was made at that time on the structure of the set of transformations based on the composition of transformations. Introduced during that period as point transformations of the plane (or space), they were still considered, in the first part of secondary school, as point transformations until 1986. After 1986, they have been presented as figure transformations at this school level.

France shared with Germany the concentration on the content to be taught in the New Math reform. In France and Germany, basic changes in the curricula can only be made "within administrative frameworks which have hindered any independent curriculum development on a rather major scale" (Tietze 1994, p.42) as opposed to more comprehensive British or American curriculum projects. This may explain why the focus on subject matter has been so strong in both countries. The question of the choice of content to be taught was viewed as the question of what is fundamental in mathematics at a given period of time and in a given societal context. The question of the teaching in school came later and was considered as an elementarizing process.

3 Reactions in France to the exclusive focus on subject matter

At the end of the 1960s and in the 1970s, another approach was undertaken by the French association of the mathematics teachers (APMEP). The commitment of its members to everyday teaching made the leaders of the association be unsatisfied with the mere change of the content to be taught. They wanted to radically change the teaching methods, to leave the world of principles behind and set up practical modalities of action in the classrooms (de Cointet, president of the association, 1975). The association advocated for a change in the study program. The program should contain:

- A list of core subjects consisting of the concepts to be acquired by students in the school year.
- A list of themes to be chosen by the teacher either for introducing new knowledge or for illustrating the usefulness of already introduced knowledge, or for nurturing new and free investigations. In particular applications related to real life were considered as providing fruitful themes.

The need to align mathematics with reality and the modeling role of mathematics were already expressed in the declaration of the APMEP. This may be linked with the notion of the basic ideas used by several German authors: "Basic ideas are aimed at describing adequate real-life contexts which represent the 'heart' (or 'essence') of the respective mathematical contents in a way understandable for the student" (vom

Hofe 1995, p. 320). In the same vein, Winter (1975) considered that the teaching of mathematics should develop (a) the ability to cognitively structure situations of everyday experience and to describe them in mathematical terms as well as (b) the creativity for coping with unknown problems.

Members of the association and the IREMs undertook a huge task on the links between core knowledge and themes. Many innovative proposals of teaching by means of core themes (noyaux-thèmes) were published in the journal of the association, in booklets, or in textbooks (see in particular *Bulletin de l'APMEP*, no.300, September 1975).

The failure of the New Math reform provided evidence of the need to take into account more than the mathematical content, to make use of general pedagogical and psychological principles (Artigue, op. cit.) in the search for improving teaching, and to know more about the learning processes and the links between teaching and learning. This certainly contributed to the need of developing research on mathematics teaching and learning that included also empirical elements. Some of the innovations taking place in schools during the 1970s were forerunners of the method of didactical engineering.

4 Toward research based on a dialectic between theory and empirical investigations

In France, long-term teaching projects covering almost all mathematics teaching on numbers and measurement were designed and experimented in primary school by Brousseau, Douady, and Perrin Glorian from the early 1970s. From an institutional point of view, it was easier for French primary schools to be places for long-term experimentation than it was for secondary schools. The culture in primary schools differed from that in secondary schools. In particular, primary school teacher education was not carried out in universities at that time. The situation in France was similar to the one in Germany in that the specificity of primary schools was different from secondary schools in many ways. In the Federal Republic of Germany, primary school teacher education focused on psychological and pedagogical aspects, whereas secondary school teacher education dealt essentially with mathematics (Sträßer 1994), and subject matter didactics started by exerting an influence especially on the *Gymnasien* (academically oriented secondary school).

The aforementioned projects were motivated by mathematical choices. In Brousseau's project about decimal numbers (1981, 1997, Chaps. 3 and 4), the epistemological rationale is to introduce decimal numbers as economic tools through which comparing, adding, and subtracting fractions can be done more quickly and with fewer errors. In particular, some types of problems – such as finding a new fraction lying between two given fractions – could also be solved more easily. The main idea of the teaching process involves constructing rational numbers as tools for measuring, and then decimal numbers as tools for approximating rational numbers. The final part of the teaching sequence focuses on rational numbers as operators, culminating in the construction of the product of two rational numbers in terms of the composition of two mappings. In the project of Douady and Perrin-Glorian (Douady 1980, 1986), decimals were introduced as approximating real numbers in measures of lengths, which were supposed to be concepts known to the students.

