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ORIGINAL ARTICLE

Knowledge measure for intuitionistic fuzzy sets with attitude towards non-specificity

Kaihong Guo[1] · Hao Xu[1]

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Abstract
This paper is devoted to an entropy-independent measure of knowledge in the context of intuitionistic fuzzy sets (IFSs). We point out and justify that there are at least two facets of knowledge associated with an IFS, namely the information content and the information clarity. Having this in mind, we put forward a novel axiomatic definition of knowledge measure for IFSs. More specifically, a set of new axioms is presented with which knowledge measure should comply in the context of IFSs. A parametric model following these axioms is then developed to realize this measure. Both facets mentioned above are simultaneously taken into account in the axioms and the model so as to better capture the unique features of an IFS. In particular, we suggest the concept of the amount of potential knowledge related to the hesitancy or non-specificity of an IFS. This allows us to introduce the idea of an attitudinal-based knowledge measure for IFSs. We believe that the knowledge measure provided in this manner could truly reflect the nature of IFSs, and what people really want with different attitudes towards the unknown. Finally, a practical application of the developed technique to decision making under uncertainty is illustrated. As such, the developed measure could be considered as a safe and effective alternative to help tackle some special problems that are difficult to handle by using entropy alone, especially when dealing with the complex situation in which different attitudes of users have to be considered.

Keywords Intuitionistic fuzzy sets · Amount of knowledge · Knowledge measure · Knowledge personalization · Uncertainty modeling

1 Introduction

Atanassov [1–4] introduced the concept of intuitionistic fuzzy sets (IFSs), in which each element is characterized by a membership degree and a non-membership degree, thus generalizing Zadeh's [34] fuzzy sets (FSs) that only assign to each element a membership degree. Motivated by interval-valued fuzzy sets (IVFSs) conceived by Zadeh [36], Atanassov and Gargov [5] further extended IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs), in which the membership degree and the non-membership degree of each element are expressed as intervals rather than real numbers. For its excellent flexibility and agility in coping with vagueness or uncertainty, the theory of FSs/IFSs/ IVIFSs has been widely investigated and applied to a variety of fields [12, 15, 24–27, 29, 31–33, 36]. As an active research topic, the investigation of fuzzy entropy has been receiving much attention from researchers since it was first mentioned by Zadeh [35]. In the current study, the non-probabilistic-type entropy has become a popular research trend, and a detailed overview can be found in [6, 13, 18]. The fact is, this type of entropy, derived directly from the context of FSs, may not be totally reasonable in the context of IFSs and therefore it could suffer from some limitations in real-world applications [13, 40]. In this paper, we are interested only in the seeming dual problem of this type of entropy and dedicated to a measure of knowledge in the context of IFSs, with the aim of tackling some complex problems that are difficult to handle by using entropy alone. With this motivation, in what follows we focus our attention on the axiom and modeling of an entropy-independent, flexible knowledge measure in the context of IFSs and shall not go into further details of fuzzy entropy.

Knowledge is related to the information considered useful in a particular context and it would play an important role in

✉ Kaihong Guo
guokh@126.com

[1] School of Information, Liaoning University, Shenyang 110036, China

many areas such as information theory and decision making under uncertainty. It is generally believed that a measure of knowledge can be viewed as a dual measure of entropy in a fuzzy system. Guo [8] pointed out that this simply couldn't reflect the reality in the context of IFSs as there is no natural logic between these two kinds of measures with the introduction of hesitancy. Nor in the context of IVIFSs [17, 20, 28, 29, 39], for that matter. The fact is, an entropy measure answers the question about the fuzziness but does not consider any peculiarities of how the fuzziness is distributed [22]. Consider, for instance, such a collection of IFSs whose membership degrees are equal to the corresponding non-membership degrees. According to the existing models [15, 16, 21, 23, 37, 38], these IFSs are all characterized by the maximal entropy equal to one, thus leading to the minimal amount of knowledge (equal to zero) associated with them. This is obviously counter-intuitive because there are many different possibilities for a membership degree equal to a non-membership degree and thus these IFSs may differ significantly from each other from a practical point of view. Thus it is necessary and seen as significant work to develop an independent technique with robust properties to take measurements of the amount of knowledge in the context of IFSs/IVIFSs to distinguish between them. The pioneering work by Szmidt, Kacprzyk and Bujnowski [22] on this topic has generated considerable interest. Nguyen [19] developed a knowledge measure with full consideration of the information content conveyed by the membership and non-membership functions of an IFS, but not giving enough attention to the inherent fuzziness of an IFS. Thus the measure may suffer from great limitations for practical use. Nguyen [18] further extended this measure into the context of IVIFSs. Das, Dutta, and Guha [6] also extended the work of [22] into the context of IVIFSs and proposed a set of entropy-based axioms of knowledge measure for IFSs, which is obviously a non-independent axiomatic system and difficult to achieve the goal fully and accurately. Recently, Guo [8] presented, by using the normalized Hamming distance in combination with some related axiom of fuzzy entropy, an axiomatic definition of knowledge measure in the context of IFSs, including a set of axioms and a concrete model following these axioms. It seems as if he put more emphasis on the inherent fuzziness of an IFS during the process of developing that kind of measure.

Given the nature of an IFS, it can be understood that there are at least two facets of knowledge associated with an IFS, one of which is the information content conveyed by the membership and non-membership functions, while the other is related to the inherent fuzziness of specificity with which we introduce the notion of the information clarity. Intuitively, the more information content and the greater information clarity an IFS has, the larger amount of knowledge it will carry. This work is devoted to the introduction, based on the belief above, of a novel axiomatic definition of knowledge measure in the context of IFSs, and focused on the problem of how close an IFS is to a crisp set from the perspective of certainty. In order to do that, we propose a set of new axioms with which knowledge measure should comply in the context of IFSs, and then develop a parametric model to realize the measure under this axiomatic framework. Both facets mentioned above are simultaneously taken into account in the axioms and the model. With such a parametric model, we provide an available solution to deal with the amount of potential knowledge related to the hesitancy or non-specificity of an IFS and thus introduce the idea of an attitudinal-based knowledge measure for IFSs. We further show that the concrete model presented in former work [8] is just an instance of our parametric model with a particular attitude. This is a remarkable coincidence, because what we previously did differs significantly from this research and we don't really expect at all that would happen during the process of the modeling in this paper. It can also be regarded as a convincing evidence to show the generality of the developed technique in this work. Our aim is to provide an entropy-independent, effective, and flexible technique for measuring the amount of knowledge associated with an IFS, with which to help tackle some special problems that are difficult to handle by using entropy alone, especially in the complex situation in which different attitudes of users have to be considered. These are our motivation and target, too.

The rest of the paper is organized as follows: Sect. 2 briefly recalls some basic notions of IFSs. In Sect. 3, a novel axiomatic definition of knowledge measure for IFSs is introduced, including a set of new axioms with which knowledge measure should comply in the context of IFSs, and a parametric model following these axioms. Concrete models guided by different attitudes are also discussed in detail. Section 4 shows some properties and features of the developed technique, including the connection between knowledge measure and fuzzy entropy under a unified framework. Section 5 provides a series of examples to examine the performance of the developed technique, and Sect. 6 illustrates the application of the developed measure to decision making under uncertainty, followed by conclusions in Sect. 7.

