International Journal of Machine Learning and Cybernetics (2018) 9:1857–1867
https://doi.org/10.1007/s13042-018-0857-y

ORIGINAL ARTICLE

L-fuzzy concept analysis for three-way decisions: basic definitions and fuzzy inference mechanisms

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Received: 4 December 2017 / Accepted: 24 July 2018 / Published online: 2 August 2018

Abstract

In this paper, the basic ideas underlying fuzzy logic are introduced into the study of three-way formal concept analysis. This leads naturally to the notion of *L*-fuzzy three-way concepts. The *L*-fuzzy three-way operators and their inverse are defined and their properties are given. Based on these operators, two types of *L*-fuzzy three-way concepts are defined and the corresponding three-way concept lattices are constructed. A possibility theory reading of *L*-fuzzy three-way concepts is also provided. Moreover, the corresponding fuzzy inference method is studied. Two coherent fuzzy inference methods, the lower approximate fuzzy inference and the upper approximate fuzzy inference, are proposed, respectively.

Keywords Three-way decisions · Concept lattice · *L*-fuzzy three-way concept · Fuzzy inference

1 Introduction

The theory of formal concept analysis [11] was proposed by Wille in 1982. It exploits the duality between objects and attributes in a Boolean data table, and leads to an original and practical view of the notion of a formal concept with application to data mining [1, 16, 26]. In classical formal concept analysis, a concept is a pair made of a set of objects and a set of attributes possessed by all the objects, that are in mutual correspondence, through an antitone Galois connection.

This work is partially supported by the National Natural Science Foundation of China (Grant Nos. 61772021, 11371014, 61472471 and 11531009) and the Innovation Talent Promotion Plan of Shaanxi Province for Young Sci-Tech New Star (No.2017KJXX-60).

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It is important to note that in formal concept analysis, each element in the extension possesses all elements in the intension and each element in the intension is shared by all elements in the extension. This is a type of two-way decision. However, a difficulty with two-way decisions in formal concept analysis is that one is only concerned with the commonly-shared attributes of a concept. This represents a type of positive information. It seems more reasonable to consider simultaneously both positive and negative information. Such a consideration divides a whole into three parts and helps us to gain a thorough understanding of formal concepts. Indeed, similar principles and methods are commonly used in everyday life. So far, there still does not exist a unified formal description of three-way decisions. For this reason, a theory of three-way decisions (3WD) has been proposed by Yang and Yao [38] as an effective way of problem solving, which turns complexity into simplicity in many situations. Since its introduction, there has been a fast growing interest, resulting in extensive research that extends and applies three-way decisions. Explorations of three-way decisions have been made in relation to several other theories, including, for example, rough sets [17–19, 38, 40, 41], interval sets [42], [45], three-way approximations of fuzzy sets [9], shadowed sets [27], orthopairs (i.e., a pair of disjoint sets) [7], and squares of oppositions [44]. The theory of three-way decisions embraces ideas from these theories and introduces its own notions, concepts, methods, and tools.

Research in the framework of formal concept analysis also progresses rapidly. In [8], a possibility-theoretic view of formal concept analysis has been introduced. In [20], Li et al. considered the issues of approximate concept construction, rule acquisition and knowledge reduction in incomplete decision contexts. Moreover, they also focused on clarifying the relationship among the existing reduction methods in formal decision contexts [21]. In [28], a novel concept formation and novel concept lattices were developed by Qi et al. with respect to a Boolean data table to support three-way decisions. Precisely, the given concept not only describe those objects (attributes) shared by all elements in the intension (the extension), but also those objects (attributes) not possessed by all elements in the intension (the extension). In subsequent studies, three-way concept analysis has been investigated from various view of points. In [29], Qi et al. systematically analysed the connections between object (attribute) induced three-way concept lattices and classical concept lattices. In [14, 22–24], Li et al. mainly focused on three-way concept learning via multi-granularity from the viewpoint of information fusion and cognition, respectively. In [15], Hu et al. generalized measurement on decision conclusion in three-way decision spaces from fuzzy lattices to partially ordered sets. In [25], Li and Wang focused on two issues: approximate concept construction with three-way decisions and attribute reduction in incomplete contexts. In [32], formal concept analysis based on bidirectional associative memory was extended to three-way formal concept analysis (3WFCA). Qian et al. [30] focused on approaches to construct the three-way concept lattices of a given formal context. In [33, 34], the authors considered the issue of three-way fuzzy concept lattice representation using neutrosophic set. In [35, 36], the authors proposed an algorithm for generating the bipolar fuzzy formal concepts and two methods based on the properties of next neighbors and Euclidean distance for knowledge extraction. In [42, 43], Yao presented a common conceptual framework of the notions of interval sets and incomplete formal contexts for representing partially-known concepts.

Classical concept lattice has also been studied from the viewpoint of fuzzy logic. For instance, Burusco and Fuentes-Gonzales [2] studied concept lattices in fuzzy formal contexts, Bělohlávek [3–6] introduced fuzzy concepts in fuzzy formal contexts by employing a residuated lattice. Georgescu and Popescu [12] introduced and studied fuzzy conjugated pairs, with their underlying closure operators and hierarchical structure. Shao et al. [31] studied rough set approximations within formal concept analysis in fuzzy environment. Fan et al. [10] introduced a fuzzy inference method based on the notion of fuzzy concept lattice.

The present study is undertaken to introduce fuzzy logic into three-way concept analysis, and a type of L-fuzzy three-way concept analysis is thus provided. The rest of this paper is structured as follows. In Sect. 2, we recall some basic notions on residuated lattices and three-way concept lattices. In Sect. 3, based on the formal context (U, V, I), two types of three-way Galois connections are introduced, and the collection of its fixed points is shown to be a complete lattice. Then in Sect. 4, a fuzzy inference method based on the notion of L-fuzzy three-way concept lattices is studied. Some concluding remarks are presented in Sect. 5.

2 Preliminaries

In this section, we briefly recall some basic notions that will be used in this paper.

2.1 Residuated lattices

Definition 2.1 [37] A tuple $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called a complete residuated lattice, if

(1) $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1;
(2) $(L, \otimes, 1)$ is a commutative monoid;
(3) $(\otimes, \rightarrow)$ is an adjoint pair in L, i.e. $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c, a, b, c \in L$.

In a residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, the negation operator $\neg$ is defined by $\neg a = a \rightarrow 0, a \in L$. A residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is said to be involutive if it satisfies the condition $\forall a \in L, a = \neg\neg a$.

Proposition 2.1 [37] *The notions of residuated lattices obey the following properties*:

(i) $a \rightarrow b = 1 \Leftrightarrow a \leq b$,
(ii) $1 \rightarrow a = a$,
(iii) $a \leq b \rightarrow c \Leftrightarrow b \leq a \rightarrow c$,
(iv) $a \otimes (\bigvee_{i \in I} a_i) = \bigvee_{i \in I}(a \otimes a_i), a \otimes (\bigwedge_{i \in I} a_i) \leq \bigwedge_{i \in I}(a \otimes a_i)$,
(v) $(\bigvee_{i \in I} a_i) \rightarrow a = \bigwedge_{i \in I}(a_i \rightarrow a)$,
(vi) $a \rightarrow (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I}(a \rightarrow a_i)$,
(vii) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$,
(viii) $\rightarrow$ *is antitone in the first argument and isotone in the second argument*,
(ix) $\otimes$ *is isotone in both arguments*,
(x) $a \otimes b \leq a, a \otimes b \leq b$,
(xi) $a \rightarrow b \leq \neg b \rightarrow \neg a$,
(xii) $a \rightarrow \neg b = b \rightarrow \neg a$,
(xiii) $a \otimes b \rightarrow c = a \rightarrow (b \rightarrow c), a, a_i (i \in I), b, c \in L$.

Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. An L-fuzzy set [13] in a universe set U is a function $X : U \rightarrow L$ and the collection of L-fuzzy sets on U is denoted by L^U. For $x \in U$ and $X \in L^U$, the value $X(x)$ is called the membership degree of x in X and it is interpreted as the truth value of "x is element of X". Similarly, an L-fuzzy relation between U and V is a function $I : U \times V \to L$. The set of L-fuzzy relations between U and V is denoted by $L^{U\times V}$.

Given two fuzzy sets $X_1, X_2 \in L^U$, the famous subsethood degree [12] was defined as follows:

$$S(X_1, X_2) = \bigwedge_{x\in U} \left(X_1(x) \to X_2(x)\right). \quad (1)$$

It expresses the truth value of "each element of X_1 is an element of X_2".

Definition 2.2 [3, 5] An L-fuzzy Galois connection between the sets U and V is a pair $(\uparrow, \downarrow)$ of mapping $\uparrow : L^U \to L^V, \downarrow : L^V \to L^U$, satisfying

(i) $S(X_1, X_2) \le S(X_2^{\uparrow}, X_1^{\uparrow})$,
(ii) $S(B_1, B_2) \le S(B_2^{\downarrow}, B_1^{\downarrow})$,
(iii) $X \subseteq X^{\uparrow\downarrow}$,
(iv) $B \subseteq B^{\downarrow\uparrow}$,

for all $X_1, X_2, X \in L^U, B_1, B_2, B \in L^V$.

Theorem 2.1 [3, 5] *A pair $(\uparrow, \downarrow)$ forms an L-fuzzy Galois connection between the sets U and V if and only if*

$$S(X, B^{\downarrow}) = S(B, X^{\uparrow}).$$

For a relation I, a pair of operators $(\uparrow_I, \downarrow_I)$ ($\uparrow_I : L^U \to L^V, \downarrow_I : L^V \to L^U$) is given in the following manner:

$$X^{\uparrow_I}(y) = \bigwedge_{x\in U} (X(x) \to I(x, y)), X \in L^U, y \in V,$$
$$Y^{\downarrow_I}(x) = \bigwedge_{y\in V} (Y(y) \to I(x, y)), Y \in L^V, x \in U.$$

Proposition 2.2 [3, 5] *Let $(\uparrow_I, \downarrow_I)$ be the pair of operators defined as above, then we have*

(i) $S(X, Y^{\downarrow_I}) = S(Y, X^{\uparrow_I})$,
(ii) $(\bigvee_{i\in I} X_i)^{\uparrow_I} = \bigwedge_{i\in I} X_i^{\uparrow_I}, (\bigvee_{i\in I} Y_i)^{\downarrow_I} = \bigwedge_{i\in I} Y_i^{\downarrow_I}$.

Given two fuzzy sets $(Y_1^+, Y_1^-), (Y_2^+, Y_2^-) \in L^V \times L^V$, we define

$$S^*\left(\left(Y_1^+, Y_1^-\right), \left(Y_2^+, Y_2^-\right)\right) = \bigwedge_{y\in V} \left(Y_1^+(y) \to Y_2^+(y)\right) \wedge \bigwedge_{y\in V} \left(Y_1^-(y) \to Y_2^-(y)\right).$$

Clearly, for $(Y_1^+, Y_1^-), (Y_2^+, Y_2^-) \in L^V \times L^V$, $S^*((Y_1^+, Y_1^-), (Y_2^+, Y_2^-)) = S(Y_1^+, Y_2^+) \wedge S(Y_1^-, Y_2^-)$.

Proposition 2.3 [3, 5] *Let $(\uparrow, \downarrow)$ be an L-fuzzy Galois connection between U and V. Then there exists an L-fuzzy relation $I \in L^{U\times V}$ such that for the induced mappings $\uparrow_I$ and $\downarrow_I$ it holds that $(\uparrow, \downarrow) = (\uparrow_I, \downarrow_I)$.*

2.2 Three-way concept analysis

In three-way concept analysis, the three-way operators and their inverse are defined and their properties were given. Based on these operators, two types of three-way concepts were defined and the corresponding three-way concept lattices were constructed.

In a formal context (U, V, I), for $X \subseteq U$, $Y \subseteq V$ and $I \subseteq U \times V$, a pair of operators, $* : 2^U \to 2^V$ and $* : 2^V \to 2^U$, are defined by

$$X^* = \{y \in V \mid \forall x \in X, (x, y) \in I\},$$
$$Y^* = \{x \in U \mid \forall y \in Y, (x, y) \in I\}.$$

This pair of operators is called derivation operators in formal concept analysis [11]. Sometimes, to stress the underlying binary relation, we write $(*, *)$ as $(*_I, *_I)$.

For $X \subseteq U$ and $Y \subseteq V$, a pair of three-way operators $(*_T, *_T)$ are defined as follows: $*_T : 2^U \to 2^V \times 2^V, *_T : 2^V \times 2^V \to 2^U$ as follows:

$X^{*_T} = (Y^+, Y^-)$, where $Y^+ = \{y \in V \mid \forall x \in X, (x, y) \in I\}$, $Y^- = \{y \in V \mid \forall x \in X, (x, y) \in I^c\}$.

$(Y^+, Y^-)^{*_T} = \{x \in U \mid \forall y \in Y^+, (x, y) \in I\} \cap \{x \in U \mid \in Y^-, (x, y) \in I^c\}$.

Indeed, a trivial verification shows that $X^{*_T} = (X^{*_I}, X^{*_{I^c}})$ and $(Y^+, Y^-)^{*_T} = (Y^+)^{*_I} \cap (Y^-)^{*_{I^c}}$. We can thus conclude that the proposed three-way concept analysis is obtained by combining derivation operators in classical formal concept analysis.

For $X \in 2^U$ and $(Y^+, Y^-) \in 2^V \times 2^V$, if $X^{*_T} = (Y^+, Y^-)$ and $(Y^+, Y^-)^{*_T} = X$, then the pair $(X, (Y^+, Y^-))$ is called a three-way concept.

Define a binary relation $\le$ on the collection of three-way concepts by

$$(X_1, (Y_1^+, Y_1^-)) \le (X_2, (Y_2^+, Y_2^-)) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow (Y_2^+, Y_2^-) \subseteq (Y_1^+, Y_1^-),$$

then it has been shown in [28] that $\le$ is a partial order, and the collection of three-way concepts forms a complete lattice with respect to $\le$. The infimum and supremum are given by

$$\big(X_1,(Y_1^+,Y_1^-)\big)\vee\big(X_2,(Y_2^+,Y_2^-)\big) = \big((X_1\cup X_2)^{*_T*_T},(Y_1^+,Y_1^-)\cap(Y_2^+,Y_2^-)\big),$$
$$\big(X_1,(Y_1^+,Y_1^-)\big)\wedge\big(X_2,(Y_2^+,Y_2^-)\big) = \big(X_1\cap X_2,\big((Y_1^+,Y_1^-)\cup(Y_2^+,Y_2^-)\big)^{*_T*_T}\big).$$

3 *L*-fuzzy concept lattices based on three-way decisions

In this section, we aim to introduce the basic ideas underlying fuzzy logic into the study of three-way formal concept analysis. This leads naturally to the notion of L-fuzzy three-way concepts. As will be shown below, the collection of L-fuzzy three-way concepts forms a lattice under the usual partial order. The properties of L-fuzzy three-way concept lattices are also investigated in detail.

Definition 3.1 Define a pair of operators $(\uparrow_T,\downarrow_T)$ between L^U and $L^V\times L^V$ ($\uparrow_T: L^U\to L^V\times L^V, \downarrow_T: L^V\times L^V\to L^U$) as follows: $\forall X\in L^U,(Y^+,Y^-)\in L^V\times L^V, x\in U, y\in V$,

$$X^{\uparrow_T}(y)=\big(Z^+(y),Z^-(y)\big),\ \big(Y^+,Y^-\big)^{\downarrow_T}(x) = \bigwedge_{y\in V}\big(Y^+(y)\to I(x,y)\big)\wedge\bigwedge_{y\in V}(Y^-(y)\to\neg I(x,y)), \tag{2}$$

where $Z^+(y)=\bigwedge_{x\in U}(X(x)\to I(x,y)), Z^-(y)=\bigwedge_{x\in U}\to\neg I(x,y)), (Z^+,Z^-)\in L^V\times L^V$.