However, although mathematical choices prevailed in these projects, their design contained theoretical aspects that were later formulated and theorized by their authors. For example, a lever used by Douady in the sequencing of the tasks was the interplay between the numerical and the geometrical settings on the one hand, and between settings and registers on the other hand. The notion of setting introduced by Douady refers to a set of objects and relationships between them belonging to a domain of mathematics. A setting includes not only objects and relationships but also various formulations and mental images. Examples of settings are the numerical setting, the geometrical setting, and the algebraic setting. The term "register" denotes here a semiotic register, i.e., a semiotic system for representing objects and relationships (Duval 2006). The role of visual representation was not only to express and support mathematical thinking as in basic ideas. It was also meant for posing problems that cannot be solved in the register in which it was expressed, in order to necessitate a move to another setting and/or register.

One of the problems posed in the teaching sequence by Douady for introducing decimal numbers was to find the length of the side of a square with a given area equal to a whole number. The 8- to 9-year-old students could not solve the problem in the numerical setting. A graphical register with a system of axes was introduced. Students had to represent a rectangle with dimensions a and b by means of a point (a, b) on this system. A whole number p was given. Students had to represent many rectangles and then to color each obtained point in red if the area of the corresponding rectangle is larger than p, in blue if it is less than p, and in black if it is equal to p. They then had to find points representing rectangles with area equal to p. The initial question became: "Is there a square among the rectangles and what is the length of its sides?"

In the design of the teaching project about decimals, Brousseau examined how decimal numbers had evolved within the wider field of mathematics in order to identify the key mathematical problems that gave rise to decimal numbers, and to clarify the relationships between decimal numbers and other types of numbers, especially rational numbers, typically expressed in the form a/b. He also investigated the former and current presentations of decimals in teaching. These studies were published later (available in English in Brousseau 1997, Chap. 3). Brousseau started from the notion of obstacles proposed by Bachelard and distinguished between three origins of obstacles: ontogenetic, epistemological, or didactical. After Bachelard (1938), who introduced the notion of epistemological obstacle, i.e., an obstacle constitutive of the way of knowing, Brousseau considered that knowledge is simultaneously a support and an obstacle (Brousseau 1997, p. 84). Obstacles are made apparent by errors or inefficient processes. Such errors and processes are not due to chance but are persistent and reproducible. Knowledge is very efficient in certain situations but can be inappropriate for others. Whereas epistemological obstacles can be found in the history of the concepts themselves, didactical obstacles stem from the presentation of a concept and the way of using it in teaching. Inherited from a long tradition, a widespread presentation of decimal numbers is associated to measurement and related to technical operations on whole numbers. "As a result, for students today, decimal numbers are whole numbers with a change of units" (Brousseau ibid., p. 87). Overcoming obstacles means transforming knowledge acquired by the learner. Exposing the learners to problem situations and organized "milieux" with which the learner interacts is the way proposed by Brousseau.

The authors of these two long-term projects did not claim to have used a didactic engineering method because the term was not yet coined. However, it is very clear that problems or rather problem situations to which the students were exposed played a fundamental role in creating the conditions for the students' development and transformation of knowledge.

5 Development of students' conceptions when faced with problems

In the mid-1970s, after the New Math reform, research on the development of students' conceptions and understandings was undertaken simultaneously with the design and experimentation of these long teaching sequences.

Feeling that it was not enough to change the curricula without knowing more about the ways students understood mathematical concepts, several researchers investigated the solving strategies and erroneous procedures of students faced with well-chosen problems in order to propose models of their thinking processes, understanding, and conceptualization (Vergnaud 1991).

Student conceptions of specific mathematical notions could be identified through situations students were faced with, as expressed by Rouchier et al. (1980) and Artigue and Robinet (1982). When presenting conceptions about the notion of a circle at primary school, Artigue and Robinet wrote that they did not want to analyze the students' conceptions independently of a precise study of situations in which these conceptions were involved.

Situations are chosen according to the conceptions they may favor, and if they are in a sequence their order is also chosen in the same way. Several research projects studied how students' knowledge developed in a sequence of problem situations carried out in a classroom with teachers' interventions and collective phases under the guidance of the teacher.