2 Preliminaries

Zadeh [34] defined the concept of FSs as follows.

Definition 1 [34] A FS A' in a finite set X is an object having the following form:

$$A' = \left\{ \langle x, \mu_{A'}(x) \rangle | x \in X \right\},$$

where $\mu_{A'} : X \to [0, 1]$ is the membership function of A', denoting the degree of membership of $x \in A'$.

Atanassov [1–4] generalized Zadeh's [34] concept of FSs and defined the notion of IFSs below.

Definition 2 [1] An IFS A in a finite set X is an object having the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\},$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ such that $\mu_A(x) + \nu_A(x) \le 1$ for $\forall x \in X$, and they denote the degree of x satisfying and dissatisfying the property A, respectively.

Another parameter of an IFS is $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, known as intuitionistic fuzzy index (or hesitation margin) of $x \in A$, which expresses a lack of knowledge of whether x belongs to A or not. It is clear that for $\forall x \in X, 0 \le \pi_A(x) \le 1$. Obviously, when $\mu_A(x) = 1 - \nu_A(x)$ for all elements of the universe of discourse, the concept of ordinary FSs is recovered.

For any two IFSs A, B in X, the following relations and operations can be defined [30]:

1. $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for $\forall x \in X$;
2. $A = B$ iff $A \subseteq B$ and $A \supseteq B$;
3. The complement set $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$;
4. The triplet $(\mu_A(x), \nu_A(x), \pi_A(x))$ is called an intuitionistic fuzzy value (IFV) .

3 Axioms and parametric model

3.1 Axioms

Let $X = \{x_i | i = 1, 2, \ldots, n\}$ be a universe of discourse. Denote by $FS(X)$ the family of all FSs in X, and by $IFS(X)$ the family of all IFSs in X. Let $A_i = \langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle$ be the i-th element from an IFS $A \in IFS(X)$. Inspired in part by the spirit in [8], we describe some intrinsic properties that a knowledge measure should have for a separate element A_i as follows.

1. It is a non-negative function, and symmetric between any A_i and its complement.
2. The amount of knowledge associated with A_i should reach the maximum equal to one if and only if A_i is a crisp element (i.e., we know everything for sure).
3. The amount of knowledge associated with A_i should reach the minimum equal to zero if and only if $\mu_A(x_i) = \nu_A(x_i) = 0$ (i.e., we know absolutely nothing).
4. In the case of $\mu_A(x_i) = \nu_A(x_i) \neq 0$ (i.e., we have a number of arguments in favor but an equally number of arguments in disapproval), our rule is that the less value of $\pi_A(x_i)$ the element A_i has, the more information content it contains, and thus the greater amount of knowledge it conveys. In particular, when $\mu_A(x_i) = \nu_A(x_i) = 0.5$ we take the value equal to 0.5 since it has both the greatest information content and the least information clarity, in other words, we seriously have no particular preference here between the information content and the information clarity. In fact, we can also take other values in the interval (0,1) to reflect the different preferences between the information content and the information clarity. However, that will lead to a rather complex situation involving an additional parameter β, which is planned to be separately discussed in another paper.
5. For a fixed value of hesitation margin, our rule is that the more the element A_i differs from its complement, the greater information clarity it has, and thus the larger amount of knowledge it conveys.

Given the arguments above, especially the last two properties, we strongly believe that there are at least two facets of knowledge associated with an IFS, one of which is the information content conveyed by the membership and non-membership functions while the other is related to the information clarity of specificity. With this understanding, we formally present below an axiomatic definition of knowledge measure for IFSs, with the aim of ensuring the effectiveness of the measure.

Definition 3 Let $A, B \in IFS(X)$. A mapping $K : IFS(X) \to [0,1]$ is called a knowledge measure on $IFS(X)$, if K has the following properties:

($KP_{IFS}1$) $K(A) = 1$ iff A is a crisp set;

($KP_{IFS}2$) $K(A) = 0$ iff $\pi_A(x_i) = 1$ for $\forall x_i \in X$;

($KP_{IFS}3$) $K(A) \ge K(B)$ if A contains more information content with greater information clarity in comparison with B, i.e.,

$\mu_A(x_i) + \nu_A(x_i) \ge \mu_B(x_i) + \nu_B(x_i)$ and $|\mu_A(x_i) - \nu_A(x_i)| \ge |\mu_B(x_i) - \nu_B(x_i)|$, for $\forall x_i \in X$;

$(KP_{IFS}4) K(A^c) = K(A)$.

Note that this notion only defines a standard partial order between IFSs in the measurement of knowledge, which contains two facets of knowledge associated with an IFS, i.e., the information content and the information clarity, denoted respectively by $\mu_A(x_i) + \nu_A(x_i)$ and $|\mu_A(x_i) - \nu_A(x_i)|$. It can be easily understood that there is no need to consider the role of $\pi_A(x_i)$ in formalizing the facet of the information clarity because it just hides inside the formulation for the information content given that $\mu_A(x_i) + \nu_A(x_i) = 1 - \pi_A(x_i)$. So we can see that to be an effective knowledge measure, these two facets should always work together in the context of IFSs. Let's look at the relationship between the two facets. It is clear that $0 \le |\mu_A(x_i) - \nu_A(x_i)| \le \mu_A(x_i) + \nu_A(x_i) \le 1$. Note that when $\mu_A(x_i) \ge \nu_A(x_i)$ we get $\mu_A(x_i) \in [\nu_A(x_i), 1 - \nu_A(x_i)]$ and $\nu_A(x_i) \in [0, 0.5]$; when $\mu_A(x_i) < \nu_A(x_i)$ we get

$\mu_A(x_i) \in [0, 0.5]$ and $\nu_A(x_i) \in [\mu_A(x_i), 1-\mu_A(x_i)]$. This means for any virtual values of $|\mu_A(x_i) - \nu_A(x_i)|$ and $\mu_A(x_i) + \nu_A(x_i)$, there always exists $A_i \in A \in IFS(X), i = 1, 2, \dots, n$. Thus we may say for sure that there is no other numerical connection between $|\mu_A(x_i) - \nu_A(x_i)|$ and $\mu_A(x_i) + \nu_A(x_i)$ except for $0 \le |\mu_A(x_i) - \nu_A(x_i)| \le \mu_A(x_i) + \nu_A(x_i) \le 1$. It is clear from the axiom of $KP_{IFS}3$ that the measure K should be monotonically non-decreasing with respect to $|\mu_A(x_i) - \nu_A(x_i)|$ and $\mu_A(x_i) + \nu_A(x_i)$, respectively.

