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ORIGINAL ARTICLE

Three-way conflict analysis with similarity degree on an issue set

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Abstract

The present paper introduces a new model of three-way conflict analysis with similarity degree on an issue set. Specifically, we introduce an evaluation of similarity degree, from a relative quantitative point of view, to evaluate the attitude similarity between any two agents. Based on similarity degree, we define a trisection of all pairs of agents on an issue set, and propose a three-level conflict model induced by such a trisection. More importantly, we solve the threshold-selection problem for three-level conflict analysis on multiple issues. We prove that the trisection model (resp. the three-level conflict model) defined in this paper is a conservative extension of the corresponding trisection model (resp. three-level conflict model) defined in Yao 2019 on multiple issues. Therefore, the present paper extends and improves the results of Yao 2019 on multiple issues.

Keywords Three-way decision · Conflict analysis · Threshold selection · Similarity degree

1 Introduction

Conflicts occur naturally in the real world at all levels of individual and society. So the study and resolution of conflicts is crucial in both theory and practice. In particular, conflict analysis plays an important role in political and lawsuits disputes [23], labor-management negotiations [24], military operations [20, 21] and so on [22]. In recent years, many scholars have constructed different types of conflict analysis models based on their own knowledge backgrounds [16–19].

In the beginning, Pawlak [20] mapped each agent's attitude towards issues into three values $\{-1, 0, +1\}$, divided all agents and all pairs of agents into three regions, respectively, with the help of auxiliary function and distance function, and introduced the most basic conflict model; then Deja [2] argued that Pawlak's conflict model is limited to the outermost layer and does not take into account the essential causes of conflict; subsequently, Sun et al. [25] developed a rough set-based conflict analysis model along with Deja's thinking and solved the problem of how to find a feasible consensus strategy. In addition, Yang et al. [30] investigated evidence conflict and belief convergence based on the analysis of the degree of coherence between two sources of evidence and illustrated the stochastic interpretation for the basic probability assignment; Yu et al. [37] proposed the supporting probability distance to characterize the differences among bodies of evidence, and defined a new combination rule for the combination of conflicting-evidence; by combining game theory with conflict analysis, Zia et al. [1] constructed a new conflict analysis model based on game-theoretic rough sets; Zhi et al. [42] proposed a model of multilevel conflict analysis based on fuzzy concept lattices, which incorporated two types of uncertainties into a unified framework.

In recent years, the theory of three-way decision has also received more and more attention [5, 11, 14, 27–29, 33, 34, 36, 39, 40]. It was originally proposed by Yao [31, 32] for thinking, problem solving and information processing in three levels. Generally speaking, there are mainly two types of models of three-way decision: one is based on inclusion relations and the other is based on one or two evaluations. With the in-depth research, we can find that three-way decision and conflict analysis are closely related to each other. Therefore, a growing number of scholars have applied the idea of three-way decision to conflict analysis, and established many different types of three-way conflict analysis models [3, 4, 6–8, 15, 38]. For

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example, Lang et al. [9] combined decision-theoretic rough sets with three-way decision to design probabilistic conflict, neutral and coalition sets on dynamic information systems; by reformulating and generalizing Pawlak's conflict model, Yao [35] introduced three levels of conflict: strong conflict, weak conflict and non-conflict; Sun et al. [26] established an improved Pawlak conflict model by combining three-way decision with probabilistic rough sets on dual universes; Zhi et al. [41] considered alliance, conflict, and neutrality attributes of cliques under a one-vote veto based on approximate three-way concept lattice. In addition, some conflict models based on fuzzy information systems, such as [10, 12, 13], were also constructed from different points of view.

The threshold-selection problem has always been a fundamental issue in three-way conflict analysis. However, many existing three-way conflict analysis models have not yet solved the threshold-selection problem. This is one of the motivations of the present paper. Another motivation of this paper is from the fifth open problem proposed by Xu et al. in [29]. In [29], Xu et al. introduced a new method of three-way decision on hybrid information tables, and suggested in the fifth open problem that: "Study conflict problems by using $\Phi_{(\alpha,\beta)}$ or $\Phi_{(\alpha,\beta)}^{\langle\rho_1:\rho_2\rangle}$. One may introduce importance ratio into conflict tables and define the model of conflict analysis induced by the trisections of $\Phi_{(\alpha,\beta)}$ or $\Phi_{(\alpha,\beta)}^{\langle\rho_1:\rho_2\rangle}$. We claim that this work will extend the conflict analysis model of Yao [35]." So, motivated by this suggestion and following Yao's work in [35], the present paper proposes a new method of three-way conflict analysis with similarity degree so as to extend the work of Yao in [35]. However, we did not use the trisection model $\Phi_{(\alpha,\beta)}$ or $\Phi_{(\alpha,\beta)}^{\langle\rho_1:\rho_2\rangle}$ proposed in [29] but a newly defined one to induce the conflict model. Below we summarize the main contribution of this article.

- We propose the evaluation of similarity degree, in a three-valued situation table, to quantitatively evaluate the attitude similarity between any two agents, and propose the evaluation of difference degree to quantitatively evaluate the attitude difference between any two agents. The proposed difference degree has a different formulation from the aggregated conflict function proposed by Yao in [35] on multiple issues, but it is proved to be equivalent to the latter. Moreover, it also resolves the inconsistency of Pawlak's treatment of any two agents who have the same attitude "neutral" on a single issue.
- Based on the above evaluations, we introduce a trisection of all pairs of agents on an issue set, and introduce a three-level conflict model induced by such a trisection. More importantly, we successfully solve the threshold-selection problems for both the trisections of all pairs of agents and the three-level conflicts on multiple issues, which are achieved by defining unique measure functions for the trisections and the three-level conflicts, respectively.
- We prove that the trisection model (resp. the three-level conflict model) proposed in this paper is a conservative extension of the corresponding trisection model (resp. three-level conflict model) proposed by Yao [35]. Therefore, we solve the threshold-selection problem for Yao's three-level conflict analysis on multiple issues.

The rest of this paper is organized as follows. In Sect. 2, we give a brief review of two models of conflict analysis to which the later parts of the paper will relate. In Sect. 3, we introduce the basic concepts of this paper, define the trisection of all pairs of agents and the three-level conflict induced by such a trisection, and discuss the properties of these trisections and three-level conflicts; at the same time, we propose two ways of finding the optimal three-level conflict in finite steps, and therefore solve the threshold-selection problem for three-level conflict analysis on multiple issues. Finally, in Sect. 4 we summarize the results of this paper and look forward to the future work.

2 Preliminaries

In this section, we briefly review two models of conflict analysis to which the later parts of this paper will relate, i.e., the model proposed by Pawlak [20] and the model proposed by Yao [35].

Definition 2.1 [20] A ternary conflict information system is defined as a triple $S = (U, V, f)$, where $U = \{u_1, u_2, \cdots, u_m\}$ is a finite non-empty set of agents, $V = \{v_1, v_2, \cdots, v_n\}$ is a finite non-empty set of issues, $f : U \times V \to \{-1, 0, +1\}$ is a three-valued evaluation that maps a pair of an agent and an issue to a value in $\{-1, 0, +1\}$. The meaning of the mapping is interpreted as follows:

$$f(u,v) = \begin{cases} -1, & \text{agent } u \text{ is negative about issue } v, \\ 0, & \text{agent } u \text{ is neutral about issue } v, \\ +1, & \text{agent } u \text{ is positive about issue } v. \end{cases} \tag{1}$$

In [20], Pawlak first proposed the above definition of conflict information system which is now also called three-valued situation table. In the rest of this paper, when we mention a three-valued situation table, we always mean $S = (U, V, f)$ defined above. For simplicity, in a three-valued situation table we will use $\{-, 0, +\}$ instead of $\{-1, 0, +1\}$ to represent the agent's attitude on the issue. Based on Definition 2.1, Pawlak defined a distance function to measure the distance between any two agents in a three-valued situation table, as shown below.

Definition 2.2 [20] In a ternary conflict information system $S = (U, V, f)$, a distance function $d(u_i, u_j)$, for any two objects $u_i, u_j \in U$, is defined as follows

$$d(u_i, u_j) = \frac{\sum_{v \in V} \varphi_v^*(u_i, u_j)}{|V|}, \tag{2}$$

where

$$\varphi_v^*(u_i, u_j) = \frac{1 - \varphi_v(u_i, u_j)}{2} = \begin{cases} 0, & \text{if } f(u_i, v) \cdot f(u_j, v) = 1 \vee u_i = u_j \\ 0.5, & \text{if } f(u_i, v) \cdot f(u_j, v) = 0 \wedge u_i \neq u_j \\ 1, & \text{if } f(u_i, v) \cdot f(u_j, v) = -1 \end{cases} \tag{3}$$

and

$$\varphi_v(u_i, u_j) = \begin{cases} 1, & \text{if } f(u_i, v) \cdot f(u_j, v) = 1 \vee u_i = u_j \\ 0, & \text{if } f(u_i, v) \cdot f(u_j, v) = 0 \wedge u_i \neq u_j \\ -1, & \text{if } f(u_i, v) \cdot f(u_j, v) = -1 \end{cases}. \tag{4}$$

By using the above distance function d, Pawlak further defined three relation sets, namely alliance, conflict and neutrality, as follows:

$$\begin{cases} A = \{(u_i, u_j) | d(u_i, u_j) < 0.5\}, \\ C = \{(u_i, u_j) | d(u_i, u_j) > 0.5\}, \\ N = \{(u_i, u_j) | d(u_i, u_j) = 0.5\}. \end{cases} \tag{5}$$

According to the above Pawlak's definitions, agents u_i and u_j are in an alliance relation on a single issue v when $f(u_i, v) = f(u_j, v) = 0$ and $u_i = u_j$; while agents u_i and u_j are in a neutrality relation when $f(u_i, v) = f(u_j, v) = 0$ and $u_i \neq u_j$. Therefore, Pawlak's definitions have inconsistency in the treatment of $f(u_i, v) = f(u_j, v) = 0$ on a single issue v. In addition, the three relation sets are defined by the value of 0.5, which may be further improved by introducing appropriate thresholds. In 2019, Yao [35] reformulated and extended the above Pawlak's model to a more general level, and resolved the inconsistency of Pawlak's treatment of $f(u_i, v) = f(u_j, v) = 0$ on a single issue v. Below we specifically review and explain the results of Yao [35].

In [35], Yao proposed two types of trisections, namely the trisection of all agents and the trisection of all pairs of agents, on a single issue (resp. on multiple issues); at the same time, Yao proposed two types of three-level conflicts, namely the one induced by a trisection of all agents and the one induced by a trisection of all pairs of agents on a single issue (resp. on multiple issues). It can be verified that the two types of trisections on a single issue induce the same three-level conflicts, but the two types of trisections on multiple issues do not seem to have such result. We firstly reproduce the two types of trisections on a single issue defined in [35].

Definition 2.3 [35] In a three-valued situation table $S = (U, V, f)$, the trisection of all agents on a single issue $v \in V$, denoted by $\ll A_v^-, A_v^0, A_v^+ \gg$, is defined by

$$\begin{cases} A_v^- = \{u \in U | f(u, v) = -1\}, \\ A_v^0 = \{u \in U | f(u, v) = 0\}, \\ A_v^+ = \{u \in U | f(u, v) = +1\}. \end{cases} \tag{6}$$

Definition 2.4 [35] In a three-valued situation table $S = (U, V, f)$, the trisection of all pairs of agents on a single issue $v \in V$, denoted by $\ll R_v^=, R_v^\approx, R_v^\asymp \gg$, is defined by

$$\begin{cases} R_v^= = \{(u_i, u_j) \in U \times U \mid |f(u_i, v) - f(u_j, v)|/2 = 0\} \\ \quad = \{(u_i, u_j) \in U \times U | f(u_i, v) = f(u_j, v)\}, \\ R_v^\approx = \{(u_i, u_j) \in U \times U \mid |f(u_i, v) - f(u_j, v)|/2 = 0.5\} \\ \quad = \{(u_i, u_j) \in U \times U | f(u_i, v) \neq f(u_j, v) \wedge f(u_i, v) \cdot f(u_j, v) = 0\}, \\ R_v^\asymp = \{(u_i, u_j) \in U \times U \mid |f(u_i, v) - f(u_j, v)|/2 = 1\} \\ \quad = \{(u_i, u_j) \in U \times U | f(u_i, v) \cdot f(u_j, v) = -1\}. \end{cases} \tag{7}$$

Based on the above two types of trisections, Yao proposed two types of three-level conflicts on a single issue as follows.

Definition 2.5 [35] In a three-valued situation table $S = (U, V, f)$, for a single issue $v \in V$, the three-level conflict with respect to the trisection $\ll A_v^-, A_v^0, A_v^+ \gg$ is defined by

$$\begin{cases} \text{strong conflict:} \quad SC(A_v^-, A_v^+), \\ \text{weak conflict:} \quad WC(A_v^-, A_v^0), \quad WC(A_v^0, A_v^+), \\ \text{non-conflict:} \quad NC(A_v^-, A_v^-), \quad NC(A_v^0, A_v^0), \quad NC(A_v^+, A_v^+). \end{cases} \tag{8}$$

Definition 2.6 [35] In a three-valued situation table $S = (U, V, f)$, for a single issue $v \in V$, the three-level conflict with respect to the trisection $\ll R_v^=, R_v^\approx, R_v^\asymp \gg$ is defined by

$$\begin{cases} \text{strong conflict:} \quad SC = R_v^{\asymp}, \\ \text{weak conflict:} \quad WC = R_v^{\approx}, \\ \text{non-conflict:} \quad NC = R_v^{=}. \end{cases} \tag{9}$$

One can verify that the above two three-level conflict models are equivalent to each other when they are induced by $\ll A_v^-, A_v^0, A_v^+ \gg$ and $\ll R_v^=, R_v^{\approx}, R_v^{\asymp} \gg$, respectively, on the same issue $v \in V$. Furthermore, suppose that in a three-valued situation table $f(u_1, v) = 0, f(u_2, v) = 0$ and $u_1 \neq u_2$. Then by the above method of Yao, $(u_1, u_2) \in R_v^=$ and hence they are in non-conflict, i.e., in alliance, while they are in neutrality relation by the method of Pawlak. Therefore, the above method of Yao resolved the inconsistency of Pawlak's treatment of $f(u_i, v) = f(u_j, v) = 0$ on a single issue v. Yao [35] further extended the above trisections and three-level conflict models to the level of multiple issues, and we reproduce them as follows.