A trivial verification shows that for each $X\in L^U$, $X^{\uparrow_T}=(X^{\uparrow_I},X^{\uparrow_{\neg I}})$, that is, $\uparrow_T$ is obtained by considering both $\uparrow_I$ and $\uparrow_{\neg I}$ simultaneously.

In the case of $L=\{0,1\}$, it can be easily checked that $Y^+=\{y\in V\mid \forall x\in X,(x,y)\in I\}, Y^-=\{y\in V\mid\forall x\in X,(x,y)\in I^c\}$. Similarly, we have $(Y^+,Y^-)^{\downarrow_T}=\{x\in U\mid \forall y_1\in Y^+,(x,y_1)\in I,\forall y_2\in Y^-,(x,y_2)\in I^c\}$. That is, $(\uparrow_T,\downarrow_T)=(*_T,*_T)$. And thus, the proposed L-fuzzy three-way concept analysis is a natural generalization of three-way concept analysis in [20].

The pair of operators $(\uparrow_T,\downarrow_T)$ forms a type of L-fuzzy Galois connection, as will be shown below.

Proposition 3.1 *Let $(\uparrow_T,\downarrow_T)$ be the pair of operators defined by formula* (2), *then for any $X\in L^U,(Y^+,Y^-)\in L^V\times L^V$, we have*

$$S(X,(Y^+,Y^-)^{\downarrow_T})=S^*((Y^+,Y^-),X^{\uparrow_T}). \tag{3}$$

Proof According to formula (1), we have

$$\begin{aligned}
S(X,(Y^+,Y^-)^{\downarrow_T}) &= \bigwedge\nolimits_{x\in U}\big(X(x)\to(Y^+,Y^-)^{\downarrow_T}(x)\big)\\
&=\bigwedge\nolimits_{x\in U}\Big(X(x)\to\bigwedge\nolimits_{y\in V}((Y^+(y)\to I(x,y))\wedge(Y^-(y)\to\neg I(x,y)))\Big)\\
&=\bigwedge\nolimits_{x\in U}\Big(X(x)\to\bigwedge\nolimits_{y\in V}(Y^+(y)\to I(x,y))\wedge\bigwedge\nolimits_{y\in V}(Y^-(y)\to\neg I(x,y))\Big)\\
&=\bigwedge\nolimits_{x\in U}\Big((X(x)\to\bigwedge\nolimits_{y\in V}(Y^+(y)\\
&\quad\to I(x,y)))\wedge\Big(X(x)\to\bigwedge\nolimits_{y\in V}(Y^-(y)\to\neg I(x,y)))\Big)\\
&=\bigwedge\nolimits_{x\in U}\big((X(x)\to(Y^+)^{\downarrow_I}(x))\wedge(X(x)\to(Y^-)^{\downarrow_{\neg I}}(x))\big)\\
&=\bigwedge\nolimits_{x\in U}\Big(X(x)\to(Y^+)^{\downarrow_I}(x))\wedge\bigwedge\nolimits_{x\in U}(X(x)\to(Y^-)^{\downarrow_{\neg I}}(x)\Big)\\
&=S(X,(Y^+)^{\downarrow_I})\wedge S(X,(Y^-)^{\downarrow_{\neg I}})\\
&=S(Y^+,X^{\uparrow_I})\wedge S(Y^-,X^{\uparrow_{\neg I}})\\
&=S^*((Y^+,Y^-),(X^{\uparrow_I},X^{\uparrow_{\neg I}}))\\
&=S^*((Y^+,Y^-),X^{\uparrow_T}).
\end{aligned}$$

Thus, the proposition is proved. □

Contrary to the fact that for any L-fuzzy Galois connection $(\uparrow, \downarrow)$, there exists an L-fuzzy relation I such that $(\uparrow, \downarrow) = (\uparrow_I, \downarrow_I)$, the converse of Proposition 3.1 is not necessarily true, that is, for $(\uparrow_T, \downarrow_T)$ satisfying formula (3), it does not necessarily enjoy the form of formula (2). Indeed, for any two L-fuzzy relations I_1 and I_2, we define operators $\uparrow_T : L^U \to L^V \times L^V$ and $\downarrow_T : L^V \times L^V \to L^U$ by $X^{\uparrow_T} = (Y_1, Y_2)$ and $(Y^+, Y^-)^{\downarrow_T}(x) = \bigwedge_{y\in V}(Y^+(y) \to I_1(x, y)) \wedge \bigwedge_{y\in V}(Y^-(y) \to I_2(x, y))$, where $Y_1(y) = \bigwedge_{x\in U}(X(x) \to I_1(x, y)), y \in V$ and $Y_2(y) = \bigwedge_{x\in U}(X(x) \to I_2(x, y)), y \in V$. An easy verification shows that $(\uparrow_T, \downarrow_T)$ also satisfies the formula (3).

The pair of operators $(\uparrow_T, \downarrow_T)$ defined by formula (2) satisfies the following properties.

Proposition 3.2 *For* $X, X_1, X_2 \in L^U, (Y^+, Y^-), (Y_1^+, Y_1^-), (Y_2^+, Y_2^-) \in L^V \times L^V$,

(i) $X_1 \subseteq X_2$ *implies that* $X_2^{\uparrow_T} \subseteq X_1^{\uparrow_T}$,
(ii) $(Y_1^+, Y_1^-) \subseteq (Y_2^+, Y_2^-)$ *implies that* $(Y_2^+, Y_2^-)^{\downarrow_T} \subseteq (Y_1^+, Y_1^-)^{\downarrow_T}$,
(iii) $X \subseteq X^{\uparrow_T\downarrow_T}$,
(iv) $X^{\uparrow_T} = X^{\uparrow_T\downarrow_T\uparrow_T}, (Y^+, Y^-)^{\downarrow_T} = (Y^+, Y^-)^{\downarrow_T\uparrow_T\downarrow_T}$,
(v) $(Y^+, Y^-) \subseteq (Y^+, Y^-)^{\downarrow_T\uparrow_T}$,
(vi) $(\bigvee_{i\in I} X_i)^{\uparrow_T} = \bigwedge_{i\in I} X_i^{\uparrow_T}$,
(vii) $(\bigvee_{i\in I}(Y_i^+, Y_i^-))^{\downarrow_T} = \bigwedge_{i\in I}(Y_i^+, Y_i^-)^{\downarrow_T}$.

We use L_o^3 to denote the collection of fixed points of $(\uparrow_T, \downarrow_T)$, that is,

$$L_o^3 = \{(X, (Y^+, Y^-)) \mid X^{\uparrow_T} = (Y^+, Y^-), (Y^+, Y^-)^{\downarrow_T} = X\}.$$

For any $(X, (Y^+, Y^-)) \in L_o^3$, since a pair of L-fuzzy sets (Y^+, Y^-) on V (the set of attributes) is associated with an L-fuzzy set X on U (the set of objects), we therefore call $(X, (Y^+, Y^-))$ an object-induced L-fuzzy three-way concept. Define a binary relation $\leq_o$ on L_o^3 as follows:

$(X_1, (Y_1^+, Y_1^-)) \leq_o (X_2, (Y_2^+, Y_2^-)) \Leftrightarrow X_1 \subseteq X_2$, or equivalently, $(Y_2^+, Y_2^-) \subseteq (Y_1^+, Y_1^-)$.

It can be shown that $\leq_o$ satisfies the reflexivity condition, anti-symmetry condition and the transitivity condition, and thus, $\leq_o$ is a partial order on L_o^3. We are now in a position to show whether L_o^3 forms a lattice under the partial order $\leq_o$ or not. The corresponding result is summarized in the following proposition.

