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ORIGINAL ARTICLE

Novel fusion strategies for continuous interval-valued *q*-rung orthopair fuzzy information: a case study in quality assessment of SmartWatch appearance design

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Abstract
The notion of Yager's *q*-rung orthopair fuzzy set (QROFS) have gained considerable and continuously increasing attention as a useful tool for imprecision and uncertainty representation due to its capability to discard the constraints on the membership and nonmembership functions as generally required by its intuitionistic fuzzy counterpart. Among the generalizations and variants established in the past few years, the interval-valued QROFSs (IVQROFSs) have been diffusely considered to be a powerful generalization of the interval-valued fuzzy sets. The continuous ordered weighted averaging (COWA) operator has been extended successfully to some special cases of IVQROFSs, including interval-valued intuitionistic and Pythagorean fuzzy sets. Thus, to expand on previous studies, several continuous IVQROF (C-IVQROF) aggregation operators are proposed in this study. First, the dual C-GOWA operator is defined on the basis of the continuous generalized ordered weighted averaging (C-GOWA) operator and Yager class of fuzzy negation. Subsequently, the C-IVQROFOWA operator with two independent parameters is constructed, and the weighted C-IVQROFOWA operator is then proposed for aggregating a collection of IVQROFSs. The C-IVQROFOWA operator and its weighted version can model commendably the attitudinal characteristics of the decision-maker. Second, a parameter optimization model and its algorithm-solving strategy driven by consensus measures are built to develop a group decision-making method. Finally, a case study to evaluate the SmartWatch design alternatives is provided to demonstrate the proposed approach, and the results of a comparative analysis verify the rationality and efficiency of the proposed operators.

Keywords Interval-valued *q*-rung orthopair fuzzy sets · Aggregation operators · Group decision making · Product appearance design evaluation

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1 Introduction

Q-rung orthopair fuzzy sets [1–3] (QROFSs) are fuzzy sets [4] in which the membership grades of an element are the pairs of values in the unit interval, $(\mu(x), \nu(x))$, one of which indicates support for membership in the fuzzy set and the other represents support against membership. QROFS presents the following prominent features:

- *Large space for membership* the constraining relationship between the support for and against memberships is $(\mu(x))^q + (\nu(x))^q \leq 1$ with $q \geq 1$;
- *Strong degeneracy of sets* the Atanassov's intuitionistic fuzzy sets [5] and what Yager called Pythagorean fuzzy sets [6] are special cases of QROFSs with $q = 1$ and $q = 2$, respectively;
- *Flexibility in the application* it allows the decision-maker the freedom to provide information in support or against the membership of an element in a set.

QROFSs have received considerable critical attention from researchers since its introduction. *Operational laws and aggregation operators* are proposed to aggregate q-rung orthopair fuzzy (QROF) numbers [7–16], including QROF Heronian mean, Hamy mean, Bonferroni mean, Maclaurin symmetric mean, and Power mean operators. *Information measures*, such as distance measure, similarity measure, entropy, and inclusion measure, are used to analyze the relationship between different QROFSs [17–22]; the intrinsic correlation of these measures have also been explored [23]. *QROF integrals and differentials* are constructed to aggregate QROF continuous information [24–27]. *QROF preference relation* has been proposed to deal with the QROF decision-making problems of the comparison matrix [28, 29]. *QROF multiple attribute decision-making (MADM) method* are developed to solve MADM problems with hierarchical interacting criteria under QROF environment [30]. Existing research results are used to consolidate the theoretical construction of QROFSs and promote the application of QROFSs in various decision-making problems.

With the continuous progress of society and the rapid development of the economy, decision-making problems have become increasingly complicated with their noticeable uncertainty and fuzzy human thinking [31–33]. Therefore, in the decision-making process, unlike the formal representation of precise values, the representation of input parameters in the form of interval values is more suitable in the current decision-making environment. For this reason, a generalization of QROFSs has been introduced by some researchers [34, 35] in terms of interval-valued fuzzy sets called interval-valued QROFSs (IVQROFSs). To date, existing studies include mainly the operations of union, intersection, addition, and multiplication (with real scalars) of IVQROFSs. On this basis, the interval-valued q-rung orthopair fuzzy weighted averaging (IVQROFWA) [35] and Maclaurin symmetric mean operators are developed [36] to aggregate different IVQROFSs. Moreover, a new notion of the IVQROF graph (IVQROFG) [37] is introduced to investigate the common problem of determining the shortest path in a traffic network. The interval-valued QROF (IVQROF) operators mentioned above are constructed by directly extending the QROF operators to the interval-valued fuzzy environment. The characteristics of its aggregation process are that the endpoints of interval-valued membership (or non-membership) are directly aggregated, and the information aggregation results are dominated by the interval endpoint values, and the contribution of other points in the interval is rarely considered.

The continuous ordered weighted averaging (COWA) operator was introduced by Yager in [38] and is a continuation of the OWA operator [39, 40] when the given argument is a continuous-valued interval rather than an exact argument. Since its introduction, the COWA operator has been used in many fields, including preference relations [41, 42], information measures [43–46], and aggregation operators [47–50]. The COWA operator is applied to propose the weighted continuous interval-valued intuitionistic fuzzy ordered weighted averaging (WC-IVIFOWA) operator [47–49]. Another interesting extension of the OWA operator is the continuous ordered weighted quadratic averaging (C-OWQA) operator [51], which is applied to propose the weighted continuous interval-valued Pythagorean fuzzy ordered weighted averaging (WC-IVPFOWA) operator [52]. As mentioned earlier, intuitionistic and Pythagorean fuzzy sets are two special cases of IVQROFSs and the COWA and C-OWQA operators are the core tools for constructing WC-IVIFOWA and WC-IVPFOWA operators, respectively. Correspondingly, the COWA and C-OWQA operators are two special cases of a generalization of ordered weighted generalized averaging (OWGA) operator [53] called the continuous OWGA (COWGA) operator [51]. The aggregation of group information and multi-attribute information is a necessary process to obtain comprehensive information of alternatives and an important stage to deal with multi-attribute group decision-making (MAGDM) problems. For the MAGDM approaches proposed in references [47–49, 52], as the core tools, the WC-IVPFOWA and WC-IVIFOWA operators can effectively aggregate weighted group evaluation information, reduce the computational complexity of information aggregation and improve the accuracy of decision-making.However, the COWA operator has not been extended to the IVQROF environment. Furthermore, for any given interval-valued intuitionistic fuzzy numbers (IVIFNs), decision makers can not obtain any IFNs contained in the IVIFNs by using the C-IVIFOWA operator. The

C-IVPFOWA operator in interval-valued Pythagorean fuzzy environment exhibits the same shortcoming alike. From a mathematical point of view, the reason for the aforementioned shortcoming is that the values of attitudinal characteristics of interval-valued membership and interval-valued non-membership are the same, and there is no independent setting. The deficiency of the operators makes it easier for decision makers to ignore most of the information in the given interval-valued fuzzy numbers, and then it is difficult to accurately model the attitude preference of decision makers.

These facts motivate us to design a robust decision-making framework with the application of continuous ordered weighted averaging operator in solving the group decision-making (GDM) scenarios under IVQROF environment. Concretely, this study proposes continuous IVQROF aggregation operators by combining the COWGA operator with IVQROFSs. The dual COWGA (DCOWGA) is constructed using Yager's class negation [54, 55] and the COWGA operator. With COWGA and DCOWGA operators as the aggregation tools, the continuous IVQROF ordered weighted averaging (C-IVQROFOWA) and weighted C-IVQRFOWA (WC-IVQROFOWA) operators are proposed to aggregate IVQROF numbers. Based on these operators, a GDM method with consensus-improving is developed with the help of a parameter optimization model. Several novel consensus measures are defined in the use of the proposed WC-IVQROFOWA operator. This method is used to solve problems in the evaluation of SmartWatch appearance design alternatives.

The remainder of this paper is organized as follows. In Sect. 2, we briefly review basic concepts, such as the IVQROFSs, GOWA operator, and DGOWA operator. In Sect. 3, the WC-IVQROFOWA operator is proposed. A decision-making approach based on the WC-IVQROFOWA operator and parameter optimization model is presented in Sect. 4, and an illustrative example is examined in Sect. 5. Comparison analysis is performed between the aggregation operators proposed in this study and the existing aggregation operators in Sect. 6. The main conclusions of the study are drawn in Sect. 7.

2 Preliminaries

In this section, we provide the necessary background for our subsequent developments.

2.1 Q-rung orthopair fuzzy set interval-valued *q*-rung orthopair fuzzy set

Yager [1] generalizes Atanassov's intuitionistic fuzzy set theory [5] with the concept of QROFS as defined below.

Definition 1 [1] Let X be a universe of discourse. A QROFS P in X is expressed as

$$P = \left\{ \langle x, \mu_P(x), \nu_P(x) \rangle | x \in X \right\}, \tag{1}$$

where the function $\mu_P : X \to [0,1]$ defines the degree of membership and $\nu_P : X \to [0,1]$ defines the degree of non-membership of the element $x \in X$ to P, respectively. For every $x \in X$, it holds that $(\mu_P(x))^q + (\nu_P(x))^q \le 1$. The degree of indeterminacy is $\pi_P(x) = (1 - (\mu_P(x))^q - (\nu_P(x))^q)^{1/q}$.

For simplicity, we define $(\mu_P(x), \nu_P(x))$ as a q-rung orthopair fuzzy number (QROFN) as denoted by $P = (\mu_P, \nu_P)$, where $\mu_P, \nu_P \in [0,1], \pi_P = (1 - \mu^q - \nu^q)^{1/q}$ and $\mu^q + \nu^q \le 1$. It can be conveniently observed that the intuitionistic fuzzy sets (IFSs) [5] are QROFS with $q = 1$ and Yager's Pythagorean fuzzy sets (PFSs) [6] are QROFS with $q = 2$.

Theorem 1 [1] *If α is a QROFN on X and if $q_1 > q$, then α is also a QROFN on X.*

Corollary 1 [1] (1) *Any IFS is a QROFS for all $q \ge 2$;*

(2) *An IFS is a PFS;*

(3) *Any PFS is a QROFS for $q \ge 2$.*

Definition 2 [17] Let $\alpha = (\mu, \nu)$ be a QROFN solid at q, then $s(\alpha) = \mu^q - \nu^q$ is the score function of α and $h(\alpha) = \mu^q + \nu^q$ is the accuracy function of α.

Based on the defined score function and the accuracy function, a ranking method was proposed in the following definition.

Definition 3 [17] Let $\alpha_i = (\mu_i, \nu_i)$ be two QROFNs solid at q.

1. If $s(\alpha_1) < s(\alpha_2)$, then $\alpha_1 \prec \alpha_2$;
2. If $s(\alpha_1) = s(\alpha_2)$, (i) $h(\alpha_1) < h(\alpha_2)$, $\alpha_1 \prec \alpha_2$, and (ii) if $s(\alpha_1) = s(\alpha_2)$ and $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 = \alpha_2$.

Definition 4 [34, 35] Let X be a universe of discourse. An IVQROFS $\tilde{P}$ in X is expressed as

$$\tilde{P} = \left\{ \langle x, \tilde{\mu}_P(x), \tilde{\nu}_P(x) \rangle | x \in X \right\}, \tag{2}$$

where the function $\tilde{\mu}_P(x) = [\mu_P^-(x), \mu_P^+(x)]$ defines the degree of membership and $\tilde{\nu}_P(x) = \left[\nu_P^-(x), \nu_P^+(x)\right]$ defines the degree of non-membership of the element $x \in X$ to P, respectively; and for every $x \in X$, it holds that $(\mu_P^+(x))^p + (\nu_P^+(x))^p \le 1$. The degree of indeterminacy is

$$\tilde{\pi}_P(x) = [\pi_P^-(x), \pi_P^+(x)] = \left[\left(1 - \left(\mu_P^+(x)\right)^p - \left(v_P^+(x)\right)^p\right)^{1/p}, \left(1 - \left(\mu_P^-(x)\right)^p - \left(v_P^-(x)\right)^p\right)^{1/p}\right].$$

For simplicity, we call $(\tilde{\mu}_P(x), \tilde{v}_P(x))$ an interval-valued q-rung orthopair fuzzy number (IVQROFN) denoted by $P = (\tilde{\mu}_P, \tilde{v}_P) = ([\mu_P^-, \mu_P^+], [v_P^-, v_P^+])$, where $(\mu_P^+)^p + (v_P^+)^p \leq 1$. Let $P = ([\mu^-, \mu^+], [v^-, v^+])$ be an IVQROFN solid at q, then

$$S(\alpha)=\frac{1}{2} + \frac{(\mu^-)^q + \left(\mu^+\right)^q - (v^-)^q - \left(v^+\right)^q}{4}$$

is the score function [35] of P and

$$H(\alpha)=\frac{(\mu^-)^q + \left(\mu^+\right)^q+(v^-)^q+\left(v^+\right)^q}{2}$$

is the accuracy function [35].

A ranking method of IVQROFNs is proposed based on the above score and the accuracy functions [35]. Let $\tilde{\alpha}_i = ([\mu_i^-, \mu_i^+], [v_i^-, v_i^+])(i = 1, 2)$ be two IVQROFN solids at q.

1. If $S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \prec \tilde{\alpha}_2$;
2. If $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$, (i) $H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2)$, $\tilde{\alpha}_1 \prec \tilde{\alpha}_2$, and (ii) if $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$ and $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \sim \tilde{\alpha}_2$.

2.2 GOWA and DGOWA operators

Definition 5 [56] Fuzzy negation is a mapping denoted by $\mathcal{N} : [0, 1] \to [0, 1]$, which satisfies the following properties:

1. Boundary conditions: $\mathcal{N}(0) = 1$ and $N(1) = 0$;
2. Monotonicity: for all $a, b \in [0, 1]$, if $a \leq b$, then $\mathcal{N}(a) \geq \mathcal{N}(b)$;
3. Continuity;
4. Involution: $\mathcal{N}(\mathcal{N}(a)) = a$ for all $a \in [0, 1]$.