Definition 2.7 [35] In a three-valued situation table $S = (U, V, f)$, let $f(u, J) \in [-1, 1]$ be a function with $f(u, J) = \frac{1}{|J|}\sum_{v \in J} f(u, v)$, called the aggregated rating function of agent $u \in U$ on multiple issues of $J \subseteq V$. Given a pair of thresholds (β, α) with $-1 \leq \beta < 0 < \alpha \leq 1$, the trisection of all agents on multiple issues J, denoted by $\ll A_J^{[-1,\beta]}, A_J^{[\beta,\alpha]}, A_J^{[\alpha,1]} \gg$, is defined by

$$\begin{cases} A_J^{[-1,\beta]} = \{u \in U \mid f(u, J) \leq \beta\}, \\ A_J^{[\beta,\alpha]} = \{u \in U \mid \beta < f(u, J) < \alpha\}, \\ A_J^{[\alpha,1]} = \{u \in U \mid f(u, J) \geq \alpha\}. \end{cases} \tag{10}$$

Definition 2.8 [35] In a three-valued situation table $S = (U, V, f)$, let $c_J(u_i, u_j) \in [0, 1]$ be a function with $c_J(u_i, u_j) = \frac{1}{2|J|}\sum_{v \in J} |f(u_i, v) - f(u_j, v)|$, called the aggregated conflict function of agents $u_i, u_j \in U$ on multiple issues of $J \subseteq V$. Given a pair of thresholds (ξ, η) with $0 \leq \xi < 0.5 < \eta \leq 1$, the trisection of all pairs of agents on multiple issues J, denoted by $\ll C_J^{[0,\xi]}, C_J^{[\xi,\eta]}, C_J^{[\eta,1]} \gg$, is defined by

$$\begin{cases} C_J^{[0,\xi]} = \{(u_i, u_j) \in U \times U \mid c_J(u_i, u_j) \leq \xi\}, \\ C_J^{[\xi,\eta]} = \{(u_i, u_j) \in U \times U \mid \xi < c_J(u_i, u_j) < \eta\}, \\ C_J^{[\eta,1]} = \{(u_i, u_j) \in U \times U \mid c_J(u_i, u_j) \geq \eta\}. \end{cases} \tag{11}$$

Based on the above two types of trisections, the two types of three-level conflicts on multiple issues are defined as follows.

Definition 2.9 [35] In a three-valued situation table $S = (U, V, f)$, for multiple issues of $J \subseteq V$, the three-level conflict with respect to the trisection $\ll A_J^{[-1,\beta]}, A_J^{[\beta,\alpha]}, A_J^{[\alpha,1]} \gg$ is defined by

$$\begin{cases} \text{strong conflict:} \quad SC(A_J^{[-1,\beta]}, A_J^{[\alpha,1]}), \\ \text{weak conflict:} \quad WC(A_J^{[-1,\beta]}, A_J^{[\beta,\alpha]}), \quad WC(A_J^{[\beta,\alpha]}, A_J^{[\alpha,1]}), \\ \text{non-conflict:} \quad NC(A_J^{[-1,\beta]}, A_J^{[-1,\beta]}), \quad NC(A_J^{[\beta,\alpha]}, A_J^{[\beta,\alpha]}), \quad NC(A_J^{[\alpha,1]}, A_J^{[\alpha,1]}). \end{cases} \tag{12}$$

Definition 2.10 [35] In a three-valued situation table $S = (U, V, f)$, for multiple issues of $J \subseteq V$, the three-level conflict with respect to the trisection $\ll C_J^{[0,\xi]}, C_J^{[\xi,\eta]}, C_J^{[\eta,1]} \gg$ is defined by

$$\begin{cases} \text{strong conflict:} \quad SC = C_J^{[\eta,1]}, \\ \text{weak conflict:} \quad WC = C_J^{[\xi,\eta]}, \\ \text{non-conflict:} \quad NC = C_J^{[0,\xi]}. \end{cases} \tag{13}$$

As we see, Yao [35] proposed a framework of three-level conflict analysis on multiple issues, and there are still some problems needing to be further resolved, such as the threshold-selection problems for both the two types of trisections and the two types of three-level conflicts on multiple issues. Following the work of Yao [35], the present paper proposes new evaluations and a whole set of methods to successfully resolve the threshold-selection problems for both the trisections of all pairs of agents and the three-level conflicts on multiple issues. Note that the forthcoming similarity degree, difference degree, probability functions and measure of trisections or three-level conflicts are specially defined on three-valued situation tables, though the related notations and symbol-manipulation techniques are similar to those of [27–29].

3 Three-way conflict analysis on an issue set

In this section, we introduce the main work of this paper, including the basic concepts, notations and conclusions of this paper.

3.1 Similarity degree and difference degree

In this paper, we have two basic concepts, i.e., the evaluation of similarity degree and the evaluation of difference degree between any two agents. Below we introduce them in turn.

Definition 3.1 Let $S=(U,V,f)$ be a three-valued situation table, and $Y \subseteq V$ be a non-empty subset of V. For any two agents $u_i, u_j \in U$, the **similarity degree** of u_i and u_j on the issue set Y is defined by an evaluation function:

$$e_Y(u_i,u_j)=\frac{\sum_{v\in Y}\rho_v(u_i,u_j)}{|Y|}, \tag{14}$$

where

$$\rho_v(u_i,u_j) = \begin{cases} 1, & f(u_i,v)=f(u_j,v) \\ 0.5, & f(u_i,v)\neq f(u_j,v)\wedge f(u_i,v)\cdot f(u_j,v)=0 \\ 0, & f(u_i,v)\neq f(u_j,v)\wedge f(u_i,v)\cdot f(u_j,v)=-1 \end{cases}. \tag{15}$$

Obviously, $e_Y(u_i,u_j)\in[0,1]$, and the value of $e_Y(u_i,u_j)$ describes the similarity degree between u_i and u_j concerning the attitudes to the issues of Y. Specifically, in a three-valued situation table $e_Y(u_i,u_j)=1$ means that u_i and u_j have the same attitude on every issue of Y and hence have the highest similarity degree on Y; $e_Y(u_i,u_j)=0$ means that u_i and u_j have clear opposite attitude on every issue of Y and hence have the lowest similarity degree on Y; the other cases of $0<e_Y(u_i,u_j)<1$ mean that u_i and u_j may have the same attitudes on some issues of Y and have different attitudes on the other issues of Y. For instance, if the attitudes of u_1 on the issue set $Y=\{v_1,\ldots,v_5\}$ are $+,+,-,0,+$ respectively, and the attitudes of u_2 on Y are $+,-,-,0,0$ respectively, then $e_Y(u_1,u_2)=\frac{1+0+1+1+0.5}{5}=0.7$. Therefore, the concept of similarity degree captures the attitude similarity of any two agents on multiple issues from a relative quantitative point of view.

Note that, although Definition 3.1 of similarity degree has a similar form to that of "matching degree" in [29], they are technically different in the following aspects: (1) matching degree is defined based on general binary relations in an information table with the numerator $\phi(u_i,v_i)\in\{0,1\}$, while similarity degree is defined based on the equality relation in a three-valued situation table with the numerator $\rho_v(u_i,u_j)\in\{0,0.5,1\}$; (2) matching degree is used to evaluate the matching degree of any object u and the given reference tuple τ in each rank, while similarity degree is used to evaluate the attitude similarity of any two agents on multiple issues.

Definition 3.2 Let $S=(U,V,f)$ be a three-valued situation table, and $Y \subseteq V$ be a non-empty subset of V. For any two agents $u_i, u_j \in U$, the **difference degree** of u_i and u_j on the issue set Y is defined by

$$d_Y(u_i,u_j)=1-e_Y(u_i,u_j), \tag{16}$$

where $e_Y(u_i,u_j)$ is the similarity degree of u_i and u_j on Y.

Obviously, $d_Y(u_i,u_j)\in[0,1]$, and the value of $d_Y(u_i,u_j)$ describes the difference degree between u_i and u_j concerning the attitudes to the issues of Y. For instance, if the attitudes of u_1 on the issue set $Y=\{v_1,\ldots,v_5\}$ are $+,+,-,0,+$ respectively, and the attitudes of u_2 on Y are $+,-,-,0,0$ respectively, then $d_Y(u_1,u_2)=1-e_Y(u_1,u_2)=0.3$. Therefore, the concept of difference degree captures the attitude difference of any two agents on multiple issues from a relative quantitative point of view. In addition, this definition of $d_Y(u_i,u_j)$ is different from the distance function defined by Pawlak [20] or Yao [35]. Specifically, when two agents u_i and u_j have the same attitude "0" on a single issue v, Pawlak's distance function treats them as $d(u_i,u_j)=0.5$ while our difference degree treats them as $d_v(u_i,u_j)=0$[1]; our definition of difference degree is defined on multiple issues while Yao's distance function is defined on a single issue.

3.2 Trisection of all pairs of agents on an issue set

In this part, we introduce the trisection of all pairs of agents on an issue set, and discuss the properties of such trisections.

Definition 3.3 Let $S=(U,V,f)$ be a three-valued situation table, $Y\subseteq V$ be a non-empty subset of V, and (α,β) be a pair of thresholds with $0\leq\beta<\alpha\leq 1$. The **trisection of all pairs of agents on the issue set** Y, denoted by $Q_{(\alpha,\beta)}(Y)=\langle P^{=}_{\alpha}(Y),P^{\asymp}_{\beta}(Y),P^{\approx}_{(\alpha,\beta)}(Y)\rangle$, is defined by

$$\begin{cases} P^{=}_{\alpha}(Y)=\{(u_i,u_j)\in U\times U \mid e_Y(u_i,u_j)\geq\alpha\}, \\ P^{\asymp}_{\beta}(Y)=\{(u_i,u_j)\in U\times U \mid e_Y(u_i,u_j)\leq\beta\}, \\ P^{\approx}_{(\alpha,\beta)}(Y)=\{(u_i,u_j)\in U\times U \mid \beta<e_Y(u_i,u_j)<\alpha\}. \end{cases} \tag{17}$$

An equivalent representation of the above trisection is by using the difference degree as follows:

$$\begin{cases} P^{=}_{\alpha}(Y)=\{(u_i,u_j)\in U\times U \mid d_Y(u_i,u_j)\leq 1-\alpha\}, \\ P^{\asymp}_{\beta}(Y)=\{(u_i,u_j)\in U\times U \mid d_Y(u_i,u_j)\geq 1-\beta\}, \\ P^{\approx}_{(\alpha,\beta)}(Y)=\{(u_i,u_j)\in U\times U \mid 1-\alpha<d_Y(u_i,u_j)<1-\beta\}. \end{cases} \tag{18}$$

Remark 3.4

[1] When $Y=\{v\}$, the difference degree (resp. similarity degree) of u_i and u_j on Y should be technically denoted by $d_{\{v\}}(u_i,u_j)$ (resp. $e_{\{v\}}(u_i,u_j)$). However, we will simply use $d_v(u_i,u_j)$ (resp. $e_v(u_i,u_j)$), when such expression does not cause ambiguity.

(i) Obviously, the three sets $P^{=}_{\alpha}(Y), P^{\asymp}_{\beta}(Y), P^{\approx}_{(\alpha,\beta)}(Y)$ are pair-wise disjoint and their union is the entire product space $U \times U$. In particular, when $\alpha = 1$, $\beta = 0$ and $Y = \{v\}$ for some $v \in V$, the trisection of Definition 3.3 is the trisection of all pairs of agents on the single issue v.

(ii) Since the value of $e_Y(u_i, u_j)$ is equal to the value of $e_Y(u_j, u_i)$ for any (u_i, u_j), the pairs (u_i, u_j) and (u_j, u_i) are certain to be in the same set of $P^{=}_{\alpha}(Y), P^{\asymp}_{\beta}(Y)$ or $P^{\approx}_{(\alpha,\beta)}(Y)$. Therefore, in this paper we will not differentiate (u_i, u_j) and (u_j, u_i). In other words, suppose that $|U| = m$, we only consider $m \cdot (m+1)/2$ different pairs of agents.

Next, we discuss the properties of the trisections defined in Definition 3.3, which will lay the theoretical foundations for later discussion.