Proposition 3.3 $(L_o^3, \leq_o)$ *forms a complete lattice, where for* $\{(X_i, (Y_i^+, Y_i^-))\}_{i\in I} \subseteq L_o^3$,

$$\bigvee \{(X_i, (Y_i^+, Y_i^-))\}_{i\in I} = \left((\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T}, \bigwedge_{i\in I}(Y_i^+, Y_i^-) \right),$$

$$\bigwedge \{(X_i, (Y_i^+, Y_i^-))\}_{i\in I} = \left(\bigwedge_{i\in I} X_i, (\bigvee_{i\in I}(Y_i^+, Y_i^-))^{\downarrow_T\uparrow_T} \right).$$

Proof We have from Proposition 3.2 that $(\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T} = (\bigwedge_{i\in I} X_i^{\uparrow_T})^{\downarrow_T} = (\bigwedge_{i\in I}(Y_i^+, Y_i^-))^{\downarrow_T}$ and $(\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T\uparrow_T} = (\bigvee_{i\in I} X_i)^{\uparrow_T} = \bigwedge_{i\in I}(Y_i^+, Y_i^-)$, which shows that $((\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T}, \bigwedge_{i\in I}(Y_i^+, Y_i^-))$ is an L-fuzzy three-way concept. Similarly, we can also show that $(\bigwedge_{i\in I} X_i, (\bigvee_{i\in I}(Y_i^+, Y_i^-))^{\downarrow_T\uparrow_T})$ is an L-fuzzy three-way concept. Moreover, observe from the partial order defined on L_o^3 that $\forall i \in I, (X_i, (Y_i^+, Y_i^-)) \leq_o ((\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T}, \bigwedge_{i\in I}(Y_i^+, Y_i^-))$, that is, $((\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T}, \bigwedge_{i\in I}(Y_i^+, Y_i^-))$ is a upper bound of $\{(X_i, (Y_i^+, Y_i^-)) \mid i \in I\}$. For any concept $(C, (D^+, D^-))$ satisfying $(X_i, (Y_i^+, Y_i^-)) \leq_o (C, (D^+, D^-)), i \in I$, we have $((\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T}, \bigwedge_{i\in I}(Y_i^+, Y_i^-)) \leq_o (C, (D^+, D^-))$, that is, $((\bigvee_{i\in I} X_i)^{\uparrow_T\downarrow_T}$, $\bigwedge_{i\in I}(Y_i^+, Y_i^-))$ is the least L-fuzzy three-way concept larger than or equal to $\{(X_i, (Y_i^+, Y_i^-))\}_{i\in I}$, that is, it is the supremum of $\{(X_i, (Y_i^+, Y_i^-))\}_{i\in I}$. The other equality can be shown similarly. □

Example 3.1 Let $L = [0, 1]$ and (U, V, I) be an L-fuzzy formal context with $U = \{o_1, o_2, o_3\}, V = \{a, b, c, d\}$, the binary relation I is depicted in Table 1. Take $\to$ as the R_0 implication operator, which is given by

$$x \to y = \begin{cases} 1, & x \leq y, \\ (1-x) \vee y, & x > y. \end{cases} \quad x, y \in [0, 1]$$

Take $X = \frac{0.1}{o_1} + \frac{0.1}{o_2} + \frac{0.1}{o_3}$, then by computing, we have that $X^{\uparrow_T} = (\frac{1.0}{a} + \frac{0.9}{b} + \frac{1.0}{c} + \frac{1.0}{d}, \frac{0.9}{a} + \frac{1.0}{b} + \frac{1.0}{c} + \frac{1.0}{d})$. Similarly, let $(Y^+, Y^-) = (\frac{1.0}{a} + \frac{0.9}{b} + \frac{1.0}{c} + \frac{1.0}{d}, \frac{0.9}{a} + \frac{1.0}{b} + \frac{1.0}{c} + \frac{1.0}{d})$, then we have from Definition 3.1 that $(Y^+, Y^-)^{\downarrow_T} = \frac{0.1}{o_1} + \frac{0.1}{o_2} + \frac{0.1}{o_3}$. That is, $X^{\uparrow_T} = (Y^+, Y^-)$ and $(Y^+, Y^-)^{\downarrow_T} = X$, then according to Definition 3.1, we conclude that $(X, (Y^+, Y^-))$ is an object-induced L-fuzzy three-way concept.

Using an analogous manner, we can also obtain that $(\frac{0.8}{o_1} + \frac{0.7}{o_2} + \frac{0.6}{o_3}, (\frac{0.5}{a} + \frac{0.3}{b} + \frac{0.4}{c} + \frac{0.2}{d}, \frac{0.2}{a} + \frac{0.7}{b} + \frac{0.3}{c} + \frac{1.0}{d}))$ is also an object-induced L-fuzzy three-way concept.

Table 1 A fuzzy formal context

	a	b	c	d
o_1	1.0	0.3	0.7	0.1
o_2	0.5	0.0	0.4	0.2
o_3	0.7	0.1	0.2	0.2

So far we have mainly focused on the notions of object-induced L-fuzzy three-way concept lattices. By using a similar method, one can also construct attribute-induced L-fuzzy three-way concept lattices, as specified below.

Definition 3.2 Define a pair of operators $(\uparrow_T, \downarrow_T)$ between $L^U \times L^U$ and L^V ($\uparrow_T: L^U \times L^U \to L^V, \downarrow_T: L^V \to L^U \times L^U$) as follows: $\forall Y \in L^V, (X^+, X^-) \in L^U \times L^U$,

$$Y^{\downarrow_T} = (Z^+, Z^-), (X^+, X^-)^{\uparrow_T}(y) = \bigwedge_{x \in U}(X^+(x) \to I(x, y)) \wedge \bigwedge_{x \in U}(X^-(x) \to \neg I(x, y)),$$

where $Z^+(x) = \bigwedge_{y \in V}(Y(y) \to I(x, y)), Z^-(x) = \bigwedge_{y \in V}(Y(y) \to \neg I(x, y)), x \in U, (Z^+, Z^-) \in L^U \times L^U$.

The pair of operators $(\uparrow_T, \downarrow_T)$ satisfies a type of L-Galois connection, as will be shown below.

Proposition 3.4 *Let $(\uparrow_T, \downarrow_T)$ be the pair of operators given in Definition 3.2, then for any $(X^+, X^-) \in L^U \times L^U$ and $Y \in L^V$, we have $S^*((X^+, X^-), Y^{\downarrow_T}) = S(Y, (X^+, X^-)^{\uparrow_T})$.*

Proposition 3.5 *For $(X^+, X^-), (X_1^+, X_1^-), (X_2^+, X_2^-) \in L^U \times L^U, Y, Y_1, Y_2 \in L^V$,*

(i) $(X_1^+, X_1^-) \subseteq (X_2^+, X_2^-)$ implies that $(X_2^+, X_2^-)^{\uparrow_T} \subseteq (X_1^+, X_1^-)^{\uparrow_T}$,
(ii) $Y_1 \subseteq Y_2$ implies that $Y_2^{\downarrow_T} \subseteq Y_1^{\downarrow_T}$,
(iii) $(X^+, X^-) \subseteq (X^+, X^-)^{\uparrow_T \downarrow_T}$,
(iv) $(X^+, X^-)^{\uparrow_T} = X^{\uparrow_T \downarrow_T \uparrow_T}, Y^{\downarrow_T} = Y^{\downarrow_T \uparrow_T \downarrow_T}$,
(v) $Y \subseteq Y^{\downarrow_T \uparrow_T}$,
(vi) $(\bigvee_{i \in I}(X_i^+, X_i^-))^{\uparrow_T} = \bigwedge_{i \in I}(X_i^+, X_i^-)^{\uparrow_T}$,
(vii) $(\bigvee_{i \in I} Y_i)^{\downarrow_T} = \bigwedge_{i \in I} Y_i^{\downarrow_T}$.