The Yager class of fuzzy negation [54, 55] is defined by $\mathcal{N}_q(a) = (1 - a^q)^{1/q}$, where $q \in (0, \infty)$. When $q = 1$, this function becomes the classical fuzzy negation $\mathcal{N}_1(a) = 1 - a$, whereas $q = 2$ will make this function become the Pythagorean negation [6] $\mathcal{N}_2(a) = \sqrt{1 - a^2}$.

Definition 6 [56] An aggregation function is a function of $n > 1$ arguments that maps the (n-dimensional) unit cube onto the unit interval $\mathsf{F} : [0, 1]^n \to [0, 1]$ with the following properties:

1. $\mathsf{F}(0, 0, \ldots, 0) = 0$ and $\mathsf{F}(1, 1, \ldots, 1) = 1$;
2. $a_j \geq b_j$implies $\mathsf{F}(a_1, a_2, \ldots, a_n) \geq \mathsf{F}(b_1, b_2, \ldots, b_n)$ for all j.

The dual of aggregation function with respect to fuzzy negation is given as follows:

Definition 7 [56] Let $\mathsf{F} : [0, 1]^n \to [0, 1]$ be an aggregation function. Then, the aggregation function F_d is given by

$$\mathsf{F}_d\left(x_1, x_2, \ldots, x_n\right) = \mathcal{N}\left(\mathsf{F}\left(\mathcal{N}\left(x_1\right), \mathcal{N}\left(x_2\right), \ldots, \mathcal{N}\left(x_n\right)\right)\right), \quad (3)$$

which is the dual of F with respect to $\mathcal{N}$, where $\mathcal{N}$ is a negation function.

Definition 8 [53] A mapping $\mathsf{GOWA} : R^n \to R$ is called a generalized ordered weighted averaging (GOWA) operator of dimension n if

$$\mathsf{GOWA}\left(x_1, x_2, \ldots, x_n\right) = \left(\sum_{i=1}^{n} w_j x_{\sigma(j)}^p\right)^{1/p}, \quad (4)$$

where σ is an index function to ensure that $x_{\sigma(j)}$ is the j-th largest of the x_i and the W with components w_j for $j = 1$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. p is a parameter with $p \in [-\infty, +\infty]$ and $p \neq 0$.

We can obtain the OWA weights by using

$$w_j = Q(\frac{j}{n}) - Q(\frac{j-1}{n}), \quad j = 1, 2, \ldots, n, \quad (5)$$

where basic unit-interval monotonic (BUM) function [40] $Q : [0, 1] \to [0, 1]$ is monotonic with the following properties: (1) $Q(0) = 0$; (2) $Q(1) = 1$; and (3) $Q(x) \geq Q(y)$ if $x > y$. Hence, these weights satisfy the conditions $w_j \in [0, 1] (j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. When $p = 1$, the GOWA operator becomes the OWA operator as follows [39]:

$$\mathsf{OWA}\left(x_1, x_2, \ldots, x_n\right) = \sum_{i=1}^{n} w_j x_{\sigma(j)}. \quad (6)$$

Definition 9 [1] Let $\mathsf{GOWA} : R^n \to R$ be a generalized ordered weighted averaging aggregation operator of dimension n and its dual generalized ordered weighted averaging aggregation (DGOWA) operator with respect to a q-rung negation $\mathcal{N}_q$ is defined as follows:

$$\mathsf{DGOWA}\left(x_1, x_2, \ldots, x_n\right) = \mathcal{N}_q\left(\mathsf{GOWA}\left(\mathcal{N}_q\left(x_1\right), \ldots, \mathcal{N}_q\left(x_n\right)\right)\right). \quad (7)$$

According to the above definitions, we can obtain

$$\mathrm{DGOWA}(x_1, x_2, \dots, x_n) = \mathcal{N}_q\left(\left(\sum_{i=1}^{n} w_j(\mathcal{N}_q(x_{\delta(j)}))^p\right)^{1/p}\right), \tag{8}$$

where δ is an index function to ensure that $\mathcal{N}_q(x_{\delta(j)})$ is the j-th largest of $\mathcal{N}_q(x_i)$.

3 WC-IVQROFWA operator

This section defines the weighted continuous interval-valued q-rung orthopais fuzzy ordered weighted averaging (C-IVQROFOWA) and studies its properties. Based on this operator, a new score and accuracy functions are introduced.

3.1 Dual continuous generalized ordered weighted averaging (DC-GOWA) operator

Definition 10 [51] The continuous generalized ordered weighted averaging (C-GOWA) operator is defined as follows:

$$\phi_Q([a,b]) = \left(\int_0^1 \frac{dQ(y)}{dy}(b^p - y(b^p - a^p))dy\right)^{1/p}. \tag{9}$$

When $p = 1$ the C-GOWA operator becomes the C-OWA operator, it can be defined as [38]

$$\mathrm{C-OWA}(x_1, x_2, \dots, x_n) = \int_0^1 \frac{dQ(y)}{dy}(b - y(b - a))dy. \tag{10}$$

In the following, we present the establishment of the C-GOWA operator.

Let Q be a BUM function and $w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)(j = 1, 2, \cdots, n)$, and $\sum_{j=1}^n w_j = 1$. Based on the GOWA operator, we can show that the discrete case is as follows:

$$\phi_Q(x_1, \cdots, x_n) = \left(\sum\nolimits_{j=1}^n \left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right) y_j^p\right)^{1/p}, \tag{11}$$

where y_j is the j-th largest of the x_i. $[a,b] \subseteq [0,1]$ is assumed to be a continuous interval-valued argument and we will take a finite approximation of $\phi_Q([a,b])$. Let $\delta = (b^p - a^p)/n$, $y_j = (b^p - j\delta)^{1/p}$, $b = y_0 \geq \cdots \geq y_n = a$, then

$$\begin{aligned}\phi_Q([a,b]) &\approx \phi_Q(y_1, \cdots, y_n) = \left(\sum\nolimits_{j=1}^n w_j y_j^p\right)^{1/p} \\ &= \left(\sum\nolimits_{j=1}^n \left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right) \times \left(b^p - \frac{j(b^p - a^p)}{n}\right)\right)^{1/p}.\end{aligned} \tag{12}$$

Let $\Delta y = 1/n$, then

$$\phi_Q([a,b]) \approx \left(\sum\nolimits_{j=1}^n \left(\frac{Q(j\Delta y) - Q(j\Delta y - \Delta y)}{\Delta y}\right) \times (b^p - j\Delta y(b^p - a^p))\Delta y\right)^{1/p}. \tag{13}$$

When $n \to \infty$ denoting $y = j\Delta y = j/n$, then $y \in [0,1]$ and

$$\phi_Q([a,b]) = \left(\int_0^1 \frac{dQ(y)}{dy}(b^p - y(b^p - a^p))dy\right)^{1/p}. \tag{14}$$

Theorem 2 [51] (*Monotonicity*) *Let ϕ_Q be the C-GOWA operator. If $a_1 \geq a_2, b_1 \geq b_2$ for all Q, then $\phi_Q([a_1, b_1]) \geq \phi_Q([a_2, b_2])$.*

Theorem 3 [51] (*Boundness*) *Let ϕ_Q be the C-GOWA operator. For all Q, we have $a \leq \phi_Q([a, b]) \leq b$.*

Theorem 4 [51] *If $Q_1 \geq Q_2$, then $\phi_{Q_1}([a_1, b_1]) \geq \phi_{Q_2}([a_2, b_2])$*
.

Definition 11 The dual continuous generalized ordered weighted averaging (DC-GOWA) operator is defined as follows:

$$\phi_Q^d([a,b]) = \mathcal{N}_q\Bigg(\bigg(\int_0^1 \frac{dQ(y)}{dy}\big((\mathcal{N}_q(a))^p - y\big((\mathcal{N}_q(a))^p - (\mathcal{N}_q(b))^p\big)\big)dy\bigg)^{1/p}\Bigg), \tag{15}$$

where $\mathcal{N}_q$ is the q-rung negation function.

The derivative of the DC-GOWA operator is analyzed based on the establishment of the C-GOWA operator.

Theorem 5 (Monotonicity) *If $a_1 \geq a_2, b_1 \geq b_2$ for all Q, then $\phi_Q^d([a_1, b_1]) \geq \phi_Q^d([a_2, b_2])$.*

Proof According to Definitions 10 and 11, we have

$$\phi_Q^d([a,b]) = \mathcal{N}_q\big(\phi_Q\big[\mathcal{N}_q(b), \mathcal{N}_q(a)\big]\big).$$

From ***Theorem 2, if $a_1 \geq a_2, b_1 \geq b_2$ for all , then

$$\phi_Q\big(\big[a_1, b_1\big]\big) \geq \phi_Q\big(\big[a_2, b_2\big]\big).$$

Therefore, we have

$$\phi_Q\big(\big[\mathcal{N}_q(b_1), \mathcal{N}_q(a_1)\big]\big) \leq \phi_Q\big(\big[\mathcal{N}_q(b_2), \mathcal{N}_q(a_2)\big]\big)$$

and

$$\mathcal{N}_q\big(\phi_Q\big(\big[\mathcal{N}_q(b_1), \mathcal{N}_q(a_1)\big]\big)\big) \geq \mathcal{N}_q\big(\phi_Q\big(\big[\mathcal{N}_q(b_2), \mathcal{N}_q(a_2)\big]\big)\big).$$

From this and $\mathcal{N}_q(x_1) \leq \mathcal{N}_q(x_2)$ for $x_1 \geq x_2$ and $x_1, x_2 \in [0, 1]$. Thus,

$$\phi_Q^d\big(\big[a_1, b_1\big]\big) \geq \phi_Q^d\big(\big[a_2, b_2\big]\big).$$

The proof is completed. □

Based on the DGOWA operator, we can show that the discrete case is as follows:

$$\phi_Q^d(x_1, \cdots, x_n) = \mathcal{N}_q(\phi_Q(\mathcal{N}_q(x_1), \cdots, \mathcal{N}_q(x_n))). \tag{16}$$

$[a, b] \subseteq [0, 1]$ is assumed to be a continuous interval-valued argument, and we will take a finite approximation of $\phi_Q^d([a, b])$. Let $\delta = (b^p - a^p)/n$, $y_j = (b^p - j\delta)^{1/p}$, $b = y_0 \geq \cdots \geq y_n = a$, then we have

$$\phi_Q^d([a, b]) \approx \phi_Q^d(y_1, \cdots, y_n) = \mathcal{N}_q(\phi_Q(\mathcal{N}_q(y_1), \cdots, \mathcal{N}_q(y_n))). \tag{17}$$

Let $y_j^d = \mathcal{N}_q(y_{n-j+1})$, then $\mathcal{N}_q(a) \geq y_0^d \geq \cdots \geq y_n^d = \mathcal{N}_q(b)$ approximate $[\mathcal{N}_q(b), \mathcal{N}_q(a)]$

$$\phi_Q^d([a, b]) \approx \mathcal{N}_q(\phi_Q(\mathcal{N}_q(y_1), \cdots, \mathcal{N}_q(y_n))) \approx \mathcal{N}_q(\phi_Q[\mathcal{N}_q(b), \mathcal{N}_q(a)]). \tag{18}$$

According to Definitions 10 and 11, when $n \to \infty$, we have

$$\phi_Q^d([a, b]) = \mathcal{N}_q\left(\left(\int_0^1 \frac{dQ(y)}{dy}((\mathcal{N}_q(a))^p - y((\mathcal{N}_q(a))^p - (\mathcal{N}_q(b))^p))\,dy\right)^{1/p}\right). \tag{19}$$

Theorem 6 (Boundness) *For all Q, we have $a \leq \phi_Q^d([a,b]) \leq b$.*

Proof According to Definitions 10 and 11, we have

$$\phi_Q^d([a,b]) = \mathcal{N}_q\left(\phi_Q\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]\right).$$

From Theorem 3, For all Q, we have

$$\mathcal{N}_q(b) \leq \phi_Q\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right] \leq \mathcal{N}_q(a).$$

Therefore, we have

$$\begin{aligned} a = \mathcal{N}_q\left(\mathcal{N}_q(a)\right) &\leq \mathcal{N}_q\left(\phi_Q\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]\right) \\ &\leq \mathcal{N}_q\left(\mathcal{N}_q(b)\right) = b. \end{aligned}$$

From this and (i) $\mathcal{N}_q(x_1) \leq \mathcal{N}_q(x_2)$ for $x_1 \geq x_2$ and $x_1, x_2 \in [0,1]$; (ii) $\mathcal{N}_q(\mathcal{N}_q(x)) = x$ for all . Thus,

$$a \leq \phi_Q^d([a,b]) \leq b.$$

The proof is completed. $\square$

Definition 12 [38] If Q_1 and Q_2 are such that $Q_1(x) \geq Q_2(x)$ for all $x \in [0,1]$, we denote this as $Q_1 \geq Q_2$.

Theorem 7 *If $Q_1 \geq Q_2$, then $\phi_{Q_1}^d([a,b]) \leq \phi_{Q_2}^d([a,b])$.*

Proof According to Definitions 10 and 11, we have

$$\phi_Q^d([a,b]) = \mathcal{N}_q\left(\phi_Q\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]\right).$$

From Theorem 4, if $Q_1 \geq Q_2$, then

$$\phi_{Q_1}([a,b]) \geq \phi_{Q_2}([a,b]).$$

Therefore, we have

$$\phi_{Q_1}\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right] \geq \phi_{Q_2}\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]$$

and

$$\mathcal{N}_q\left(\phi_{Q_1}\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]\right) \leq \mathcal{N}_q\left(\phi_{Q_2}\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]\right).$$

From this and $\mathcal{N}_q(x_1) \leq \mathcal{N}_q(x_2)$ for $x_1 \geq x_2$ and $x_1, x_2 \in [0,1]$. Thus,

$$\phi_{Q_1}^d([a,b]) \leq \phi_{Q_2}^d([a,b]). \quad \square$$

Yager [38] pointed out that $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of Q and $\lambda \in [0,1]$. Some properties of C-GOWA and DC-GOWA operators with respect to the attitudinal character λ are discussed.