Theorem 3.5 *Let $S = (U, V, f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V with $|Y| = n$, and (α, β) be a pair of thresholds with $0 \le \beta < \alpha \le 1$. For any pair of thresholds (α, β), there exist two intervals $(\frac{i-1}{2n}, \frac{i}{2n}]$ and $[\frac{j-1}{2n}, \frac{j}{2n})$ with $i, j \in \{1, 2, \ldots, 2n\}$, such that $\alpha \in (\frac{i-1}{2n}, \frac{i}{2n}]$ and $\beta \in [\frac{j-1}{2n}, \frac{j}{2n})$, respectively, and the following conclusions hold*:

$$\begin{cases} P^{=}_{\alpha}(Y) = P^{=}_{\frac{i}{2n}}(Y), \\ P^{\asymp}_{\beta}(Y) = P^{\asymp}_{\frac{j-1}{2n}}(Y), \\ P^{\approx}_{(\alpha,\beta)}(Y) = P^{\approx}_{(\frac{i}{2n},\frac{j-1}{2n})}(Y). \end{cases} \tag{19}$$

Proof By Definition 3.1, it is easy to obtain that $e_Y(u_k, u_l) \in \{0, \frac{1}{2n}, \ldots, \frac{2n-1}{2n}, 1\}$ for any $(u_k, u_l) \in U \times U$. Then, for any pair of thresholds (α, β) with $0 \le \beta < \alpha \le 1$, we have that $0 < \alpha \le 1$ and $0 \le \beta < 1$. So it is obvious that there exists an interval $(\frac{i-1}{2n}, \frac{i}{2n}]$ such that $\alpha \in (\frac{i-1}{2n}, \frac{i}{2n}]$, and there exists an interval $[\frac{j-1}{2n}, \frac{j}{2n})$ such that $\beta \in [\frac{j-1}{2n}, \frac{j}{2n})$, where $i, j \in \{1, 2, \ldots, 2n\}$ and $i \ge j$. Below we only prove the conclusion $P^{=}_{\alpha}(Y) = P^{=}_{\frac{i}{2n}}(Y)$, with the other two being similar.

For any $(u_k, u_l) \in P^{=}_{\alpha}(Y)$, we have that $e_Y(u_k, u_l) \ge \alpha$ by Definition 3.3. The conditions $e_Y(u_k, u_l) \in \{0, \frac{1}{2n}, \ldots, \frac{2n-1}{2n}, 1\}$, $e_Y(u_k, u_l) \ge \alpha$ and $\alpha \in (\frac{i-1}{2n}, \frac{i}{2n}]$ imply that $e_Y(u_k, u_l) \ge \frac{i}{2n}$. Therefore, $(u_k, u_l) \in P^{=}_{\frac{i}{2n}}(Y)$ and hence $P^{=}_{\alpha}(Y) \subseteq P^{=}_{\frac{i}{2n}}(Y)$. Conversely, for any $(u_k, u_l) \in P^{=}_{\frac{i}{2n}}(Y)$, we have that $e_Y(u_k, u_l) \ge \frac{i}{2n}$. Since $\alpha \in (\frac{i-1}{2n}, \frac{i}{2n}]$, we obtain that $e_Y(u_k, u_l) \ge \alpha$ and hence $(u_k, u_l) \in P^{=}_{\alpha}(Y)$. This means that $P^{=}_{\frac{i}{2n}}(Y) \subseteq P^{=}_{\alpha}(Y)$. Therefore, we have that $P^{=}_{\alpha}(Y) = P^{=}_{\frac{i}{2n}}(Y)$. □

Theorem 3.6 *Let $S = (U, V, f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V with $|Y| = n$, and (α, β) be a pair of thresholds with $0 \le \beta < \alpha \le 1$. For any given Y, there are at most $n(2n+1)$ different trisections of all pairs of agents, namely the trisections $Q_{(\frac{i}{2n},\frac{j}{2n})}(Y)$ with $i = 1$ to $2n$ and $j = 0$ to $i-1$, on the issue set Y.*

Proof This is an immediate result by using Theorem 3.5. For any $\alpha \in (0, \frac{1}{2n}]$, the fact $\beta < \alpha$ implies that β is certain to be in $[0, \frac{1}{2n})$; for any $\alpha \in (\frac{1}{2n}, \frac{2}{2n}]$, β is certain to be in $[0, \frac{1}{2n})$ or $[\frac{1}{2n}, \frac{2}{2n})$, and so on. Combining the conclusions of Theorem 3.5, we immediately have that there are at most $1 + 2 + \cdots + 2n = n(1+2n)$ different trisections of all pairs of agents on the issue set Y. In other words, the finite trisections $Q_{(\frac{i}{2n},\frac{j}{2n})}(Y)$ involve all possible trisections of all pairs of agents on Y, where $i = 1$ to $2n$ and $j = 0$ to $i-1$. □

Remark 3.7 Theorems 3.5 and 3.6 describe the properties of the trisections of all pairs of agents in Definition 3.3, and they lay the theoretical foundations for later discussion. They tell us that, although there are infinite pairs of thresholds (α, β), we only need to consider a finite number of pairs $(\frac{i}{2n}, \frac{j}{2n})$ and their corresponding trisections of all pairs of agents. Moreover, these pairs $(\frac{i}{2n}, \frac{j}{2n})$ are immediately obtained by the number of issues of Y.

Example 3.8 Here we take the Middle East conflict [20] as an example which is shown in the following Table 1. As Pawlak said in [20], the data in Table 1 may not necessarily reflect the current problems in the Middle East, we are just using these data for an illustration.

Suppose that the issue set $Y = \{v_1, v_2, v_3\}$, then $|Y| = 3$ and we need to consider the attitude similarity between any two agents on the issue set Y. Since there are 6 agents in U of Table 1, we only need to consider $6 \times (6+1)/2 = 21$ pairs of agents. Table 2 shows the similarity degrees of all these pairs of agents. By Theorem 3.6, there are at most $3 \times (2 \times 3 + 1) = 21$ different trisections of all pairs of agents, and Table 3 shows all these trisections of all pairs of agents on Y. The next problem is how to evaluate these trisections so as to find the optimal one. We will solve this problem in the Sect. 3.4.

3.3 Three-level conflict model on an issue set

In this part, we introduce the three-level conflict model induced by a trisection of all pairs of agents on an issue set. Intuitively, any pair of agents $(u_i, u_j) \in P^{=}_{\alpha}(Y)$ roughly (but reasonably) has the same attitudes towards the multiple issues of Y; any pair of agents $(u_i, u_j) \in P^{\asymp}_{\beta}(Y)$ roughly (but

Table 1 Middle East conflict situation

$U \setminus V$	v_1	v_2	v_3	v_4	v_5
u_1	$-$	$+$	$+$	$+$	$+$
u_2	$+$	0	$-$	$-$	$-$
u_3	$+$	$-$	$-$	$-$	0
u_4	0	$-$	$-$	0	$-$
u_5	$+$	$-$	$-$	$-$	$-$
u_6	0	$+$	$-$	0	$+$

reasonably) has opposite attitudes towards the multiple issues of Y; any pair of agents $(u_i, u_j) \in P^{\approx}_{(\alpha,\beta)}(Y)$ has other cases of attitudes towards the multiple issues of Y. Therefore, we have the following definition of three-level conflict induced by a trisection of all pairs of agents.

Definition 3.9 Let $S = (U, V, f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V, and $Q_{(\alpha,\beta)}(Y) = \langle P^{=}_{\alpha}(Y), P^{\asymp}_{\beta}(Y), P^{\approx}_{(\alpha,\beta)}(Y)\rangle$ be a trisection of all pairs of agents on the issue set Y. The **three-level conflict induced by** $Q_{(\alpha,\beta)}(Y)$, denoted by $C_Y^{(\alpha,\beta)} = \langle SC_Y^{\beta}, WC_Y^{(\alpha,\beta)}, NC_Y^{\alpha}\rangle$, is defined by

$$\begin{cases} \textbf{strong conflict:}\ SC_Y^{\beta} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P^{\asymp}_{\beta}(Y)\}, \\ \textbf{weak conflict:}\ WC_Y^{(\alpha,\beta)} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P^{\approx}_{(\alpha,\beta)}(Y)\}, \\ \textbf{non-conflict:}\ NC_Y^{\alpha} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P^{=}_{\alpha}(Y)\}. \end{cases} \tag{20}$$

Conversely, the **three-level alliance induced by** $Q_{(\alpha,\beta)}(Y)$, denoted by $A_Y^{(\alpha,\beta)} = \langle SA_Y^{\alpha}, WA_Y^{(\alpha,\beta)}, NA_Y^{\beta}\rangle$, is defined by

$$\begin{cases} \textbf{strong alliance:}\ SA_Y^{\alpha} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P^{=}_{\alpha}(Y)\}, \\ \textbf{weak alliance:}\ WA_Y^{(\alpha,\beta)} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P^{\approx}_{(\alpha,\beta)}(Y)\}, \\ \textbf{non-alliance:}\ NA_Y^{\beta} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P^{\asymp}_{\beta}(Y)\}. \end{cases} \tag{21}$$

Table 2 Similarity degrees of all pairs of agents

$e_Y(u_i, u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1					
u_2	1/6	1				
u_3	0	5/6	1			
u_4	1/6	2/3	5/6	1		
u_5	0	5/6	1	5/6	1	
u_6	1/2	2/3	1/2	2/3	1/2	1

Remark 3.10 By this definition, we in fact have that $NC_Y^{\alpha} = P^{=}_{\alpha}(Y)$, $SC_Y^{\beta} = P^{\asymp}_{\beta}(Y)$ and $WC_Y^{(\alpha,\beta)} = P^{\approx}_{(\alpha,\beta)}(Y)$ for any given (α, β) and Y. Therefore, the properties of three-level conflicts are similar to those of trisections of all pairs of agents, and we omit them for readers to fill.

Example 3.11 Consider Example 3.8 again. In Table 3, we have obtained 21 trisections of all pairs of agents on the issue set $Y = \{v_1, v_2, v_3\}$. By Definition 3.9, we can obtain 21 three-level conflicts induced by these trisections. We present all these three-level conflicts in the following Table 4 which is the result of replacing in Table 3 all $P^{=}_{\alpha}(Y)$, $P^{\asymp}_{\beta}(Y)$ and $P^{\approx}_{(\alpha,\beta)}(Y)$ by NC_Y^{α}, SC_Y^{β} and $WC_Y^{(\alpha,\beta)}$, respectively. The next problem is how to find the optimal three-level conflict, which will be immediately solved in the following part.

3.4 Threshold-selection problem

In this part, we solve the threshold-selection problem for three-level conflict analysis on multiple issues. In other words, we provide strategies to find the final optimal three-level conflict.[2] Intuitively, there are two ways of obtaining the final optimal three-level conflicts on multiple issues, as shown in Fig. 1. The first way is firstly finding the optimal trisection of all pairs of agents and secondly obtaining the three-level conflict induced by the optimal trisection; the second way is firstly obtaining all three-level conflicts induced by all trisections of all pairs of agents and secondly finding the optimal three-level conflict. Since $SC_Y^{\beta} = P^{\asymp}_{\beta}(Y)$, $NC_Y^{\alpha} = P^{=}_{\alpha}(Y)$ and $WC_Y^{(\alpha,\beta)} = P^{\approx}_{(\alpha,\beta)}(Y)$ by Definition 3.9, the two ways of obtaining the final optimal three-level conflict are in fact equivalent to each other in this paper. Below we respectively present the two ways and show their equivalence relationship.

(1) The first way of finding the final optimal three-level conflict

As Fig. 1 shows, the first way is firstly finding the optimal trisection of all pairs of agents and secondly obtaining the three-level conflict induced by the optimal trisection by Definition 3.9. So the core of this way is defining the measure of trisections of all pairs of agents, namely the forthcoming measure H, and defining the optimal one among all trisections.

[2] Since $A_Y^{(\alpha,\beta)}$ is the result of replacing in $C_Y^{(\alpha,\beta)}$ all SC_Y^{α}, $WC_Y^{(\alpha,\beta)}$ and NC_Y^{β} by NA_Y^{α}, $WA_Y^{(\alpha,\beta)}$ and SA_Y^{β}, respectively, in this part we only provide the ways of obtaining the final optimal three-level conflict, with the ways of obtaining the final optimal three-level alliance being similar.

Table 3 All trisections of all pairs of agents on Y

(α, β)	$P^{=}_{\alpha}(Y)$	$P^{\asymp}_{\beta}(Y)$	$P^{\approx}_{(\alpha,\beta)}(Y)$
(1, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_2,u_3) (u_1,u_4) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(1, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_2,u_3) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(1, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_2,u_3) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(1, 1/2)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)	(u_2,u_3) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_2) (u_6,u_4)
(1, 2/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_2)	(u_2,u_3) (u_4,u_3) (u_5,u_2) (u_5,u_4)
(1, 5/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_2,u_3)	$\emptyset$
(5/6, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4) (u_4,u_2) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(5/6, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_4,u_2) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(5/6, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_4,u_2) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(5/6, 1/2)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)	(u_4,u_2) (u_6,u_2) (u_6,u_4)
(5/6, 2/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_2)	$\emptyset$
(2/3, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2) (u_5,u_4) (u_6,u_2)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)
(2/3, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2) (u_5,u_4) (u_6,u_2)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_6,u_1) (u_6,u_3) (u_6,u_5)
(2/3, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2) (u_5,u_4) (u_6,u_2)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_6,u_1) (u_6,u_3) (u_6,u_5)
(2/3, 1/2)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)	$\emptyset$

Table 3 (continued)

(α, β)	$P^{=}_{\alpha}(Y)$	$P^{\asymp}_{\beta}(Y)$	$P^{\approx}_{(\alpha,\beta)}(Y)$
	(u_5,u_4) (u_6,u_2)		
(1/2, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4)
(1/2, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5)	$\emptyset$
(1/2, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5)	$\emptyset$
(1/3, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4)
(1/3, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5)	$\emptyset$
(1/6, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4) (u_1,u_2) (u_1,u_4)	(u_1,u_3) (u_1,u_5)	$\emptyset$

Definition 3.12 Let $S = (U, V, f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V with $|Y| = n$. For any $(u_i, u_j) \in U \times U$, the following three functions

$$g_1(u_i,u_j) = \frac{e_Y(u_i,u_j)\cdot\left(2n\cdot e_Y(u_i,u_j)+1\right)}{2n+1}, \tag{22}$$

$$g_2(u_i,u_j) = \frac{\left(1-e_Y(u_i,u_j)\right)\cdot\left[2n\cdot\left(1-e_Y(u_i,u_j)\right)+1\right]}{2n+1}, \tag{23}$$

$$g_3(u_i,u_j) = \frac{4n\cdot e_Y(u_i,u_j)\cdot\left(1-e_Y(u_i,u_j)\right)}{2n+1} \tag{24}$$

are called $P^{=}$**-probability**, $P^{\asymp}$**-probability** and $P^{\approx}$**-probability** of (u_i, u_j), respectively.