We use L_a^3 to denote the collection of fixed points of $(\uparrow_T, \downarrow_T)$ given in Definition 3.2, that is,

$$L_a^3 = \{((X^+, X^-), Y) \mid (X^+, X^-)^{\uparrow_T} = Y, Y^{\downarrow_T} = (X^+, X^-)\}.$$

For any $((X^+, X^-), Y) \in L_a^3$, since a pair of L-fuzzy sets (X^+, X^-) on U (the set of objects) is associated with an L-fuzzy set Y on V (the set of attributes), we therefore call $((X^+, X^-), Y)$ an attribute-induced L-fuzzy three-way concept. Define a binary relation $\leq_a$ on L_a^3 as follows:

$$((X_1^+, X_1^-), Y_1) \leq_a ((X_2^+, X_2^-), Y_2) \Leftrightarrow (X_1^+, X_1^-) \subseteq (X_2^+, X_2^-), \text{ or equivalently } Y_2 \subseteq Y_1. \quad (4)$$

Proposition 3.6 *$(L_a^3, \leq_a)$ forms a complete lattice, where for $\{((X_i^+, X_i^-), Y_i)\}_{i \in I} \subseteq L_a^3$,*

$$\bigvee\{((X_i^+, X_i^-), Y_i)\}_{i \in I} = \left(\bigwedge_{i \in I}(X_i^+, X_i^-), \left(\bigvee_{i \in I} Y_i\right)^{\downarrow_T \uparrow_T}\right),$$

$$\bigwedge\{((X_i^+, X_i^-), Y_i)\}_{i \in I} = \left(\left(\bigvee_{i \in I}(X_i^+, X_i^-)\right)^{\uparrow_T \downarrow_T}, \bigwedge_{i \in I} Y_i\right).$$

The proof concerning the above results can be shown in a similar way to that in object-induced L-fuzzy three-way concept lattices, and so we omit the details here.

Example 3.2 Let $L = [0, 1]$ and (U, V, I) be an L-fuzzy formal context with $U = \{o_1, o_2, o_3\}, V = \{a, b, c, d\}$, the binary relation I is depicted in Table 1. Take $\to$ as the R_0 implication operator, which is given by

$$x \to y = \begin{cases} 1, & x \leq y, \\ (1 - x) \vee y, & x > y. \end{cases} \quad x, y \in [0, 1]$$

Take $(X^+, X^-) = (\frac{0.1}{o_1} + \frac{0.1}{o_2} + \frac{0.1}{o_3}, \frac{0.0}{o_1} + \frac{0.5}{o_2} + \frac{0.3}{o_3})$, then by computing, we have that $(X^+, X^-)^{\uparrow_T} = \frac{1.0}{a} + \frac{0.9}{b} + \frac{1.0}{c} + \frac{1.0}{d}$. Similarly, let $Y = \frac{1.0}{a} + \frac{0.9}{b} + \frac{1.0}{c} + \frac{1.0}{d}$, then we have from Definition 3.2 that $Y^{\downarrow_T} = (\frac{0.1}{o_1} + \frac{0.1}{o_2} + \frac{0.1}{o_3}, \frac{0.0}{o_1} + \frac{0.5}{o_2} + \frac{0.3}{o_3})$. That is, $(X^+, X^-)^{\uparrow_T} = Y$ and $Y^{\downarrow_T} = (X^+, X^-)$, then according to Definition 3.2, we conclude that $((X^+, X^-), Y)$ is an attribute-induced L-fuzzy three-way concept.

Take $Y = \frac{0.5}{a} + \frac{0.2}{b} + \frac{0.3}{c} + \frac{0.1}{d}$, then by computing, we obtain that $Y^{\downarrow_T} = (\frac{1.0}{o_1} + \frac{0.8}{o_2} + \frac{0.7}{o_3}, \frac{0.5}{o_1} + \frac{1.0}{o_2} + \frac{0.5}{o_3})$. Let $(X^+, X^-) = (\frac{1.0}{o_1} + \frac{0.8}{o_2} + \frac{0.7}{o_3}, \frac{0.5}{o_1} + \frac{1.0}{o_2} + \frac{0.5}{o_3})$, then we have from Definition 3.2 that $(X^+, X^-)^{\uparrow_T} = \frac{0.5}{a} + \frac{0.2}{b} + \frac{0.3}{c} + \frac{0.1}{d}$. That is, $(X^+, X^-)^{\uparrow_T} = Y$ and $Y^{\downarrow_T} = (X^+, X^-)$, then according to Definition 3.2, we conclude that $((X^+, X^-), Y)$ is an attribute-induced L-fuzzy three-way concept.

It should be noted that there exists similar research works along this research line. For instance, in [15], Hu et al. generalized measurement on decision conclusion in three-way decision spaces from fuzzy lattices to partially ordered sets. In [23], Li et al. focused on three-way concept learning via multi-granularity from the viewpoint of cognition. In [33, 34], Prem Kumar Singh analyzed the uncertainty and incompleteness in the given fuzzy attribute set characterized by truth-membership, indeterminacy-membership, and falsity membership functions of a defined single-valued neutrosophic set. In [35, 36], they also proposed an algorithm for generating the bipolar fuzzy formal concepts and two methods based on the properties of next neighbors and Euclidean distance for knowledge extraction. In [42, 43], Yao presented a common conceptual framework of the notions of interval sets and incomplete formal contexts for representing partially-known concepts.

A comparative study is performed as follows:

(1) Comparison with Hu's research work [15]: Firstly, these two research works are conducted from different viewpoints. Concretely, Hu's research work is mainly conducted from the axiomatic viewpoint, they attempt to generalize measurement on decision conclusion in three-way decision spaces from fuzzy lattices to partially ordered sets due to the fact that the family of hesitant fuzzy sets are partially ordered sets but not necessarily fuzzy lattices. Secondly, the hesitant fuzzy set considered in [15] takes a subset of [0,1] as its truth value, while in our approach, the membership degree of each L-fuzzy sets takes exactly one value.

(2) Comparison with Li's research work [23]: Both studies focus on combination of three-way decision and formal concept analysis. However, there exist visible differences between them. In [23], the authors put forward an axiomatic approach to describe three-way concepts while in the present paper, we mainly employ the constructive approach. Furthermore, three-way cognitive operators in [23] were defined in the crisp setting while the concept-forming operators in our paper are presented in L-fuzzy environment.

(3) Comparison with Prem Kumar Singh's research work: In [33, 34], a neutrosophic set of attributes (resp. objects) is characterized by a truth-membership function, a indeterminacy-membership function and a falsity-membership function independently. Contrarily, in our research work, both the extents and intents of a fuzzy concept are represented by L-fuzzy sets with L being a residuated lattice. For an L-fuzzy set X on the set of objects U and an object $o \in U$ (resp. an attribute $a \in A$), $X(o)$ denotes the membership degree of o with respect to X, according to the algebraic semantics of a residuated lattice, $\neg X(o)$ means the non-membership degree of o with respect to X. That is, we only consider the truth-membership degree and the non-membership degree. We don't consider the indeterminacy-membership degree. Moreover, in Prem Kumar Singh's research work, the neutrosophic set N satisfies the condition $0^+ \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$ while in our approach, even in case $L = [0, 1]$, such a condition does not necessarily hold. Lastly, both the approaches to generating formal concepts are based on different principle. Concretely, in [33, 34], the formal fuzzy concept can be interpreted as neutrosophic set of objects having maximal truth membership value, minimum indeterminacy and minimum falsity membership value with respect to integrating the information from the common set of fuzzy attributes in a defined three-way space $[0, 1]^3$ using component-wise Godel residuated lattice, while in our approach, extents and intents have the same relationship as in the crisp case, they are just the fuzzy generalization of classical three-way concepts. In Prem Kumar Singh's research work [35, 36], they proposed an algorithm for generating the bipolar fuzzy formal concepts and the corresponding lattice structure. The proposed approach and our approach both embody the idea of three-way decision. However, results in [35, 36] are obtained based on the bipolar formal context while our results are derived from an unipolar context. Furthermore, the construction methods for bipolar concepts and L-fuzzy three-way concepts are fundamentally different.

