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ORIGINAL ARTICLE

Tri-level thinking: models of three-way decision

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Abstract

The underlying philosophy of three-way decision is thinking in threes, namely, understanding and processing a whole through three distinct and related parts. One can formulate many concrete models of three-way decision to account for different interpretations of the three parts. By interpreting the three parts as three levels, this paper investigates tri-level thinking to build concrete models of three-way decision. We examine some fundamental issues and basic ingredients of tri-level thinking. In accordance with the data–information–knowledge–wisdom (DIKW) hierarchy, we present a perception–cognition–action (PCA) tri-level conceptual model that is applicable to studying intelligent data analytics, intelligent systems, and human understanding.

Keywords Three-way decision · Thinking in threes · Tri-level analysis · Tri-level thinking · Data–information–knowledge–wisdom hierarchy

1 Introduction

In the last few years, we have witnessed a growing interest in and a rapid development of a theory of three-way decision (3WD) [20, 22, 23, 31, 32, 34–37, 64, 66, 70–72, 74, 83]. The theory is motivated by and based on a particular way of human thinking known as thinking in threes. Three is a mystic, magic, and powerful number. It is the lowest number of appearances or instances that is needed to form patterns in our mind, the largest number of items that we can recognize without counting, and the first number for representing many [5, 6, 9, 12, 46]. It is a favourite number in literature, arts, and science, standing for solidity, balance, and completion [18]. Thinking in threes has a solid cognitive basis about a limited capacity of human information processing [10, 42, 72]. Thinking in threes reflects the human desire and universal habit of grouping things into sets of three [6]. Thinking in threes shows up in many ways, shapes, and forms [7]. A theory of three-way decision as thinking in threes will be universally applicable to many disciplines and fields.

A fundamental notion of three-way decision is a trisection that consists of three nearly independent parts of a whole, namely, a set of three parts. By attaching different meanings to trisections in different contexts, we can build a variety of concrete and special-purpose models of three-way decision. Examples of sets of threes include three parts, three elements, three components, three categories, three perspectives, three views, three dimensions, three levels, three layers, three generations, three periods, three stages, three steps, triangles, triads, triplets, and many others [74]. Recent studies along this line of thought have fostered a number of three-way approaches, for example, three-way classification [8, 30], three-way clustering [2, 62, 63, 78, 79], three-way recommendation [3, 82], three-way decision support [60, 67], three-way concept analysis [47, 49, 55–58, 73], three-way concept learning [13, 25, 31], three-way conflict analysis [14, 27, 28, 59, 75], three-way approximation [24], three-way attribute reduction [40, 48, 84], and many more [21, 33, 81].

Tri-level thinking or thinking in three levels is a special model of three-way decision proposed and briefly discussed in a recent paper [74]. Several studies have, either implicitly and explicitly, explored ideas of tri-level thinking in the context of three-way decision. Xu et al. [65] introduced a model of classification based on a three-way confusion matrix. They introduced a tri-level architecture of a three-way decisions evaluation system, consisting of a basic model layer, a measurement layer, and an application layer. Yao et al. [77] and Zhou et al. [86] investigated shadowed sets

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as three-way approximations of fuzzy sets, in which many-valued fuzzy membership grades are approximated by using three-level grades. Zhang et al. [85] studied class-specific attribute reducts at three levels, namely, macro-top, meso-middle, micro-bottom levels. Yao [75] and Lang et al. [28] considered three levels of conflict in conflict analysis, consisting of no-conflict, weak conflict, and strong conflict. Liu and Yang [38] and Jiao et al. [26] investigated three-way decision by using a single-valued neutrosophic set defined by membership functions at three levels: a truth-membership function, an indeterminacy-membership function, and a falsity-membership function. It is evident that tri-level thinking as more concrete ways of three-way decision is receiving attentions from many researchers.

The main objective of the present paper is to provide a more detailed description of tri-level thinking based on a trisecting-acting-outcome (TAO) model of three-way decision [74]. We focus on a philosophical inquiry, a conceptual formulation, and a methodological exposition of tri-level thinking. The aim is to cast some light on building new concrete models of three-way decision in general and tri-level thinking in particular. The technical details of three-way decision are available from the many references given at the end of the paper.

The rest of the paper is organized as follows. Section 2 is a brief review of the TAO model of three-way decision. Section 3 describes tri-level understanding and thinking. Three interpretations of an ordering of levels are examined. An evaluation-based tri-segment model is introduced. By using one-step three-way decision as a basis, we build models of multilevel thinking. Section 4 introduces a tri-level interpretation of the widely used data-information-knowledge-wisdom (DIKW) hierarchy. This new interpretation is used to build a perception–cognition–action (PCA) tri-level model for studying human understanding, intelligent data analysis, and intelligent systems.

2 The TAO of three-way decision

Three-way decision is a philosophy of thinking in threes, a methodology of working with threes, and a mechanism of processing through threes (i.e., triplets or sets of three items). A TAO (trisecting-acting-outcome) model of three-way decision is given in Fig. 1, which focuses on three related tasks [74]: (1) to divide the whole into three parts, (2) to devise strategies to process the three parts, and (3) to optimize a desirable outcome. The TAO model of three-way decision by itself is an example of thinking in threes, namely, trisecting, acting, and outcome evaluation. Trisecting divides the whole into three parts, acting applies a set of strategies to process the three parts, and outcome evaluation measures the effectiveness.