3.2 Parametric model

Given the roles of $|\mu_A(x_i) - \nu_A(x_i)|$ and $\mu_A(x_i) + \nu_A(x_i)$ that would play in the measurement of knowledge as shown in Definition 3, the following parametric model is presented for measuring the amount of knowledge associated with a separate element $A_i \in A \in IFS(X)(i = 1, 2, \dots, n)$:

$$K_{IFS}(A_i;\alpha,\beta,\gamma) = \alpha|\mu_A(x_i) - \nu_A(x_i)| + \beta(\mu_A(x_i) + \nu_A(x_i)) + \gamma|\mu_A(x_i) - \nu_A(x_i)|(\mu_A(x_i) + \nu_A(x_i)), \quad (1)$$

where $\alpha, \beta, \gamma \in R$ are undetermined coefficients. Let's see what relationship actually exists between these coefficients according to the properties discussed above. On one hand, when A_i is a crisp element, we get $\alpha + \beta + \gamma = 1$. On the other hand, when $\mu_A(x_i) = \nu_A(x_i) = 0.5$, then $\beta = 0.5$. Thus Eq. (1) can be rewritten as

$$K_{IFS}(A_i;\alpha) = \alpha|\mu_A(x_i) - \nu_A(x_i)| + \frac{1}{2}(\mu_A(x_i) + \nu_A(x_i)) + \left(\frac{1}{2} - \alpha\right)|\mu_A(x_i) - \nu_A(x_i)|(\mu_A(x_i) + \nu_A(x_i)). \quad (2)$$

As for the range of values chosen for α, we have the following theorem.

Theorem 1 *A necessary condition for the parametric model given by Eq. (2) to be an effective knowledge measure for A_i is that $0 \le \alpha \le 1$.*

Proof For simplicity, let $x = |\mu_A(x_i) - \nu_A(x_i)|$ and $y = \mu_A(x_i) + \nu_A(x_i)$ where $0 \le x \le y \le 1$. Equation (2) can then be rewritten as

$$K_{IFS}(x, y;\alpha) = \alpha x + \frac{1}{2}y + \left(\frac{1}{2} - \alpha\right)xy, \quad 0 \le x \le y \le 1. \quad (3)$$

To ensure the effectiveness of the model K_{IFS} given by Eq. (3), we should at least make it follow the axiom of $KP_{IFS}3$ in Definition 3. Given the monotone non-decreasing property of K_{IFS} with respect to x and y, respectively, it should be ensured that

$$\frac{\partial K_{IFS}}{\partial x} = \alpha + (\frac{1}{2} - \alpha)y \ge 0, \quad (4)$$

$$\frac{\partial K_{IFS}}{\partial y} = \frac{1}{2} + (\frac{1}{2} - \alpha)x \ge 0. \quad (5)$$

Let's explore the conditions that would make Inequalities (4), (5) simultaneously hold. On one hand, for Inequality (4), we further have

$$\alpha + (0.5 - \alpha)y = \alpha(1 - y) + 0.5y \ge \alpha(1 - y) \ge 0 \quad if \quad \alpha \ge 0,$$

which means in order to maintain Inequality (4), the range of values chosen for α is $\alpha \ge 0$.

On the other hand, for Inequality (5), we have

$$0.5 + (0.5 - \alpha)x \ge 0.5x + (0.5 - \alpha)x = (1 - \alpha)x \ge 0 \quad if \quad \alpha \le 1,$$

which means in order to maintain Inequality (5), the range of values chosen for α is $\alpha \le 1$.

Clearly, the condition that would make Inequalities (4), (5) simultaneously hold is $0 \le \alpha \le 1$, which is clearly a necessary condition for the parametric model given by Eq. (2) to be an effective knowledge measure for A_i. □

With this result, Eq. (2) can further be rewritten as

$$K_{IFS}(A_i;\alpha) = \alpha\pi_A(x_i)|\mu_A(x_i) - \nu_A(x_i)| + \frac{1}{2}(\mu_A(x_i) + \nu_A(x_i))(1 + |\mu_A(x_i) - \nu_A(x_i)|), \quad 0 \le \alpha \le 1. \quad (6)$$

It can be shown that $K_{IFS}(A_i;\alpha) \in [0, 1](0 \le \alpha \le 1 ;i = 1, 2, \dots, n)$.

3.3 Attitudinal-based knowledge measure for IFSs

Let's see what we can derive from the structure of Eq. (6). Obviously, this model includes two treatments, one of which is for the basic knowledge only from the specificity of A_i, while the other is for the potential knowledge only from the hesitancy or non-specificity of A_i. We regard here the result of $\pi_A(x_i)|\mu_A(x_i) - \nu_A(x_i)|$ as an amount of potential knowledge associated with A_i, which is obviously related to the degree of the information clarity of specificity. This is in line with such a basic fact that the more clarity on the specificity of A_i, the greater possibility of potential knowledge related to the non-specificity of A_i, if any. In other words, the familiarity with the current status is considered here for the expectation of the unknown. Suppose, for example, that you are responsible for supervising two students doing their research. Another student may now be more likely to be sent to your team. You are surely well aware of the academic performance of the current two students, but don't know the educational background of the upcoming guy. What then is your attitude towards his coming? Naturally, if both of your current students are brilliant and doing well, you may have positive expectations of this new guy, actually based on the excellent performance your current students achieve. On the other hand, if both of your students are mediocrities and performing very poorly, you may likely have negative expectations of the new guy due mainly to the poor academic performance of your current students. Another special case is that one of your students performs well while the other does poorly. Under the circumstances, you may have to admit that there is no reasonable expectation of the new guy allowing for the fuzziness about the current situation. The fact is, the expectation of the unknown may depend on many factors. Given the discussion above, one of them, we believe, is the familiarity with the current status. We use it here as an entry point for measuring the amount of potential knowledge.

The question now arises: by what ratio should we make use of the potential knowledge? Some people may take it all while others do the opposite. It depends actually on attitudes of users. Yager [32] defined the concept of attitudinal character that is associated directly with a Regular Increasing Monotone (RIM) quantifier Q, i.e.,

$$\lambda_Q = \int_0^1 Q(x)dx,$$

where $\lambda_Q \in [0,1]$, and $Q : [0,1] \to [0,1]$ is denoted by a basic unit-interval monotonic (BUM) function with the properties: (1) $Q(0) = 0$, (2) $Q(1) = 1$, (3) $Q(x) \geq Q(y)$ if $x \geq y$. In this sense, our parameter $\alpha \in [0,1]$ shown in Eq. (6) can just be regarded as an adjustable attitudinal character. Let's see how it works. When taking the value of $\alpha \in [0, 0.5)$, less than a half of the potential knowledge is accepted; thus, there is relatively more uncertainty abandoned. This is conservative behavior and clearly a pessimistic attitude. When $\alpha = 0$ (extremely pessimistic), then

$$K_{IFS}(A_i;\alpha = 0) = \frac{1}{2}\left(\mu_A(x_i) + \nu_A(x_i)\right)\left(1 + \left|\mu_A(x_i) - \nu_A(x_i)\right|\right), \tag{7}$$

which means the total knowledge is derived only from the specificity of A_i and all the potential knowledge is dumped. In other words, only the basic knowledge is desired throughout the process. This result well keeps in line with the pessimistic attitude and conservative characteristic. On the other hand, when taking the value of $\alpha \in (0.5, 1]$, more than a half of the potential knowledge is accepted; thus, relatively less uncertainty abandoned. This is progressive behavior and clearly an optimistic attitude. When $\alpha = 1$ (extremely optimistic), then

$$\begin{aligned} K_{IFS}(A_i;\alpha = 1) &= \pi_A(x_i)\left|\mu_A(x_i) - \nu_A(x_i)\right| + \frac{1}{2}\left(\mu_A(x_i) + \nu_A(x_i)\right)\left(1 + \left|\mu_A(x_i) - \nu_A(x_i)\right|\right) \\ &= 1 - \frac{1}{2}\left(1 - \left|\mu_A(x_i) - \nu_A(x_i)\right|\right)\left(1 + \pi_A(x_i)\right). \end{aligned} \tag{8}$$