(4) Comparison with Yao's research work [42]: Yao's work was conducted in the context of an incomplete formal context, which can be interpreted as a family of complete formal contexts, and any one in the family may possibly be the actual formal context when the information or knowledge becomes complete. Contrarily, in our approach, for each object o and each attribute a, $I(0, a)$ has a definite value, in this sense, the present study is conducted in the context of a complete formal context; Moreover, the formal context in [42] is a binary one while that in our paper is an L-fuzzy one, whose value is taken from a complete residuated lattice.

4 Fuzzy inference based on *L*-fuzzy three-way concept lattice

In this section, we will study the fuzzy inference based on the proposed L-fuzzy three-way concept lattices. We mainly focus on such a problem: when the L-fuzzy relation I is unknown, how to calculate new fuzzy three-way concepts from the given ones.

More precisely,

$$
\begin{array}{ll}
\text{suppose that} & (X_1, (Y_1^+, Y_1^-)) \\
 & (X_2, (Y_2^+, Y_2^-)) \\
 & \cdots\cdots\cdots\cdots\cdots\cdots \\
 & (X_n, (Y_n^+, Y_n^-))
\end{array}
$$

are n L-fuzzy three-way concepts,

$$
\begin{array}{ll}
\text{and given} & X \\
 & \cdots\cdots\cdots\cdots\cdots\cdots \\
\text{calculate} & (Y^+, Y^-),
\end{array}
$$

where $X_i, X \in L^U, (Y_i^+, Y_i^-), (Y^+, Y^-) \in L^V \times L^V$.

Our idea is to introduce two types of fuzzy inference rules, by which we can calculate new L-fuzzy three-way concepts. A natural requirement for such fuzzy inference methods is that it should "agree" with the original data, i.e. if the input set is a extent (or intent) of a known fuzzy concept, the output set should be the corresponding intent (or extent). In the following, if a fuzzy inference method exactly "agree" with the original data, the fuzzy inference method is said to be coherent.

Definition 4.1 Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice, and $(X_i, (Y_i^+, Y_i^-)) \in L_o^3(U, V, R)(i = 1, \cdots, n)$. Given $X \in L^U$, we define the lower approximation, denoted by (Y^+, Y^-), of $X^{\uparrow_T}$ as follows:

$$Y^+(y) = \bigvee_{i=1}^n (Y_i^+(y) \otimes S(X, X_i)), y \in V, Y^+ \in L^V,$$
$$Y^-(y) = \bigvee_{i=1}^n (Y_i^-(y) \otimes S(X, X_i)), y \in V, Y^- \in L^V.$$

Given $(Y^+, Y^-) \in L^U$, we define the lower approximation, denoted by X, of $(Y^+, Y^-)^{\downarrow_T}$ as follows:

$$X(x) = \bigvee_{i=1}^n (X_i(y) \otimes S^*((Y^+, Y^-), (Y_i^+, Y_i^-))), x \in U.$$

To show the coherence of the above fuzzy inference, we need the following lemma.

Lemma 4.1 *Let (U, V, I) be an L-fuzzy formal context and $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice, then for all $X_1, X_2 \in L^U, (Y_1^+, Y_2^+) \in L^V$:*

$$S(X_1, X_2) \leq S^*(X_2^{\uparrow_T}, X_1^{\uparrow_T}), S^*((Y_2^+, Y_2^-), (Y_1^+, Y_1^-)) \leq S((Y_1^+, Y_1^-)^{\downarrow_T}, (Y_2^+, Y_2^-)^{\downarrow_T}).$$

Proof Note first that in [10], it is shown that $S(X_1, X_2) \leq S(X_2^{\uparrow}, X_1^{\uparrow})$ and $S(Y_1, Y_2) \leq S(Y_2^{\downarrow}, Y_1^{\downarrow})$ for each $X_1, X_2 \in L^U$ and $Y_1, Y_2 \in L^V$, which implies that $S(X_1, X_2) \leq S(X_2^{\uparrow_I}, X_1^{\uparrow_I})$ and $S(X_1, X_2) \leq S(X_2^{\uparrow_{\neg I}}, X_1^{\uparrow_{\neg I}})$. Consequently,

$$\begin{aligned} S^*\left(X_2^{\uparrow_T}, X_1^{\uparrow_T}\right) &= S^*\left(\left(X_2^{\uparrow_I}, X_2^{\uparrow_{\neg I}}\right), \left(X_1^{\uparrow_I}, X_1^{\uparrow_{\neg I}}\right)\right) \\ &= S\left(\left(X_2^{\uparrow_I}\right), \left(X_1^{\uparrow_I}\right)\right) \wedge S\left(\left(X_2^{\uparrow_{\neg I}}\right), \left(X_1^{\uparrow_{\neg I}}\right)\right) \\ &\geq S(X_1, X_2) \wedge S(X_1, X_2) \\ &= S(X_1, X_2). \end{aligned}$$

Similarly, we have

$$\begin{aligned} S\left(\left(Y_1^+, Y_1^-\right)^{\downarrow_T}, \left(Y_2^+, Y_2^-\right)^{\downarrow_T}\right) &= S\left(\left(Y_1^+\right)^{\downarrow_I} \wedge \left(Y_1^-\right)^{\downarrow_{\neg I}}, \left(Y_2^+\right)^{\downarrow_I} \wedge \left(Y_2^-\right)^{\downarrow_{\neg I}}\right) \\ &\geq S\left(\left(Y_1^+\right)^{\downarrow_I}, \left(Y_2^+\right)^{\downarrow_I}\right) \wedge S\left(\left(Y_1^-\right)^{\downarrow_I}, \left(Y_2^-\right)^{\downarrow_I}\right) \\ &\geq S\left(Y_2^+, Y_1^+\right) \wedge S\left(Y_2^-, Y_1^-\right) \\ &= S^*\left(\left(Y_2^+, Y_2^-\right), \left(Y_1^+, Y_1^-\right)\right), \end{aligned}$$

as desired. □

Proposition 4.1 *The fuzzy inference rules defined by Definition 4.1 are coherent.*

Proof (i) We will firstly show that for all $(X_i, (Y_i^+, Y_i^-))(1 \leq i \leq n), X = X_j$ implies $(Y^+, Y^-) = (Y_j^+, Y_j^-)$.

Indeed, we have from Lemma 4.1 that for any i satisfying $1 \leq i \leq n$ and $i \neq j$, $S(X_j, X_i) \leq S(Y_i^+, Y_j^+) = \bigwedge_{y \in V}(Y_i^+(y) \rightarrow Y_j^+(y)) \leq Y_i^+(y) \rightarrow Y_j^+(y)$. Then we have from the adjoint property of $(\otimes, \rightarrow)$ that $S(X_j, X_i) \otimes Y_i^+(y) \leq Y_j^+(y)$.

Hence, $Y^+(y) = \bigvee_{i=1}^n (Y_i^+(y) \otimes S(X, X_i)) = \bigvee_{i=1}^n (Y_i^+(y) \otimes S(X_j, X_i)) = Y_j^+(y) \otimes S(X_j, X_j) = Y_j^+(y) \otimes 1 = Y_j^+(y)$.

Similarly, since $S(X_j, X_i) \leq S(Y_i^-, Y_j^-) = \bigwedge_{y \in V}(Y_i^-(y) \rightarrow Y_j^-(y)) \leq Y_i^-(y) \rightarrow Y_j^-(y)$. Then we have from the adjoint property of $(\otimes, \rightarrow)$ that $S(X_j, X_i) \otimes Y_i^-(y) \leq Y_j^-(y)$.

Hence, $Y^-(y) = \bigvee_{i=1}^n (Y_i^-(y) \otimes S(X, X_i)) = \bigvee_{i=1}^n (Y_i^-(y) \otimes S(X_j, X_i)) = Y_j^-(y) \otimes S(X_j, X_j) = Y_j^-(y) \otimes 1 = Y_j^-(y)$.