Two significant examples are given by the situations and didactic processes on rationale positive numbers (Rouchier et al. 1980) and the didactic experiment on the concept of volume (Vergnaud et al. 1983). This latter research project gave rise to a whole issue of the newly created French journal *Recherches en didactique des mathématiques* (4.1, 1983) and consisted of three articles: the first one on the conceptions and competences of students of four middle school classes when faced with tasks outside the classroom (Ricco et al. 1983), the second on a sequence of didactic situations in a Grade 7 classroom (Vergnaud et al. 1983), and the third on a comparison between students' answers to a questionnaire given before the sequence and to the same questionnaire given after the sequence (Rogalski 1983; Rogalski et al. 1983). In the introduction of the issue (pp. 23–24), Vergnaud claimed that the theory of situations, the psychogenetic complexity, and the task analyses complement each other. However, his argument reveals that the general aim of the study lies in investigating the genesis of knowledge in the short term for the teaching sequence and in the longer term for the interviews: “Il existe un temps long de la psychogenèse, bien connu des psychologues, qui se mesure en années et qui permet d’établir des hiérarchies dans la complexité des problèmes et des concepts mathématiques. Il existe aussi un temps court de la psychogenèse, moins bien étudié que le premier et pourtant essentiel en didactique, qui concerne l’évolution des conceptions et des pratiques d’un sujet ou d’un groupe de sujets face à une situation nouvelle” (p. 24). (“There is a long-term time of the psychogenesis, well known from psychologists, that is measured in years and allows [one] to establish hierarchies in the complexity of problems and mathematical concepts. There is also a short-term time of the psychogenesis, less studied than the former one but essential in didactics, that deals with the development of conceptions and practices of an individual or a group of individuals faced with a new situation”).

6 Didactical engineering

At the time of these investigations, i.e., at the beginning of the 1980s, the term “didactical engineering” appeared in articles and internal meetings of the French community of researchers in mathematics education (Artigue 1994). Some researchers like Chevallard (Artigue 1990, p. 284) urged the community to eventually cope with theorizing the critical and complex real object of the didactics of mathematics: the actual functioning of the didactic system, or in other terms the functioning of teaching sequences in classrooms with real students and real teachers. Didactical engineering refers to a method that aims at carrying out empirical studies of didactic phenomena in circumstances compatible with an ethical study of teaching.

The method consisted of four phases: design of a teaching sequence comprising a sequence of situations, experimentation in one or several classrooms, observation of the students’ activity and of the teacher’s interventions as well as of the collective discussions, and analysis of the observations.

It becomes a method and is no longer an innovation as soon as the design of the situations considers each situation as dependent on global and/or local variables. A variable of a given situation (or of a task) is a feature of the situation affecting the possible solving strategies. Playing on such a feature may make the task easier or more difficult. It is a lever in the hands of the teacher or the designer of the tasks in order to foster the construction of knowledge by the learner. For example, in an additive task, the nature of the numbers is a variable that can have different values such as integers, decimals, fractions. The task is easier with whole integers than with decimals or with fractions – and often easier with decimals than with fractions.

For each situation, the researcher analyzes the possible effect of different values of the variables on students’ solving strategies and chooses the values according to the strategies (s)he wants to favor. Each situation is not considered in isolation from the others but within the whole sequence of situations. Values of variables are chosen in order to foster an expected development of students’ strategies during the sequence. Two components of the method are critical:

- The design of situations
- The a priori analysis

6.1 The design and role of situations

A keystone is indeed the notion of a "situation" calling for a specific functioning of knowledge. The problem is the source and criterion of mathematical knowledge from both epistemological and cognitive perspectives, wrote Vergnaud (1981), who later preferred to replace the word "problem" by "situation" under the influence of the theory of didactical situations by Brousseau. Piaget's theory of "equilibration" (Piaget 1975) was a crucial source for the idea of adaptation in which students construct new knowledge by becoming directly engaged in solving a novel type of problem, refining their concepts and strategies in light of the feedback from a material and social milieu (Brousseau 1986, 1997, pp. 64, 147). Here "situation" refers to a collection of problem-solving tasks and task environments designed to evoke a particular form of "adidactical" adaptation on the part of students, and intended to help them construct some specific new knowledge. The adjective "a-didactical" refers to the fact that the students must experience the task not as intended to teach them but as if they had to cope with a real problematic situation outside the classroom and find a way to solve it with all their means.