Obviously, in addition to the basic knowledge, all the potential knowledge is accepted. This result fully keeps in line with the optimistic attitude and progressive spirit. Note that this model is exactly the same as the one for A_i in former work [8]. Thus what we previously explored is just an instance of our parametric model with a particular attitude. Finally, when taking $\alpha = 0.5$ (completely neutral), then

$$\begin{aligned} K_{IFS}(A_i;\alpha = 0.5) &= \frac{1}{2}\left(\mu_A(x_i) + \nu_A(x_i) + \left|\mu_A(x_i) - \nu_A(x_i)\right|\right) \\ &= \max\left\{\mu_A(x_i), \nu_A(x_i)\right\}. \end{aligned} \tag{9}$$

This is a quite simple form, but in most cases it may not work effectively owing largely to its insensitivity to slight variations of A_i. For further details of attitudinal character, quantifiers, and BUM functions, please refer to Refs [7, 9–12, 14, 32, 33].

One special case of Eq. (6) is $\mu_A(x_i) = \nu_A(x_i)$. In this case, Eq. (6) reduces to

$$K_{IFS}(A_i) = \mu_A(x_i), \quad 0 \leq \mu_A(x_i) \leq 0.5.$$

Thus we find an effective way, from the viewpoint of the amount of knowledge, to make a difference between such special IFSs whose membership degrees are equal to the corresponding non-membership degrees.

Let's consider another case of Eq. (6) in which $\pi_A(x_i) = 0$. This implies $\nu_A(x_i) = 1 - \mu_A(x_i)$, thus the single element A_i reduces to an ordinary fuzzy one, denoted by $A'_i = \langle x_i, \mu_{A'}(x_i)\rangle$, which is from a FS $A' \in FS(X)$. In this case, the model K_{IFS} given by Eq. (6) reduces to

$$K_{FS}(A'_i) = \frac{1}{2} + \left|\mu_{A'}(x_i) - \frac{1}{2}\right| = \max\left\{\mu_{A'}(x_i), 1 - \mu_{A'}(x_i)\right\}.$$

This is quite similar to the formula given by Eq. (9) but the essential difference between them is that the assignment to the specificity is complete and there does not exist any non-specificity in the context of FSs. This makes a lot of sense for the above formulation despite its simplicity.

Equation (6) describes a parametric model for a single element belonging to an IFS. For$\forall A \in IFS(X)$,

$$K_{IFS}(A;\alpha) = \frac{1}{n}\sum_{i=1}^{n} K_{IFS}(A_i;\alpha) = \frac{1}{n}\sum_{i=1}^{n}\left[\alpha\pi_A(x_i)\left|\mu_A(x_i)-\nu_A(x_i)\right| + \frac{1}{2}\left(\mu_A(x_i)+\nu_A(x_i)\right)\left(1+\left|\mu_A(x_i)-\nu_A(x_i)\right|\right)\right] \quad 0 \le \alpha \le 1. \tag{10}$$

Theorem 2 *Let $A \in IFS(X)$. A real parametric function $K_{IFS}(A;\alpha) \in [0,1](0 \le \alpha \le 1)$ defined by Eq. (10) is a knowledge measure for IFSs.*

Proof As a meaningful knowledge measure of IFSs, the model defined by Eq. (10) should strictly comply with the axioms of $KP_{IFS}1 \sim 4$ in Definition 3.

($KP_{IFS}1$): Let A be a crisp set. This implies $\mu_A(x_i)=1$ or $\nu_A(x_i)=1$ for $\forall x_i \in X$, thus $K_{IFS}(A;\alpha)=1$. Now let $K_{IFS}(A;\alpha)=1$. Given that $K_{IFS}(A_i;\alpha) \in [0,1](i=1,2,\ldots,n)$, then

$$\begin{aligned} K_{IFS}(A;\alpha)=1 &\Leftrightarrow K_{IFS}(A_i;\alpha)=1, \quad A_i \in A,\\ &\Leftrightarrow \alpha\pi_A(x_i)\left|\mu_A(x_i)-\nu_A(x_i)\right| + \frac{1}{2}\left(\mu_A(x_i)+\nu_A(x_i)\right)\\ &\quad \left(1+\left|\mu_A(x_i)-\nu_A(x_i)\right|\right)=1, \quad \mu_A(x_i)+\nu_A(x_i)\neq 0\\ &\Leftrightarrow \alpha\left[1-\left(\mu_A(x_i)+\nu_A(x_i)\right)\right]\left|\mu_A(x_i)-\nu_A(x_i)\right|\\ &\quad + \frac{1}{2}\left(\mu_A(x_i)+\nu_A(x_i)\right)\left(1+\left|\mu_A(x_i)-\nu_A(x_i)\right|\right)\\ &\quad = 1-\left(\mu_A(x_i)+\nu_A(x_i)\right)+\left(\mu_A(x_i)+\nu_A(x_i)\right),\\ &\quad \mu_A(x_i)+\nu_A(x_i)\neq 0,\\ &\Leftrightarrow \left[1-\left(\mu_A(x_i)+\nu_A(x_i)\right)\right]\left(\alpha\left|\mu_A(x_i)-\nu_A(x_i)\right|-1\right)\\ &\quad = \frac{1}{2}\left(\mu_A(x_i)+\nu_A(x_i)\right)\left(1-\left|\mu_A(x_i)-\nu_A(x_i)\right|\right),\\ &\quad \mu_A(x_i)+\nu_A(x_i)\neq 0. \end{aligned} \tag{11}$$

It is certain that

$$\left[1-\left(\mu_A(x_i)+\nu_A(x_i)\right)\right]\left(\alpha\left|\mu_A(x_i)-\nu_A(x_i)\right|-1\right) \le 0,$$

while

$$\left(\mu_A(x_i)+\nu_A(x_i)\right)\left(1-\left|\mu_A(x_i)-\nu_A(x_i)\right|\right) \ge 0.$$

To make Eq. (11) hold, we have to let $\left|\mu_A(x_i)-\nu_A(x_i)\right|=1$, i.e., $\mu_A(x_i)=1$ or $\nu_A(x_i)=1$ for $\forall x_i \in X$, which implies that A is a crisp set.