Theorem 8 [51] *If λ is the attitudinal character of Q, then*

$$\phi_Q([a,b]) = (\lambda b^p + (1-\lambda)a^p)^{1/p}. \tag{20}$$

Theorem 9 *If λ is the attitudinal character of Q, then*

$$\phi_Q^d([a,b]) = \mathcal{N}_q\left(\lambda\left(\mathcal{N}_q(a)\right)^p + (1-\lambda)\left(\mathcal{N}_q(b)\right)^p\right)^{1/p}. \tag{21}$$

Proof According to Definitions 10 and 11, we have

$$\phi_Q^d([a,b]) = \mathcal{N}_q\left(\phi_Q\left[\mathcal{N}_q(b), \mathcal{N}_q(a)\right]\right).$$

From Theorem 8, we have

$$\phi_Q^d([a,b]) = \mathcal{N}_q\left(\lambda\left(\mathcal{N}_q(a)\right)^p + (1-\lambda)\left(\mathcal{N}_q(b)\right)^p\right)^{1/p}. \quad \square$$

Remark 1 For convenience, we denote ϕ_Q and ϕ_Q^d as ϕ_λ and ϕ_λ^d, respectively.

3.2 C-IVQROFOWA operator

Consider the situation where $\tilde{\alpha} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$ is a QROFN. Use ϕ_Q and ϕ_Q^d to aggregate the continuous interval-valued $[\mu^-, \mu^+]$ and $[\nu^-, \nu^+]$, respectively. Further, assume the power of $p = q$ in this case, then

$$\phi_Q([a,b]) = \left(\int_0^1 \frac{dQ(y)}{dy}(b^q - y(b^q - a^q))dy\right)^{1/q} \tag{22}$$

and

$$\phi_Q^d([a,b]) = \left(1 - \int_0^1 \frac{dQ(y)}{dy}(1 - a^q - y(b^q - a^q))dy\right)^{1/q} \tag{23}$$

If λ is the attitudinal character of Q, then

$$\begin{aligned} \phi_\lambda([a,b]) &= (\lambda b^q + (1-\lambda)a^q)^{1/q}, \phi_\lambda^d([a,b]) \\ &= (\lambda a^q + (1-\lambda)b^q)^{1/q}. \end{aligned} \tag{24}$$

Definition 13 Let $\tilde{\alpha} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$ be an IVQROFN solid at q and the continuous interval-valued q-rung orthopair fuzzy ordered weighted averaging (C-IVQROFOWA) operator is defined as follows:

$$G_{\lambda_1,\lambda_2}(\tilde{\alpha}) = \left(\phi_{\lambda_1}\left(\left[\mu^-, \mu^+\right]\right), \phi_{\lambda_2}^d\left(\left[\nu^-, \nu^+\right]\right)\right), \tag{25}$$

where λ_1 and λ_2 are the attitudinal character of BUM function Q_1 and Q_2, respectively.

Example 1 Let $\tilde{\alpha} = ([0.5, 0.6], [0.7, 0.8])$ be an IVQROFN with $q = 2$. Let $Q_1(x) = x^2$ and $Q_2(x) = x$ be the related BUM functions. The attitudinal character of $Q_i (i = 1, 2)$ and the

C-IVQROFOWA operator of $\tilde{\alpha}$ can be computed on the basis of Definition 13 as follows:

$$\begin{cases} \lambda_1 = \int_0^1 Q_1(x)d_x = \frac{1}{3}, \lambda_2 = \int_0^1 Q_2(x)d_x = \frac{1}{2}, \\ G_{\lambda_1,\lambda_2}(\alpha) = \left(\left(\frac{1}{3}0.6^2 + \frac{2}{3}0.5^2 \right)^{1/2}, \left(\frac{1}{2}0.7^2 + \frac{1}{2}0.8^2 \right)^{1/2} \right) = (0.5354, 0.7517). \end{cases}$$

Theorem 10 *Let $\tilde{\alpha} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$ be an IVQROFN solid at q, then $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$ is also an IVQROFN solid at q.*

Proof According to Definition 11, we have

$\phi_{\lambda_1}([\mu^-, \mu^+]) \in [\mu^-, \mu^+], \phi^d_{\lambda_2}([\nu^-, \nu^+]) \in [\nu^-, \nu^+]$,

where $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$ is also an IVQROFN solid at q. □

Definition 14 Let $\tilde{\alpha} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$ be an IVQROFN solid at q. For any QROFN α, if $\mu \in [\mu^-, \mu^+]$ and $\nu \in [\nu^-, \nu^+]$, then α belongs to $\tilde{\alpha}$. We write $\alpha \in \tilde{\alpha}$ and read "α is a q-rung orthopair fuzzy element of $\tilde{\alpha}$".

Remark 2 According to the ranking method of IVQROFNs in [35], we have

$$\tilde{\alpha}_1 \prec \tilde{\alpha}_2 \Leftrightarrow \tilde{\alpha}_1 \prec_\lambda \tilde{\alpha}_2 (\lambda_1 = \lambda_2 = 0.5) \tag{27}$$

with the following

$$S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2) \Leftrightarrow Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}_1) < Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}_2),$$
$$H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2) \Leftrightarrow Acc_{\lambda_1,\lambda_2}(\tilde{\alpha}_1) < Acc_{\lambda_1,\lambda_2}(\tilde{\alpha}_2).$$

Theorem 11 *Let $\tilde{\alpha} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$ be an IVQROFN solid at q, then $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$ increases with respect to λ_1 and λ_2, and $G_{0,0}(\tilde{\alpha}) \le G_{\lambda_1,\lambda_2}(\tilde{\alpha}) \le G_{1,1}(\tilde{\alpha})$.*

Proof Suppose that $\lambda_1 < \bar{\lambda}_1$ and $\lambda_2 < \bar{\lambda}_2$. According to Definition 15, we have

$$Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}) - Sco_{\bar{\lambda}_1,\bar{\lambda}_2}(\tilde{\alpha}) \\ = \phi_{\lambda_1}([\mu^-, \mu^+]) - \phi_{\bar{\lambda}_1}([\mu^-, \mu^+]) + \phi^d_{\bar{\lambda}_2}([\nu^-, \nu^+]) - \phi^d_{\lambda_2}([\nu^-, \nu^+]).$$

Given that ϕ_λ increases with respect to λ and ϕ^d_λ decreases with respect to λ, then

$$\phi_{\lambda_1}([\mu^-, \mu^+]) < \phi_{\bar{\lambda}_1}([\mu^-, \mu^+]),$$
$$\phi^d_{\bar{\lambda}_2}([\nu^-, \nu^+]) < \phi^d_{\lambda_2}([\nu^-, \nu^+]).$$

Therefore, $Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}) - Sco_{\bar{\lambda}_1,\bar{\lambda}_2}(\tilde{\alpha}) < 0$, then $G_{\lambda_1,\lambda_2}(\tilde{\alpha}) \prec G_{\bar{\lambda}_1,\bar{\lambda}_2}(\tilde{\alpha})$. Thus, $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$ increases with respect to λ_1 and λ_2.

The proof is completed. □

3.3 New score and accuracy functions based on the C-IVQROFOWA operator

For any IVPFN $\tilde{\alpha}$, its related C-IVQROFOWA operator $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$ is QROFN. Based on the score and accuracy functions of QROF, the new score and accuracy functions of IVQROF are defined as follows.

Definition 15 Let $\tilde{\alpha} = (\tilde{\mu}, \tilde{\nu}) = ([\mu^-, \mu^+], [\nu^-, \nu^+])$ be an IVQROFN solid at q and the new score function $Sco_{\lambda_1,\lambda_2}(\tilde{\alpha})$ and accuracy function $Acc_{\lambda_1,\lambda_2}(\tilde{\alpha})$ are defined as follows:

$$Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}) = \left(\phi_{\lambda_1}(\tilde{\mu})\right)^q - \left(\phi^d_{\lambda_2}(\tilde{\nu})\right)^q,$$
$$Acc_{\lambda_1,\lambda_2}(\tilde{\alpha}) = \left(\phi_{\lambda_1}(\tilde{\mu})\right)^q + \left(\phi^d_{\lambda_2}(\tilde{\nu})\right)^q, \tag{26}$$

where $G_{\lambda_1,\lambda_2}(\tilde{\alpha}) = (\phi_{\lambda_1}(\tilde{\mu}), \phi^d_{\lambda_2}(\tilde{\nu}))$ is the C-IVQROFOWA operator of $\tilde{\alpha}$.

The ranking method is proposed based on the score function and accuracy function. Assume that $\tilde{\alpha}_i = ([\mu_i^-, \mu_i^+], [\nu_i^-, \nu_i^+]) (i = 1, 2)$ are two IVPFNs solid at q.

(1) If $Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}_1) < Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \prec_\lambda \tilde{\alpha}_2$.

(2) If $Sco_{\lambda_1,\lambda_2}(\tilde{\alpha}_1) = Sco_{\lambda_1,\lambda_2}$and $Acc_{\lambda_1,\lambda_2}(\tilde{\alpha}_1) < Acc_{\lambda_1,\lambda_2}(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \prec_\lambda \tilde{\alpha}_2$.

3.4 Weighted continuous interval-valued *q*-rung orthopair fuzzy aggregation operators

Definition 16 [1] Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i}) (i = 1, 2, \ldots, n)$ be a collection of QROFNs, then

$$\mathrm{Agg}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ = \left(\mathbb{E}(\mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_n}), \mathbb{E}_d(\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n})\right), \tag{28}$$

is their aggregation operator, where

$$\mathbb{E}_d(v_{\alpha_1}, v_{\alpha_2}, \dots, v_{\alpha_n}) = \mathcal{N}_q(\mathbb{E}(\mathcal{N}_q(v_{\alpha_1}), \mathcal{N}_q(v_{\alpha_2}), \dots, \mathcal{N}_q(v_{\alpha_n}))) \tag{29}$$

with a negation function $\mathcal{N}_q$.

Example 2 Let $\tilde{\alpha}_1 = ([0.5, 0.6], [0.7, 0.8])$, $\tilde{\alpha}_2 = ([0.2, 0.4], [0.8, 0.9])$, $\tilde{\alpha}_3 = ([0.7, 0.8], [0.4, 0.6])$ be three IVQROFNs solid at $q = 2$,and $w = (0.2, 0.3, 0.5)^T$ be the weighting vector. The WC-IVQRFOWA operator is conputed as foolows with $Q_1(x) = x^2$ and $Q_2(x) = x$:

$$\begin{aligned} &\mathrm{WC-IVQRFOWA}(\alpha_1, \alpha_2, \alpha_3) \\ &= \left(\begin{array}{c} \left(0.2\left(\frac{1}{3}0.6^2 + \frac{2}{3}0.5^2\right) + 0.3\left(\frac{1}{3}0.4^2 + \frac{2}{3}0.2^2\right) + 0.5\left(\frac{1}{3}0.8^2 + \frac{2}{3}0.7^2\right)\right)^{1/2} \\ \left(0.2\left(\frac{1}{2}0.7^2 + \frac{1}{2}0.8^2\right) + 0.3\left(\frac{1}{2}0.8^2 + \frac{1}{2}0.9^2\right) + 0.5\left(\frac{1}{2}0.4^2 + \frac{1}{2}0.6^2\right)\right)^{1/2} \end{array} \right) \\ &= (0.3513, 0.4605). \end{aligned}$$

Theorem 12 [1] *If each α_1 is an orthopair fuzzy set solid at rung q, which is a QROFN and the negation used to define E is taken with respect to q, which is the q-th rung negation function N_q, then Agg is an orthopair fuzzy set solid at rung q, which is a QROFN.*

By using the general form in Definition 16, Yager [1] introduced some important mean-type aggregation operators.

Definition 17 [1] Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1, 2, \dots, n)$ be a collection of QROFNs and $w = (w_1, w_2, \dots, w_n)^T$ be the weighting vector to satisfy $\sum_{i=1}^n w_i = 1$ and $w_i \geq 0 (i = 1, 2, \dots, n)$. The q-rung orthopair fuzzy weighted averaging (Q-ROFWA) operator is presented as follows:

$$\mathrm{QROFWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\left(\sum_{i=1}^n w_i \mu_i^q \right)^{1/q}, \left(\sum_{i=1}^n w_i v_i^q \right)^{1/q} \right). \tag{30}$$

Some limiting cases are also provided.

- If $q = 1$, then $\mathrm{QROFWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = (\sum_{i=1}^n w_i \mu_i, \sum_{i=1}^n w_i v_i)$.
- If $q = 2$, then $\mathrm{QROFWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = ((\sum_{i=1}^n w_i \mu_i^2)^{1/2}, (\sum_{i=1}^n w_i v_i^2)^{1/2})$.