Designing a situation not only means designing a problem but also determining the conditions under which it will be solved, the means of action by the students, and the feedback they will receive from the environment in the solving process. Conditions, means of actions, and feedback depend on variables on which researchers can play in order to favor an expected development of the students' strategies.

In the teaching sequences about circles at primary school, Artigue and Robinet (1982) exposed students to three situations so that each situation called on more than the preceding one for using the invariance of the curvature of a circle. They did it by playing on the possibility of using the center of a circle.

In the first situation, students had to build three discs with various radiuses from a set of circular sectors in cardboard and then to create a missing sector (Fig. 1). The second situation was similar with parts of rings (Fig. 2). In the third situation, students had to reconstruct four circles with various radiuses from 14 arcs (Fig. 3). In the first situation, students could rotate an existing sector around its center for building the missing sector. In the second one, as the center was no longer available, students drew by hand the missing annulus. Because obviously it was not satisfactory, they looked for another strategy and moved to the use of the constant curvature. This strategy was confirmed in the third situation in which it was too tedious to find the center of 14 arcs. In this sequence, it was expected that a drawing by hand would not be precise enough and would be rejected by the other students without the need for an intervention by the teacher. Means of action were limited by the researchers with the assumption that visual feedback would be sufficient to make students reject poor drawings by hand and look for a more precise process.

In geometry the move from a property used in action by students to another one less familiar can also be organized by a play on available instruments. A good example is given by the didactic engineering on reflection in Grade 6 as proposed by Grenier (1990). Paper folding is given at first for introducing symmetry lines but then paper folding only plays the role of empirical checking of the validity of a construction. Students are rapidly asked to construct symmetry lines of figures without resorting to

Fig. 1 The discs

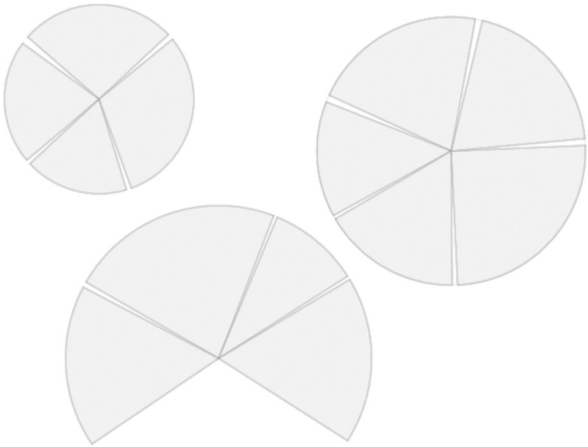

Fig. 2 The parts of the rings

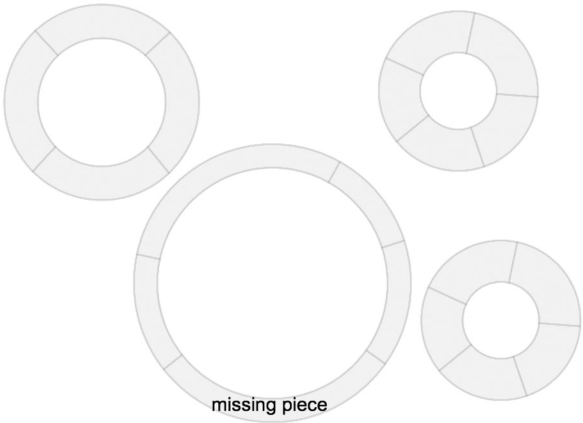

Fig. 3 The arcs of the circles

paper folding and with specific instruments in order to favor the use of mathematical properties. The play on instruments is systematically used to hinder the use of certain properties and favor the use of others. Only later does the teacher formulate the properties used in action by students, after (s)he has gathered the various strategies.

Didactical engineering places importance on problems and organized milieus. The construction of these problems and milieu is accomplished by means of an a priori analysis.