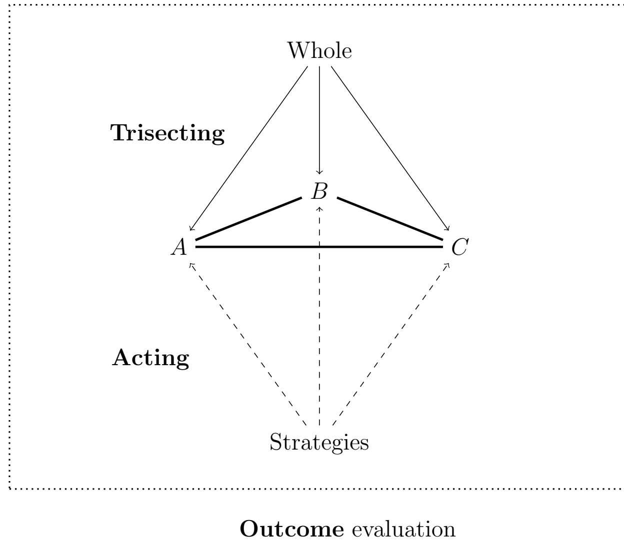


Fig. 1 TAO model of three-way decision

A trisection of the whole is a fundamental notion of three-way decision. Each part represents a particular aspect, perspective, or component of the whole; their integration represents and covers the whole. Each part is distinct from and nearly independent of the other two. At the same time, the three parts are connected. Fig. 1 is a refinement of Fig. 2 of a previous paper [74]. We explicitly bring in the triangle of three-way decision labeled by the triplet (A, B, C), which indicates that the three parts are both separated and connected. The combination of trisecting and acting forms the diamond of three-way decision for the purpose of producing a desired outcome.

In a set-theoretical setting, we may explain the notion of a trisection as follows. The whole is a universal set and a part is a subset. A trisection is a triplet of three subsets that are pairwise disjoint and their union is the universal set. Since one or two subsets may be the empty set, the triplet only forms a weak tri-partition of the universal set. In general, it may be meaningful to allow some overlap between subsets. In this case, the triplet is a weak tri-covering of the universal set.

We demonstrate the TAO of three-way decision by using three practical examples.

Example 1 The first example is an ancient Chinese wisdom about planning presented in Guanzi, which is quoted here [50]:

> "When planning for one year, there is nothing better than planting grain. When planning for ten years, there is nothing better than planting trees. When planning

> for a lifetime, there is nothing better than planting men. Grain is something that is planted once and produces only a single harvest. Trees are things that are planted once but may produce ten harvests. Men are things that are planted once but may produce a hundred harvests."

This example provides a beautiful demonstration of the TAO of three-way decision by seamlessly integrating the three ingredients. For trisecting, we divide time into one year, ten years, and lifetime, which figuratively represents short, medium, and long term planning. For acting, we devise strategies of planting grain, planting trees, and planting men, respectively, for the three time scales. For outcome, the returns of three different strategies are one fold, ten folds, and a hundred folds, which figuratively indicates low, moderate, and high returns, respectively.

Example 2 Benjamin Franklin is considered to be a founding father of philanthropy. Our second example is his advice for fundraising, which is quoted as follows [15]:

> "In the first place, I advise you to apply to all those whom you know will give something; next, to those whom you are uncertain whether they will give anything or not, and show them the list of those who have given; and lastly, do not neglect those whom you are sure will give nothing, for in some of them you may be mistaken."

In this example, the criterion for trisecting is simple and straightforward. According to our belief about their likelihood of giving, we divide a set of potential donors into three groups. The wisdom of Franklin's fundraising advice can be seen from several aspects in terms of strategies and actions. First, a strategy of prioritization: we move from the most promising group, to the doubtful group, and finally to the least promising group. Second, a strategy of social influence: by showing names of donors, we may influence people in other groups to donate. Third, a strategy of potential mistake avoidance: by recognizing a possibility of an incorrect trisection, we should not ignore the least promising group. In addition, the strategy of prioritization is effective in two aspects. One is to properly allocate our efforts according to the potential benefits of different groups, which is related to the outcome of three-way decision. The other is to apply the results from earlier steps to influence the later steps. This example clearly shows that different strategies of action for the three parts are often interrelated and may not be simply separated.

Example 3 The third example is taken from a book by Daniel Levitin [29]. When discussing how Highly Successful Persons (HSPs) organize tasks, he states,

> "HSPs create systems to automatically divide up paperwork and projects temporally, based on how urgent they are. A small category of 'now' items, things that they need to deal with right away, is close by. A second category of 'near-term' items is a little farther away, perhaps on the other side of the office or down the hall. A third category of reference or archival papers can be even farther away, maybe on another floor or off-site."

This example again demonstrates the principles of thinking in threes. HSPs divide paperwork and projects temporally into three groups. In order to ensure that proper attentions are paid to the three groups for achieving a desired outcome, they use physical distances so that "now" items are nearby, "near-term" items are a little farther away, and the rest are farthest away. Levitin's model not only trisects tasks, but also enforces proper actions on the three groups of tasks.

The three examples capture some of the most important aspects and fundamental ideas of three-way decision. To gain further insights, we adapt the architecture of a house of influence from a book by McIntosh and Luecke [43] to give the corresponding architecture of a house of three-way decision, as shown by Fig. 2. The three parts of the whole form a solid foundation, the interior of the house is occupied by strategies for acting on the three parts, and the roof of the house is a desirable outcome. In the light of three-way thinking, we may take a closer look at McIntosh and Luecke's model of influence as a concrete and useful example of three-way decision. Conceptually, they view influence as "a structure built on a solid foundation of personal attributes and supportive tactics." In particular, they consider three attributes: trustworthiness, reliability, and assertiveness. The personal attributes are associated with an array of supportive tactics, for example,

- creating reciprocal credits,
- being a source of expertise, information, and resources,
- helping people find common ground,
- framing the issue your way,

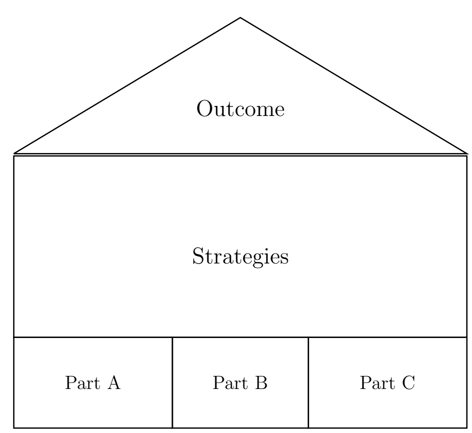


Fig. 2 The architecture of three-way decision

- building a network of support, and
- employing persuasive communication.