Definition 18 Let $\tilde{\alpha}_i = (\tilde{\mu}_i, \tilde{v}_i) = ([\mu_i^-, \mu_i^+], [v_i^-, v_i^+])(i = 1, \dots, n)$ be a collection of IVQROFNs solid at q, and the weighted C-IVQROFOWA (WC-IVQROFOWA) operator is defined as follows:

$$\mathrm{WC-IVQROFOWA}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n) = \left(\left(\sum_{i=1}^n w_i (\phi_{\lambda_1}(\tilde{\mu}_i))^q \right)^{1/q}, \left(\sum_{i=1}^n w_i \left(\phi_{\lambda_2}^d(\tilde{v}_i)\right)^q \right)^{1/q} \right). \tag{31}$$

Theorem 13 *Let $\tilde{\alpha}_i = (\tilde{\mu}_i, \tilde{v}_i) = ([\mu_i^-, \mu_i^+], [v_i^-, v_i^+])(i = 1, \dots, n)$ be a collection of IVQROFNs solid at q, then $\mathrm{WC-IVQROFOWA}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$ is a QROFN.*

Theorem 14 *Let $\tilde{\alpha}_i = (\tilde{\mu}_i, \tilde{v}_i) = ([\mu_i^-, \mu_i^+], [v_i^-, v_i^+])(i = 1, \dots, n)$ be a collection of IVQROFNs solid at q, then we have*

$$\begin{aligned} s(\mathrm{WC-IVQROFOWA}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)) &= \sum_{i=1}^n w_i Sco_{\lambda_1, \lambda_2}(\tilde{\alpha}_i) \\ h(\mathrm{WC-IVQROFOWA}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)) &= \sum_{i=1}^n w_i Acc_{\lambda_1, \lambda_2}(\tilde{\alpha}_i) \end{aligned} \tag{32}$$

Proof According to the definition of the score and accuracy functions, we have

$$\begin{aligned} &s(\mathrm{WC-IVQROFOWA}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)) \\ &= \left(\left(\sum_{i=1}^n w_i (\phi_{\lambda_1}(\tilde{\mu}_i))^q \right)^{1/q} \right)^q - \left(\left(\sum_{i=1}^n w_i \left(\phi_{\lambda_2}^d(\tilde{v}_i)\right)^q \right)^{1/q} \right)^q \\ &= \sum_{i=1}^n w_i \left((\phi_{\lambda_1}(\tilde{\mu}_{\tilde{\alpha}_i}))^q - \left(\phi_{\lambda_2}^d(\tilde{v}_{\tilde{\alpha}_i})\right)^q \right) \\ &= \sum_{i=1}^n w_i Sco_{\lambda_1, \lambda_2}(\tilde{\alpha}_i) \end{aligned}$$

Similarly, we can obtain

$$Acc_{\lambda_1, \lambda_2}(\mathrm{WC-IVQROFOWA}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)) = \sum_{i=1}^n w_i Acc_{\lambda_1, \lambda_2}(\tilde{\alpha}_i).$$

The proof is completed. □

Theorem 15 (Monotonicity) *Let* $\tilde{\alpha}_i = \left(\tilde{\mu}_{\tilde{\alpha}_i}, \tilde{\nu}_{\tilde{\alpha}_i}\right)$ *and* $\tilde{\beta}_i = \left(\tilde{\mu}_{\tilde{\beta}_i}, \tilde{\nu}_{\tilde{\beta}_i}\right)(i = 1, \ldots, n)$ *be two collections of IVPFNs solid at q. If* $\tilde{\alpha}_i \prec_\lambda \tilde{\beta}_i (i = 1, \ldots, n)$, *then*

$$\begin{aligned} &\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) \\ &\quad \prec \mathrm{WC-IVQROFOWA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right). \end{aligned} \tag{33}$$

Proof If $\tilde{\alpha}_i \prec_\lambda \tilde{\beta}_i (i = 1, \ldots, n)$,

Case 1. $Sco_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_i\right) < Sco_{\lambda_1,\lambda_2}\left(\tilde{\beta}_i\right)(i = 1, \ldots, n)$, by Theorem 14,

$$\begin{aligned} &s\left(\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right)\right) \\ &\quad - s\left(\mathrm{WC-IVQROFOWA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right)\right) \\ &= \sum_{i=1}^{n} w_i \left(Sco_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_i\right) - Sco_{\lambda_1,\lambda_2}\left(\tilde{\beta}_i\right)\right) < 0 \end{aligned}$$

then $\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) \prec \mathrm{WC-IVQROFOWA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right)$.

Case 2. $Sco_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_i\right) = Sco_{\lambda_1,\lambda_2}\left(\tilde{\beta}_i\right)$ and $Acc_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_i\right) < Acc_{\lambda_1,\lambda_2}\left(\tilde{\beta}_i\right)$, by Theorem 14,

$$\begin{aligned} &h\left(\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right)\right) \\ &\quad - h\left(\mathrm{WC-IVQROFOWA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right)\right) \\ &= \sum_{i=1}^{n} w_i \left(Acc_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_i\right) - Acc_{\lambda_1,\lambda_2}\left(\tilde{\beta}_i\right)\right) < 0 \end{aligned}$$

then $\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) \prec \mathrm{WC-IVQROFOWA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right)$.

The proof is completed. □

Theorem 16 (Idempotency) *Let* $\tilde{\alpha}_i = \left(\tilde{\mu}_i, \tilde{\nu}_i\right) = \left(\left[\mu_i^-, \mu_i^+\right], \left[\nu_i^-, \nu_i^+\right]\right)(i = 1, \ldots, n)$ *be a collection of IVPFNs. If* $\tilde{\alpha}_i = \tilde{\alpha} = \left(\left[\mu^-, \mu^+\right], \left[\nu^-, \nu^+\right]\right)$ *for all i, then*

$$\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) = \tilde{\alpha}. \tag{34}$$

Theorem 17 (Boundness) *Let* $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[\nu_i^-, \nu_i^+\right]\right)$ $(i = 1, \ldots, n)$ *be a collection of IVPFNs,*

$$\tilde{\alpha}_{\min} \le \mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) \le \tilde{\alpha}_{\max}, \tag{35}$$

where $\tilde{\alpha}_{\min} = \left(\min_i\left\{\mu_i^-\right\}, max_i\left\{\nu_i^+\right\}\right)$ *and* $\tilde{\alpha}_{\max} = \left(\max_i\left\{\mu_i^-\right\}, \min_i\left\{\nu_i^+\right\}\right)$.

Proof According to Definition 18 and Theorem 17, we have

$$\begin{aligned} &\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) \\ &\quad = \mathrm{QROFWA}\left(G_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_1\right), \ldots, G_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_n\right)\right) \end{aligned}$$

and

$$\begin{aligned} &\tilde{\alpha}_{\min} \le \left(\mu_i^-, \nu_i^+\right) = G_{0,0}\left(\tilde{\alpha}_i\right) \\ &\quad \le G_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_i\right) \le G_{1,1}\left(\tilde{\alpha}_i\right) = \left(\mu_i^+, \nu_i^-\right) \le \tilde{\alpha}_{\max}. \end{aligned}$$

Therefore,

$$\tilde{\alpha}_{\min} \le \mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) \le \tilde{\alpha}_{\max}.$$

The proof is completed. □

The WC-IVQROFOWA operator proposed in this section is used mainly to aggregate interval-valued q-rung orthopair fuzzy numbers and support the research and construction of the decision-making method in the next section.

4 Decision-making approach based on the WC-IVQROFOWA operator

To select the best alternative for a given application, we put forward a novel framework based on the WC-IVQROFOWA operator for solving fuzzy multiple attributes decision-making problems. Hence, the proposed approach consists of the following main stages: *(1) determining the parameter value of aggregation operators and (2) obtaining the collective matrix, computing the collective evaluation values of alternatives, and obtaining the ranking orders of alternatives.*

In the first stage, the performance rating of alternatives of each attribute provided by decision-makers is linguistic terms expressed in IVQROFNs. Given that the WC-IVQROFOWA operator is used to aggregate the individual

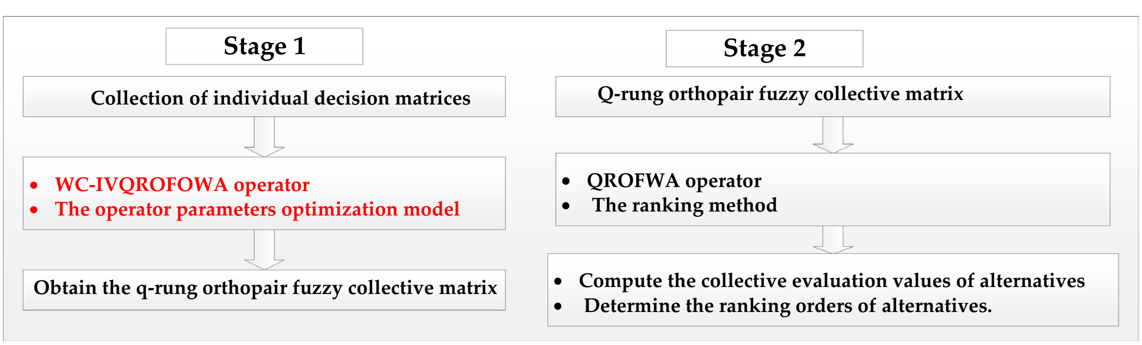

Fig. 1 Flowchart of the proposed decision-making approach

decision matrix, the parameter value of the aggregation operator is calculated by using an optimization model and then the QROF collective matrix can be obtained. *In the second stage,* the q-rung orthopair fuzzy weighted averaging (QROFWA) operator is used to compute the collective evaluation values of alternatives, and the ranking method used to determine the ranking orders of alternatives.

Figure 1 delineates the flowchart of the proposed IVQROF decision-making approach and its detailed explanations are given in the following subsections.

4.1 Determining the parameter value and obtaining the collective matrix

Considering the decision-making problems of general fuzzy multiple attributes with a set of alternatives $\{x_1, \ldots, x_m\}$, which are evaluated by t decision-makers $e_k(k = 1, \ldots, t)$ on a set of attributes $\{c_1, \ldots, c_n\}$. Suppose that $\tilde{R}_k = \left(\tilde{\alpha}_{ij}^k\right)_{m\times n}$ is a decision matrix, where $\tilde{\alpha}_{ij}^k$ is the performance or rating of alternative x_i with respect to attribute c_j and given by the decision-maker e_k and $\tilde{\alpha}_{ij}^k = \left(\tilde{\mu}_{ij}^k, \tilde{v}_{ij}^k\right) = \left(\left[\mu_{ij}^{k+}, \mu_{ij}^{k+}\right], \left[v_{ij}^{k+}, v_{ij}^{k+}\right]\right)$ is an IVQROFN indicating the range of degrees to which the alternative x_i satisfies and dissatisfies attribute c_j, respectively. Given that decision-makers may come from different departments and have different backgrounds and expertise, each decision-maker is given a weight w_k, $k = 1, \ldots, t$, (where $\sum_{k=1}^t w_k = 1$ and $w_k > 0$) to reflect his/her influence on the overall decision results. Accordingly, the IVQROFS theory is adopted to handle the uncertain assessments of alternatives provided by the decision-makers.

(1) Consensus degree (CD) with interval-valued q-rung orthopair fuzzy sets

Before using the WC-IVQROFOWA operator to aggregate t decision matrices $\tilde{R}_k$,$k = 1, \ldots, t$, the parameters λ_1 and λ_2 need to be determined.

Consensus measure plays an important role in the MAGDM process [57–60]. In the process of GDM, different experts having different opinions on certain attributes of evaluation alternatives are unavoidable because of the differences in their professional fields [61]. Therefore, defining the *element, attribute, and group consensus measures* is necessary to identify the decision-making results with low group consensus effectively and lay the foundation for the follow-up model.

This study constructs the consensus evaluation index and defines the CD of each attribute, $c_j(j = 1, \ldots, n)$ based on the proximity index between opinions.

- The CD on the evaluation element $\tilde{\alpha}_{ij}^k$.

Definition 19 Let $\tilde{R}_k = \left(\tilde{\alpha}_{ij}^k\right)_{m\times n}(k \in \{1, \ldots, t\})$ be an individual evaluation matrix and $\tilde{R}_c = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ is the collective matrix aggregated by the WC-IVQROFOWA operator, then their CD on the alternative x_i for the attribute c_j is

$$CE_{ij}\left(\tilde{R}_k, \tilde{R}_c\right) = 1 - \frac{1}{2}\left(\left|\left(\phi_{\lambda_1}\left(\tilde{\mu}_{ij}^k\right)\right)^q - \left(\mu_{ij}\right)^q\right| + \left|\left(\phi_{\lambda_2}^d\left(\tilde{v}_{ij}^k\right)\right)^q - \left(v_{ij}\right)^q\right|\right), \tag{36}$$

where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$, which satisfies

$$\alpha_{ij} = \mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_{ij}^1, \ldots, \tilde{\alpha}_{ij}^t\right) = \left(\left(\sum_{l=1}^t w_l\left(\phi_{\lambda_1}\left(\tilde{\mu}_{ij}^l\right)\right)^q\right)^{1/q}, \left(\sum_{l=1}^t w_l\left(\phi_{\lambda_2}^d\left(\tilde{v}_{ij}^l\right)\right)^q\right)^{1/q}\right) \tag{37}$$

and w_l is the weight of expert e_l, and $G_{\lambda_1,\lambda_2}\left(\tilde{\alpha}_{ij}^k\right) = \left(\phi_{\lambda_1}\left(\tilde{\mu}_{ij}^k\right), \phi_{\lambda_2}^d\left(\tilde{v}_{ij}^k\right)\right)$ satisfies

$$\phi_{\lambda_1}\left(\tilde{\mu}_{ij}^k\right) = \left(\lambda_1\left(\mu_{ij}^{k+}\right)^q + \left(1 - \lambda_1\right)\left(\mu_{ij}^{k-}\right)^q\right)^{1/q}, \tag{38}$$

$$\phi_{\lambda_2}^d\left(\tilde{v}_{ij}^k\right) = \left(\lambda_2\left(v_{ij}^{k-}\right)^q + \left(1 - \lambda_2\right)\left(v_{ij}^{k+}\right)^q\right)^{1/q}. \tag{39}$$

Therefore, we have

$$\begin{aligned} CE_{ij}\left(\tilde{R}_k, \tilde{R}_c\right) &= 1 - \frac{1}{2}\left(\left|\sum_{l=1}^t w_l\left(\lambda_1\left(\left(\mu_{ij}^{k+}\right)^q - \left(\mu_{ij}^{l+}\right)^q\right) + \left(1 - \lambda_1\right)\left(\left(\mu_{ij}^{k-}\right)^q - \left(\mu_{ij}^{l-}\right)^q\right)\right)\right|\right) \\ &\quad - \frac{1}{2}\left(\left|\sum_{l=1}^t w_l\left(\lambda_2\left(\left(v_{ij}^{k-}\right)^q - \left(v_{ij}^{l-}\right)^q\right) + \left(1 - \lambda_2\right)\left(\left(v_{ij}^{k+}\right)^q - \left(v_{ij}^{l+}\right)^q\right)\right)\right|\right) \end{aligned} \tag{40}$$

- The CD on the attribute c_j.