($KP_{IFS}2$): Let $\pi_A(x_i)=1$ for $\forall x_i \in X$. This implies $\mu_A(x_i)=\nu_A(x_i)=0$ for $\forall x_i \in X$, thus $K_{IFS}(A;\alpha)=0$. Now let $K_{IFS}(A;\alpha)=0$. Given that $K_{IFS}(A_i;\alpha) \in [0,1](i=1,2,\ldots,n)$, then

$$\begin{aligned} K_{IFS}(A;\alpha)=0 &\Leftrightarrow K_{IFS}(A_i;\alpha)=0, \quad A_i \in A,\\ &\Leftrightarrow \alpha\pi_A(x_i)\left|\mu_A(x_i)-\nu_A(x_i)\right|\\ &\quad + \frac{1}{2}\left(\mu_A(x_i)+\nu_A(x_i)\right)\left(1+\left|\mu_A(x_i)-\nu_A(x_i)\right|\right)=0,\\ &\Leftrightarrow \alpha\pi_A(x_i)\left|\mu_A(x_i)-\nu_A(x_i)\right|=0,\\ &\quad \frac{1}{2}\left(\mu_A(x_i)+\nu_A(x_i)\right)\left(1+\left|\mu_A(x_i)-\nu_A(x_i)\right|\right)=0,\\ &\Leftrightarrow \mu_A(x_i)=\nu_A(x_i)=0, \quad \pi_A(x_i)=1\\ &\quad \text{for} \quad \forall x_i \in X. \end{aligned}$$

($KP_{IFS}3$): Let $A, B \in IFS(X)$ where $A_i \in A$, $B_i \in B$, $i=1,2,\ldots,n$. The proof of Theorem 1 shows that this axiom is definitely fulfilled for A_i and B_i with $0 \le \alpha \le 1$, given the monotone non-decreasing property of K_{IFS} with respect to $\mu_A(x_i)+\nu_A(x_i)$ and $\left|\mu_A(x_i)-\nu_A(x_i)\right|$, respectively. It can then be deduced from Eq. (10) that $K_{IFS}(A;\alpha) \ge K_{IFS}(B;\alpha)$ $(0 \le \alpha \le 1)$ with the same conditions.

($KP_{IFS}4$): Trivial from the definition of A^c. □

4 Properties and features

It is clear from Definition 3 that we present here an entropy-independent axiomatic framework for knowledge measure in the context of IFSs. Both facets of knowledge associated with an IFS, i.e., the information content and the information clarity, are taken into account in the framework, along with the treatment of hesitancy or non-specificity of an IFS. This may actually help to capture the intrinsic features of IFSs. We also believe that any model complying with the axioms in Definition 3 can be regarded as an instance of implementation of this framework.

Before discussing the connection between knowledge measure and fuzzy entropy under this framework, let's recall a classic axiomatic definition of fuzzy entropy first [21], which has been widely used in [6, 8, 13, 15, 16, 19, 22, 23, 37, 38].

Definition 4 [21]: Let $A, B \in IFS(X)$. A real function $E: IFS(X) \to [0,1]$ is called an entropy on $IFS(X)$, if E has the following properties:

($EP_{IFS}1$) $E(A) = 0$ iff A is a crisp set;
($EP_{IFS}2$) $E(A) = 1$ iff $\mu_A(x_i) = \nu_A(x_i)$ for $\forall x_i \in X$;
($EP_{IFS}3$) $E(A) \le E(B)$ if A is less fuzzy than B, i.e.,
$A \subseteq B \quad \text{for} \quad \mu_B(x_i) \le \nu_B(x_i), \quad \forall x_i \in X,$
or
$A \supseteq B$ for $\mu_B(x_i) \ge \nu_B(x_i)$, $\forall x_i \in X$;
$\left(EP_{IFS}4\right) E(A) = E(A^c)$.

Comparison of Definition 3 with Definition 4 seems to show that these two kinds of measures differ significantly from each other. Still, there are some subtle connections between them. Note that there are two optional conditions in $EP_{IFS}3$, one of which, $A \subseteq B$ for $\mu_B(x_i) \le \nu_B(x_i)$, implies

$$\mu_A(x_i) \le \mu_B(x_i) \le \nu_B(x_i) \le \nu_A(x_i),$$

while the other, $A \supseteq B$ for $\mu_B(x_i) \ge \nu_B(x_i)$, implies

$$\mu_A(x_i) \ge \mu_B(x_i) \ge \nu_B(x_i) \ge \nu_A(x_i).$$

In either case, it can easily be deduced that $\left|\mu_A(x_i) - \nu_A(x_i)\right| \ge \left|\mu_B(x_i) - \nu_B(x_i)\right|$, thus we get one condition of $KP_{IFS}3$ shown in Definition 3. Going on this premise, if another condition of $KP_{IFS}3$ is taken into account, i.e., $\mu_A(x_i) + \nu_A(x_i) \ge \mu_B(x_i) + \nu_B(x_i)$, then the two core axioms, $EP_{IFS}3$ and $KP_{IFS}3$, are both followed. This makes it possible to establish a direct connection between these two kinds of measures under a unified framework in an axiomatic manner. Consider two single-element IFSs, $A_1 = \{\langle x, 0.1, 0.5\rangle\}$ and $A_2 = \{\langle x, 0.2, 0.3\rangle\}$, for example. It is clear that

$$A_1 \subseteq A_2, \quad \mu_{A_2}(x) \le \nu_{A_2}(x).$$

According to $EP_{IFS}3$, we surely have $Entropy(A_1) \le Entropy(A_2)$ even without need for a specific calculation. We also notice that

$$\left|\mu_{A_1}(x) - \nu_{A_1}(x)\right| \ge \left|\mu_{A_2}(x) - \nu_{A_2}(x)\right|,$$
$$\mu_{A_1}(x) + \nu_{A_1}(x) \ge \mu_{A_2}(x) + \nu_{A_2}(x).$$

According to $KP_{IFS}3$ and Theorem 2, there is no doubt that $K_{IFS}(A_1;\alpha) \ge K_{IFS}(A_2;\alpha)(0 \le \alpha \le 1)$. Thus we can say for sure that for two IFSs, the one with less entropy may always carry the greater amount of knowledge provided ONLY that both $KP_{IFS}3$ and $EP_{IFS}3$ are strictly followed.

On the other hand, if the above conditions fail to be fulfilled, then the notion of the aforementioned connection between fuzzy entropy and knowledge measure goes beyond the scope of our defined axioms. Under the circumstances, we cannot derive directly from the designated axioms the numerical relationship between these two kinds of measures. It actually depends on the specific calculations on the given IFSs by concrete measures or models. In this sense, our developed measure can provide a total order that extends the usual partial order between IFSs in the measurement of knowledge as shown in Definition 3. By this means, we can find some special cases in which some IFSs with less entropy may also carry less amount of knowledge. Take another two single-element IFSs for example, i.e., $B_1 = \{\langle x, 0.1, 0.3\rangle\}$ and $B_2 = \{\langle x, 0.2, 0.3\rangle\}$. It is clear that

$$B_1 \subseteq B_2, \quad \mu_{B_2}(x) \le \nu_{B_2}(x).$$

According to $EP_{IFS}3$, we surely have $Entropy(B_1) \le Entropy(B_2)$. But note that

$$\left|\mu_{B_1}(x) - \nu_{B_1}(x)\right| \ge \left|\mu_{B_2}(x) - \nu_{B_2}(x)\right|,$$
$$\mu_{B_1}(x) + \nu_{B_1}(x) < \mu_{B_2}(x) + \nu_{B_2}(x).$$

Obviously, the two IFSs fail to satisfy the conditions of $KP_{IFS}3$, thus we cannot make a direct comparison between $K_{IFS}(B_1;\alpha)$ and $K_{IFS}(B_2;\alpha)$ only from Definition 3. In this case, there is really a need for specific calculations with concrete measuring models. Using the two different models presented respectively by Nguyen [19], and Szmidt, Kacprzyk, and Bujnowski [22], we get

$$K_n(B_1) = 0.361 < 0.436 = K_n(B_2),$$
$$K_{skb}(B_1) = 0.311 < 0.313 = K_{skb}(B_2).$$