The combination of personal attributes and supportive tactics results in high influence. One may argue that there are more personal attributes to be considered in a model of influence. The choice of three major attributes in McIntosh and Luecke's model conforms to the principles of thinking in threes, leading to an argument that is easy to understand, memorable, and convincing.

3 Tri-level understanding and processing

The abstract TAO model gives only the bare bones of a theory of three-way decision. To apply it in particular situations, it is necessary to attach concrete interpretations to basic notions such as trisection, strategies, and outcome. By interpreting a trisection of the whole as three levels, we examine tri-level thinking as a special model of three-way decision.

3.1 A formulation of tri-level thinking

Yao [74] introduced and briefly discussed the basic ideas of tri-level thinking or thinking in three levels. We review the relevant results and give a further exposition.

The three basic tasks of tri-level thinking are (1) to divide a whole into three levels, (2) to explore a natural ordering of the three levels, and (3) to ask different questions at the three levels. As shown in Fig. 3, tri-level thinking approaches a complex whole from three relative simpler and specifically focused levels: a top level, a middle level, and a bottom level. Separation and integration are two important features of tri-level thinking. By separation, each level focuses on a particular aspect of the whole, uses a different language for description and representation, and offers an explanation and understanding of the whole. By integration, a synthesis of the investigations at three levels may provide a full understanding of the whole. Separating the whole into three levels reduces the complexity of investigating the whole; integrating the results from the three levels provides insights about the whole.

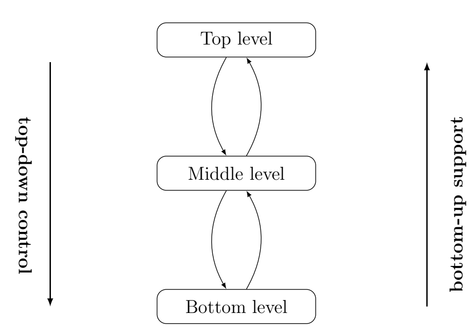


Fig. 3 Tri-level thinking

A trisection of the whole is guided by and, at the same time, reflects a natural ordering of the three levels, namely, top-down control and bottom-up support. The three levels form a simple hierarchy in which a higher level determines a lower level and a lower level supports a higher level. The top level is indirectly connected to the bottom level through the middle level. There are multiple interpretations of the ordering of the three levels [69]. We examine three most commonly used interpretations.

Three levels of abstraction. Levels of abstraction may also be interpreted as levels of granularity and levels of detail. As we move from the top level down to the bottom level, we make abstract concepts more concrete by adding more details or by employing a lower granularity. Conversely, as we move from the bottom level up to the top level, we remove less relevant details and keep the most essential elements. The two directions of movement may also be viewed as process of generalization and specialization, abstraction and concretization, or zooming-out and zooming-in. The roles of control and support as suggested by the natural ordering are apparent. An abstract idea at a higher level determines possible multiple concretizations; concrete ideas at a lower level support and explain an abstract idea at a higher level. An excellent example of levels of abstraction is the tri-level model proposed by Marr [41]. A full understanding of any information processing system requires understandings at a computational theory level, an algorithm level, and a hardware implementation level. There are one-to-many mappings from top down. That is, we can realize a computational theory by using multiple different algorithms, and we can implement an algorithm by using multiple different physical devices. As we move up, we have device-independent and algorithm-independent abstractions.

Three levels of scope and scale. Scopes and scales define the contexts of investigations. Looking at the whole through the prisms of scopes and scales may bring insights that cannot be seen at a single scale. In order to have a deeper understanding of the whole, investigations with three levels of scope and scale are perhaps one of the most common practices. There are abundant examples of tri-level thinking and understanding with various scopes and under differing scales. In philosophy, ethics is studied at three levels of meta-ethics, normative ethics, and applied ethics. Physics has three levels of micro-, meso-, and macro-approaches. Economics investigates microeconomics, mesoeconomics, and macroeconomics. Biology looks at biodiversity at the three levels of genetic diversity, species diversity, and ecosystem diversity. Management typically involves low-level management, middle-level management, and top-level management. Personal development and health are examined at the personal, interpersonal, and societal levels.

Three levels of complexity. In situations where it is difficult to solve a complex and ill-defined problem, we may decompose the problem into three sub-problems with increasing complexity. Solving simpler sub-problems at a lower level provides hints and supports for solving a more complex problem at a higher level. Therefore, levels of complexity enables us to progressively solve the problem. We may explain the concept of three levels of complexity by using the three levels of communication problems classified by Weaver [53]. This tri-level model decomposes communication problems into technical problems, semantics problems, and effectiveness problems. The three levels address three questions: How accurately can the symbols of communication be transmitted? How precisely do the transmitted symbols convey the desired meaning? How effectively does the received meaning affect the receiver's conduct in the desired way? The three questions are of progressive complexity. Shannon [53] had successfully solved the technical problems at the bottom level by introducing the information theory. Continuing research efforts have been made to answer the other two questions.