(ii) We will then show that for all $(X_i, (Y_i^+, Y_i^-))(1 \leq i \leq n)$, if $(Y^+, Y^-) = (Y_j^+, Y_j^-)$, then $X = X_j$ below.

Indeed, for $i \neq j$, since $S^*((Y^+, Y^-), (Y_i^+, Y_i^-)) = S^*((Y_j^+, Y_j^-), (Y_i^+, Y_i^-)) \leq S((Y_i^+, Y_i^-)^{\downarrow_T}, (Y_j^+, Y_j^-)^{\downarrow_T}) = S(X_i, X_j) = \bigwedge_{x \in U}(X_i(x) \rightarrow X_j(x)) \leq X_i(x) \rightarrow X_j(x)$, then we have from the adjoint property of $(\otimes, \rightarrow)$ that $S^*((Y^+, Y^-), (Y_i^+, Y_i^-)) \otimes X_i(x) \leq X_j(y)$.

Consequently, $X(x) = \bigvee_{i=1}^n (X_i(y) \otimes S((Y^+, Y^-), (Y_i^+, Y_i^-))) = \bigvee_{i=1}^n (X_i(y) \otimes S((Y_j^+, Y_j^-), (Y_i^+, Y_i^-))) = X_j(x) \otimes 1 = X_j(x), x \in U$. □

Definition 4.2 Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a complete involutive residuated lattice, and $(X_i, (Y_i^+, Y_i^-)) \in L_o^3(U, V, I), (i = 1, \cdots, n)$. Given $X \in L^U$, we define the upper approximation, denoted by (Y^+, Y^-), of $X^{\uparrow_T}$ as follows:

$$Y^+(y) = \bigwedge_{i=1}^n (\neg Y_i^+(y) \rightarrow T^{\neg}(X_i, X)),$$
$$Y^-(y) = \bigwedge_{i=1}^n (\neg Y_i^-(y) \rightarrow T^{\neg}(X_i, X)), \text{ where } T^{\neg}(X_i, X) = \bigvee_{x \in U} (X_i(x) \otimes \neg X(x)).$$

Given (Y^+, Y^-), we define the upper approximation, denoted by X, of $(Y^+, Y^-)^{\downarrow_T}$ as follows:

$X(x) = \bigwedge_{i=1}^{n} (\neg X_i(x) \to T^{\neg *}((Y_i^+, Y_i^-), (Y^+, Y^-)))$, where $T^{\neg *}((Y_i^+, Y_i^-), (Y^+, Y^-)) = \bigvee_{y \in U} (Y_i^+(x) \otimes \neg Y^+(x)) \vee \bigvee_{y \in U} (Y_i^-(x) \otimes \neg Y^-(x))$.

To prove the coherence of the above inference method, we need the following lemma.

Lemma 4.2 *Let (U, V, I) be an L-fuzzy formal context and $(L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete involutive residuated lattice, then* for $X_1, X_2 \in L^U, (Y_1^+, Y_1^-), (Y_2^+, Y_2^-) \in L^V \times L^V$:

$$T^{\neg}(X_1, X_2) \geq T^{\neg *}\left(X_2^{\uparrow_T}, X_1^{\uparrow_T}\right),$$

$$T^{\neg *}\left(\left(Y_1^+, Y_1^-\right), \left(Y_2^+, Y_2^-\right)\right) \geq T^{\neg}\left(\left(Y_2^+, Y_2^-\right)^{\downarrow_T}, \left(Y_1^+, Y_1^-\right)^{\downarrow_T}\right).$$

Proof Note first that in [9], it is proved that $T^{\neg}(X_1, X_2) \geq T^{\neg}(X_2^{\uparrow}, X_1^{\uparrow})$ and $T^{\neg}(Y_1, Y_2) \geq T^{\neg}(Y_2^{\uparrow}, Y_1^{\uparrow})$ for $X_1, X_2 \in L^U$ and $Y_1, Y_2 \in L^V$, independent on the underlying binary relation in the L-fuzzy context.

Then we have

$$\begin{aligned} T^{\neg *}\left(X_2^{\uparrow_T}, X_1^{\uparrow_T}\right) &= T^{\neg *}\left(\left(X_2^{\uparrow_I}, X_2^{\uparrow_{\neg I}}\right), \left(X_1^{\uparrow_I}, X_1^{\uparrow_{\neg I}}\right)\right) \\ &= T^{\neg}\left(X_2^{\uparrow_I}, X_1^{\uparrow_I}\right) \vee T^{\neg}\left(X_2^{\uparrow_{\neg I}}, X_1^{\uparrow_{\neg I}}\right) \\ &\leq T^{\neg}\left(X_1, X_2\right) \vee T^{\neg}\left(X_1, X_2\right) \\ &= T^{\neg}\left(X_1, X_2\right). \end{aligned}$$

Similarly,

$$\begin{aligned} & T^{\neg}\left(\left(Y_2^+, Y_2^-\right)^{\downarrow_T}, \left(Y_1^+, Y_1^-\right)^{\downarrow_T}\right) \\ &= T^{\neg}\left(Y_2^{+\downarrow_I} \wedge Y_2^{-\downarrow_{\neg I}}, Y_1^{+\downarrow_I} \wedge Y_1^{-\downarrow_{\neg I}}\right) \\ &= \bigvee_{x \in U} \left(\left(Y_2^{+\downarrow_I} \wedge Y_2^{-\downarrow_{\neg I}}\right)(x) \otimes \neg\left(Y_1^{+\downarrow_I} \wedge Y_1^{-\downarrow_{\neg I}}\right)(x)\right) \\ &= \bigvee_{x \in U} \left(\left(Y_2^{+\downarrow_I}(x) \wedge Y_2^{-\downarrow_{\neg I}}(x)\right) \otimes \left(\neg Y_1^{+\downarrow_I}(x) \vee \neg Y_1^{-\downarrow_{\neg I}}(x)\right)\right) \\ &= \bigvee_{x \in U} \left(\left(Y_2^{+\downarrow_I}(x) \wedge Y_2^{-\downarrow_{\neg I}}(x)\right) \otimes \left(\neg Y_1^{+\downarrow_I}(x)\right)\right) \\ &\vee \bigvee_{x \in U} \left(\left(Y_2^{+\downarrow_I}(x) \wedge Y_2^{-\downarrow_{\neg I}}(x)\right) \otimes \left(\neg Y_1^{-\downarrow_{\neg I}}(x)\right)\right) \\ &\leq \bigvee_{x \in U} \left(\left(Y_2^{+\downarrow_I}\right)(x) \otimes \left(\neg Y_1^{+\downarrow_I}(x)\right)\right) \\ &\vee \bigvee_{x \in U} \left(\left(Y_2^{-\downarrow_{\neg I}}\right)(x) \otimes \left(\neg Y_1^{-\downarrow_{\neg I}}(x)\right)\right) \\ &= T^{\neg}\left(Y_2^{+\downarrow_I}, Y_1^{+\downarrow_I}\right) \vee T^{\neg}\left(Y_2^{-\downarrow_I}, Y_1^{-\downarrow_I}\right) \\ &\leq T^{\neg}\left(Y_1^+, Y_2^+\right) \vee T^{\neg}\left(Y_1^-, Y_2^-\right) \\ &= T^{\neg *}\left(\left(Y_1^+, Y_1^-\right), \left(Y_2^+, Y_2^-\right)\right). \end{aligned}$$

□

Proposition 4.2 *The fuzzy inference rules defined by Definition 4.2 are coherent.*

Proof

(1) We will prove that $X = X_j (1 \leq j \leq n)$ implies $(Y^+, Y^-) = (Y_j^+, Y_j^-)$. To do this, we need show the following preliminary results:

(i) For all $1 \leq i \leq n$ and $i \neq j$, $Y_j^+(x) = \neg Y_j^+(x) \to 0 = \neg Y_j^+(x) \to T^{\neg}(X_j, X_j)$,

(ii) $Y_j^+(x) \leq \neg Y_i^+(x) \to T^{\neg}(X_i, X_j)$,

(iii) For all $1 \leq i \leq n$ and $i \neq j$, $Y_j^-(x) = \neg Y_j^-(x) \to 0 = \neg Y_j^-(x) \to T^{\neg}(X_j, X_j)$,

(iv) $Y_j^-(x) \leq \neg Y_i^-(x) \to T^{\neg}(X_i, X_j)$.