Remark 3.13 For any given issue set Y, the functions $g_1(u_i,u_j)$, $g_2(u_i,u_j)$ and $g_3(u_i,u_j)$ describe the probabilities of (u_i,u_j)'s being in the sets $P^{=}_{\alpha}(Y)$, $P^{\asymp}_{\beta}(Y)$ and $P^{\approx}_{(\alpha,\beta)}(Y)$, respectively. For example, the probability of (u_1,u_2)'s being in the set $P^{=}_{\alpha}(Y)$ is $\frac{1}{21}$ in Table 3 of Example 3.8; one can verify that the value of $g_1(u_1,u_2)$ in Definition 3.12 is

Table 4 All three-level conflicts on Y

(α, β)	NC_Y^{α}	SC_Y^{β}	$WC_Y^{(\alpha,\beta)}$
(1, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_2,u_3) (u_1,u_4) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(1, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_2,u_3) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(1, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_2,u_3) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(1, 1/2)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)	(u_2,u_3) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_6,u_2) (u_6,u_4)
(1, 2/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_2)	(u_2,u_3) (u_4,u_3) (u_5,u_2) (u_5,u_4)
(1, 5/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_2) (u_4,u_3) (u_5,u_2) (u_5,u_4) (u_2,u_3)	$\emptyset$
(5/6, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4) (u_4,u_2) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(5/6, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_4,u_2) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(5/6, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_4,u_2) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5)
(5/6, 1/2)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)	(u_4,u_2) (u_6,u_2) (u_6,u_4)
(5/6, 2/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_4,u_3) (u_3,u_5) (u_2,u_3) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_2)	$\emptyset$
(2/3, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2) (u_5,u_4) (u_6,u_2)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)
(2/3, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2) (u_5,u_4) (u_6,u_2)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_6,u_1) (u_6,u_3) (u_6,u_5)
(2/3, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2) (u_5,u_4) (u_6,u_2)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4)	(u_6,u_1) (u_6,u_3) (u_6,u_5)
(2/3, 1/2)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_4,u_3) (u_4,u_2) (u_6,u_4) (u_5,u_2)	(u_1,u_3) (u_1,u_5) (u_1,u_2) (u_1,u_4) (u_6,u_1) (u_6,u_3) (u_6,u_5)	$\emptyset$

Table 4 (continued)

(α, β)	NC_Y^{α}	SC_Y^{β}	$WC_Y^{(\alpha,\beta)}$
	(u_5,u_4) (u_6,u_2)		
(1/2, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4)
(1/2, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5)	$\emptyset$
(1/2, 1/3)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5)	$\emptyset$
(1/3, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_3) (u_1,u_5)	(u_1,u_2) (u_1,u_4)
(1/3, 1/6)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4)	(u_1,u_2) (u_1,u_3) (u_1,u_4) (u_1,u_5)	$\emptyset$
(1/6, 0)	(u_1,u_1) (u_2,u_2) (u_3,u_3) (u_4,u_4) (u_5,u_5) (u_6,u_6) (u_3,u_5) (u_2,u_3) (u_6,u_1) (u_6,u_2) (u_6,u_3) (u_6,u_4) (u_6,u_5) (u_4,u_3) (u_4,u_2) (u_5,u_2) (u_5,u_4) (u_1,u_2) (u_1,u_4)	(u_1,u_3) (u_1,u_5)	$\emptyset$

exactly $\frac{1}{21}$, where $n=3$ and $e_Y(u_1,u_2)=\frac{1}{6}$. In addition, one can verify that $0 \le g_k(u_i,u_j) \le 1 \quad (k=1,2,3)$ and $g_1(u_i,u_j)+g_2(u_i,u_j)+g_3(u_i,u_j)=1$ for any pair (u_i,u_j).

Definition 3.14 Let $S=(U,V,f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V, (α,β) be a pair of thresholds with $0 \le \beta < \alpha \le 1$, and $Q_{(\alpha,\beta)}(Y)=\langle P_{\alpha}^{=}(Y), P_{\beta}^{\asymp}(Y), P_{(\alpha,\beta)}^{\approx}(Y)\rangle$ be a trisection of all pairs of agents on the issue set Y. The **measure** H of $Q_{(\alpha,\beta)}(Y)$ is defined by:

$$H\big(Q_{(\alpha,\beta)}(Y)\big)=\textstyle\sum_{i=1}^{3} q_i, \tag{25}$$

where

$$\begin{aligned} q_1 &= \frac{\sum_{(u_i,u_j)\in P_{\alpha}^{=}(Y)} g_1(u_i,u_j)}{\sum_{(u_i,u_j)\in U\times U} g_1(u_i,u_j)},\\ q_2 &= \frac{\sum_{(u_i,u_j)\in P_{\beta}^{\asymp}(Y)} g_2(u_i,u_j)}{\sum_{(u_i,u_j)\in U\times U} g_2(u_i,u_j)},\\ q_3 &= \frac{\sum_{(u_i,u_j)\in P_{(\alpha,\beta)}^{\approx}(Y)} g_3(u_i,u_j)}{\sum_{(u_i,u_j)\in U\times U} g_3(u_i,u_j)}. \end{aligned} \tag{26}$$

Definition 3.15 Let $S=(U,V,f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V, $Q_{(\alpha,\beta)}(Y)$ be any trisection of all pairs of agents on the issue set Y. A

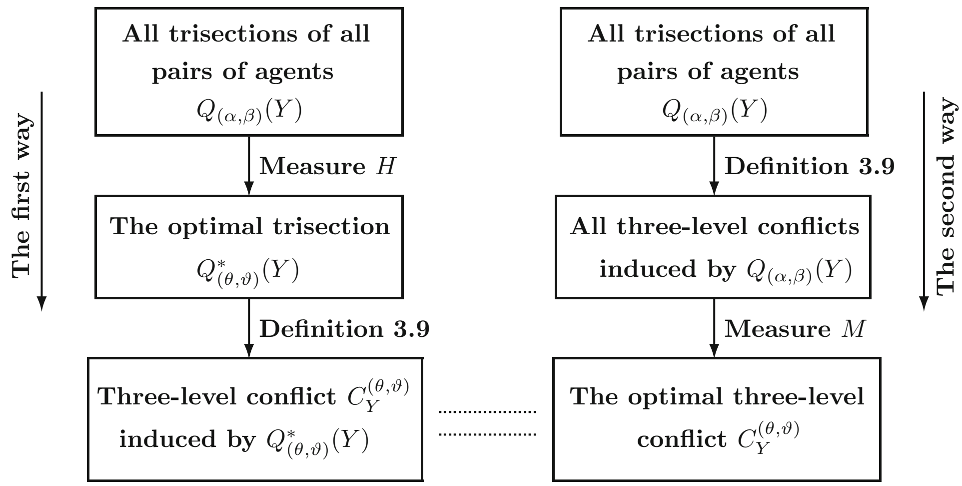

Fig. 1 Two ways of finding the final optimal three-level conflict

Table 5 $P^{=}$-probabilities of all pairs

$g_1(u_i,u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1					
u_2	1/21	1				
u_3	0	5/7	1			
u_4	1/21	10/21	5/7	1		
u_5	0	5/7	1	5/7	1	
u_6	2/7	10/21	2/7	10/21	2/7	1

trisection $Q^*_{(\theta,\vartheta)}(Y)$ of all pairs of agents on the issue set Y is **optimal** when it satisfies the condition that

$$H(Q^*_{(\theta,\vartheta)}(Y)) = \max_{\forall Q_{(\alpha,\beta)}(Y)} H(Q_{(\alpha,\beta)}(Y)). \tag{27}$$

Remark 3.16

(i) It is obvious that $0 \le q_i \le 1$ $(i = 1,2,3)$ and therefore $0 \le H\big(Q_{(\alpha,\beta)}(Y)\big) \le 3$ for any trisection $Q_{(\alpha,\beta)}(Y)$ in Definition 3.14.

(ii) We think that the whole trisection of all pairs of agents gets an optimal state when its measure is maximal. Note that this does not ensure any one set of the trisection is separately optimal.

(iii) By the previous Theorems 3.5 and 3.6, here we only need to compare a finite number of $H\big(Q_{(\alpha,\beta)}(Y)\big)$ to find the maximal one.

Example 3.17 Continue with Example 3.8. By Definition 3.12, we can compute the $P^{=}$-probabilities, $P^{\asymp}$-probabilities and $P^{\approx}$-probabilities, respectively, for all pairs of agents as shown in Tables 5, 6 and 7.

By Definitions 3.14 and 3.15, we can further compute the measures of all trisections that have been listed in Table 3 so as to find the optimal one. Table 8 shows such results, from which we find that the optimal trisections of all pairs of agents are $Q_{(1,1/6)}(Y)$ and $Q_{(1,1/3)}(Y)$ with $H(Q_{(1,1/6)}(Y)) = H(Q_{(1,1/3)}(Y)) \approx 2.1475$.

Example 3.18 In the above example, we have obtained the optimal trisection $Q_{(1,1/6)}(Y)$ or $Q_{(1,1/3)}(Y)$ with $H(Q_{(1,1/6)}(Y)) = H(Q_{(1,1/3)}(Y)) \approx 2.1475$ on the issue set $Y = \{v_1, v_2, v_3\}$. According to the first way of obtaining the final optimal three-level conflict, the three-level conflict induced by $Q_{(1,1/6)}(Y)$ or $Q_{(1,1/3)}(Y)$ is what we find. So, by Definition 3.9 and Table 3, we obtain the final optimal three-level conflict induced by $Q_{(1,1/6)}(Y)$ as follows.

$$\begin{cases} \text{strong conflict:} \quad SC_Y^{1/6} = \{(u_1,u_3),(u_1,u_5),(u_1,u_2),(u_1,u_4)\}, \\ \text{weak conflict:} \quad WC_Y^{(1,1/6)} = \{(u_2,u_3),(u_4,u_2),(u_4,u_3),(u_5,u_2),(u_5,u_4),(u_6,u_1),(u_6,u_2),(u_6,u_3),(u_6,u_4),(u_6,u_5)\}, \\ \text{non-conflict:} \quad NC_Y^{1} = \{(u_1,u_1),(u_2,u_2),(u_3,u_3),(u_4,u_4),(u_5,u_5),(u_6,u_6),(u_3,u_5)\}. \end{cases}$$

Table 6 $P^{\asymp}$-probabilities of all pairs

$g_2(u_i,u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0					
u_2	5/7	0				
u_3	1	1/21	0			
u_4	5/7	1/7	1/21	0		
u_5	1	1/21	0	1/21	0	
u_6	2/7	1/7	2/7	1/7	2/7	0

Table 7 $P^{\approx}$-probabilities of all pairs

$g_3(u_i,u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0					
u_2	5/21	0				
u_3	0	5/21	0			
u_4	5/21	8/21	5/21	0		
u_5	0	5/21	0	5/21	0	
u_6	3/7	8/21	3/7	8/21	3/7	0

Conversely, the final optimal three-level alliance induced by $Q_{(1,1/6)}(Y)$ is the following three sets:

$$\begin{cases} \text{strong alliance:} \quad SA_Y^1 = \{(u_1,u_1),(u_2,u_2),(u_3,u_3),(u_4,u_4), \\ (u_5,u_5),(u_6,u_6),(u_3,u_5)\}, \\ \text{weak alliance:} \quad WA_Y^{(1,1/6)} = \{(u_2,u_3),(u_4,u_2),(u_4,u_3), \\ (u_5,u_2),(u_5,u_4),(u_6,u_1),(u_6,u_2),(u_6,u_3),(u_6,u_4),(u_6,u_5)\}, \\ \text{non-alliance:} \quad NA_Y^{1/6} = \{(u_1,u_3),(u_1,u_5),(u_1,u_2),(u_1,u_4)\}. \end{cases}$$

To sum up the first way of finding the optimal three-level conflict, we give Algorithm 1 which contains the following three main steps.