The three interpretations are not exclusive. In some situations, two or three interpretations are applicable. For example, levels of abstraction may be related to levels of complexity, so are the levels of scope and scale. That is, an analysis at a higher abstract level may be more complex than an analysis at a lower level. In some sense, we may interpret the levels of scope and the levels of complexity as special cases of the levels of abstraction. We explicitly separate out these two interpretations to emphasize their uses in a wide range of disciplines and domains.

Example 4 This example, taken from [74], demonstrates that a trisection leads naturally to tri-level thinking and tri-level analysis. Figure 1 shows that a trisection consists of three parts A, B, and C. In order to have a full understanding of the trisection, it is necessary to consider three levels of analysis as shown in Fig. 4. The bottom level deals with analysis of individual parts independently. The middle level is about comparative analysis of pairs of parts. The top level integrates all three parts. The ordering of three levels may be interpreted in terms of either scope or complexity. At each level, we ask a different type of questions. A tri-level analysis of a trisection may help us to devise and select the best strategies.

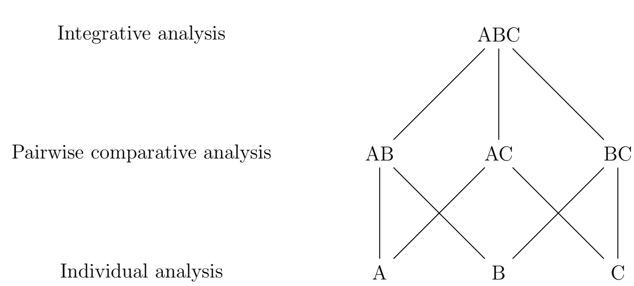


Fig. 4 Tri-level analysis in three-way decision

This example, Marr's tri-level model, and Weaver's tri-level model provide useful hints on an important issue in tri-level thinking, namely, asking the right questions at the right level. Recall that each level may use a different language and vocabulary suitable for answering a specific question. We seek for explanations that are appropriate at different levels. Furthermore, we study transformations between different representations at different levels. The links, as given by transformations, provide a way to integrate results from different levels. Three levels of understanding and explanation are, more often than not, superior to an explanation at a single level.

3.2 A formulation of tri-level thinking with three segments

There are situations where we divide a universal set into three levels and rank the three levels to form a tri-level structure. However, the ordering reflects the ranking of measurement or evaluation values, rather than a "control-support" relationship in a hierarchy. For example, many test results such as blood pressures and body temperatures are typically categorized as the low, normal, and high three segments. There does not exist a "control-support" relationship among the three segments. These examples call for another type of tri-level thinking.

One can formulate a model of tri-level thinking with three segments based on an evaluation-based model of three-way decision [71]. Let $e : U \longrightarrow \mathfrak{R}$ be an evaluation function that maps an object in U to a real number in $\mathfrak{R}$. Given a pair of threshold (l, u) with $l < u$, we can easily divide U into three segments as follows:

$$\begin{aligned} \mathbb{L}_{(l,u)} &= \{x \in U \mid e(x) \leq l\}, \\ \mathbb{M}_{(l,u)} &= \{x \in U \mid l < e(x) < u\}, \\ \mathbb{H}_{(l,u)} &= \{x \in U \mid e(x) \geq u\}, \end{aligned} \tag{1}$$

where $\mathbb{L}$, $\mathbb{M}$, and $\mathbb{H}$ denote, respectively, the sets of objects with low, medium, and high evaluation values. In this model, the construction of an evaluation and the determination of the pair of thresholds are some of the fundamental issues.

Typically, the problem of determining a pair of thresholds may be formulated as an optimization problem. Suppose the triplet $(\mathbb{L}_{(l,u)}, \mathbb{M}_{(l,u)}, \mathbb{H}_{(l,u)})$ represents three non-overlap lower, middle, and upper segments of objects of the ranking produced by the evaluation values. Suppose $Q(\mathbb{L}_{(l,u)}, \mathbb{M}_{(l,u)}, \mathbb{H}_{(l,u)})$

denotes a quality measure of the triplet and assume that a larger value indicates a better triplet. The problem of determining a pair of thresholds is to solve the following optimization problem:

$$\arg \max_{l,u \in \Re, l<u} Q(\mathbb{L}_{(l,u)}, \mathbb{M}_{(l,u)}, \mathbb{H}_{(l,u)}), \tag{2}$$

where arg takes the argument that maximizes the objective function Q.

Yao and Gao [76] suggested a statistical method for determining the pair of thresholds when the evaluation values are, in fact, measurements of a particular quantity. Let μ and σ denote, respectively, the mean value and the standard deviation. Two positive numbers k_l and k_h represent the positions of the two thresholds away from the mean in terms of the number of times of the standard deviation. The pair of thresholds can be constructed as follows: for $k_l > 0$ and $k_h > 0$,

$$\begin{aligned} l &= \mu - k_l \sigma, \\ h &= \mu + k_h \sigma. \end{aligned} \tag{3}$$

Generally, k_l and k_h do not need to be equal. This model provides a good explanation of tri-level categorization of many medical testing results. For example, the blood pressure is normal if it is around the mean blood pressure of a very large population, it is low if it is far lower than the mean, and it is high if it is far higher than the mean. This type of statistical tri-level thinking is effective and easy to explain in many situations.

3.3 Three-way thinking as a basis of multilevel thinking

In many situations, tri-level thinking leads to a simple and yet powerful conceptual model for human understanding. By using three-way thinking as a basis, we suggest two ways of multilevel thinking, namely, hierarchical thinking and multilevel thinking with tri-level windows.