Definition 20 Let $\tilde{R}_k(k \in \{1, \ldots, t\})$ be an individual evaluation matrix and $\tilde{R}_c$ be the collective matrix, then their CD on attribute c_j is

$$CC_j\left(\tilde{R}_k, \tilde{R}_c\right) = \frac{1}{m}\sum_{i=1}^m CE_{ij}\left(\tilde{R}_k, \tilde{R}_c\right), \tag{41}$$

which is the CD on attribute c_j with the group.

Definition 21 Let $\tilde{R}_k (k = 1, \dots, t)$ be a collection of individual evaluation matrices, then the CD on attribute c_j with the group is defined as

$$G\text{-}CC_j = \frac{1}{t}\sum_{k=1}^{t} CC_j(\tilde{R}_k, \tilde{R}_c). \tag{42}$$

Moreover, the CD on all attributes $c_j (j = 1, \dots, n)$ with the group is

$$G\text{-}CD = \frac{1}{n}\sum_{j=1}^{n} G\text{-}CC_j. \tag{43}$$

Notice that

$$G\text{-}CD = \frac{1}{mtn}\sum_{j=1}^{n}\sum_{k=1}^{t}\sum_{i=1}^{m} CE_{ij}(\tilde{R}_k, \tilde{R}_c). \tag{44}$$

(2) Initial parameter optimization model driven by the group consensus measure

Based on the consensus measure defined above, a parameter optimization model is constructed to maximize the consensus measure *G-CD*.

In [40], Yager proposed the following parameterized family of regular increasing monotone (RIM) quantifiers $Q(x) = x^\alpha$ with $\alpha \in [0, +\infty)$ as an attitudinal parameter that reflects the attitude of the expert. In this study, RIM quantifiers $Q(x) = x^\alpha$ are used to construct the following parameter optimization model:

Model (M1)

$$\max G\text{-}CD = \sum_{j=1}^{n}\sum_{k=1}^{t}\sum_{i=1}^{m} CE_{ij}(\tilde{R}_k, \tilde{R}_c)/mtn$$

$$s.t.\begin{cases} 0 \le \kappa_\mu^* \le 1, 0 \le \kappa_v^* \le 1, \\ \lambda_\mu^* = \int_0^1 x^{\kappa_\mu^*}dx = \frac{1}{1+\kappa_\mu^*}, \lambda_v^* = \int_0^1 x^{\kappa_v^*}dx = \frac{1}{1+\kappa_v^*} \\ CE_{ij}(\tilde{R}_k, \tilde{R}_c) = 1 - \frac{1}{2}\Big(\Big|\Big(\phi_{\lambda_\mu^*}\big(\tilde{\mu}_{ij}^k\big)\Big)^q - (\mu_{ij})^q\Big| \\ \quad + \Big|\Big(\phi_{\lambda_v^*}^d\big(\tilde{v}_{ij}^k\big)\Big)^q - (v_{ij})^q\Big|\Big) \\ i = 1, \dots, m; j = 1, \dots, n; k = 1, \dots, t. \end{cases} \tag{45}$$

The attributes or criteria are the basis of evaluation alternatives. Different experts may have different opinions on the determined attributes or criteria in the evaluation process. In the original model M1, we set the uniform parameters (λ_μ^* and λ_v^*), and assume that $0.5 \le \lambda_\mu^*, \lambda_v^* \le 1$. We wish to obtain the maximum consensus by adjusting the parameters flexibly to improve the credibility of the decision results based on this model.

(3) Determining the parameter value and obtaining the collective matrix

Using the proposed original optimization model, the following Algorithm 1 is constructed to determine the parameter values and obtain the collective matrix $\tilde{R}_c$.

Algorithm 1 Obtain the collective matrix $\tilde{R}_c$ based on the optimization model M1

Input: a collection of individual evaluation matrices $\tilde{R}_k = \left(\tilde{\alpha}_{ij}^k\right)_{m\times n} (k = 1, \cdots, t)$ solid at q

Step1: Obtain the consensus measure $G\text{-}CD^*$ and uniform parameters $\left(\lambda_\mu^*, \lambda_v^*\right)$ by solving model **M1**; Set the threshold value θ. If $G\text{-}CD^* > \theta$, go to Step 5 otherwise, go to Step 2;

Step2: Compute and rank attribute consensus measure $G\text{-}CC_{\sigma(j)} \ge G\text{-}CC_{\sigma(j+1)}\ (j = 1, \cdots, n)$;

Step3: Replace the uniform parameters $\left(\lambda_\mu^*, \lambda_v^*\right)$ with element parameters $\left(\lambda_\mu^{ijk}, \lambda_v^{ijk}\right)$;

Step4: Execution iteration for $\lambda_{\mu(z)}^{i\sigma(j)k}$, $\lambda_{v(z)}^{i\sigma(j)k}$, $G\text{-}CD^{(z)}$, $z = 1, \cdots, n$, and let $G\text{-}CD^{(1)} = G\text{-}CD^*$. $\lambda_{\mu(1)}^{i\sigma(j)k} = \lambda_\mu^*$, $\lambda_{v(1)}^{i\sigma(j)k} = \lambda_v^*$;

Step 4.1: If $G\text{-}CD^{(z)} < \theta$, then set $\begin{cases} \lambda_{\mu(z)}^{i\sigma(j)k} = \lambda_\mu^*, \lambda_{v(z)}^{i\sigma(j)k} = \lambda_v^*, j = 1, \cdots, n - z \\ \lambda_{\mu(z)}^{i\sigma(j)k}, \lambda_{v(z)}^{i\sigma(j)k} \in [0, 1], else \end{cases}$, and go to Step 4.2

Otherwise, stop iteration, output $G\text{-}CD^{(z)}$, $\left(\lambda_{\mu(z)}^{i\sigma(j)k}, \lambda_{v(z)}^{i\sigma(j)k}\right)$and go to Step 5

Step 4.2: Obtain $G\text{-}CD^{(z+1)}$ and parameters $\left(\lambda_{\mu(z+1)}^{i\sigma(j)k}, \lambda_{v(z+1)}^{i\sigma(j)k}\right)$ by solving model M1 If $z + 1 < n$ then go to Step 4.1 and let $z = z + 1$ Otherwise, stop iteration, output $G\text{-}CD^{(n)}$, $\left(\lambda_{\mu(n)}^{i\sigma(j)k}, \lambda_{v(n)}^{i\sigma(j)k}\right)$, and go to Step 5;

Step5: Compute the collective matrix $\tilde{R}_c$.

Output: the consensus measure $G\text{-}CD$ and the collective matrix $\tilde{R}_c$.

4.2 Computing the comprehensive evaluation values and ranking orders of alternatives

From Sect. 4.2, the collective matrix $\tilde{R}_c = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ is obtained, where $\tilde{\alpha}_{ij} = \left(\mu_{ij}, v_{ij}\right)$. Assume that the weights of attributes c_j are $\omega_j (j = 1, \ldots, n)$, where $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j > 0$ for all $j = 1, \ldots, n$. Use the QROFWA operator to compute the comprehensive evaluation values α_i of alternatives $X_i (i = 1, \ldots, m)$, where

$$\begin{aligned}\tilde{\alpha}_i &= \mathrm{QROFWA}\left(\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2}, \ldots, \tilde{\alpha}_{in}\right) = \left(\mu_i, v_i\right) \\ &= \left(\left(\sum_{j=1}^{n} \omega_j \mu_{ij}^q\right)^{1/q}, \left(\sum_{j=1}^{n} \omega_j v_{ij}^q\right)^{1/q}\right).\end{aligned} \tag{46}$$

Compute the score $s\left(\alpha_i\right) = \mu_i^q - v_i^q$ and accuracy $h\left(\alpha_i\right) = \mu_i^q + v_i^q$ values as follows:

(i) If $s\left(\alpha_{i_1}\right) < s\left(\alpha_{i_2}\right)$, then $\alpha_{i_1} \prec \alpha_{i_2}$;
(ii) If $s\left(\alpha_{i_1}\right) = s\left(\alpha_{i_2}\right)$ and $h\left(\alpha_{i_1}\right) < h\left(\alpha_{i_2}\right)$, then $\alpha_{i_1} \prec \alpha_{i_2}$;
(iii) If $s\left(\alpha_{i_1}\right) = s\left(\alpha_{i_2}\right)$ and $h\left(\alpha_{i_1}\right) = h\left(\alpha_{i_2}\right)$, then $\alpha_{i_1} = \alpha_{i_2}$.

According to the above method, rank the comprehensive evaluation values $\alpha_{\sigma(1)} \prec \cdots \prec \alpha_{\sigma(n)}$. Then, all the alternatives are ranked based on the increasing order of their related comprehensive evaluation values (Fig. 2).

5 Numerical examples

In this section, the background in [62] is utilized to demonstrate the implementation process and effectiveness of the proposed decision approach in the previous section, which is an evaluation problem of SmartWatch design.

A team of six experts and decision-makers is gathered to conduct the performance assessment and determine the most suitable alternative. These decision-makers include a product manager, a customer manager, a product designer, a senior user, an R&D manager, and a senior expert. The authors also assume the relative weights of experts are $w = (0.1, 0.2, 0.1, 0.4, 0.1, 0.1)^T$. The semantic difference method is used in the evaluation stage, as shown in Fig. 3. These decision-makers expressed their preferences on the rating of candidate design alternatives of SmartWatch appearance (Fig. 4) with respect to the four essential attributes ("Fashionability (C_1) ", "Science and Technology (C_2)", "Friendliness (C_3)," and "Comfort (C_4)") by using linguistic variables according to their domain knowledge. Table 1 shows the linguistic variables used by the decision-makers and their corresponding IVQROFNs (solid at $q = 2$). The obtained linguistic assessments of the 10 alternatives provided by the decision-makers are listed in Tables 2, 3 and 4.

We then apply the developed method in this study to derive the order relation of $A_i (i = 1, 2, \ldots, 10)$. The necessary steps of the method are provided as follows.

Stage 1. Obtain the q-rung orthopair fuzzy collective matrix.

- **Stage 1.1.** According to the linguistic variables and their corresponding IVQROFNs shown in Table 1 and the ratings of design alternatives with respect to attribute by the decision-makers shown in Table 2, 3 and 4, the IVQROFNs of the decision matrix assigned by the decision-makers are obtained through $\tilde{R}_k = \left(\tilde{\alpha}_{ij}^k\right)_{10\times 4} (k \in \{1, \ldots, 6\})$. These fuzzy matrices are readily available in Tables 1, 2, 3 and 4. Hence, no specific data are provided in this study.
- **Stage 1.2.** Obtain a collective matrix $\tilde{R}_c$ (Table 5) based on **Algorithm 2**.