Step 1 (lines 2–5): For the given Y, let $n=|Y|$ and compute the values of $e_Y(u_k,u_l)$, $g_1(u_k,u_l)$, $g_2(u_k,u_l)$ and $g_3(u_k,u_l)$ for each pair of agents $(u_k,u_l)\in U\times U$;

Step 2 (lines 6–24): For $i=1$ to $2n$ and $j=0$ to $i-1$, compute all possible trisections $Q_{(\frac{i}{2n},\frac{j}{2n})}$ of all pairs of agents, find the first emerging maximal measure $H(Q_{(\frac{i}{2n},\frac{j}{2n})})$ among all $n(2n+1)$ trisections, and regard this trisection as the final determined optimal one, namely $Q^*_{(\theta,\vartheta)}$ in Algorithm 1;

Table 8 The measures of all trisections of all pairs of agents on Y

$Q_{(\alpha,\beta)}(Y)$	q_1	q_2	q_3	$H(Q_{(\alpha,\beta)}(Y))$
$Q_{(1,0)}$	0.5720	0.4078	1	1.9798
$Q_{(1,1/6)}$	0.5720	0.6990	0.8765	**2.1475**
$Q_{(1,1/3)}$	0.5720	0.6990	0.8765	**2.1475**
$Q_{(1,1/2)}$	0.5720	0.8738	0.5432	1.9890
$Q_{(1,2/3)}$	0.5720	0.9612	0.2469	1.7801
$Q_{(1,5/6)}$	0.5720	1.0000	0	1.5720
$Q_{(5/6,0)}$	0.8054	0.4078	0.7531	1.9663
$Q_{(5/6,1/6)}$	0.8054	0.6990	0.6296	2.1340
$Q_{(5/6,1/3)}$	0.8054	0.6990	0.6296	2.1340
$Q_{(5/6,1/2)}$	0.8054	0.8738	0.2963	1.9755
$Q_{(5/6,2/3)}$	0.8054	0.9612	0	1.7666
$Q_{(2/3,0)}$	0.9222	0.4078	0.4568	1.7868
$Q_{(2/3,1/6)}$	0.9222	0.6990	0.3333	1.9545
$Q_{(2/3,1/3)}$	0.9222	0.6990	0.3333	1.9545
$Q_{(2/3,1/2)}$	0.9222	0.8738	0	1.7960
$Q_{(1/2,0)}$	0.9922	0.4078	0.1235	1.5235
$Q_{(1/2,1/6)}$	0.9922	0.6990	0	1.6912
$Q_{(1/2,1/3)}$	0.9922	0.6990	0	1.6912
$Q_{(1/3,0)}$	0.9922	0.4078	0.1235	1.5235
$Q_{(1/3,1/6)}$	0.9922	0.6990	0	1.6912
$Q_{(1/6,0)}$	1	0.4078	0	1.4078

The numbers in bold are the maximal measures of all trisections of all pairs of agents on Y

Step 3 (lines 25–26): By Definition 3.9, compute and output the optimal three-level conflict induced by $Q^{*}_{(\theta,\vartheta)}$, namely $\langle SC^{\vartheta}_{Y}, WC^{(\theta,\vartheta)}_{Y}, NC^{\theta}_{Y}\rangle$, and its thresholds (θ,ϑ).

Remark 3.19 Suppose that $|U| = m$. The time complexity of Step 1 is obviously $O(m^2)$. As Algorithm 1 shows, Step 2 mainly contains three iterations, where the outermost and the 2nd iterations terminate in $n(2n+1)$ steps, and the innermost iteration terminates in m^2 steps. So Step 2 terminates in $n(2n+1)m^2$ steps. However, n is the number of issues of Y and m is the number of agents of U, and we usually have that $n \ll m$. Therefore, the time complexity of Step 2 is $O(m^2)$, and hence the time complexity of Algorithm 1 is $O(m^2)$.

(2) The second way of finding the final optimal three-level conflict

As Fig. 1 shows, the second way is firstly obtaining all three-level conflicts induced by all trisections of all pairs of agents by Definition 3.9 and secondly finding the optimal three-level conflict. So the core of this way is defining the measure of three-level conflicts, namely the forthcoming measure M, and defining the optimal one among these three-level conflicts.

Definition 3.20 Let $S = (U, V, f)$ be a three-valued situation table, $Y \subseteq V$ be a non-empty subset of V with $|Y| = n$. For any $(u_i, u_j) \in U \times U$, the following three functions

Algorithm 1: An algorithm for computing the optimal three-level conflict in the first way

Input: A three-valued situation table $S = (U, V, f)$, an issue set $Y \subseteq V$
Output: The optimal three-level conflict and its thresholds

1 **begin**
2 **let:** $n = |Y|$;
3 **foreach** $(u_k, u_l) \in U \times U$ **do**
4 **compute:** $e_Y(u_k, u_l)$, $g_1(u_k, u_l)$, $g_2(u_k, u_l)$, $g_3(u_k, u_l)$;
5 **end**
6 **let:** $P^{=} = \emptyset$, $P^{\asymp} = \emptyset$, $P^{\approx} = \emptyset$, $Max = 0$;
7 **for** $i = 1$ *to* $2n$ **do**
8 **for** $j = 0$ *to* $i - 1$ **do**
9 **foreach** $(u_k, u_l) \in U \times U$ **do**
10 **if** $e_Y(u_k, u_l) \geq \frac{i}{2n}$ **then**
11 $P^{=} = P^{=} \cup \{(u_k, u_l)\}$;
12 **else if** $e_Y(u_k, u_l) \leq \frac{j}{2n}$ **then**
13 $P^{\asymp} = P^{\asymp} \cup \{(u_k, u_l)\}$;
14 **else** $P^{\approx} = P^{\approx} \cup \{(u_k, u_l)\}$;
15 ;
16 **end**
17 **let:** $Q_{(\frac{i}{2n}, \frac{j}{2n})} = \langle P^{=}, P^{\asymp}, P^{\approx}\rangle$;
18 **compute:** $H(Q_{(\frac{i}{2n}, \frac{j}{2n})})$;
19 **if** $H(Q_{(\frac{i}{2n}, \frac{j}{2n})}) > Max$ **then**
20 $Max = H(Q_{(\frac{i}{2n}, \frac{j}{2n})})$, $(\theta, \vartheta) = (\frac{i}{2n}, \frac{j}{2n})$, $P^{=}_{\theta} = P^{=}$, $P^{\asymp}_{\vartheta} = P^{\asymp}$, $P^{\approx}_{(\theta,\vartheta)} = P^{\approx}$,
21 $Q^{*}_{(\theta,\vartheta)} = Q_{(\frac{i}{2n}, \frac{j}{2n})}$;
22 **end**
23 **let:** $P^{=} = \emptyset$, $P^{\asymp} = \emptyset$, $P^{\approx} = \emptyset$;
24 **end**
25 **end**
26 **let:** $SC^{\vartheta}_{Y} = P^{\asymp}_{\vartheta}$, $WC^{(\theta,\vartheta)}_{Y} = P^{\approx}_{(\theta,\vartheta)}$, $NC^{\theta}_{Y} = P^{=}_{\theta}$;
27 **return:** SC^{ϑ}_{Y}, $WC^{(\theta,\vartheta)}_{Y}$, NC^{θ}_{Y}, (θ, ϑ)
28 **end**

$$f_1(u_i,u_j) = \frac{\left(1-e_Y(u_i,u_j)\right)\cdot\left[2n\cdot\left(1-e_Y(u_i,u_j)\right)+1\right]}{2n+1}, \tag{28}$$

$$f_2(u_i,u_j) = \frac{4n\cdot e_Y(u_i,u_j)\cdot\left(1-e_Y(u_i,u_j)\right)}{2n+1}, \tag{29}$$

$$f_3(u_i,u_j) = \frac{e_Y(u_i,u_j)\cdot\left(2n\cdot e_Y(u_i,u_j)+1\right)}{2n+1} \tag{30}$$

are called **strong-conflict probability**, **weak-conflict probability** and **non-conflict probability** of (u_i,u_j), respectively.

Remark 3.21 For any given issue set Y, the functions $f_1(u_i,u_j)$, $f_2(u_i,u_j)$ and $f_3(u_i,u_j)$ describe the probabilities of (u_i,u_j)'s being in the sets SC_Y^{β}, $WC_Y^{(\alpha,\beta)}$ and NC_Y^{α}, respectively. For example, the probability of (u_1,u_2)'s being in the set SC_Y^{β} is $\frac{15}{21}=\frac{5}{7}$ in Table 4 of Example 3.11; one can verify that the value of $f_1(u_1,u_2)$ in Definition 3.20 is exactly $\frac{5}{7}$, where $n=3$ and $e_Y(u_1,u_2)=\frac{1}{6}$. It is easy to verify that $0\le f_k(u_i,u_j)\le 1$ $(k=1,2,3)$ and $f_1(u_i,u_j)+f_2(u_i,u_j)+f_3(u_i,u_j)=1$ for any pair $(u_i,u_j)\in U\times U$.

Definition 3.22 Let $S=(U,V,f)$ be a three-valued situation table, $Y\subseteq V$ be a non-empty subset of V, (α,β) be a pair of thresholds with $0\le\beta<\alpha\le 1$, and $C_Y^{(\alpha,\beta)}=\langle SC_Y^{\beta},WC_Y^{(\alpha,\beta)},NC_Y^{\alpha}\rangle$ be a three-level conflict on the issue set Y. The **measure** M of $C_Y^{(\alpha,\beta)}$ is defined by:

$$M\left(C_Y^{(\alpha,\beta)}\right)=\textstyle\sum_{i=1}^{3}p_i, \tag{31}$$

where

$$p_1=\frac{\sum_{(u_i,u_j)\in SC_Y^{\beta}}f_1(u_i,u_j)}{\sum_{(u_i,u_j)\in U\times U}f_1(u_i,u_j)},\quad p_2=\frac{\sum_{(u_i,u_j)\in WC_Y^{(\alpha,\beta)}}f_2(u_i,u_j)}{\sum_{(u_i,u_j)\in U\times U}f_2(u_i,u_j)},\quad p_3=\frac{\sum_{(u_i,u_j)\in NC_Y^{\alpha}}f_3(u_i,u_j)}{\sum_{(u_i,u_j)\in U\times U}f_3(u_i,u_j)}. \tag{32}$$

Definition 3.23 Let $S=(U,V,f)$ be a three-valued situation table, $Y\subseteq V$ be a non-empty subset of V, and $C_Y^{(\alpha,\beta)}$ be any three-level conflict on the issue set Y. A three-level conflict $C_Y^{(\theta,\vartheta)}$ on the issue set Y is **optimal** when it satisfies the condition that

$$M\left(C_Y^{(\theta,\vartheta)}\right)=\max_{\forall C_Y^{(\alpha,\beta)}}M\left(C_Y^{(\alpha,\beta)}\right). \tag{33}$$

Table 9 Strong-conflict probabilities of all pairs

$f_1(u_i,u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0					
u_2	5/7	0				
u_3	1	1/21	0			
u_4	5/7	1/7	1/21	0		
u_5	1	1/21	0	1/21	0	
u_6	2/7	1/7	2/7	1/7	2/7	0

Table 10 Weak-conflict probabilities of all pairs

$f_2(u_i,u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0					
u_2	5/21	0				
u_3	0	5/21	0			
u_4	5/21	8/21	5/21	0		
u_5	0	5/21	0	5/21	0	
u_6	3/7	8/21	3/7	8/21	3/7	0

Table 11 Non-conflict probabilities of all pairs

$f_3(u_i,u_j)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1					
u_2	1/21	1				
u_3	0	5/7	1			
u_4	1/21	10/21	5/7	1		
u_5	0	5/7	1	5/7	1	
u_6	2/7	10/21	2/7	10/21	2/7	1

Remark 3.24

(i) It is obvious that $0\le p_i\le 1$ $(i=1,2,3)$ and therefore $0\le M(C_Y^{(\alpha,\beta)})\le 3$ for any three-level conflict $C_Y^{(\alpha,\beta)}$ in Definition 3.22.

(ii) We think that the whole three-level conflict $C_Y^{(\alpha,\beta)}$ gets an optimal state when its measure is maximal. Note that this does not ensure any one of the three sets $SC_Y^{\beta},WC_Y^{(\alpha,\beta)},NC_Y^{\alpha}$ is separately optimal.

(iii) Since the properties of three-level conflicts are similar to those of trisections of all pairs of agents, here we only need to compare a finite number of $M(C_Y^{(\alpha,\beta)})$ to find the maximal one.

Table 12 The measures of all three-level conflicts on Y

$C_Y^{(\alpha,\beta)}$	p_1	p_2	p_3	$M(C_Y^{(\alpha,\beta)})$
$C_Y^{(1,0)}$	0.4078	1	0.5720	1.9798
$C_Y^{(1,1/6)}$	0.6990	0.8765	0.5720	**2.1475**
$C_Y^{(1,1/3)}$	0.6990	0.8765	0.5720	**2.1475**
$C_Y^{(1,1/2)}$	0.8738	0.5432	0.5720	1.9890
$C_Y^{(1,2/3)}$	0.9612	0.2469	0.5720	1.7801
$C_Y^{(1,5/6)}$	1.0000	0	0.5720	1.5720
$C_Y^{(5/6,0)}$	0.4078	0.7531	0.8054	1.9663
$C_Y^{(5/6,1/6)}$	0.6990	0.6296	0.8054	2.1340
$C_Y^{(5/6,1/3)}$	0.6990	0.6296	0.8054	2.1340
$C_Y^{(5/6,1/2)}$	0.8738	0.2963	0.8054	1.9755
$C_Y^{(5/6,2/3)}$	0.9612	0	0.8054	1.7666
$C_Y^{(2/3,0)}$	0.4078	0.4568	0.9222	1.7868
$C_Y^{(2/3,1/6)}$	0.6990	0.3333	0.9222	1.9545
$C_Y^{(2/3,1/3)}$	0.6990	0.3333	0.9222	1.9545
$C_Y^{(2/3,1/2)}$	0.8738	0	0.9222	1.7960
$C_Y^{(1/2,0)}$	0.4078	0.1235	0.9222	1.5235
$C_Y^{(1/2,1/6)}$	0.6990	0	0.9222	1.6912
$C_Y^{(1/2,1/3)}$	0.6990	0	0.9222	1.6912
$C_Y^{(1/3,0)}$	0.4078	0.1235	0.9222	1.5235
$C_Y^{(1/3,1/6)}$	0.6990	0	0.9222	1.6912
$C_Y^{(1/6,0)}$	0.4078	0	1	1.4078