3.3.1 Tri-level thinking for building simple powerful conceptual models

Too often, our acceptance and appreciation of a novel and creative idea depends on not only the idea itself but also the way in which the idea is presented. This calls for a simple and yet powerful conceptual model for explanation. Resting on a sound cognitive basis, three-way decision in general and tri-level thinking in specific may offer a solution.

The power of tri-level lies in its simplicity. The idea of simplification is the very first trick for creative thinking suggested by Shannon [52]:

> "The first one that I might speak of is the idea of simplification. Suppose that you are given a problem to solve, ... probably a very powerful approach to this is to attempt to eliminate everything from the problem except the essentials; that is, cut it down to size. Almost every problem that you come across is befuddled with all kinds of extraneous data of one sort or another; and if you can bring this problem down into the main issues, you can see more clearly what you're trying to do and perhaps find a solution."

To some extent, Shannon's information theory may be viewed as a result of applying this principle of simplicity. For conceptualizing the complex problem of communication, Shannon used a simple schematic diagram. We regroup

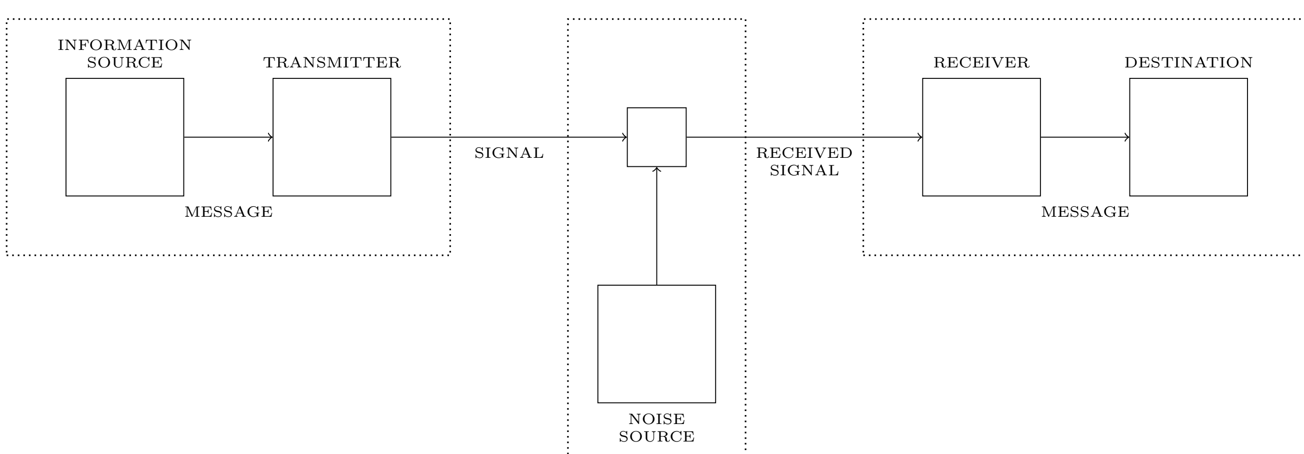


Fig. 5 Three-way interpretation of Shannon's diagram of a general communication system

Shannon's original five parts into three for a three-way interpretation as shown in Fig. 5. The input end or level is represented by the pair (information source, transmitter), the output end or level is represented by the pair (receiver, destination), and the two are linked by a channel with noise. The three-part re-interpretation enables us to connect Shannon's diagram to other commonly used three-part diagrams.

By abstracting from, or simplifying, Shannon's diagram, we have a three-part conceptual model known as the input-process-output (IPO) model. This high-level abstract model has been widely used across many disciplines. In the model, a process is abstractly viewed as a transformation from the input (i.e., what are given) to the output (i.e., what are required outcome). The details are normally not given, but to be added in further development or articulation. The simple IPO model provides a conceptual view without possible distractions from details.

The IPO-like tri-level structures can be found in many places. For example, for understanding artificial neural networks, there is a tri-layer model that consists of the input layer, the hidden layers, and the output layer. The hidden layers play the role of a process in the IPO model. For understanding the mind, one may use a tri-level model consisting of perception, cognition, and action [45], in which cognition mediates between sensory input and motor output. These models are conceptual models for a quick, intuitive, and high-level understanding. The middle level/element may be viewed as a placeholder that connects input and output levels and whose details are not precisely given.

As a final note, we perhaps should point out that the number three in tri-level thinking should be read in its figurative sense of "a few" [74], although our discussions have been focused on exactly three. In some situations, it may be better to think in twos, fours, fives, etc. We may view tri-level thinking as thinking in a few levels with three as the pivot number of levels.

3.3.2 Hierarchical thinking in threes

In many situations, a trisection of a whole is only a first order approximation or a very brief description. One may trisect some or all three parts into more sub-parts. In other words, we treat a part as a whole and divide it further into three new parts. This progressive process continues until certain conditions are met [68, 69]. The result is a ternary tree in which each node has zero or three children. Internal nodes of the tree form intermediate levels used to produce the final level given by the leaves. In the top-down construction of multilevel hierarchical structure, as we move from the top level of the root downwards to the levels of leaves, we add more details. Alternatively, one may also construct a hierarchical structure in a bottom-up manner. In this case, one extracts the commonalities and essential features from smaller parts and merges smaller parts to form a larger part progressively. Both top-down and bottom-up construction methods have proved to be very effective in structured programming and many other fields. In the figurative sense of the number three, each node has about three children. By considering about three parts each time, we have an easy-to-understand hierarchical structure.