6.2 A priori analysis

The design of the teaching sequence in didactical engineering is based on an "a priori analysis" that plays a critical role, since the a priori analysis is contrasted with the a posteriori analysis of the observations of the implementation in the classroom. "A priori" does not refer to a temporal place (i.e., prior to the experimentation) but to the independence of the analysis from any empirical fact arising from the experimentation. The discrepancies between a priori and a posteriori analyses lead us to reconsider the hypotheses on which the a priori analysis is based and allow us to refine or even modify the theoretical approach underpinning the research work.

As explained earlier in the example of the decimal-teaching project by Brousseau, a priori analysis devotes a large part to the epistemological analysis of the mathematical content involved in the teaching project but also to a cognitive analysis of the available knowledge of the students based on research results and to an analysis linked to the institutional functioning of the teaching. These two latter analyses are critical for the functioning of the situations in classrooms. The a priori analysis must take into account the teaching constraints coming from the program of studies as well as the possibility for the teacher to manage the project in the classroom. At the cognitive level, an adidactical situation is expected to foster the emergence of new solving procedures that will be the seed of new knowledge. Students must be able to start solving the task in this situation but with an incomplete or tedious procedure. If not, the adidactical situation cannot play its role. The design of such situations must optimize the choice of the variables of the situation in order to secure as much as possible the expected processes of the students and the adequacy of the teaching project with the usual teaching that takes place in the classroom.

A priori analyses and subject matter analyses share the mathematical, epistemological, and possibly historical analysis of the content to be taught (Sträßer 2013; Winslow 2013). It does not mean that subject matter didactics did not take into consideration cognitive aspects. In an article (1977 and seminar translated into English in 2000) based on his plenary lecture at ICME 3, Kirsch carried out an analysis of modes of simplification, developed in view of making content accessible and understandable by students, and claimed that the preexisting knowledge of students should be taken into account when designing teaching: "Above and beyond internal mathematical considerations, the didactician and mathematician must show imagination, and take into account the pupils' background knowledge" (Kirsch, 2000, p. 270). But the traditional German subject matter didactics did not look into the consequences of choices in actual teaching sequences in classrooms. Holland (1982, p. 297) expressed very clearly the role and place of cognitive aspects with regard to a subject analysis:

"The psychological components in concept acquisition are here in parentheses, not because they are not important but because they can be object of (empirical) investigations only after a successful subject analysis".

7 Development of didactical engineering

Initially, didactical engineering investigated the teaching of specific concepts or, as stated previously, the development of students' conceptions in a sequence of problems, generally at primary or secondary school.

Later the method shed light on components of the teaching process that were not extensively investigated and theorized. Finally, it was used for studying general didactical phenomena. Let us give some examples.

Grenier (op. cit.) experimented for the first time on a teaching sequence on reflection in Grade 6. Contrasting the a priori analysis with the a posteriori analysis, she observed that the play on instruments did not necessarily lead to a change of solving strategies. When they did not have measurement tools, the students tried to estimate measures by eye or using a pen as a measurement unit, instead of using geometrical properties. The interventions of the teacher seemed to have no effect on students' strategies. Grenier modified the situations for another teaching experiment the following year but even if the trajectories of the students were closer to the expected ones, the analysis of the observations revealed a strong resistance both in the students' conceptions and in the teacher interventions. In collective debriefings of the group work, the teacher ignored some popular strategies and focused on strategies used by a small number of students because they were the expected ones. The teacher rejected strategies of measurement with a pen or with the section of a ruler, by saying that it was not precise enough. This argument was not understood by students who thought that using a measurement was more precise than using the fact that points are collinear. This research showed that a priori analysis could deal not only with situations but also with teachers' interventions and decisions. The a priori analysis also had to take into account phenomena related to the didactical contract. Some behaviors of students and teachers can be explained only by the fact that there are implicit rules underlying the progress of the classroom. This research showed very clearly how much a teaching sequence results from a balance between two poles: the adidactical pole and the pole related to the didactical contract.

The didactical engineering method was used at tertiary level (Robert 1992; Dorier et al. 1994) and questioned the construction of knowledge as a tool for solving problems at that level. More than efficient tools for solving a class of problems, concepts taught at tertiary level own a power of generalization and unification of different strategies and methods. It seems difficult that students can construct such concepts on their own from adidactical situations.