The numbers in bold are the maximal measures of all three-level conflicts on Y

Example 3.25 We have obtained all three-level conflicts on the issue set Y in Table 4 of Example 3.11. By Definition 3.20, we can compute the strong-conflict, weak-conflict and non-conflict probabilities, respectively, for all pairs of agents as shown in Tables 9, 10 and 11.
Furthermore, by Definitions 3.22 and 3.23, we can compute the measures of all three-level conflicts that have been listed in Table 4. Table 12 shows such results, from which we find that the optimal three-level conflict is $C_Y^{(1,1/6)}$ or $C_Y^{(1,1/3)}$ with $M(C_Y^{(1,1/6)}) = M(C_Y^{(1,1/3)}) \approx 2.1475$. According to the second way of finding the final optimal three-level conflict, the three-level conflict $C_Y^{(1,1/6)}$ or $C_Y^{(1,1/3)}$ is what we find, and we list it below by Table 4.

$$\begin{cases} \text{strong conflict:} \quad SC_Y^{1/6} = \{(u_1,u_3),(u_1,u_5),(u_1,u_2),(u_1,u_4)\}, \\ \text{weak conflict:} \quad WC_Y^{(1,1/6)} = \{(u_2,u_3),(u_4,u_2),(u_4,u_3),(u_5,u_2),(u_5,u_4),(u_6,u_1),(u_6,u_2),(u_6,u_3),(u_6,u_4),(u_6,u_5)\}, \\ \text{non-conflict:} \quad NC_Y^{1} = \{(u_1,u_1),(u_2,u_2),(u_3,u_3),(u_4,u_4),(u_5,u_5),(u_6,u_6),(u_3,u_5)\}. \end{cases}$$

Similarly, we can obtain the final optimal three-level alliance as follows:

$$\begin{cases} \text{strong alliance:} \quad SA_Y^{1} = \{(u_1,u_1),(u_2,u_2),(u_3,u_3),(u_4,u_4),(u_5,u_5),(u_6,u_6),(u_3,u_5)\}, \\ \text{weak alliance:} \quad WA_Y^{(1,1/6)} = \{(u_2,u_3),(u_4,u_2),(u_4,u_3),(u_5,u_2),(u_5,u_4),(u_6,u_1),(u_6,u_2),(u_6,u_3),(u_6,u_4),(u_6,u_5)\}, \\ \text{non-alliance:} \quad NA_Y^{1/6} = \{(u_1,u_3),(u_1,u_5),(u_1,u_2),(u_1,u_4)\}. \end{cases}$$

To sum up the second way of finding the optimal three-level conflict, we give the following Algorithm 2. It is similar to Algorithm 1 to some extent, with the differences that: (1) here we need to compute the values of $f_1(u_k, u_l)$, $f_2(u_k, u_l)$ and $f_3(u_k, u_l)$; (2) we compute all possible three-level conflicts right after the innermost iteration, and find the first emerging maximal measure $M(C_Y^{(\frac{i}{2n},\frac{j}{2n})})$ and regard this three-level conflict as the final determined optimal one, namely $C_Y^{(\theta,\vartheta)} = \langle SC_Y^{\vartheta}, WC_Y^{(\theta,\vartheta)}, NC_Y^{\theta}\rangle$ in Algorithm 2. Obviously, the time complexity of Algorithm 2 is also $O(m^2)$ with $m = |U|$.

Algorithm 2: An algorithm for computing the optimal three-level conflict in the second way

Input: A three-valued situation table $S = (U, V, f)$, an issue set $Y \subseteq V$
Output: The optimal three-level conflict and its thresholds

begin
  let: $n = |Y|$;
  foreach $(u_k, u_l) \in U \times U$ **do**
    compute: $e_Y(u_k, u_l)$, $f_1(u_k, u_l)$, $f_2(u_k, u_l)$, $f_3(u_k, u_l)$;
  end
  let: $P^{=} = \emptyset$, $P^{\asymp} = \emptyset$, $P^{\approx} = \emptyset$, $Max = 0$;
  for $i = 1$ *to* $2n$ **do**
    for $j = 0$ *to* $i - 1$ **do**
      foreach $(u_k, u_l) \in U \times U$ **do**
        if $e_Y(u_k, u_l) \geq \frac{i}{2n}$ **then**
          $P^{=} = P^{=} \cup \{(u_k, u_l)\}$;
        else if $e_Y(u_k, u_l) \leq \frac{j}{2n}$ **then**
          $P^{\asymp} = P^{\asymp} \cup \{(u_k, u_l)\}$;
        else $P^{\approx} = P^{\approx} \cup \{(u_k, u_l)\}$;
        ;
      end
      let: $C_Y^{(\frac{i}{2n},\frac{j}{2n})} = \langle P^{\asymp}, P^{\approx}, P^{=}\rangle$;
      compute: $M(C_Y^{(\frac{i}{2n},\frac{j}{2n})})$;
      if $M(C_Y^{(\frac{i}{2n},\frac{j}{2n})}) > Max$ **then**
        $Max = M(C_Y^{(\frac{i}{2n},\frac{j}{2n})})$, $(\theta, \vartheta) = (\frac{i}{2n}, \frac{j}{2n})$, $SC_Y^{\vartheta} = P^{\asymp}$, $WC_Y^{(\theta,\vartheta)} = P^{\approx}$, $NC_Y^{\theta} = P^{=}$,
        $C_Y^{(\theta,\vartheta)} = C_Y^{(\frac{i}{2n},\frac{j}{2n})}$;
      end
      let: $P^{=} = \emptyset$, $P^{\asymp} = \emptyset$, $P^{\approx} = \emptyset$;
    end
  end
  return: SC_Y^{ϑ}, $WC_Y^{(\theta,\vartheta)}$, NC_Y^{θ}, (θ, ϑ)
end

(3) Conclusion of the threshold-selection problem

According to the above discussion, we immediately have the following conclusion:

Theorem 3.26 *For any given three-valued situation table $S = (U, V, f)$ and non-empty issue set $Y \subseteq V$, the two ways of finding the final optimal three-level conflict on Y can be finished in finite steps, and they bring about the same result in this paper.*

Proof For any given three-valued situation table $S = (U, V, f)$ and non-empty issue set $Y \subseteq V$ with $|Y| = n$, there are at most $n(2n + 1)$ different trisections of all pairs of agents on Y by Theorem 3.6, and the finite trisections are determined by known pairs of thresholds (α, β). Therefore, the optimal trisection of all pairs of agents can be found in finite steps by the measure H. This shows that the first way of finding the final optimal three-level conflict can be finished in finite steps. For the second way, we firstly compute all three-level conflicts induced by all trisections of all pairs of agents on Y. So there are also at most $n(2n + 1)$ different three-level conflicts on Y. For these finite three-level conflicts, we use the measure M to compute their measures and find the maximal one among these finite measures. Therefore, this procedure can also be finished in finite steps.

For any given $Y \subseteq V$ and pair of thresholds (α, β), we have that $SC_Y^{\beta} = P_{\beta}^{\asymp}(Y)$, $NC_Y^{\alpha} = P_{\alpha}^{=}(Y)$ and $WC_Y^{(\alpha,\beta)} = P_{(\alpha,\beta)}^{\approx}(Y)$ by Definition 3.9. At the same time, we have that $H(Q_{(\alpha,\beta)}(Y)) = M(C_Y^{(\alpha,\beta)})$ by Definitions 3.15 and 3.23, where $Q_{(\alpha,\beta)}(Y) = \langle P^{=}_{\alpha}(Y), P^{\asymp}_{\beta}(Y), P^{\approx}_{(\alpha,\beta)}(Y)\rangle$ and $C^{(\alpha,\beta)}_{Y} = \langle SC^{\beta}_{Y}, WC^{(\alpha,\beta)}_{Y}, NC^{\alpha}_{Y}\rangle$. Therefore, the two ways proposed in this paper bring about the same resulting three-level conflict on an issue set Y. □

3.5 Comparison and discussion

In this part, we mainly compare our approach with the approach of Yao [35] from different aspects, and give more discussions about the three-level conflicts on a single issue.

(1) The three-level conflict induced by $Q_{(\alpha,\beta)}(Y)$ versus the three-level conflict induced by $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$. Firstly, we compare our evaluation of similarity degree (resp. deference degree) with the aggregated conflict function in Definition 2.8, and compare the three-level conflict induced by $Q_{(\alpha,\beta)}(Y)$ with the three-level conflict induced by $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$.

Proposition 3.27 *Let $S = (U, V, f)$ be a three-valued situation table, and $Y \subseteq V$ be a non-empty issue set. For any $(u_i, u_j) \in U \times U$, the following conclusion holds:*

$$c_Y(u_i, u_j) = 1 - e_Y(u_i, u_j) = d_Y(u_i, u_j). \tag{34}$$

Proof By Definition 2.8, the aggregated conflict function $c_Y(u_i, u_j) = \frac{1}{2|Y|}\sum_{v\in Y} |f(u_i, v) - f(u_j, v)|$ for any agents $u_i, u_j \in U$ on an issue set $Y \subseteq V$. For any $v \in Y$, $|f(u_i, v) - f(u_j, v)|/2 = 1$ if and only if $f(u_i, v) \neq f(u_j, v)$ and $f(u_i, v)\cdot f(u_j, v) = -1$. At the same time, $f(u_i, v) \neq f(u_j, v)$ and $f(u_i, v)\cdot f(u_j, v) = -1$ if and only if $\rho_v(u_i, u_j) = 0$ by Definition 3.1. Therefore, $|f(u_i, v) - f(u_j, v)|/2 = 1$ if and only if $\rho_v(u_i, u_j) = 0$. Similarly, $|f(u_i, v) - f(u_j, v)|/2 = 0.5$ if and only if $f(u_i, v) \neq f(u_j, v)$ and $f(u_i, v)\cdot f(u_j, v) = 0$, which in turn if and only if $\rho_v(u_i, u_j) = 0.5$. $|f(u_i, v) - f(u_j, v)|/2 = 0$ if and only if $f(u_i, v) = f(u_j, v)$, which in turn if and only if $\rho_v(u_i, u_j) = 1$. Therefore, we have that $c_Y(u_i, u_j) = 1 - e_Y(u_i, u_j) = d_Y(u_i, u_j)$ for any $(u_i, u_j) \in U \times U$. □

Proposition 3.28 *Given a three-valued situation table $S = (U, V, f)$ and a non-empty issue set $Y \subseteq V$, the trisection model of all pairs of agents $Q_{(\alpha,\beta)}(Y)$ in this paper is a conservative extension of the trisection model of all pairs of agents $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$ in Yao [35].*

Proof For a given three-valued situation table $S = (U, V, f)$ and a given non-empty issue set Y, we have that $c_Y(u_i, u_j) = d_Y(u_i, u_j)$ for any $(u_i, u_j) \in U \times U$ by Proposition 3.27. Furthermore, the scope of thresholds is $0 \le \xi < 0.5 < \eta \le 1$ in the model $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$, while the scope of thresholds is $0 \le \beta < \alpha \le 1$ in the model $Q_{(\alpha,\beta)}(Y)$. Therefore, the trisections generated by the model $Q_{(\alpha,\beta)}(Y)$ includes the trisections generated by the model $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$. □

Proposition 3.29 *Given a three-valued situation table $S = (U, V, f)$ and a non-empty issue set $Y \subseteq V$, the three-level conflict model $C^{(\alpha,\beta)}_{Y}$ in this paper is a conservative extension of the three-level conflict model induced by $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$ in Yao [35].*

Proof Since the three-level conflict model is induced by $\ll C^{[0,\xi]}_{Y}, C^{[\xi,\eta]}_{Y}, C^{[\eta,1]}_{Y} \gg$ by Definition 2.10, i.e., $SC = C^{[\eta,1]}_{J}$, $WC = C^{[\xi,\eta]}_{J}$ and $NC = C^{[0,\xi]}_{J}$, we immediately obtain the above conclusion by Proposition 3.28. □

Conclusion: Proposition 3.28 tells us that we remove the limitation of $0 \le \xi < 0.5 < \eta \le 1$ and extend the trisection model of Yao [35], namely Definition 2.8, to a more general level; Proposition 3.29 tells us that we extend the three-level conflict model of Yao [35], namely Definition 2.10, to a more general level. In the previous part, we have solved the threshold-selection problem for three-level conflict analysis on multiple issues in two ways. So in this paper we in fact solve the threshold-selection problem for three-level conflict analysis proposed by Yao [35] on multiple issues.

(2) Trisection of all pairs of agents on a single issue. Now we focus on three-level conflict analysis on a single issue, through which one can see the relationship between our approach and Yao's approach in [35] on a single issue. Firstly, it is easy to obtain the following equivalent definition of trisection of all pairs of agents on a single issue $v \in V$, when we limit the issue set Y to a single-issue set $\{v\}$ in Definition 3.3.