An example of hierarchical three-way thinking is structured writing. A simple and abstract structure of an article consists of an introduction, a body, and a conclusion. The body is further expanded into sections, a section is expanded into sub-sections, and a sub-section is expanded into a number of paragraphs. A paragraph typically consists of three elements: an opening sentence, supporting sentences, and a concluding sentence. Thinking in threes is an effective strategy in writing and speech. We, both consciously and subconsciously, use three words, three phrases, three sentences, three paragraphs, three examples, three points, and many such three-element structures in writing and speech [4, 6]. Hierarchical three-way thinking is powerful for producing an impactful article and a memorable speech.

The three-segment model introduced in Sect. 3.2 can be used to construct multi-segment models. We may trisect some and all three segments to produce more smaller segments. For a segment, say $\mathbb{L}$, we can trisect it into three smaller segments $\mathbb{LL}$, $\mathbb{ML}$, and $\mathbb{HL}$. If we trisect all three segments, we would have a total of nine segments. The segments are ordered naturally by the values of an evaluation function. By interpreting each segment as representing a particular level evaluation, we have a linearly ordered multilevel structure. In the figurative sense of the number three, one may divide a segment into two to four segments. This enables us to explain commonly used multi-scale or multi-point ratings such as a five-point system with ratings from A to E, a ten-point system with ratings from 1 to 10, and many more. By using a pair of thresholds, we have three segments. With m thresholds, we have $m + 1$ segments. It may be argued that one can build $m + 1$ segments simply by using m thresholds. A possible difficulty with this method is the understandability. It is a cognitive load to consider m thresholds and their interdependencies when $m \geq 3$. Progressively segmenting in terms of threes is much easier to do and to interpret.

In summary, a one-step three-way thinking produces a trisection. Sequential and progressive three-way thinking is a basic strategy for constructing multilevel hierarchical structures, such as a ternary tree or a linearly ordered sequence of segments.

3.3.3 Tri-level windows for multilevel thinking

Hierarchical multilevel thinking involves both creating and working with multiple levels. There are many ways to construct multilevel structures. When the number of levels is greater than three, it might be difficult to simultaneously consider all levels. Tri-level thinking may potentially overcome this difficulty.

We assume that interactions only happen between two adjacent levels. Information exchange and transmission between two non-adjacent levels are achieved through consecutively adjacent levels between the two levels. Under this assumption, we may use a window containing three levels to process multilevel. The middle level in the window is the current level. The upper and lower levels give the contexts of the current level. The current level is guided by and supports the upper level. At the same time, the current level guides and is supported by the lower level. By moving the window upwards and downwards in a multilevel hierarchical structure, we transform the problem of multilevel thinking into a series of tri-level thinking.

A window of three levels is appropriate for piece-wise approximations of multilevel thinking. The upper and lower levels in the window provide the necessary contexts of the middle/current level from both above and below. In other words, a window of size three enables us to have a full understanding of the current level with guidance from the upper level and support from the lower level. If a window of size two is used, we would miss one of the two contexts. If a window of size greater than three is used, we again run into the problem of complexity. In summary, a window of size three is both simple and sufficient.

4 A PCA tri-level model based on DIKW hierarchy

The data-information-knowledge-wisdom (DIKW) hierarchy is one of the fundamental concepts in information science, computer science, management science, psychology, cognitive science, philosophy, and many others [1, 51, 61, 80]. There are many interpretations and re-interpretations of the DIKW. An example of the latest interpretation was given in the context of big data [19, 39]. In this section, we give a tri-level interpretation of the DIKW hierarchy, which naturally corresponds to a perception–cognition–action (PCA) tri-level model of human understanding.

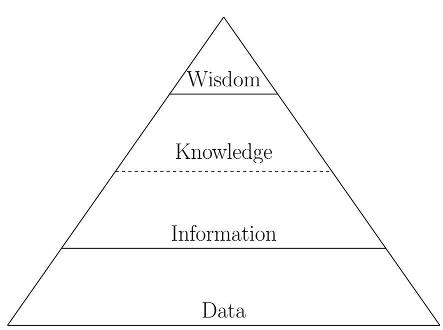
4.1 A tri-level interpretation of DIKW hierarchy

The data-information-knowledge-wisdom (DIKW) hierarchy, as shown in Fig. 6, consists of four levels. There are several variations of the hierarchy with different numbers of levels, for example, the data-information-knowledge (DIK) tri-level hierarchy [51, 61], the data-information-knowledge-understanding-wisdom five-level hierarchy [1], and the data-information-knowledge-wisdom-enlightenment five-level hierarchy [80]. The tri-level DIK hierarchy serves as a basis to build other hierarchies by adding new levels. A hierarchy with more than three levels, although may provide more information or a more detailed description, may suffer from a difficulty in understanding. It might be useful to seek for a tri-level interpretation of the DIKW hierarchy.

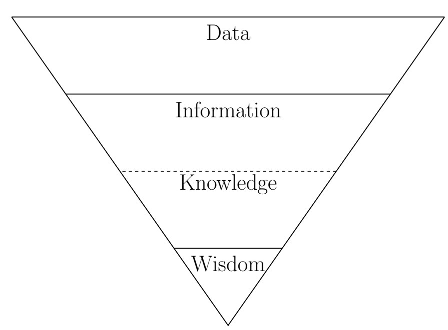
Studies on the DIKW hierarchy generally agree on the hierarchical transformation of data-to-information-to-knowledge-to-wisdom [51, 61]. That is, data create information, information creates knowledge, and knowledge creates wisdom [51]. There are two possible ways to view the DIKW hierarchy. The most commonly used pyramid view, as shown in Fig. 6a, considers data as the broad base and wisdom as the much smaller summit of a pyramid [61]. Reading bottom-up, we see an increase in complexity, abstraction, meaning, and value, as well as a decrease in automation and programmability [51]. Rowley [51] provided a more evocative funnel view, as shown in Fig. 6b. The DIKW funnel vividly shows that data naturally becomes more concentrated when moving down the funnel. There is a need for big data in order to obtain sufficient wisdom.