Indeed, since $T^{\neg}(X_j, X_j) = \bigvee_{x \in U} (X_j(x) \otimes \neg X_j(x)) = 0$, (i) clearly holds. For the latter one, due to the adjoint property of $(\otimes, \to)$, it is equivalent to show that $Y_j^+(x) \otimes \neg Y_i^+(x) \leq T^{\neg}(X_i, X_j)$. Indeed, $T^{\neg}(X_i, X_j) \geq T^{\neg}(X_j^{\uparrow_I}, X_i^{\uparrow_I}) = T^{\neg}(Y_j^+, Y_i^+) = \bigvee_{x \in U} (Y_j^+(x) \otimes \neg Y_i^+(x)) \geq Y_j^+(x) \otimes \neg Y_i^+(x)$, as desired.

Both (iii) and (iv) can be shown in a similar way.

(2) We will then prove that $(Y^+, Y^-) = (Y_j^+, Y_j^-) (1 \leq j \leq n)$ implies $X = X_j$. As before, we need show the following preliminary results:

(i) $X_j = \neg X_j \to 0 = \neg X_j \to T^{\neg *}((Y_j^+, Y_j^-), (Y^+, Y^-))$,

(ii) For $i \neq j, X_j \leq \neg X_i(x) \to T^{\neg *}((Y_i^+, Y_i^-), (Y_j^+, Y_j^-))$.

Indeed, since $T^{\neg *}((Y_j^+, Y_j^-), (Y_j^+, Y_j^-)) = \bigvee_{y \in U} (Y_j^+(x) \otimes \neg Y_j^+(x)) \vee \bigvee_{y \in U} (Y_j^-(x) \otimes \neg Y_j^-(x)) = 0$, (i) clearly holds. For the latter one, we have from Lemma 4.2 that $T^{\neg *}((Y_i^+, Y_i^-), (Y_j^+, Y_j^-)) \geq T^{\neg}((Y_j^+, Y_j^-)^{\downarrow_T}, (Y_i^+, Y_i^-)^{\downarrow_T}) = T^{\neg}(X_j, X_i) = \bigvee_{x \in U} (X_j(x) \otimes \neg X_i(x)) \geq X_j(x) \otimes \neg X_i(x)$, which together with the adjoint property of $(\otimes, \to)$ implies (ii).

□

Proposition 4.3 *Let $L = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete involutive residuated lattice, for all $X \in L^U, (Y^+, Y^-) \in L^V \times L^V$ and $\{(X_i, (Y_i^+, Y_i^-))\} \subseteq L_o^3(U, V, I)$, we have*

Table 2 Given L-fuzzy three-way concepts

X_i	(Y_i^+, Y_i^-)
$\frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3}$	$(\frac{1.0}{a} + \frac{0.9}{b} + \frac{1.0}{c} + \frac{1.0}{d}, \frac{0.9}{a} + \frac{1.0}{b} + \frac{1.0}{c} + \frac{1.0}{d})$
$\frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3}$	$(\frac{1.0}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{0.5}{d}, \frac{0.5}{a} + \frac{1.0}{b} + \frac{1.0}{c} + \frac{1.0}{d})$
$\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3}$	$(\frac{0.5}{a} + \frac{0.3}{b} + \frac{0.4}{c} + \frac{0.2}{d}, \frac{0.2}{a} + \frac{0.7}{b} + \frac{0.3}{c} + \frac{1.0}{d})$
$\frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.2}{x_3}$	$(\frac{1.0}{a} + \frac{0.5}{b} + \frac{1.0}{c} + \frac{0.5}{d}, \frac{0.5}{a} + \frac{1.0}{b} + \frac{0.5}{c} + \frac{1.0}{d})$

(i) $(\bigvee_{i=1}^{n}(Y_i^+(y)\otimes S(X,X_i)), \bigvee_{i=1}^{n}(Y_i^-(y)\otimes S(X,X_i))) \leq X^{\uparrow_T} \leq (\bigwedge_{i=1}^{n}(\neg Y_i^+(y)\rightarrow T^{\neg}(X_i,X), \bigwedge_{i=1}^{n}(\neg Y_i^-(y)\rightarrow T^{\neg}(X_i,X)),$

(ii) $\bigvee_{i=1}^{n}(X_i(y)\otimes S^*((Y^+,Y^-),(Y_i^+,Y_i^-))) \leq (Y^+,Y^-)^{\downarrow_T} \leq \bigwedge_{i=1}^{n}(\neg X_i(x)\rightarrow T^{\neg *}((Y_i^+,Y_i^-),(Y^+,Y^-))).$

Proof It follows immediately from the proof of both Propositions 4.1 and 4.2. □

Example 4.1 Suppose $A=\{a,b,c,d\}$ is a set of skills and $U=\{x_1,x_2,x_3\}$ is a set of problems, with which the skills in A should be tested.

Assume that the implication is the R_0 operator

$$x\rightarrow y=\begin{cases}1, & x\leq y,\\ (1-x)\vee y, & x>y.\end{cases} \quad x,y\in[0,1]$$

and the complete lattice $L=\{0.0,0.1,0.2,\cdots,1\}$. It is easy to verify that $L=(L,\wedge,\vee,\otimes,\rightarrow,0,1)$ is a complete involutive residuated lattice. Given a set of L-fuzzy three-way concepts $(X_i,(Y_i^+,Y_i^-))(i=1,\cdots,6)$ (Table 2), where $X_i(\cdot)$ is the corresponding degree of problem's similar level and $Y_i^+(\cdot), Y_i^-(\cdot)$ are the corresponding degree of skill level, then (1) X_i is the collection of all problems such that all the skills in Y_i^+ are necessary to solve any problem in X_i, and all the skills in Y_i^- are unnecessary to solve any problem in X_i , (2) Y_i^+ is the collection of all skills that are necessary for all problems in X_i, Y_i^- is the collection of all skills that are unnecessary for all problems in X_i.

In what follows, we list some inferred values depicted from the given L-fuzzy concepts above.

For a set of problems $X=(0.5,0.5,0.4)$, Y^+ is the collection of all skills that are necessary for all problems in X. We calculate by Proposition 4.3 that $(\frac{0.5}{a}+\frac{0.5}{b}+\frac{0.5}{c}+\frac{0.5}{d},\frac{0.5}{a}+\frac{0.5}{b}+\frac{0.5}{c}+\frac{1.0}{d})\leq X^{\uparrow_T}\leq(\frac{1}{a}+\frac{0.5}{b}+\frac{1}{c}+\frac{0.5}{d},\frac{1.0}{a}+\frac{1.0}{b}+\frac{1.0}{c}+\frac{1.0}{d})$.

5 Concluding remarks

In this paper, a modest attempt has been made to introduce the tool of fuzzy logic into three-way formal concept analysis, which leads to the theory of L-fuzzy three-way concept analysis. It is shown that the collection of L-fuzzy three-way concepts form a lattice under the usual order. Moreover, a possibility-theoretic view of L-fuzzy three-way concept lattice is provided. It is shown that the basic operators in L-fuzzy three-way concept analysis can be understood in terms of one of four set-functions in possibility theory. Lastly, two types of fuzzy inference methods based on the notion of L-fuzzy three-way concepts are studied.

It is important to notice that the proposed fuzzy inference approach is based on some known L-fuzzy three-way concepts, another approach based on similarity relations in formal concept analysis [5] can also be expected, which will be reported in our forthcoming papers.

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