Definition 3.30 Let $S = (U, V, f)$ be a three-valued situation table, $v \in V$ be an issue in V. The **trisection of all pairs of agents on the single issue** v, denoted by $Q_v = \langle P^{=}_{v}, P^{\asymp}_{v}, P^{\approx}_{v}\rangle$ is defined by

$$\begin{cases} P^{=}_{v} = \{(u_i, u_j) \in U \times U \mid e_v(u_i, u_j) = 1\}, \\ P^{\asymp}_{v} = \{(u_i, u_j) \in U \times U \mid e_v(u_i, u_j) = 0\}, \\ P^{\approx}_{v} = \{(u_i, u_j) \in U \times U \mid e_v(u_i, u_j) = 0.5\}. \end{cases} \tag{35}$$

An equivalent representation of the above trisection is by using the difference degree as follows:

$$\begin{cases} P^{=}_{v} = \{(u_i, u_j) \in U \times U \mid d_v(u_i, u_j) = 0\}, \\ P^{\asymp}_{v} = \{(u_i, u_j) \in U \times U \mid d_v(u_i, u_j) = 1\}, \\ P^{\approx}_{v} = \{(u_i, u_j) \in U \times U \mid d_v(u_i, u_j) = 0.5\}. \end{cases} \tag{36}$$

Remark 3.31 Obviously, the three sets $P^{=}_{v}, P^{\asymp}_{v}, P^{\approx}_{v}$ are pair-wise disjoint and their union is the entire product space $U \times U$. At the same time, by Definition 3.30, it is easy to obtain the following facts[3]:

$$\begin{cases} P_v^{=} = \{(+,+),(-,-),(0,0)\}, \\ P_v^{\asymp} = \{(+,-)\}, \\ P_v^{\approx} = \{(+,0),(-,0)\}. \end{cases}$$

Example 3.32 Here we give an example to compare Definition 3.30 with Definition 2.4 of Yao [35]. Consider Table 1 of Example 3.8 again. We take the issue v_1 to discuss, with the other issues being similar. Firstly, by Definition 3.30, we can obtain the trisection of all pairs of agents on the single issue v_1 as follows:

$$\begin{cases} P_{v_1}^{=} = \{(u_2,u_3),(u_2,u_5),(u_3,u_5),(u_4,u_6),(u_1,u_1), \\ \quad (u_2,u_2),(u_3,u_3),(u_4,u_4),(u_5,u_5),(u_6,u_6)\}, \\ P_{v_1}^{\asymp} = \{(u_1,u_2),(u_1,u_3),(u_1,u_5)\}, \\ P_{v_1}^{\approx} = \{(u_1,u_4),(u_1,u_6),(u_2,u_4),(u_2,u_6),(u_3,u_4), \\ \quad (u_3,u_6),(u_5,u_4),(u_5,u_6)\}. \end{cases}$$

Secondly, by Definition 2.4 we can obtain the trisection of all pairs of agents on the single issue v_1 in Yao [35] as follows:

$$\begin{cases} R_{v_1}^{=} = \{(u_2,u_3),(u_2,u_5),(u_3,u_5),(u_4,u_6),(u_1,u_1), \\ \quad (u_2,u_2),(u_3,u_3),(u_4,u_4),(u_5,u_5),(u_6,u_6)\}, \\ R_{v_1}^{\asymp} = \{(u_1,u_2),(u_1,u_3),(u_1,u_5)\}, \\ R_{v_1}^{\approx} = \{(u_1,u_4),(u_1,u_6),(u_2,u_4),(u_2,u_6),(u_3,u_4), \\ \quad (u_3,u_6),(u_5,u_4),(u_5,u_6)\}. \end{cases}$$

Conclusion: We see that Definitions 3.30 and 2.4 generate the same trisection of all pairs of agents on a single issue. This conclusion is not accidental, and it is due to the fundamental fact — $e_v(u_i,u_j)=1$ iff $|f(u_i,v)-f(u_j,v)|/2=0$, $e_v(u_i,u_j)=0$ iff $|f(u_i,v)-f(u_j,v)|/2=1$, and $e_v(u_i,u_j)=0.5$ iff $|f(u_i,v)-f(u_j,v)|/2=0.5$.

(3) Trisection of all agents on a single issue. Following the work of Yao [35], below we aim to propose a trisection of all agents on a single issue. For this purpose, we will have to introduce the following definition of "reference agent".

Definition 3.33 Let $S=(U,V,f)$ be a three-valued situation table, and $Y \subseteq V$ be a non-empty subset of V. For any given Y, a **reference agent on** Y is an agent $\tau \in U$ satisfying the condition that there exists at least one issue $v \in Y$ such that $f(\tau,v) \neq 0$.

Intuitively, a reference agent τ on an issue set Y is any agent as long as it does not have the attitudes "0" on all issues of Y, and it will be used to induce a trisection of all agents on Y.[4] When $Y=\{v\}$ in particular, the reference agent τ on Y can be any agent that has the attitude "+" or "−" on the issue v by Definition 3.33; the reason why τ can not be the agent that has the attitude "0" on v is that, when τ is such an agent, we will not be able to differentiate the attitudes "+" and "−" in the trisection of all agents induced by τ on v.

Definition 3.34 Let $S=(U,V,f)$ be a three-valued situation table, $v \in V$ be an issue in V, and τ be a reference agent on $\{v\}$. The **trisection of all agents induced by** τ **on the single issue** v, denoted by $T_v(\tau)=\langle Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)\rangle$, is defined by

$$\begin{cases} Ag_v^{=}(\tau) = \{u \in U | e_v(u,\tau)=1\}, \\ Ag_v^{\asymp}(\tau) = \{u \in U | e_v(u,\tau)=0\}, \\ Ag_v^{\approx}(\tau) = \{u \in U | e_v(u,\tau)=0.5\}. \end{cases} \tag{37}$$

An equivalent representation of the above trisection is by using the difference degree as follows:

$$\begin{cases} Ag_v^{=}(\tau) = \{u \in U | d_v(u,\tau)=0\}, \\ Ag_v^{\asymp}(\tau) = \{u \in U | d_v(u,\tau)=1\}, \\ Ag_v^{\approx}(\tau) = \{u \in U | d_v(u,\tau)=0.5\}. \end{cases} \tag{38}$$

Note that here different reference agents may induce different trisections of all agents on the single issue v. Hence this definition generates more trisections than that of Yao [35]. Below we give an example to further illustrate this point.

Example 3.35 Consider Table 1 of Example 3.8 again. We still select the issue v_1, i.e., let $Y=\{v_1\}$, with the other issues being similar. Then, by our approach of Definition 3.34, there are the following two situations. Note that the reference agent τ on $\{v_1\}$ can not be u_4 or u_6, because u_4 or u_6 has the attitude "0" on the single issue v.

(i) If $\tau = u_2$, then we can compute the similarity degree of each agent and τ as Table 13 shows. Obviously, the case of $\tau=u_3$ or $\tau=u_5$ is similar to this case.

Then, by Definition 3.34, we have the following trisection of all agents induced by $\tau=u_2$ on the single issue v_1:

[3] Here we informally express the three sets of pairs of agents, the meaning of which should be clear.

[4] As we see, the concept of reference agent, namely Definition 3.33, is defined on any issue set Y, and Y may contain multiple issues. However, in this paper we will only use this concept to induce the trisection of all agents on a single issue. In the future, we will use Definition 3.33 to induce the trisection of all agents on multiple issues.

Table 13 Similarity degree of each agent and $\tau = u_2$ on $\{v_1\}$

	u_1	u_2	u_3	u_4	u_5	u_6
v_1	$-$	$+$	$+$	0	$+$	0
$e_{v_1}(u_i,\tau)$	0	1	1	0.5	1	0.5

$$\begin{cases} Ag^{=}_{v_1}(u_2) = \{u_2, u_3, u_5\}, \\ Ag^{\asymp}_{v_1}(u_2) = \{u_1\}, \\ Ag^{\approx}_{v_1}(u_2) = \{u_4, u_6\}. \end{cases}$$

(ii) If $\tau = u_1$, then we can similarly obtain the following trisection of all agents induced by $\tau = u_1$ on the single issue v_1:

$$\begin{cases} Ag^{=}_{v_1}(u_1) = \{u_1\}, \\ Ag^{\asymp}_{v_1}(u_1) = \{u_2, u_3, u_5\}, \\ Ag^{\approx}_{v_1}(u_1) = \{u_4, u_6\}. \end{cases}$$

So we obtain the above two trisections of all agents by our approach, and the two trisections are meaningful from different trisecting points of view. Let us now consider the approach of Yao [35]. By Definition 2.3, we can obtain the following trisection of all agents on the single issue v_1:

$$\begin{cases} A^{+}_{v_1} = \{u \in U | f(u, v_1) = +\} = \{u_2, u_3, u_5\}, \\ A^{-}_{v_1} = \{u \in U | f(u, v_1) = -\} = \{u_1\}, \\ A^{0}_{v_1} = \{u \in U | f(u, v_1) = 0\} = \{u_4, u_6\}. \end{cases}$$

Conclusion: One can see that Yao's trisection of all agents on a single issue is exactly one case of our trisections, namely the case of $\tau = u_2$. Therefore, our approach of Definition 3.34 produces more trisections than Yao's approach of Definition 2.3 on a single issue. In other words, the trisection model of all agents on a single issue in this paper is a conservative extension of the trisection model of all agents in Yao [35] on the same single issue.

(4) The relationship between the two types of trisections on a single issue. Next, we present a theorem to show the relationship between trisections of all agents and trisections of all pairs of agents on a single issue.

Theorem 3.36 *Let $S = (U, V, f)$ be a three-valued situation table, $v \in V$ be an issue in V, and τ be a reference agent on $\{v\}$. The relationship of the six sets $Ag^{=}_{v}(\tau)$, $Ag^{\asymp}_{v}(\tau)$, $Ag^{\approx}_{v}(\tau)$, $P^{=}_{v}$, $P^{\asymp}_{v}$ and $P^{\approx}_{v}$ is as follows:*

$$\begin{aligned} P^{=}_{v} &= (Ag^{=}_{v}(\tau) \times Ag^{=}_{v}(\tau)) \cup (Ag^{\asymp}_{v}(\tau) \times Ag^{\asymp}_{v}(\tau)) \\ &\quad \cup (Ag^{\approx}_{v}(\tau) \times Ag^{\approx}_{v}(\tau)), \\ P^{\asymp}_{v} &= Ag^{=}_{v}(\tau) \times Ag^{\asymp}_{v}(\tau), \\ P^{\approx}_{v} &= (Ag^{=}_{v}(\tau) \times Ag^{\approx}_{v}(\tau)) \cup (Ag^{\asymp}_{v}(\tau) \times Ag^{\approx}_{v}(\tau)). \end{aligned} \tag{39}$$

Proof Suppose that $f(\tau, v) = +$, then we have $Ag^{=}_{v}(\tau) = \{u \in U | f(u, v) = +\}$, $Ag^{\asymp}_{v}(\tau) = \{u \in U | f(u, v) = -\}$, $Ag^{\approx}_{v}(\tau) = \{u \in U | f(u, v) = 0\}$. On the other hand, by Definition 3.30, we have that $P^{=}_{v} = \{(+,+), (-,-), (0,0)\}$, $P^{\asymp}_{v} = \{(+,-)\}$ and $P^{\approx}_{v} = \{(+,0), (-,0)\}$. Hence the conclusion holds for the case of $f(\tau, v) = +$.

Suppose that $f(\tau, v) = -$, then we have $Ag^{=}_{v}(\tau) = \{u \in U | f(u, v) = -\}$, $Ag^{\asymp}_{v}(\tau) = \{u \in U | f(u, v) = +\}$, $Ag^{\approx}_{v}(\tau) = \{u \in U | f(u, v) = 0\}$. Similar to the case of $f(\tau, v) = +$, it is easy to obtain that the conclusion also holds for the case of $f(\tau, v) = -$. □

Remark 3.37 This theorem also explains why we agree that $f(\tau, v) = 0$ is illegal on a single issue v. Assume that $f(\tau, v) = 0$, then we have $Ag^{=}_{v}(\tau) = \{u \in U | f(u, v) = 0\}$, $Ag^{\asymp}_{v}(\tau) = \emptyset$, and $Ag^{\approx}_{v}(\tau) = \{u \in U | f(u, v) = +\} \cup \{u \in U | f(u, v) = -\}$. Then, $Ag^{=}_{v}(\tau) \times Ag^{=}_{v}(\tau) = \{(0,0)\}$, $Ag^{\asymp}_{v}(\tau) \times Ag^{\asymp}_{v}(\tau) = \emptyset$, and $Ag^{\approx}_{v}(\tau) \times Ag^{\approx}_{v}(\tau) = \{(+,+), (-,-), (+,-), (-,+)\}$. So in this case we have that $(Ag^{=}_{v}(\tau) \times Ag^{=}_{v}(\tau)) \cup (Ag^{\asymp}_{v}(\tau) \times Ag^{\asymp}_{v}(\tau)) \cup (Ag^{\approx}_{v}(\tau) \times Ag^{\approx}_{v}(\tau)) = \{(0,0), (+,+), (-,-), (+,-), (-,+)\}$ and $Ag^{=}_{v}(\tau) \times Ag^{\asymp}_{v}(\tau) = \emptyset$. Thus, the conclusion of Theorem 3.36 does not hold when $f(\tau, v) = 0$. In other words, we are not able to differentiate the attitudes $+$ and $-$ within the trisection set $Ag^{\approx}_{v}(\tau)$ or $P^{=}_{v}$ when $f(\tau, v) = 0$ on the single issue v.

(5) Two types of three-level conflicts on a single issue. Now, we discuss two types of three-level conflicts on a single issue: one is induced by a trisection of all agents, and the other is induced by a trisection of all pairs of agents. We have defined the trisection of all agents $\langle Ag^{=}_{v}(\tau), Ag^{\asymp}_{v}(\tau), Ag^{\approx}_{v}(\tau)\rangle$ for any given τ and v. The three sets are pair-wise disjoint, and their union is the set of all agents. Furthermore, we have obtained that: with $f(\tau, v) = +$ or $f(\tau, v) = -$, all the agents in any one of the three sets have the same attitudes towards the issue v. So, for a single issue, it is easy to define the three-level conflict induced by a trisection of all agents.