On the other hand, there does not exist an agreement on definitions and interpretations of data, information, knowledge, and wisdom. There may not exist a clear-cut boundary between any two adjacent levels. It is perhaps more difficult to distinguish and separate the information and knowledge levels. There are many types of knowledge [61]. Information is sometimes considered to be a type of weak

Fig. 6 DIFW hierarchy



(a) DIKW pyramid



(b) DIKW funnel

knowledge [16]. There is, at the same time, some degree of agreement on the separation of data and information, and the separation of knowledge and wisdom. From these observations, we combine the information level and knowledge level into a single information/knowledge level. The result is a tri-level data-information/knowledge-wisdom (D-I/K-W) hierarchy. In Fig. 6, we use a dashed line to separate the information/knowledge level into an information level and a knowledge level. That is, we construct a four-level DIKW model by bisecting the information/knowledge level in a three-level interpretation of the D-I/K-W hierarchy. This constructive model may be viewed as a special case of the multi-segment models discussed in Sect. 3.3.2.

4.2 A conceptual PCA tri-level model

The new tri-level interpretation of the DIKW hierarchy provides us a basis for building a conceptual perception–cognition–action (PCA) tri-level model. The model is useful to study human understanding, information processing, data analysis, intelligent systems, and so on.

We first take a look at the tri-level interpretation of the DIKW hierarchy in the input-process-output (IPO) three-element framework. We view data as raw and unprocessed symbols, signals, quantities, and such. Data are the input in the IPO triplet. We view wisdom as the quality of sound judgement and action through right and wise use of knowledge and experiences. In other words, one can appreciate the wisdom of a person or a system based on the output in the IPO triplet. The information/knowledge level corresponds to the process in the IPO triplet and is only a placeholder that accounts for whatever happens between the data and wisdom levels. This tri-level model, without a detailed and accurate description of information and knowledge, as well as mechanisms for transforming information to knowledge, is sufficient for a high level investigation. Information/knowledge level is a mediator that connects the data level and the wisdom level. Its function is to facilitate data-driven wisdom (i.e., data-driven decision-making). The details of the information/knowledge level are to be added in particular domains and for specific applications.

The Oxford English Dictionary gives two definitions, among several others, of wisdom: (1) “The quality or character of being wise, or something in which this is exhibited. ... Capacity of judging rightly in matters relating to life and conduct; soundness of judgement in the choice of means and ends; sometimes, less strictly, sound sense, esp. in practical affairs: opposed to folly.” (2) “Knowledge (esp. of a high or abstruse kind) ... ” If one takes a progressive view of the DIKW hierarchy, the second definition seems more appropriate. That is, data are a kind of weakest knowledge, information is a type of weak knowledge, and wisdom is a high kind of knowledge. In this way, the DIKW hierarchy is a knowledge hierarchy with four types of knowledge. Many studies in fact take the progressive view of the DIKW hierarchy but (somewhat inconsistently) use the first sense of wisdom. In our interpretation of the DIKW hierarchy, the first definition is indeed appropriate. Under the interpretation of information as a kind of weak knowledge, we can simply label the information/knowledge level as knowledge level. The level of wisdom is more about the values of knowledge, rather than knowledge itself. Therefore, we may have two alternative representations of the tri-level interpretation of the DIKW hierarchy: one is the data–knowledge–wisdom (DKW) hierarchy and the other is the data–knowledge–value (DKV) hierarchy. In particular, DKV hierarchy is consistent with the tri-level model of communication problems suggested by Weaver [53]. The three levels correspond, respectively, to symbols (i.e., data), meaning (i.e., knowledge), and effectiveness (i.e., value) involved in communication problems. We keep the name DIKW hierarchy because of its wide use, although the tri-level interpretation offers a new angle of the hierarchy.

Perception, cognition, and action are three fundamental concepts in psychology and cognitive science [45]. In this paper, instead of using their full meanings and interrelationships, we simply rely on their common everyday understanding and one possible tri-level structure given by a perception–cognition–action (PCA) hierarchy. In building a conceptual PCA model of understanding, we take the metaphorical power of the D-I/K-W hierarchy as a paradigm of thinking in threes. As mentioned earlier, the PCA hierarchy

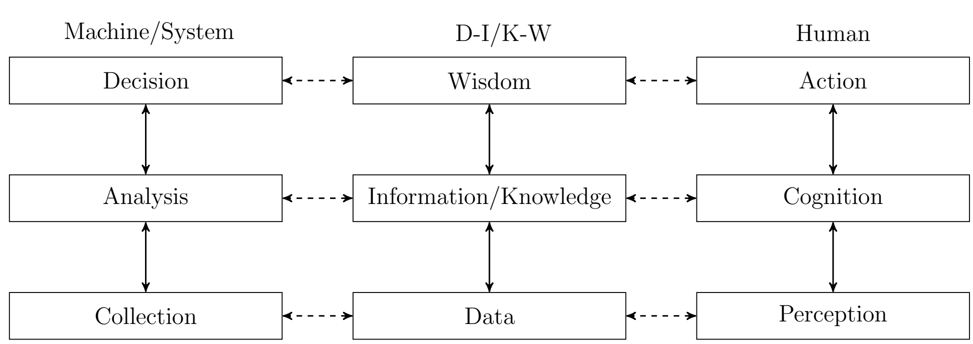


Fig. 7 Perception–cognition–action (PCA) model

may also be interpreted in terms of the input-process-output (IPO) model. Metaphorically speaking, perception gathers the input, action is the output, and cognition transforms data into knowledge to form a basis for producing the desirable output from the given input. Therefore, there is a close connection between the D-I/K-W hierarchy and the PCA hierarchy, as shown by double-arrowed dashed lines in Fig. 7. Data are obtained from perception, information/knowledge is the result from cognition, and wisdom is exhibited in the action or decision. It might be commented that, in some situations, not taking any action may actually be the most powerful action. The treatment of the information/knowledge level as a placeholder corresponds to cognition as a placeholder for whatever happens between perception and action [44, 45].