Indeed, the IFS B_1 with less entropy also carries less amount of knowledge. Please note B_1 has a greater amount of potential knowledge than B_2—0.12 vs. 0.05, to be exact. Thus how to deal with that part of potential knowledge associated respectively with B_1 and B_2 will depend on attitudes of users. In fact, by using our developed measure K_{IFS} given by Eq. (10), we get

$$K_{IFS}(B_1;\alpha) = 0.12\alpha + 0.24, \quad K_{IFS}(B_2;\alpha) = 0.05\alpha + 0.275,$$

and

$$K_{IFS}(B_1;\alpha) - K_{IFS}(B_2;\alpha) = 0.07\alpha - 0.035, \quad 0 \le \alpha \le 1.$$

Clearly, when $0 \le \alpha < 0.5$ (a pessimistic attitude, the potential knowledge will be either dumped all or accepted in small part), then $K_{IFS}(B_1;\alpha) < K_{IFS}(B_2;\alpha)$; when $0.5 < \alpha \le 1$ (an optimistic attitude, the potential knowledge will be accepted in large part or even all), then $K_{IFS}(B_1;\alpha) > K_{IFS}(B_2;\alpha)$; when $\alpha = 0.5$ (a completely neutral attitude, the potential knowledge will be accepted just by half), then $K_{IFS}(B_1;\alpha) = K_{IFS}(B_2;\alpha)$. We believe that the knowledge measure provided in this manner could truly reflect the nature of IFSs, and what people really desire with different attitudes towards the unknown. With the help of these analyses, we can safely draw a conclusion that in the context of IFSs, there is generally NO fixed numerical relationship between fuzzy entropy and knowledge measure UNLESS the designated axioms and conditions are followed. These arguments are entirely consistent with our earlier research [8].

Let's look now at some special IFSs. Consider four single-element ones $C_i(i = 1, 2, 3, 4)$:

$C_1 = \{\langle x, 0.5, 0.5\rangle\}, \quad C_2 = \{\langle x, 0.3, 0.3\rangle\},$
$C_3 = \{\langle x, 0.2, 0.2\rangle\}, \quad C_4 = \{\langle x, 0, 0\rangle\}.$

According to some existing entropy models in the context of IFSs [15, 16, 21, 23, 37, 38], these IFSs would consistently, indeed without exception, have the maximal entropy of one. Thus a dual measure of entropy leads to the minimal amount of knowledge (equal to zero) associated with each one of them. However, in view of the different combinations of $\mu = \nu$ with the increasing values of π from C_1 to C_4, we have good reason to believe that these IFSs differ considerably from each other just from the perspective of the amount of knowledge. Since using fuzzy entropy alone is difficult to make the difference between them, we employ our developed measure K_{IFS} given by Eq. (10) to tackle this issue. Note that $\left|\mu_{C_i}(x) - \nu_{C_i}(x)\right| = 0 (i = 1, 2, 3, 4)$, which means for each C_i, there is no available potential knowledge for users. Then,

$K_{IFS}(C_1) = 0.5, \quad K_{IFS}(C_2) = 0.3, \quad K_{IFS}(C_3) = 0.2, \quad K_{IFS}(C_4) = 0.$

These results are clearer and more intuitive, and can surely be useful to distinguish between these IFSs in terms of the amount of knowledge associated with them, just as we would expect. This actually helps find a way to deal with such special cases in which there are a large number of arguments in favor but an equally large number of arguments in disapproval at the same time.

5 Experimental study

This section aims at examining, by three groups of IFSs with respective features, the performance of the developed measure K_{IFS} through a comparative analysis of other measuring models.

Example 1 The first group of IFSs: characterized by the fixed values of the information content (equal to 0.9) and the decreasing degrees of the information clarity. For simplicity, all these single-element IFSs are expressed as IFVs, denoted by $A_i (i = 1, 2, \dots, 10)$. The comparative results are given in Table 1.

It is clear from Table 1 that with the fixed values of the hesitancy and the decreasing degrees of the information clarity from A_1 to A_{10}, the entropies of these IFSs gradually increase to the maximum, as shown by the entropy model E_{sk}. While the amount of knowledge associated with these IFSs follows a decreasing trend from A_1 to A_{10}, as shown by the other three measuring models. Note that the model K_n produces some much larger values, especially of A_6~A_{10}, compared with those by K_{skb} and K_{IFS}. This is because the model K_n only considers the information content of an IFS but neglects the information clarity. It must be emphasized that both facets of knowledge associated with an IFS, i.e., the information content and the information clarity, are equally important and should be simultaneously taken into account in measuring models. By contrast, our developed model K_{IFS} performs quite well given the changing trend of the information clarity from A_1 to A_{10}. Moreover, with the help of $\alpha \in [0, 1]$, an adjustable parameter for attitudinal character, the model K_{IFS} can provide an available treatment for the potential knowledge in accordance with different attitudes of users. Obviously, for a fixed IFS, the more optimistic attitude (the greater value of α), the more potential knowledge accepted, if any, and thus the greater amount of knowledge associated with this IFS. This actually offers us a more flexible approach in real-world applications involving IFSs.

Example 2 The second group of IFSs: characterized by the fixed degrees of the information clarity (equal only to 0.1) and the increasing values of the information content. All these single-element IFSs are also expressed as IFVs, denoted by $B_i (i = 1, 2, \dots, 10)$. The comparative results are given in Table 2.

Table 1 Comparative results by different measuring models for the 1st group of IFSs

IFSs	E_{sk} [21]	K_{skb} [22]	K_n [19]	K_{IFS}, $\alpha=0$	K_{IFS}, $\alpha=0.25$	K_{IFS}, $\alpha=0.5$	K_{IFS}, $\alpha=0.75$	K_{IFS}, $\alpha=1$ [8]
$A_1 = (0.90,0.00,0.1)$	0.100	0.900	0.900	0.855	0.878	0.900	0.922	0.945
$A_2 = (0.85,0.05,0.1)$	0.158	0.871	0.876	0.810	0.830	0.850	0.870	0.890
$A_3 = (0.80,0.10,0.1)$	0.222	0.839	0.854	0.765	0.783	0.800	0.818	0.835
$A_4 = (0.75,0.15,0.1)$	0.294	0.803	0.835	0.720	0.735	0.750	0.765	0.780
$A_5 = (0.70,0.20,0.1)$	0.375	0.762	0.819	0.675	0.687	0.700	0.712	0.725
$A_6 = (0.65,0.25,0.1)$	0.467	0.717	0.805	0.630	0.640	0.650	0.660	0.670
$A_7 = (0.60,0.30,0.1)$	0.571	0.664	0.794	0.585	0.592	0.600	0.607	0.615
$A_8 = (0.55,0.35,0.1)$	0.692	0.604	0.786	0.540	0.545	0.550	0.555	0.560
$A_9 = (0.50,0.40,0.1)$	0.833	0.533	0.781	0.495	0.497	0.500	0.502	0.505
$A_{10} = (0.45,0.45,0.1)$	1.000	0.450	0.779	0.450	0.450	0.450	0.450	0.450

Table 2 Comparative results by different measuring models for the 2nd group of IFSs