Definition 3.38 Let $S=(U,V,f)$ be a three-valued situation table, $v \in V$ be an issue in V, τ be a reference agent on $\{v\}$, and $\langle Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)\rangle$ be a trisection of all agents induced by τ on $\{v\}$. The **three-level conflict induced by** $\langle Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)\rangle$ is defined by

$$\begin{cases} \textbf{strong conflict: } \mathbf{SC}(Ag_v^{=}(\tau), \\ \quad Ag_v^{\asymp}(\tau)), \\ \textbf{weak conflict: } \mathbf{WC}(Ag_v^{=}(\tau), Ag_v^{\approx}(\tau)), \\ \quad \mathbf{WC}(Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)), \\ \textbf{non-conflict: } \mathbf{NC}(Ag_v^{=}(\tau), Ag_v^{=}(\tau)), \mathbf{NC}(Ag_v^{\asymp}(\tau), \\ \quad Ag_v^{\asymp}(\tau)), \mathbf{NC}(Ag_v^{\approx}(\tau), Ag_v^{\approx}(\tau)). \end{cases} \tag{40}$$

Conversely, the **three-level alliance induced by** $\langle Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)\rangle$ is defined by

$$\begin{cases} \textbf{strong alliance: } \mathbf{SA}(Ag_v^{=}(\tau), Ag_v^{=}(\tau)), \mathbf{SA}(Ag_v^{\asymp}(\tau), \\ \quad Ag_v^{\asymp}(\tau)), \mathbf{SA}(Ag_v^{\approx}(\tau), Ag_v^{\approx}(\tau)), \\ \textbf{weak alliance: } \mathbf{WA}(Ag_v^{=}(\tau), Ag_v^{\approx}(\tau)), \\ \quad \mathbf{WA}(Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)), \\ \textbf{non-alliance: } \mathbf{NA}(Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau)). \end{cases} \tag{41}$$

Remark 3.39 Although our trisection model $\langle Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)\rangle$ is an extension of the trisection model $\ll A_v^{-}, A_v^{0}, A_v^{+} \gg$ of Yao [35], the three-level conflict model defined above is equivalent to Yao's three-level conflict model of Definition 2.5 on a single issue. One may verify this conclusion through the following Example 3.40.

Example 3.40 Consider Table 1 again. In Example 3.35, we have obtained the trisections $\langle Ag_{v_1}^{=}(u_2)$, $Ag_{v_1}^{\asymp}(u_2)$, $Ag_{v_1}^{\approx}(u_2)\rangle$ and $\langle Ag_{v_1}^{=}(u_1)$, $Ag_{v_1}^{\asymp}(u_1)$, $Ag_{v_1}^{\approx}(u_1)\rangle$. Then, by Definition 3.38, we obtain that the three-level conflict induced by $\langle Ag_{v_1}^{=}(u_2)$, $Ag_{v_1}^{\asymp}(u_2)$, $Ag_{v_1}^{\approx}(u_2)\rangle$ is

$$\begin{cases} \text{strong conflict: } \mathbf{SC}(\{u_2,u_3,u_5\}, \\ \quad \{u_1\}), \\ \text{weak conflict: } \mathbf{WC}(\{u_2,u_3,u_5\}, \{u_4,u_6\}), \\ \quad \mathbf{WC}(\{u_1\}, \{u_4,u_6\}), \\ \text{non-conflict: } \mathbf{NC}(\{u_2,u_3,u_5\}, \{u_2,u_3,u_5\}), \\ \quad \mathbf{NC}(\{u_1\}, \{u_1\}), \mathbf{NC}(\{u_4,u_6\}, \{u_4,u_6\}). \end{cases}$$

Similarly, the three-level conflict induced by $\langle Ag_{v_1}^{=}(u_1)$, $Ag_{v_1}^{\asymp}(u_1)$, $Ag_{v_1}^{\approx}(u_1)\rangle$ is

$$\begin{cases} \text{strong conflict:} \\ \mathbf{SC}(\{u_1\}, \{u_2,u_3,u_5\}), \\ \text{weak conflict: } \mathbf{WC}(\{u_1\}, \{u_4,u_6\}), \\ \mathbf{WC}(\{u_2,u_3,u_5\}, \{u_4,u_6\}), \\ \text{non-conflict: } \mathbf{NC}(\{u_1\}, \{u_1\}), \\ \mathbf{NC}(\{u_2,u_3,u_5\}, \{u_2,u_3,u_5\}), \mathbf{NC}(\{u_4,u_6\}, \{u_4,u_6\}). \end{cases}$$

Conclusion: We can see that the two different trisections in fact induce the same three-level conflicts. Therefore, our conclusion is—for any given single issue $v \in V$, the two different trisections $\langle Ag_{v_1}^{=}(u_2)$, $Ag_{v_1}^{\asymp}(u_2)$, $Ag_{v_1}^{\approx}(u_2)\rangle$ and $\langle Ag_{v_1}^{=}(u_1)$, $Ag_{v_1}^{\asymp}(u_1)$, $Ag_{v_1}^{\approx}(u_1)\rangle$ induce the same three-level conflicts.

We have defined the trisection of all pairs of agents $\langle P_v^{=}$, $P_v^{\asymp}$, $P_v^{\approx}\rangle$ on a single issue v in Definition 3.30. The three sets are also pair-wise disjoint, and their union is the set of all pairs of agents. By the previous discussion, we have obtained that: for any pair of agents $(u_i,u_j) \in P_v^{=}$, u_i and u_j have the same attitudes towards issue v; for any pair of agents $(u_i,u_j) \in P_v^{\asymp}$, u_i and u_j have clearly opposite attitudes towards issue v; for any pair of agents $(u_i,u_j) \in P_v^{\approx}$, u_i and u_j have other cases of attitudes towards issue v. Therefore, we have the following definition of three-level conflict induced by a trisection of all pairs of agents.

Definition 3.41 Let $S=(U,V,f)$ be a three-valued situation table, $v \in V$ be an issue in V, and $\langle P_v^{=}, P_v^{\asymp}, P_v^{\approx}\rangle$ be a trisection of all pairs of agents on v. The **three-level conflict induced by** $\langle P_v^{=}, P_v^{\asymp}, P_v^{\approx}\rangle$ is defined by

$$\begin{cases} \textbf{strong conflict: } \mathbf{SC} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P_v^{\asymp}\}, \\ \textbf{weak conflict: } \mathbf{WC} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P_v^{\approx}\}, \\ \textbf{non-conflict: } \mathbf{NC} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P_v^{=}\}. \end{cases} \tag{42}$$

Conversely, the **three-level alliance induced by** $\langle P_v^{=}, P_v^{\asymp}, P_v^{\approx}\rangle$ is defined by

$$\begin{cases} \textbf{strong alliance: } \mathbf{SA} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P_v^{=}\}, \\ \textbf{weak alliance: } \mathbf{WA} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P_v^{\approx}\}, \\ \textbf{non-alliance: } \mathbf{NA} = \{(u_i,u_j) \in U \times U \mid (u_i,u_j) \in P_v^{\asymp}\}. \end{cases} \tag{43}$$

Example 3.42 In Example 3.32, we have obtained the trisection of all pairs of agents on v_1, namely $\langle P_{v_1}^{=}, P_{v_1}^{\asymp}, P_{v_1}^{\approx}\rangle$. Now, by Definition 3.41, we obtain the three-level conflict induced by $\langle P_{v_1}^{=}, P_{v_1}^{\asymp}, P_{v_1}^{\approx}\rangle$ is

$$\begin{cases} \text{strong conflict:} \\ \mathbf{SC} = P_{v_1}^{\asymp} = \{(u_1,u_2),(u_1,u_3),(u_1,u_5)\}, \\ \text{weak conflict:} \quad \mathbf{WC} = P_{v_1}^{\approx} = \{(u_1,u_4),(u_1,u_6), \\ (u_2,u_4),(u_2,u_6),(u_3,u_4),(u_3,u_6),(u_5,u_4),(u_5,u_6)\}, \\ \text{non-conflict:} \quad \mathbf{NC} = P_{v_1}^{=} = \{(u_2,u_3),(u_2,u_5), \\ (u_3,u_5),(u_4,u_6),(u_1,u_1),(u_2,u_2),(u_3,u_3),(u_4,u_4), \\ (u_5,u_5),(u_6,u_6)\}. \end{cases}$$

Conclusion: By comparing this resulting three-level conflict with that of Example 3.40, we can see that they in fact represent equivalent three-level conflicts in two different forms. Therefore, our conclusion is—for any given single issue $v \in V$, the two types of trisections $\langle Ag_v^{=}(\tau), Ag_v^{\asymp}(\tau), Ag_v^{\approx}(\tau)\rangle$ and $\langle P_v^{=}, P_v^{\asymp}, P_v^{\approx}\rangle$ induce the equivalent three-level conflicts.

(6) Three-way strong alliance on a single issue. Finally, we give one more discussion about the strong alliance on a single issue, through which one can see the role of reference agent in trisecting process. Since $P_v^{=} = \{(+,+),(-,-),(0,0)\}$ according to the above discussion, we further define the following trisection of $P_v^{=}$ so as to differentiate the pairs $(+,+),(-,-)$ and (0, 0).

Definition 3.43 Let $S = (U,V,f)$ be a three-valued situation table, $v \in V$ be an issue in V, and τ be a reference agent on $\{v\}$ satisfying the condition that $f(\tau, v) = +$. The **trisection of** $P_v^{=}$ is defined by

$$\begin{cases} PP_v^{=} = \{(u_i,u_j) \in P_v^{=} \mid e_v(u_i,\tau) = 1\}, \\ NP_v^{=} = \{(u_i,u_j) \in P_v^{=} \mid e_v(u_i,\tau) = 0\}, \\ BP_v^{=} = \{(u_i,u_j) \in P_v^{=} \mid e_v(u_i,\tau) = 0.5\}. \end{cases} \tag{44}$$

One can see that we can define a trisection flexibly by employing different appropriate τ on $\{v\}$. And this is exactly what the reference agent plays.

Example 3.44 By Definition 3.43, we can further compute the trisection of $P_{v_1}^{=}$ as follows, where $P_{v_1}^{=}$ has been computed in Example 3.32:

$$\begin{cases} PP_{v_1}^{=} = \{(u_2,u_3),(u_2,u_5),(u_3,u_5),(u_2,u_2), \\ (u_3,u_3),(u_5,u_5)\}, \\ NP_{v_1}^{=} = \{(u_1,u_1)\}, \\ BP_{v_1}^{=} = \{(u_4,u_6),(u_4,u_4),(u_6,u_6)\}. \end{cases}$$

One can see that $PP_{v_1}^{=} = \{(+,+)\}$, $NP_{v_1}^{=} = \{(-,-)\}$ and $BP_{v_1}^{=} = \{(0,0)\}$. Therefore, the pairs $(+,+),(-,-)$ and (0, 0) are effectively differentiated from each other by Definition 3.43.

By employing Definition 3.43, we can further divide the strong alliance $\mathbf{SA} = P_v^{=}$ into the following three parts so as to differentiate the types of alliances in **SA**.

Definition 3.45 Let $S = (U,V,f)$ be a three-valued situation table, $v \in V$ be an issue in V, $\langle P_v^{=}, P_v^{\asymp}, P_v^{\approx}\rangle$ be the trisection of all pairs of agents on v, and $\langle PP_v^{=}, NP_v^{=}, BP_v^{=}\rangle$ be the trisection of $P_v^{=}$. The **three-way strong alliance induced by** $\langle PP_v^{=}, NP_v^{=}, BP_v^{=}\rangle$ is defined by

$$\begin{cases} \textbf{support-strong alliance}: \ \mathbf{SSA} \\ = \{(u_i,u_j) \in P_v^{=} | (u_i,u_j) \in PP_v^{=}\}, \\ \textbf{opposition-strong alliance}: \\ \mathbf{OSA} = \{(u_i,u_j) \in P_v^{=} | (u_i,u_j) \in NP_v^{=}\}, \\ \textbf{neutrality-strong alliance}: \\ \mathbf{NSA} = \{(u_i,u_j) \in P_v^{=} | (u_i,u_j) \in BP_v^{=}\}. \end{cases} \tag{45}$$

Example 3.46 In Example 3.44, we have obtained the trisection $\langle PP_{v_1}^{=}, NP_{v_1}^{=}, BP_{v_1}^{=}\rangle$, and therefore we immediately obtain the following three-way strong alliance by Definition 3.45:

$$\begin{cases} \text{support-strong alliance:} \quad \mathbf{SSA} \\ = \{(u_2,u_3),(u_2,u_5),(u_3,u_5),(u_2,u_2),(u_3,u_3),(u_5,u_5)\}, \\ \text{opposition-strong alliance:} \quad \mathbf{OSA} = \{(u_1,u_1)\}, \\ \text{neutrality-strong alliance:} \\ \mathbf{NSA} = \{(u_4,u_6),(u_4,u_4),(u_6,u_6)\}. \end{cases}$$

4 Conclusion

A new model of three-way conflict analysis, namely the three-level conflict induced by a trisection of all pairs of agents on an issue set, is introduced in this paper. The threshold-selection problem for the defined three-level conflicts is successfully solved in two different ways. By comparing the approach of Yao [35], the present paper proves that the trisection model (resp. the three-level conflict model) defined in this paper is a conservative extension of the corresponding trisection model (resp. three-level conflict model) in Yao [35]. Therefore, the present paper extends and improves the results of Yao [35] on multiple issues.

When we limit three-level conflict model to the one on a single issue, we find that the three-level conflict induced by a trisection of all agents is equivalent to the three-level conflict induced by a trisection of all pairs of agents. However, we fail to obtain a similar result on multiple issues. So this problem will be further studied in the future work. In addition, there are usually importance difference among multiple issues. So we will introduce importance into the multiple issues, and study three-way conflict analysis with importance on multiple issues in the future.

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