While the PCA tri-level model is suitable for studying human understanding and action, we can similarly build a collection-analysis-decision (CAD) tri-level model for machine and system. As shown in the left column of Fig. 7, the CAD model consists of data collection, data analysis, and decision-making. Data can be collected by various types of sensors and transformed into many types of knowledge by using different analytic tools and algorithms, and knowledge is used to support wise and intelligent decision-making.

The double-arrowed solid lines in Fig. 7 reflect two working modes of tri-level thinking. Bottom-up data-driven approaches start with data and transform data into knowledge to support wise decision-making. Top-down decision-driven approaches use a decision as a goal to guide what types of knowledge to discover and use, which in turn guides what types of data to be collected. It is also possible to combine the two modes iteratively to collect the right data, use the right knowledge, and make the right decision.

5 Conclusion

We discussed a method for constructing new and concrete models of three-way decision. The basic idea is to search for, in particular contexts and applications, interpretations of a trisection of a whole. We demonstrated that, by interpreting a trisection as three levels, one can construct models of tri-level thinking and tri-level analysis. As an example of illustration, we built a perception–cognition–action (PCA) tri-level model based on a new tri-level interpretation of the widely used data–information–knowledge–wisdom (DIKW) hierarchy as the data–information/knowledge–wisdom (D–I/K–W) hierarchy. When the PCA model is applied to intelligent data analysis, we have a collection-analysis-decision (CAD) tri-level model. The power of tri-level thinking, as shown by the D-I/K-W hierarchy, the PCA tri-level model, and the CAD tri-level model, stems from a simple conceptualization, an effective methodology, and a mechanism of working with threes. Tri-level thinking is one of the useful concrete models of a theory of three-way decision.

To illustrate the value of this study, we discuss its three possible implications for future research. First, this study may be viewed as a demonstration of how to move from a narrow sense of three-way decision to a wide sense. The concept of three-way decision was first introduced in an attempt to provide a sound semantical interpretation of three types of rules constructed from the three probabilistic regions of rough sets [70]. The name "three-way decision" reflects such a narrow sense. Subsequent studies show that it is necessary to consider a wide sense of three-way decision, which leads to a conception of three-way decision as thinking in threes. In some sense, the wide sense of three-way decision may be named "three-way computing," joining the family of scientific computing, social computing, crowd computing, granular computing, cognitive computing, rough computing, fuzzy computing, soft computing, evolutionary computing, and many more. The trisecting-acting-outcome (TAO) model is an abstract and conceptual model, in which the whole and three parts can be attached with different interpretations. In this way, we are able to construct many concrete models of three-way decision. We have demonstrated that tri-level thinking or computing in three levels can be constructed by interpreting three parts as three levels. The intention of the paper is not to have a complete description of tri-level thinking, and the paper is far from it. Instead, the discussions have been meant to motivate future studies towards the long-term goal of developing a wide sense of three-way decision.

Second, in this paper we actually attempted to study three-way decision in the spirit of thinking in threes. In this case, the three consists of the philosophy, the methodology, and the mechanism of three-way decision. In the light of tri-level thinking, we can arrange the three into the philosophical, the methodological, and procedural levels. A limitation of this paper is its focus on the first two levels, although the results support the third level. To compensate for this shortcoming, we give a long list of references at the end, from which more detailed information on the mechanism of three-way decision can be found. As future research, we need to study each of the three levels and their integration, as indicated by Fig. 4. The methodology of three-way decision as working with threes is universally applicable. Ideas extracted from the examples in the paper may not only enable us to better understand three-way decision but also empower us to better solve new problems.

Third, granular computing is about thinking in granules and cognitive computing is human inspired computing. Three-way decision, as three-way computing, adds a third angle. Three-way decision has a solid cognitive basis on one hand and offers cognitive advantages on the other [72].

Three-way decision may be viewed as computing in three granules [74]. As future research, it may be worthwhile to pursue three-way decision as an interdisciplinary study by combining results from, for example, cognitive science, psychology, computer science, and many others. In a series of recently published papers, Li et al. [25, 31, 54, 64] initiated a study of three-way concept-cognitive learning (CCL) that integrates results from cognitive science, granular computing, machine learning, and three-way decision. By following their methodological paradigms, one may combine three-way decision with other fields to generate new and useful ideas, notions, and theories.

The desire to know is a basic driving force of the human pursuit of knowledge [11]. By the principle of three-way decision as thinking in threes, we may divide our territory of exploration into three regions: the known, the partially known, and the unknown. According to Gleiser's island of knowledge metaphor [17], the island of knowledge represents the known in the ocean of the unknown. Furthermore, "... as the Island of Knowledge grows, so do the shores of our ignorance – the boundary between the known and the unknown." The boundary represents the partially known and the exploration of the boundary would produce fruitful results. The three regions may also be viewed as a tri-level structure to indicate that our exploration takes the path from known to partially known, and from partially known to unknown. As a result, we change unknown into partially known, and change partially known into known. The current research of three-way decision is in the region of the partially known. With further understanding of three-way decision, we may be in a position to build new methods, algorithms, and tools in many domains, disciplines, and applications. At the same time, we may open up new frontier for further exploration.

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