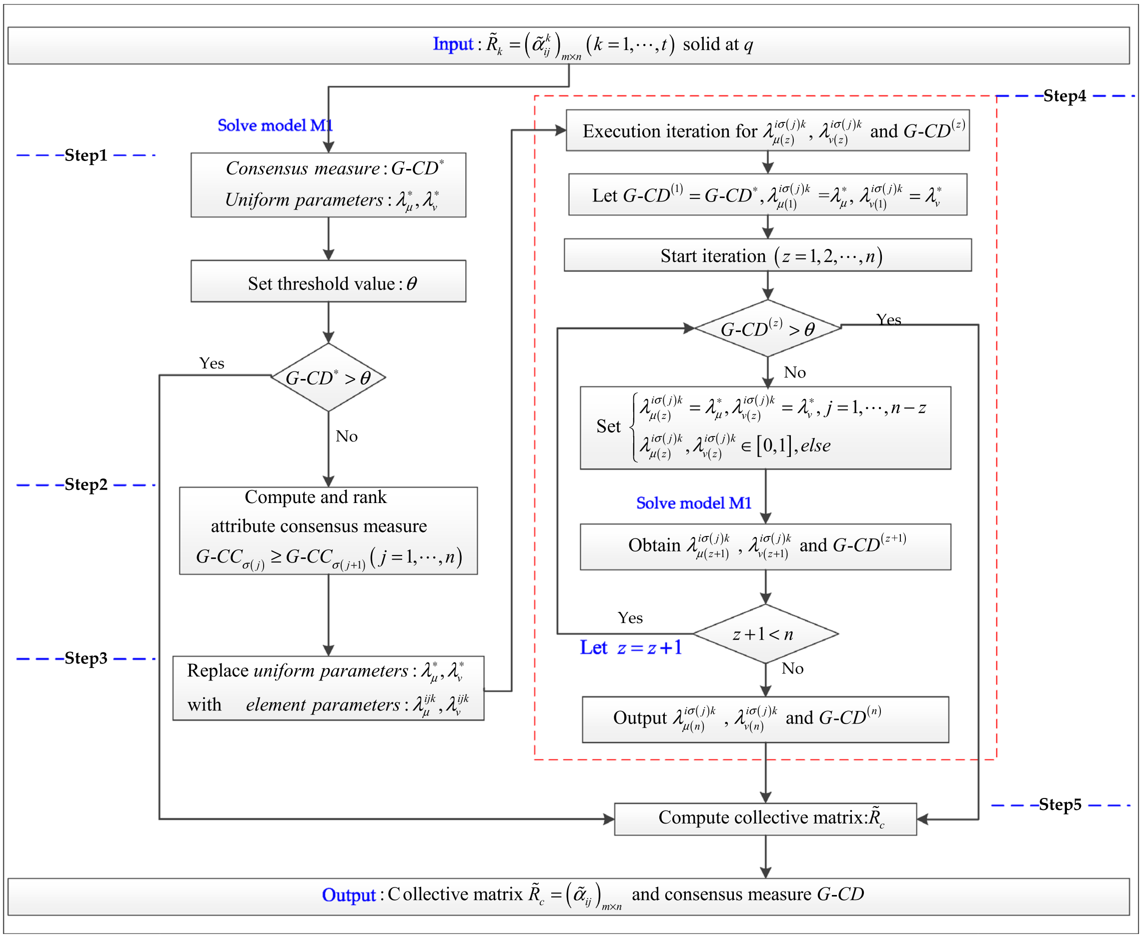

Fig. 2 Flowchart of Algorithm 1

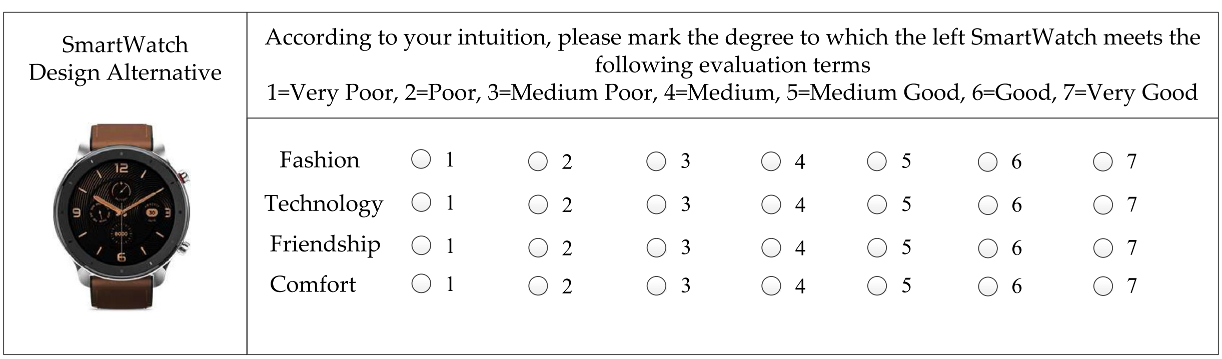

Fig. 3 Semantic difference method

Algorithm 2 Obtain the collective matrix $\tilde{R}_c$ based on the optimization model M1

Input: a collection of individual evaluation matrices $\tilde{R}_k = \left(\tilde{\alpha}_{ij}^k\right)_{10\times 4} (k = 1, \cdots, 6)$ solid at $q = 2$

Step 1: Solve model **M1** and obtain $G\text{-}CD^* = 0.7849$ and $\left(\lambda_\mu^*, \lambda_v^*\right) = (1.0, 0.5)$. Set the threshold value to $\theta = 0.8$. Given that $G\text{-}CD^* < \theta$, then go to Step 2;

Step 2: Compute and rank attribute consensus measure $G\text{-}CC_4 = 0.7732 < G\text{-}CC_2 = 0.7742 < G\text{-}CC_3 = 0.7941 < G\text{-}CC_1 = 0.7980$;

Step 3: Replace the uniform parameters $\left(\lambda_\mu^*, \lambda_v^*\right)$ with element parameters $\left(\lambda_\mu^{ijk}, \lambda_v^{ijk}\right)$;

Step 4: Execute the iteration, and let $G\text{-}CD^{(1)} = G\text{-}CD^*$, $\lambda_{\mu(1)}^{i\sigma(j)k} = \lambda_\mu^*$, $\lambda_{v(1)}^{i\sigma(j)k} = \lambda_v^*$;

Step 4.1-1: $G\text{-}CD^{(1)} < \theta$, then set $\begin{cases} \lambda_{\mu(1)}^{ijk} = \lambda_\mu^*, \lambda_{v(1)}^{ijk} = \lambda_v^*, j = 1, 2, 3 \\ \lambda_{\mu(1)}^{ijk}, \lambda_{v(1)}^{ijk} \in [0, 1], \ j = 4 \end{cases}$, and go to Step 4.2;

Step 4.2-1: Obtain $G\text{-}CD^{(2)} = 0.7948$ by solving model M1. since $2 < n = 4$, then go to Step 4.1 and let $z = 2$

Step 4.1-2: $G\text{-}CD^{(2)} < \theta$, then set $\begin{cases} \lambda_{\mu(2)}^{ijk} = \lambda_\mu^*, \lambda_{v(2)}^{ijk} = \lambda_v^*, j = 1, 3 \\ \lambda_{\mu(2)}^{ijk}, \lambda_{v(2)}^{ijk} \in [0, 1], \ j = 2, 4 \end{cases}$, and go to Step 4.2;

Step 4.2-2: Obtain $G\text{-}CD^{(3)} = 0.8042$ by solving model M1. Given that $3 < n = 4$, go to Step 4.1 and let $z = 3$;

Step 4.1-3: $G\text{-}CD^{(3)} > \theta$, then stop the iteration and output $G\text{-}CD^{(z)}$, and go to Step 5;

Step 5: Compute the collective matrix $\tilde{R}_c$ (Table 5).

Output: the consensus measure $G\text{-}CD$ and the collective matrix $\tilde{R}_c$.

Fig. 4 SmartWatch appearance design alternatives

Table 1 Linguistic terms and their corresponding IVQROFNs [63]

Linguistic terms	Corresponding IVQROFNs
Very good (VG)	([0.80,0.95], [0.00,0.15])
Good (G)	([0.70,0.80], [0.15,0.25])
Medium good (MG)	([0.55,0.70], [0.25,0.40])
Medium (M)	([0.45,0.55], [0.40,0.55])
Medium poor (MP)	([0.30,0.45], [0.55,0.70])
Poor (P)	([0.20,0.30], [0.70,0.80])
Very poor (VP)	([0.00,0.20], [0.80,0.95])

Thus, according to the proposed method, the suggested design alternative is A_1.

The relationship between the parameters and consensus measure is analyzed according to Algorithm 2.

- According to Step 1, the consensus measure $G\text{-}CD^{*} = 0.7849$ and parameter values $\left(\lambda^{*}_{\mu}, \lambda^{*}_{v}\right) = (1.0, 0.5)$ under the initial iteration state $(z = 1)$ can be obtained. Attribute consensus measure $G\text{-}CC^{(1)}_{j}$ and group CD $G\text{-}CD^{(1)} = 0.7849$ obtained are lower than the threshold value $\theta = 0.8$. The relevant results are presented in Table 6.
- Given that the consensus measure of the attribute C_4 is the lowest ($G\text{-}CC^{(1)}_{4} = 0.7732$), the parameters of C_4 are first released with constraints $\lambda^{i4k}_{\mu(1)}, \lambda^{i4k}_{v(1)} \in [0, 1]$. By solving the model M1, the new group consensus measure ($G\text{-}CD^{(2)} = 0.7948$) and the corresponding consensus measure ($G\text{-}CC^{(2)}_{4} = 0.8130$) of the attribute C_4 are obtained. The relevant results can be found in Table 6. The parameter values $\lambda^{i2k}_{\mu(2)}$ and $\lambda^{i2k}_{v(2)}$ are shown in Table 7.
- After the second iteration, the group consensus measure $G\text{-}CD^{(2)} = 0.7948$ is lower than the threshold and the consensus measure of the attribute C_2 is the lowest. Hence, the parameter constraints of attribute C_2 are released ($\lambda^{i2k}_{\mu(2)}, \lambda^{i2k}_{v(2)} \in [0, 1]$). By solving the model M1, the new group consensus measure ($G\text{-}CD^{(3)} = 0.8042$) and the corresponding consensus measure ($G\text{-}CC^{(3)}_{2} = 0.8117$) of the attribute C_2 are obtained. The relevant results can be seen in Table 6. The parameter values $\lambda^{i2k}_{\mu(3)}$ and $\lambda^{i2k}_{v(3)}$ are shown in Table 8, respectively. The new group consensus $G\text{-}CD^{(3)} = 0.8042$ is greater than the threshold, and thus, the iteration stops, and the calculation results are considered the output.

Table 2 Ratings of alternatives with respect to evaluation attribute

e_1	C_1	C_2	C_3	C_4	e_2	C_1	C_2	C_3	C_4
A_1	VG	M	MP	MG	A_1	G	MP	MG	VG
A_2	M	MG	VP	P	A_2	VP	M	VG	P
A_3	MP	VG	M	G	A_3	MP	P	MG	G
A_4	MG	M	G	VG	A_4	MG	VG	MP	MG
A_5	G	P	VG	M	A_5	VG	M	MG	G
A_6	MP	P	MP	G	A_6	MP	P	MP	M
A_7	P	VG	M	MG	A_7	M	MP	G	P
A_8	MG	VP	G	P	A_8	G	MG	G	P
A_9	VG	MG	M	VG	A_9	VG	MP	P	G
A_{10}	MG	MP	VG	MP	A_{10}	MP	VP	MG	P

Stage 2. Obtain the collective evaluation values and ranking orders of alternatives.

- **Stage 2.1.** Use the QROFWA operator to compute comprehensive values $\alpha_i (i = 1, \ldots, 10)$.

$$\begin{aligned} &\alpha_1 = (0.7173, 0.3597), \alpha_2 = (0.5828, 0.5531),\\ &\alpha_3 = (0.6131, 0.4918), \alpha_4 = (0.6558, 0.4216),\\ &\alpha_5 = (0.6278, 0.4704), \alpha_6 = (0.6252, 0.4817),\\ &\alpha_7 = (0.5640, 0.5145), \alpha_8 = (0.5811, 0.4896),\\ &\alpha_9 = (0.6897, 0.4099), \alpha_{10} = (0.5424, 0.5627) \end{aligned} \tag{47}$$

- **Stage 2.2.** Rank the comprehensive evaluation values $\alpha_i (i = 1, \ldots, 10)$ as follows:

$$\begin{aligned} &s\left(\alpha_1\right) = 0.3850, s\left(\tilde{\alpha}_2\right) = 0.0337, s\left(\tilde{\alpha}_3\right) = 0.1340,\\ &s\left(\tilde{\alpha}_4\right) = 0.2523, s\left(\tilde{\alpha}_5\right) = 0.1729\\ &s\left(\tilde{\alpha}_6\right) = 0.1588, s\left(\alpha_7\right) = 0.0534, s\left(\tilde{\alpha}_8\right) = 0.0980,\\ &s\left(\tilde{\alpha}_9\right) = 0.3077, s\left(\tilde{\alpha}_{10}\right) = 0.2523 \end{aligned} \tag{48}$$

Obtain the rank of $A_i (i = 1, \ldots, 10)$:

$$A_2 \prec A_7 \prec A_8 \prec A_3 \prec A_6 \prec A_5 \prec A_{10} \sim A_4 \prec A_9 \prec A_1. \tag{49}$$

Table 3 Ratings of alternatives with respect to evaluation attribute

e_3	C_1	C_2	C_3	C_4	e_4	C_1	C_2	C_3	C_4
A_1	G	VG	MP	G	A_1	MP	MG	VG	G
A_2	P	G	VP	MP	A_2	P	MP	G	VG
A_3	MG	G	MG	M	A_3	VG	P	M	P
A_4	VG	MG	VG	MP	A_4	M	P	G	MG
A_5	MP	VP	MP	VG	A_5	G	MP	MG	P
A_6	M	MP	MG	M	A_6	P	VG	VG	MG
A_7	MG	P	M	G	A_7	M	G	MP	P
A_8	G	M	MP	MG	A_8	P	M	MG	MP
A_9	P	G	MG	G	A_9	MG	G	MP	VG
A_{10}	M	VG	G	MG	A_{10}	VP	P	MG	MP

Table 4 Ratings of alternatives with respect to evaluation attribute

e_5	C_1	C_2	C_3	C_4	e_6	C_1	C_2	C_3	C_4
A_1	VG	M	P	M	A_1	MG	G	G	VG
A_2	MP	G	P	MG	A_2	P	P	MG	MG
A_3	G	VG	G	MP	A_3	MP	MP	M	MP
A_4	G	MP	VG	P	A_4	MG	MP	MG	MP
A_5	M	M	P	G	A_5	G	MP	G	P
A_6	MG	M	G	MP	A_6	P	G	G	MP
A_7	G	G	MP	MG	A_7	MP	MG	P	P
A_8	VG	VP	MP	MG	A_8	MG	M	G	MG
A_9	MP	MG	G	M	A_9	G	G	P	MP
A_{10}	MG	M	MG	MG	A_{10}	P	MP	G	G

Table 5 Pythagorean fuzzy collective matrix $\tilde{R}_c$

	C_1	C_2	C_3	C_4
A_1	(0.7089, 0.4242)	(0.6242, 0.3732)	(0.7566, 0.4022)	(0.7702, 0.1933)
A_2	(0.3354, 0.7431)	(0.5495, 0.4834)	(0.7089, 0.4869)	(0.6647, 0.4479)
A_3	(0.7450, 0.4160)	(0.5326, 0.5765)	(0.6265, 0.4136)	(0.5220, 0.5395)
A_4	(0.6865, 0.3702)	(0.5463, 0.5392)	(0.7681, 0.3314)	(0.6003, 0.4165)
A_5	(0.7842, 0.2945)	(0.4042, 0.5958)	(0.6910, 0.4040)	(0.5662, 0.5292)
A_6	(0.4301, 0.6569)	(0.6633, 0.4515)	(0.7728, 0.3715)	(0.5827, 0.3934)
A_7	(0.5690, 0.4952)	(0.6727, 0.3750)	(0.5466, 0.5520)	(0.4443, 0.6067)
A_8	(0.6452, 0.5086)	(0.4859, 0.5141)	(0.7018, 0.3693)	(0.4538, 0.5474)
A_9	(0.7483, 0.3785)	(0.6874, 0.3250)	(0.5012, 0.6018)	(0.7868, 0.2471)
A_{10}	(0.4402, 0.6959)	(0.4266, 0.6521)	(0.7492, 0.2872)	(0.4899, 0.5238)