The study of phenomena related to the integration of technology into the teaching of mathematics used the didactical engineering method. For example, instrumentation processes of dynamic geometry were investigated by Restrepo (2009) in a long-term didactical engineering (1 year) method.

The robustness of the didactical engineering method was also investigated by using teaching sequences designed with a didactical engineering method in other conditions. For example, Perrin Glorian (1993) showed that it is very difficult for less advanced students to engage in adidactical situations and produce new solving strategies. It was also very difficult for them to understand the institutionalization phase done by the teacher. In this phase, the teacher extracted and formulated the mathematical and official knowledge from adidactical situations. The students did not understand the link between the teachers' discourse and what they experienced in the situations.

The history of didactical engineering showed that concerns about the content to be taught at the time of the New Math reform provided a context for investigating teaching and learning phenomena beyond the pure subject matter. The method of didactic engineering started as a method for better understanding the relationships between the design of problem situations and the development of specific mathematical concepts by students. The method was then extended into several other directions: the length of the teaching experiment, teaching at tertiary level, less advanced students, and use of technology, which finally led to the study of other phenomena related to teaching. Margolinas and Drijvers (2015) recently published a paper with another perspective on didactical engineering, comparing didactical engineering with design research.

A short and subjective comparison of the traditional subject matter didactics in Germany and didactic engineering in France identifies several common features in both approaches in the 1960s and 1970s at the time of the New Math reform. In both countries, this reform resulted from and acting as a catalyst for the concentration on the teaching content through mathematical analyses and methods. This approach probably lasted longer in Germany than in France; it was more developed with many deep theoretical analyses of the mathematical content and had a stronger influence on the textbooks. In France, didactical engineering was grounded in early attempts to link the choices made on the content to be taught with the students' learning processes. It was anchored in experimental teaching projects and empirical investigations, whereas traditional subject matter didactics in Germany separated the mathematical choices from the cognitive aspects of concepts acquisition, even if both components were recognized as having equal importance: the investigation of cognitive aspects should take place in a second step. The notion of a priori analysis illustrates well the focus placed by didactical engineering on the link between task choices and students' learning as opposed to sophisticated mathematical analyses of systems of axioms in geometry or proportional reasoning on magnitudes in a traditional subject matter analysis.

8 What now?

Especially over the past 20 years, the landscape of research in mathematics education changed greatly in France and Germany, in particular through the internationalization of research. Two phenomena must be mentioned:

- The only indirect influence of didactical engineering on teachers' everyday practice and the move to second-generation didactical engineering (Perrin Glorian 2011).

- The growing use of networking of theoretical frameworks (Artigue 2009) at a national level as well as the international level.

The use of didactical engineering in everyday teacher practice was investigated by Bolon (1996). She found that teachers do not make use of original situations of didactical engineering but instead use simplified and isolated situations presented in worksheets, with the main ideas originating from situations developed in didactical engineering. Perrin Glorian (2011) investigated the transformation process of an original didactical engineering into a didactical engineering appropriate for teaching and claimed that this transformation requires work, in particular on the conditions of the transmission of the engineering.

The complexity of the processes in mathematics teaching led to the use of several theoretical frameworks in the same research project. For example, instrumentation theory was associated to the theory of didactic situations or with the anthropological theory of didactics to study the use of digital technology in mathematics teaching (Artigue 2009). Prediger (2010) illustrated the link between theoretical frameworks and scientific practices by analyzing the difficulties of problem statements through various frameworks, including the use of basic mental models.

Subject matter didactics and didactical engineering are not dead. Some calls for re-focusing more on the analysis of mathematical content have been made in Germany (Jahnke 1998; Wittmann 2014). The development of resources for mathematics teachers is a critical issue today with the increasing number of resources available on the Internet. What are the best ways of transmitting didactical engineering products in order to facilitate their use by a large number of teachers without changing their impact on the learning processes? (Perrin Glorian, op. cit.). What are the conditions for such a didactical engineering product to be really used in ordinary teacher practice? Which mathematical and didactical knowledge do teachers need to make use of such resources? Many questions remain unanswered that promote research related to subject matter didactics and didactical engineering.

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