IFSs	E_{sk} [21]	K_{skb} [22]	K_n [19]	K_{IFS}, $\alpha=0$	K_{IFS}, $\alpha=0.25$	K_{IFS}, $\alpha=0.5$	K_{IFS}, $\alpha=0.75$	K_{IFS}, $\alpha=1$ [8]
$B_1=(0.10,0.00,0.9)$	0.900	0.100	0.100	0.055	0.077	0.100	0.123	0.145
$B_2=(0.15,0.05,0.8)$	0.895	0.153	0.180	0.110	0.130	0.150	0.170	0.190
$B_3=(0.20,0.10,0.7)$	0.889	0.206	0.265	0.165	0.183	0.200	0.218	0.235
$B_4=(0.25,0.15,0.6)$	0.882	0.259	0.350	0.220	0.235	0.250	0.265	0.280
$B_5=(0.30,0.20,0.5)$	0.875	0.312	0.436	0.275	0.287	0.300	0.312	0.325
$B_6=(0.35,0.25,0.4)$	0.867	0.367	0.522	0.330	0.340	0.350	0.360	0.370
$B_7=(0.40,0.30,0.3)$	0.857	0.421	0.608	0.385	0.393	0.400	0.407	0.415
$B_8=(0.45,0.35,0.2)$	0.846	0.477	0.695	0.440	0.445	0.450	0.455	0.460
$B_9=(0.50,0.40,0.1)$	0.833	0.533	0.781	0.495	0.497	0.500	0.502	0.505
$B_{10}=(0.55,0.45,0.0)$	0.818	0.591	0.867	0.550	0.550	0.550	0.550	0.550

We can clearly see from Table 2 that with the fixed degrees of the information clarity and the decreasing values of the hesitancy from B_1 to B_{10}, the entropies of these IFSs gradually decrease, too, as shown by the entropy model E_{sk}. The reason for a small decrease in the amount of entropy is due to the less information clarity of these IFSs throughout the process. Along this line of reasoning, an increase in the amount of knowledge associated with these IFSs should also be limited, just as shown by the models K_{skb} and K_{IFS}. Note that the model K_n produces, again, some much larger values, especially of B_6~B_{10}, compared with those by the other two knowledge measures. Given the high degree of fuzziness of $B_j(j=6,7,\ldots,10)$, it is hard to accept these larger values as the actual amount of knowledge associated respectively with these IFSs.

Example 3 The third group of IFSs: characterized by the regular variations in the membership/ non-membership degree. All these single-element IFSs are still expressed as IFVs, denoted by $C_i\,(i=1,2,\ldots,11)$. Note that among these IFSs, any two neighbors just fulfill the conditions of $EP_{IFS}3$ in Definition 4, but not all of these pairs fulfill simultaneously the conditions of $KP_{IFS}3$ in Definition 3. The comparative results are given in Table 3.

From Table 3 we clearly see that with the decreasing degrees of the information clarity from C_1 to C_{11}, the entropies of these IFSs gradually increase from the minimum to the maximum, as shown by the entropy model E_{sk}. However, the amount of knowledge associated with these IFSs does not follow such a simple trend, as shown by K_{skb}, K_n, and K_{IFS}. Note that the entropy results here are merely used for reference, which does not mean we agree only with the results following monotonically entropy. According to the patterns of variations in the membership/ non-membership degree, these IFSs can be divided into two groups: Group One including C_1~C_6, and Group Two including C_7~C_{11}. In what follows, we discuss the actual changing trends of the amount of knowledge in each of the groups separately. Let's look first at the results from Group One. This is obviously a simple case as both the information content and the information clarity are decreasing from C_1 to C_6. In fact, all the results from this group by K_{skb}, K_n, and K_{IFS}, show a decreasing trend with great consistency, just as we would

Table 3 Comparative results by different measuring models for the 3rd group of IFSs

IFSs	E_{sk} [21]	K_{skb} [22]	K_n [19]	K_{IFS}, $\alpha=0$	K_{IFS}, $\alpha=0.25$	K_{IFS}, $\alpha=0.5$	K_{IFS}, $\alpha=0.75$	K_{IFS}, $\alpha=1$ [8]
$C_1=(0.0,1.0,0.0)$	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$C_2=(0.0,0.9,0.1)$	0.100	0.900	0.900	0.855	0.878	0.900	0.922	0.945
$C_3=(0.0,0.8,0.2)$	0.200	0.800	0.800	0.720	0.760	0.800	0.840	0.880
$C_4=(0.0,0.7,0.3)$	0.300	0.700	0.700	0.595	0.647	0.700	0.752	0.805
$C_5=(0.0,0.6,0.4)$	0.400	0.600	0.600	0.480	0.540	0.600	0.660	0.720
$C_6=(0.0,0.5,0.5)$	0.500	0.500	0.500	0.375	0.438	0.500	0.562	0.625
$C_7=(0.1,0.5,0.4)$	0.556	0.522	0.557	0.420	0.460	0.500	0.540	0.580
$C_8=(0.2,0.5,0.3)$	0.625	0.537	0.624	0.455	0.477	0.500	0.522	0.545
$C_9=(0.3,0.5,0.2)$	0.714	0.543	0.700	0.480	0.490	0.500	0.510	0.520
$C_{10}=(0.4,0.5,0.1)$	0.833	0.533	0.781	0.495	0.497	0.500	0.502	0.505
$C_{11}=(0.5,0.5,0.0)$	1.000	0.500	0.866	0.500	0.500	0.500	0.500	0.500

expect. Let's look now at the results from Group Two. This is a relatively complex case although the involved IFSs vary quite regularly. More specifically, with the increasing values of the information content towards one (the maximum) from C_7 to C_{11}, the degrees of the information clarity of these IFSs are going down towards zero (the minimum). Facing this challenge, the model K_{skb} performs poorly as no regularity can be reflected from its results. The model K_n produces a set of results following a steady trend upwards. Note that these results are much larger values, especially of C_9~C_{11}. Given the high degree of fuzziness of $C_j(j = 7, 8, \dots, 11)$, it is hard to accept these larger values as the actual amount of knowledge associated respectively with these IFSs. Now turn to the results by K_{IFS}. It is clear that this model produces a set of values following a steady trend upwards when $0 \le \alpha < 0.5$, and a steady trend downwards when $0.5 < \alpha \le 1$. In order to understand how this happens, let's show some intermediate results from these IFSs in Table 4.

Obviously, the magnitude of changes on the information content and on the information clarity is quite similar. Given the structure of Eq. (7), it is almost certainly the case that the variations in the information content have more significant effects than those in the information clarity. Thus there is no doubt that our model K_{IFS} produces a set of results following a steady trend upwards when $\alpha = 0$, as shown in Table 3. In fact, a slight increase in the value of α(usually $0 \le \alpha < 0.5$) may still lead to the same trend because a tiny amount of potential knowledge added extra should not make much difference to the previous trend. However, as the value of α dramatically increases, more and more amount of potential knowledge, if any, is added to the amount of basic knowledge. During this procedure, the model K_{IFS} gradually shifts the emphasis away from the amount of the basic knowledge pertaining to specificity, towards that of the potential knowledge pertaining to non-specificity. Note that Table 4 shows a downward trend of the amount of potential knowledge associated with $C_j(j = 7, 8, \dots, 11)$. Thus when the value of α is big enough (usually $0.5 < \alpha \le 1$), K_{IFS} can produce a set of results following a steady trend downwards rather than upwards in the case of $0 \le \alpha < 0.5$, just as shown in Table 3. A special case of $\alpha = 0.5$ is also worth noting.