Table 6 Consensus measure with the different iteration processes

Consensus measure	First iteration $z = 1$	Second iteration $z = 2$	Third iteration $z = 3$	Threshold
$G\text{-}CC_1^{(z)}$	0.7980	0.7980	0.7980	0.80
$G\text{-}CC_2^{(z)}$	0.7742	0.7742	0.8117	
$G\text{-}CC_3^{(z)}$	0.7941	0.7941	0.7941	
$G\text{-}CC_4^{(z)}$	0.7732	0.8130	0.8130	
$G\text{-}CD^{(z)}$	$0.7849<0.8$	$0.7948<0.8$	$0.8042>0.8$	

Table 7 Membership/non-membership element parameters $\left(\lambda_\mu^{ijk}, \lambda_v^{ijk}\right)$ with respect to attribute C_4

C_4	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
e_1	(1.0,1.0)	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)
e_2	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(0.87,0.5)	(1.0,1.0)
e_3	(0.7,0.59)	(1.0,1.0)	(0.72,0.5)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(0.87,0.5)	(0.5,0.5)
e_4	(0.5,0.99)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)	(0.6,1.0)	(0.5,0.5)	(0.67,1.0)
e_5	(1.0,1.0)	(0.76,0.5)	(1.0,1.0)	(1.0,1.0)	(0.5,0.5)	(0.84,1.0)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)
e_6	(0.5,0.5)	(0.76,0.5)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(0.84,1.0)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)

Table 8 Membership/non-membership element parameters $\left(\lambda_\mu^{ijk}, \lambda_v^{ijk}\right)$ with respect to attribute C_2

C_4	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
e_1	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(0.96,0.5)	(1.0,1.0)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)	(0.92,0.5)	(0.84,0.5)
e_2	(0.98,1.0)	(0.99,0.5)	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(1.0,1.0)	(0.5,0.5)	(1.0,1.0)	(1.0,1.0)
e_3	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
e_4	(0.5,0.5)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(0.54,0.72)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)
e_5	(1.0,1.0)	(0.5,0.5)	(0.5,0.5)	(1.0,1.0)	(0.5,0.5)	(1.0,0.66)	(0.5,0.5)	(1.0,1.0)	(0.92,0.5)	(0.5,0.5)
e_6	(0.5,0.5)	(1.0,0.82)	(1.0,1.0)	(1.0,1.0)	(0.68,0.68)	(0.5,0.5)	(0.82,0.5)	(0.5,0.5)	(0.5,0.5)	(0.84,0.5)

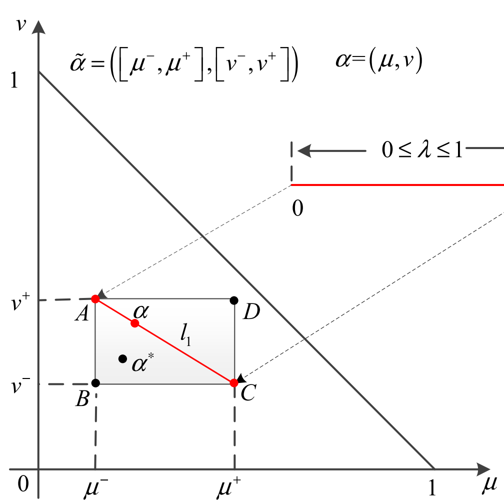

Fig. 5 IVIFN $\tilde{\alpha}$ and its related $B\text{-}F_Q(\tilde{\alpha})$

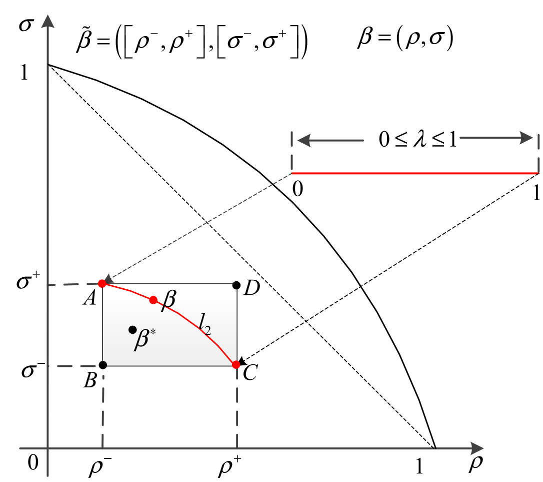

Fig. 6 IVPFN $\tilde{\beta}$ and its related $E_Q(\tilde{\beta})$

6 Comparison analysis

6.1 Existing continuous interval-valued fuzzy aggregation operators

- C-IVIFOWA and C-IVPFOWQA operators

Definition 22 [48, 49] Let $\tilde{\alpha} = (\tilde{\mu}, \tilde{v}) = \left([\mu^-, \mu^+], [v^-, v^+]\right)$ be an IVIFN and the continuous interval-valued intuitionistic fuzzy ordered weighted averaging (C-IVIFOWA) operator $A\text{-}F_Q$ is defined as follows:

$$A\text{-}F_Q(\tilde{\alpha}) = \left(f_\lambda(\tilde{\mu}), f_\lambda(\tilde{v})\right) = \left((1-\lambda)\mu^- + \lambda\mu^+, (1-\lambda)v^- + \lambda v^+\right). \tag{50}$$

However, some examples show the C-IVIFOWA operator fails in boundary accessibility and monotonicity with respect to the BUM function, and the relevant details, which are not elaborated in this study, can be found in references [48, 49]. The improved C-IVIFOWA operator $B\text{-}F_Q$ is proposed in [48, 49].

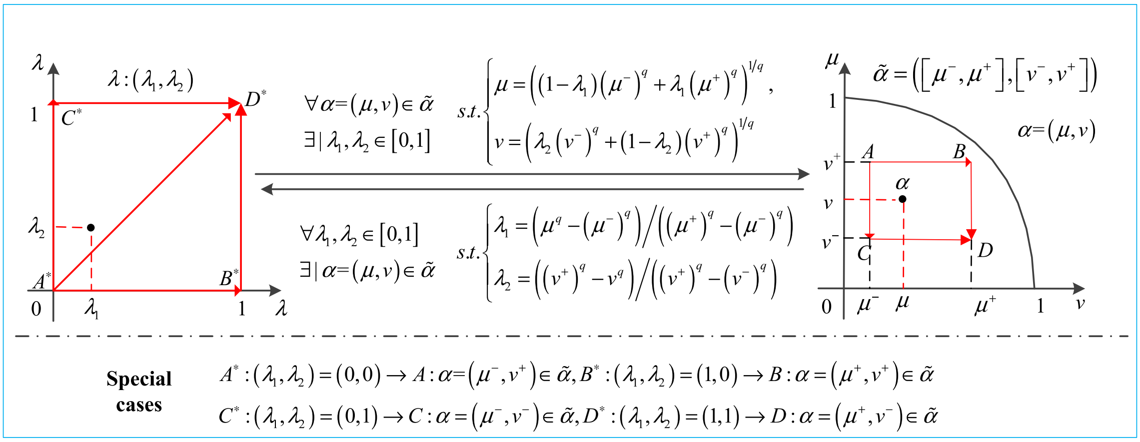

Fig. 7 IVQROFN $\tilde{\alpha}$ and its related $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$

Table 9 Comparative analysis of the three aggregation operators with given parameters

Weights/operators	IVIFNs ($q=1$)/aggregation results	IVIPNs ($q=2$)/aggregation results
$w_1=0.01$	$\tilde{\alpha}_1$=([1.0, 1.0], [0.0, 0.0])	$\tilde{\beta}_1$=([1.0, 1.0], [0.0, 0.0])
$w_2=0.29$	$\tilde{\alpha}_2$=([0.1, 0.3], [0.6, 0.7])	$\tilde{\beta}_2$=([0.5, 0.6], [0.8, 0.8])
$w_3=0.30$	$\tilde{\alpha}_3$=([0.2, 0.2], [0.7, 0.8])	$\tilde{\beta}_3$=([0.3, 0.4], [0.7, 0.8])
$w_4=0.40$	$\tilde{\alpha}_4$=([0.0, 0.1], [0.8, 0.9])	$\tilde{\beta}_4$=([0.2, 0.3], [0.6, 0.9])
WC-IVIFOWA [48, 49] (with $\lambda=0.5$)	(1.0, 0.0)	********
WC-IVPFOWQA [52] (with $\lambda=0.5$)	(1.0, 0.0)	(1.0, 0.0)
WC-IVQROFOWA (With $\lambda_1=\lambda_2=0.5$)	(0.1480, 0.7535)	(0.3782, 0.7634)

Definition 23 [48, 49] Let $\tilde{\alpha}=(\tilde{\mu},\tilde{\nu})=\left([\mu^-,\mu^+],[\nu^-,\nu^+]\right)$ be an IVIFN and the improved C-IVIFOWA operator $B\text{-}F_Q$ is defined as follows:

$$B\text{-}F_Q(\tilde{\alpha})=\left(f_\lambda(\tilde{\mu}),f_{1-\lambda}(\tilde{\nu})\right)=\left((1-\lambda)\mu^-+\lambda\mu^+,\lambda\nu^-+(1-\lambda)\nu^+\right). \tag{51}$$

For convenience, we let $B\text{-}F_Q(\tilde{\alpha})=(\mu,\nu)$. According to Definition 23, the relationship between membership degree μ and non-membership degree ν can be obtained as follows:

$$\begin{cases}\mu=(1-\lambda)\mu^-+\lambda\mu^+\\ \nu=\lambda\nu^-+(1-\lambda)\nu^+\end{cases}\Rightarrow l_1:\begin{cases}\nu=-\dfrac{\nu^+-\nu^-}{\mu^+-\mu^-}\times\mu+\dfrac{\mu^+\nu^+-\mu^-\nu^-}{\mu^+-\mu^-}\\ \mu\in[\mu^-,\mu^+],\nu\in[\nu^-,\nu^+]\end{cases} \tag{52}$$

The above results show all IFNs obtained by the operator $B\text{-}F_Q(\tilde{\alpha})$ that form the line segment l_1 when all parameters in the unit intervals are taken $0\le\lambda\le1$. The details are illustrated in Fig. 5.

Definition 24 [52] Let $\tilde{\beta}=(\tilde{\rho},\tilde{\sigma})=\left([\rho^-,\rho^+],[\sigma^-,\sigma^+]\right)$ be an IVPFN and the continuous interval-valued Pythagorean fuzzy ordered weighted averaging quadratic (C-IVIFOWQA) operator is defined as follows:

$$E_Q(\tilde{\beta})=\left(g_\lambda(\tilde{\rho}),g_{1-\lambda}(\tilde{\sigma})\right)=\left(\sqrt{(1-\lambda)(\rho^-)^2+\lambda(\rho^+)^2},\sqrt{\lambda(\sigma^-)^2+(1-\lambda)(\sigma^+)^2}\right). \tag{53}$$

For convenience, we let $E_Q(\tilde{\beta})=(\rho,\sigma)$. According to Definition 24, the relationship between membership degree ρ and non-membership degree σ can be obtained as follows:

$$\begin{cases}\rho=\sqrt{(1-\lambda)(\rho^-)^2+\lambda(\rho^+)^2}\\ \sigma=\sqrt{\lambda(\sigma^-)^2+(1-\lambda)(\sigma^+)^2}\end{cases}\Rightarrow l_2:\begin{cases}y=\sqrt{-\dfrac{(\sigma^+)^2-(\sigma^-)^2}{(\rho^+)^2-(\rho^-)^2}\rho^2+\dfrac{(\rho^+)^2(\sigma^+)^2-(\rho^-)^2(\sigma^-)^2}{(\rho^+)^2-(\rho^-)^2}}\\ \rho\in[\rho^-,\rho^+],\sigma\in[\sigma^-,\sigma^+]\end{cases} \tag{54}$$

The above results show that all PFNs obtained by the operator $E_Q(\tilde{\beta})$ form the curve l_2 when all the parameters in the unit intervals are taken $0\le\lambda\le1$. The details are demonstrated in Fig. 6.

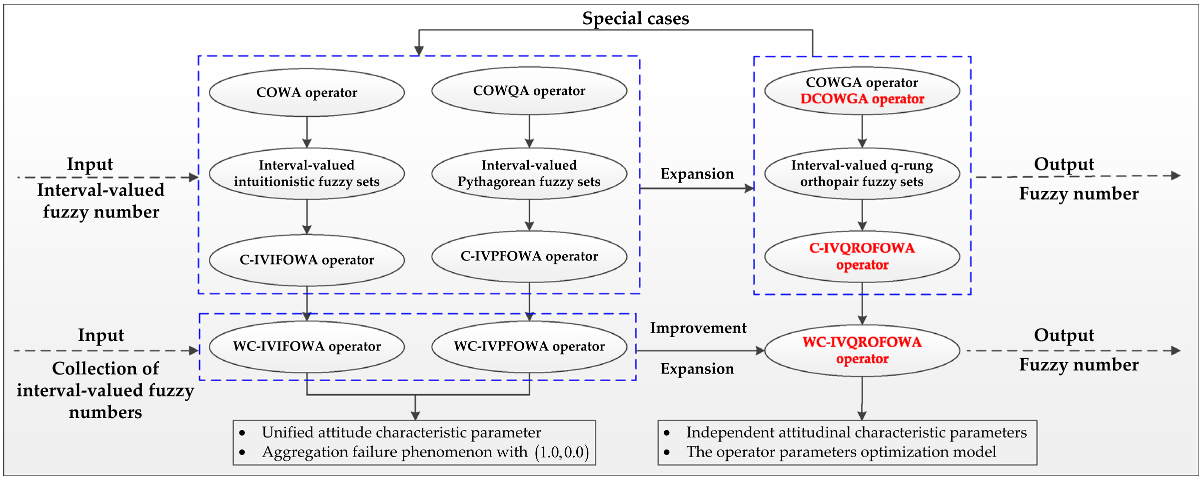

Fig. 8 Construction ideas of the continuous fuzzy operators

According to Figs. 5 and 6, we can summarize the shortcomings of the C-IVIFOWA and C-IVPFOWQA operators as follows:

- From Fig. 5, for all $\alpha \in l_1$, a parameter value $\lambda \in [0,1]$ always satisfy $B\text{-}F_Q(\tilde{\alpha}) = \alpha$; for all $\alpha^* \notin l_1$ and $\alpha^* \in \tilde{\alpha}$, no parameter value $\lambda \in [0,1]$ can satisfy $B\text{-}F_Q(\tilde{\alpha}) = \alpha^*$. However, all IFNs on all line l_1 ($\alpha \in l_1$) account for only a very small part of the region of IVIFN $\tilde{\alpha}$ (Rectangular ABCD).
- From Fig. 6, for all $\beta \in l_2$, a parameter value $\lambda \in [0,1]$ always satisfy $E_Q(\tilde{\beta}) = \beta$; for all $\beta^* \notin l_2$ and $\beta^* \in \tilde{\beta}$, no parameter value $\lambda \in [0,1]$ can satisfy $E_Q(\tilde{\beta}) = \beta^*$. However, all PFNs on all line l_2 ($\beta \in l_2$) account for only a very small part of the region of IVPFN $\tilde{\beta}$ (Rectangular ABCD).

Then, the continuous interval-valued q-rung orthopair fuzzy ordered weighted averaging (C-IVQROFOWA) operator constructed in this study is analyzed further. Let $\tilde{\alpha} = \left(\left[\mu^-, \mu^+\right], \left[\nu^-, \nu^+\right]\right)$ be an IVQROFN solid at q. By using Definition 15, we have

$$G_{\lambda_1,\lambda_2}(\tilde{\alpha}) = \left(\phi_{\lambda_1}\left(\left[\mu^-, \mu^+\right]\right), \phi^d_{\lambda_2}\left(\left[\nu^-, \nu^+\right]\right)\right) = \left(\left(\left(1-\lambda_1\right)(\mu^-)^q + \lambda_1\left(\mu^+\right)^q\right)^{1/q}, \left(\lambda_2(\nu^-)^q + \left(1-\lambda_2\right)\left(\nu^+\right)^q\right)^{1/q}\right). \tag{55}$$

First, the relationship between parameters ($\lambda_1, \lambda_2 \in [0,1]$) and the C-IVQROFOWA operator is discussed in Theorem 18. Figure 7 illustrates the results of Theorem 18.

Theorem 18 *Let* $\tilde{\alpha} = \left(\left[\mu^-, \mu^+\right], \left[\nu^-, \nu^+\right]\right)$ *be an IVQROFN solid at* q, $\lambda_1, \lambda_2 \in [0,1]$, *then*

(i) $\forall \alpha=(\mu,\nu) \in \tilde{\alpha}, \exists | \lambda_1, \lambda_2 \in [0,1]$,

$$s.t. \begin{cases} \mu = \left(\left(1-\lambda_1\right)(\mu^-)^q + \lambda_1\left(\mu^+\right)^q\right)^{1/q}, \\ \nu = \left(\lambda_2(\nu^-)^q + \left(1-\lambda_2\right)\left(\nu^+\right)^q\right)^{1/q} \end{cases}$$

(ii) $\forall \lambda_1, \lambda_2 \in [0,1], \exists | \alpha=(\mu,\nu) \in \tilde{\alpha}$,

$$s.t. \begin{cases} \lambda_1 = \left(\mu^q - (\mu^-)^q\right) / \left(\left(\mu^+\right)^q - (\mu^-)^q\right) \\ \lambda_2 = \left(\left(\nu^+\right)^q - \nu^q\right) / \left(\left(\nu^+\right)^q - (\nu^-)^q\right) \end{cases}.$$

Proof According to Definition 11, $G_{\lambda_1,\lambda_2}(\tilde{\alpha}) = \left(\phi_{\lambda_1}\left(\left[\mu^-, \mu^+\right]\right), \phi^d_{\lambda_2}\left(\left[\nu^-, \nu^+\right]\right)\right)$, where

$$\mu^- \leq \phi_{\lambda_1}\left(\left[\mu^-, \mu^+\right]\right) = \left(\lambda_1\left(\mu^+\right)^q + \left(1-\lambda_1\right)(\mu^-)^q\right)^{1/q} \leq \mu^+$$

and

$$\nu^- \leq \phi^d_{\lambda_2}\left(\left[\nu^-, \nu^+\right]\right) = \left(\left(1-\lambda_2\right)\left(\nu^+\right)^q + \lambda_2(\nu^-)^q\right)^{1/q} \leq \nu^+.$$

(i) For any $\alpha=(\mu,\nu) \in \tilde{\alpha}$, given that $\mu \in \left[\mu^-, \mu^+\right]$ and $\nu \in \left[\nu^-, \nu^+\right]$, we can find a two-tuple $\left(\lambda_1, \lambda_2\right)$ that satisfies

$$\mu = \phi_{\lambda_1}\left(\left[\mu^-, \mu^+\right]\right), \nu = \phi^d_{\lambda_2}\left(\left[\nu^-, \nu^+\right]\right).$$

Assume that another two-tuple $\left(\bar{\lambda}_1, \bar{\lambda}_2\right)$ exists to satisfy

$$\mu = \phi_{\bar{\lambda}_1}\left(\left[\mu^-, \mu^+\right]\right), \nu = \phi^d_{\bar{\lambda}_2}\left(\left[\nu^-, \nu^+\right]\right).$$

Then,

$$\begin{cases} \lambda_1\left(\mu^+\right)^q + \left(1-\lambda_1\right)(\mu^-)^q = \bar{\lambda}_1\left(\mu^+\right)^q + \left(1-\bar{\lambda}_1\right)(\mu^-)^q, \\ \left(1-\lambda_2\right)\left(\nu^+\right)^q + \lambda_2(\nu^-)^q = \left(1-\bar{\lambda}_2\right)\left(\nu^+\right)^q + \bar{\lambda}_2(\nu^-)^q. \end{cases}$$

Therefore, we have $\lambda_1 = \bar{\lambda}_1, \lambda_2 = \bar{\lambda}_2$. □

The above theorem shows that for any QROFN α in the IVQROFN $\tilde{\alpha}$ ($\alpha \in \tilde{\alpha}$), it can be obtained by selecting the appropriate parameter values ($\lambda_1, \lambda_2 \in [0,1]$) and using the operator $G_{\lambda_1,\lambda_2}(\tilde{\alpha})$. Moreover, a one-to-one correspondence exists between this QROFN α and parameters λ_1 and λ_2.

The above analysis shows the proposed C-IVQROFOWA operator can overcome the shortcomings of the existing operators C-IVIFOWA and C-IVPFOWQA and the decision-makers can acquire the corresponding fuzzy numbers by combining the C-IVQROFOWA operator with its own attitude characteristics. The operator has strong flexibility.

- WC-IVPFOWQA and WC-IVPFOWQA operators

Definition 25 [48, 49] Let $\tilde{\alpha}_i = \left(\tilde{\mu}_i, \tilde{\nu}_i\right) = \left(\left[\mu_i^-, \mu_i^+\right], \left[\nu_i^-, \nu_i^+\right]\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs and the weighted C-IVIFOWA (WC-IVIFOWA) operator is defined as follows:

$$\mathrm{WC-IVIFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) = \left(1 - \prod_{i=1}^{n} \left(1 - f_\lambda\left(\tilde{\mu}_i\right)\right)^{w_i}, \prod_{i=1}^{n} \left(f_{1-\lambda}\left(\tilde{\nu}_i\right)^{w_i}\right)\right). \tag{56}$$

Definition 26 [52] Let $\tilde{\beta}_i = \left(\tilde{\rho}_i, \tilde{\sigma}_i\right) = \left(\left[\rho_i^-, \rho_i^+\right], \left[\sigma_i^-, \sigma_i^+\right]\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVPFNs and the weighted C-IVPFOWQA (WC-IVPFOWQA) operator is defined as follows:

$$\mathrm{WC-IVPFOWQA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right) = \left(\sqrt{1 - \prod_{i=1}^{n} \left(1 - \left(g_\lambda\left(\tilde{\rho}_i\right)\right)^2\right)^{w_i}}, \prod_{i=1}^{n} \left(g_{1-\lambda}\left(\tilde{\sigma}_i\right)^{w_i}\right)\right). \tag{57}$$

The comparative analysis of the above two types of operators (WC-IVIFOWA and WC-IVPFOWQA) and the WC-IVPFOWQA operator is carried out using a simple example. Three types of operators are used to aggregate a collection of IVIFNs and a collection of IVPFNs with the assumption that the parameters and weights have been given. The details are shown in Table 9.

According to Table 9, we have

$$\mathrm{WC-IVIFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) = (1.0, 0.0), \textit{WC-IVPFOWQA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) = (1.0, 0.0).$$

Although $w_2 + w_3 + w_4 = 0.09$ and $w_1 = 0.01$, the aggregation results of operators WC-IVIFOWA and WC-IVPFOWQA are both $(1.0, 0.0)$. These results indicate IVIFNs $(\tilde{\alpha}_i(i = 2, 3, 4))$ are ignored completely in the aggregation process. This phenomenon is clearly unreasonable. Similarly, we have

$$\mathrm{WC-IVPFOWQA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right) = (1.0, 0.0).$$

This aggregation result is also unreasonable.

However, the results obtained via the WC-IVQROFOWA are reasonable.

$$\mathrm{WC-IVQROFOWA}\left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\right) = (0.1480, 0.7535)$$
$$\mathrm{WC-IVQROFOWA}\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\right) = (0.3782, 0.7634)$$

The above analysis shows the proposed WC-IVQROFOWA operator can overcome the shortcomings of WC-IVIFOWA and WC-IVPFOWQA operators, but the WC-IVQROFOWA operator can obtain more reasonable aggregation results in solving GDM problems. The construction ideas of the WC-IVIFOWA [47–49], WC-IVPFOWQA [52], and WC-IVQROFOWA operators are summarized and the details presented in Fig. 8.

The advantages and advancement of the developed approach compared with existing work are summarized as below:

- The proposed fuzzy MAGDM method develops the methodology in the use of the IVROFS and the aggregation operators. This advances its degenerate used in [30] in which the ROFS has been implemented as the ROFS is a particular case of IVROFS.
- The C-IVIFOWA and C-IVPFOWQA operators proposed in the previous studies [47–49, 52] lay a solid foundation for this study. The developed C-IVQROFOWA operator not only covers the existing two types of operators, but also realizes the independence of attitude parameters(λ_μ and λ_ν), which overcomes the shortcomings of the two types of operators as shown in Figs. 5 and 6 because of the same parameter values $\lambda_\mu = \lambda_\nu$.
- The comparative analysis in Table 9 shows that the proposed WC-IVQROFOWA operator can overcome the shortcomings of the WC-IVIFOWA [48, 49] and WC-IVPFOWQA [52] operators, and these operators are two special cases of the WC-IVQRFOWA operator with $q = 1$ and $q = 2$, respectively. Moreover, the IVQROFWA and Maclaurin symmetric mean operators developed by [35] and [36] aggregate directly the endpoints of the interval, which makes decision makers unable to obtain most of the information in the interval according to their attitude preferences. The proposed WC-IVQROFOWA operator and its counterpart can avoid successfully this deficiency as the information contained in the interval will be adequately taken into account.
- Another advantage of the proposed method upon comparison with the existing methods is that it derives the consensus measure to develop the optimization model of operator parameters and, based on which, its iteration algorithm are constructed. The consensus measures can

guide the change direction of the parameter constraints to obtain the parameter values that satisfy the threshold conditions.

7 Conclusions

In order to adapt to the rapid change of the complex MAGDM environment, the traditional fuzzy MAGDM methods have been expanded to interval-valued intuitionistic and Pythagorean fuzzy MAGDM methods. The emergence of IVQROFS greatly promotes the formation of generalized MAGDM paradigm and provides decision-makers with more flexible and broad application ideas.

This study proposes novel fusion strategies for continuous IVQROF group information and multiple attribute information, and it integrates the attitudinal characteristics of decision makers and considers the degree of consensus among individuals in the group in a bid to enhance the reliability of decision-making results. The work of this paper is mainly reflected in the following aspects. Firstly, this study revealed the shortcomings of existing operators in dealing with decision-making problems by clarifying the advantages and construction ideas of existing operators, and it proposed the WC-IVQROFOWA operator with a wider range of information processing and more flexible attitude preference, so as to consolidate the decision-making framework under the IVQROF environment. Secondly, by integrating the consensus measure between different individuals in the context of GDM, the dynamic adjustment mechanism of operator parameters is constructed, which enhances the adjustability of GDM process and improves the credibility of decision-making results; last but not least, the GDM method proposed in this study has been applied in quality assessment of SmartWatch appearance design with linguistic inputs, which shows the independence of the parameters in the WC-IVQROFOWA operator and the positive effect of the group consensus measurement on the decision-making process.

In the case study, the original evaluation information of the alternatives is obtained by semantic difference method from experts, which constitutes the linguistic evaluation matrix, and is quantified by IVQROFNs solid at $q = 2$. As regards to our future research, the applications of the developed models in this study can be used to other types of fuzzy models, for instance, the interval linguistic labels [64] and the basic uncertain information soft sets [65].

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