Table 4 Some intermediate results from the IFSs in Group Two

IFSs from group two	The information content $\mu_{Cj}+\nu_{Cj}$	The information clarity $\lvert\mu_{Cj}-\nu_{Cj}\rvert$	The amount of potential knowledge $\pi_{Cj}\lvert\mu_{Cj}-\nu_{Cj}\rvert$
$C_7 = (0.1,0.5,0.4)$	0.6	0.4	0.16
$C_8 = (0.2,0.5,0.3)$	0.7	0.3	0.09
$C_9 = (0.3,0.5,0.2)$	0.8	0.2	0.04
$C_{10} = (0.4,0.5,0.1)$	0.9	0.1	0.01
$C_{11} = (0.5,0.5,0.0)$	1.0	0.0	0.00

Given the intrinsic insensitivity of Eq. (9), there are few surprises for so many values equal to 0.5 shown in Table 3. Obviously, the model under this attitude cannot reflect the slight variations of an IFS, and therefore it fails to capture some detailed features of IFSs. By this example, we further provide some strong arguments for the connection between knowledge measure and fuzzy entropy as mentioned before.

In summary, the model E_{sk} performs quite well during the above process despite the fact that it never tells the difference between such special IFSs whose membership degrees are equal to the corresponding non-membership degrees. While the model K_{skb} is basically passable but not stable under complex situation given its unsatisfactory results in Table 3. Note that the results by K_n are always depressing, owing largely to its negligence on the part of the inherent fuzziness of an IFS. By contrast, our model K_{IFS} works effectively and flexibly throughout the process, and shows some measure of stability with the testing results that can be well explained according to different attitudinal characters. This actually provides us with a feasible analytical tool for practical applications in more complex situation in which different attitudes of users have to be considered.

6 Illustrative example

It is well known that the concept of entropy can be used in multi-attribute decision making (MADM) for the attribute weight in terms of the information content of criterion, so called the entropy weight on attribute. In fact, it can be replaced by knowledge measure in the context of IFSs, especially when different personality traits have to be considered. We may call this the knowledge weight on attribute. In this case, the more the amount of knowledge of an attribute, the greater its associated importance weight. In what follows, a real-life example (adapted from Xu and Yager [31]) is provided for an in-depth discussion on an application of the developed technique in decision making under uncertainty.

Located in Central China and the middle reaches of the Changjiang (Yangtze) River, Hubei Province is distributed in a transitional belt where physical conditions and landscapes are on the transition from north to south and from east to west. Thus, Hubei Province is well known as "a land of rice and fish" since the region enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. At the same time, however, there are also some restrictive factors for developing agriculture such as a tight man–land relation between a constant degradation of natural resources and a growing population pressure on land resource reserve. Despite cherishing a burning desire to promote their standard of living, people living in the area are frustrated because they have no ability to enhance their power to accelerate economic development because of a dramatic decline in quantity and quality of natural resources and a deteriorating environment. Based on the distinctness and differences in environment and natural resources, Hubei Province can be roughly divided into seven agro-ecological regions:

- a_1—Wuhan–Ezhou–Huanggang; a_2—Northeast of Hubei; a_3—Southeast of Hubei;
- a_4—Jianghan region; a_5—North of Hubei; a_6—Northwest of Hubei; a_7—Southwest of Hubei.

In order to prioritize the alternatives $a_i (i = 1, 2, \ldots, 7)$ in terms of their comprehensive functions, a committee comprised of three experts $d_k (k = 1, 2, 3)$ has been formed with a weighting vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T = (0.3, 0.3, 0.4)^T$ where the value of λ_k represents the importance weight of d_k. Assume that each d_k has his/her own attitude, denoted by a vector, $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T = (0.10, 0.80, 0.45)^T$, where the value of α_k expresses the attitudinal character of d_k, representing a pessimistic, optimistic, and nearly neutral attitude, respectively. The attributes which are considered here in the assessment of $a_i (i = 1, 2, \ldots, 7)$ are:

c_1 − ecological benefit; c_2 − economic benefit;
c_3 − social benefit.

Suppose that the importance of each attribute is completely unknown. The individual opinion of d_k on a_i with respect to c_j is expressed as an individual decision matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{7\times 3}$ where $r_{ij}^{(k)} = \left(\mu_{ij}^{(k)}, \nu_{ij}^{(k)}, \pi_{ij}^{(k)}\right) (i = 1, 2, \ldots, 7; j, k = 1, 2, 3)$ are IFVs, i.e.,

$$R^{(1)} = \begin{bmatrix} (0.8, 0.1, 0.1) & (0.9, 0.1, 0.0) & (0.7, 0.2, 0.1) \\ (0.7, 0.3, 0.0) & (0.6, 0.2, 0.2) & (0.6, 0.1, 0.3) \\ (0.5, 0.4, 0.1) & (0.7, 0.3, 0.0) & (0.6, 0.1, 0.3) \\ (0.9, 0.1, 0.0) & (0.7, 0.1, 0.2) & (0.8, 0.2, 0.0) \\ (0.6, 0.1, 0.3) & (0.8, 0.2, 0.0) & (0.5, 0.1, 0.4) \\ (0.3, 0.6, 0.1) & (0.5, 0.4, 0.1) & (0.4, 0.5, 0.1) \\ (0.5, 0.2, 0.3) & (0.4, 0.6, 0.0) & (0.5, 0.5, 0.0) \end{bmatrix},$$

$$R^{(2)} = \begin{bmatrix} (0.9, 0.1, 0.0) & (0.8, 0.2, 0.0) & (0.8, 0.1, 0.1) \\ (0.8, 0.2, 0.0) & (0.5, 0.1, 0.4) & (0.7, 0.2, 0.1) \\ (0.5, 0.5, 0.0) & (0.7, 0.2, 0.1) & (0.8, 0.2, 0.0) \\ (0.9, 0.1, 0.0) & (0.9, 0.1, 0.0) & (0.7, 0.3, 0.0) \\ (0.5, 0.2, 0.3) & (0.6, 0.3, 0.1) & (0.6, 0.2, 0.2) \\ (0.4, 0.6, 0.0) & (0.3, 0.4, 0.3) & (0.5, 0.5, 0.0) \\ (0.3, 0.5, 0.2) & (0.5, 0.3, 0.2) & (0.6, 0.4, 0.0) \end{bmatrix},$$

$$R^{(3)} = \begin{bmatrix} (0.7, 0.1, 0.2) & (0.9, 0.1, 0.0) & (0.9, 0.1, 0.0) \\ (0.9, 0.1, 0.0) & (0.6, 0.2, 0.2) & (0.6, 0.2, 0.2) \\ (0.4, 0.5, 0.1) & (0.8, 0.1, 0.1) & (0.7, 0.1, 0.2) \\ (0.8, 0.1, 0.1) & (0.7, 0.2, 0.1) & (0.9, 0.1, 0.0) \\ (0.6, 0.3, 0.1) & (0.8, 0.2, 0.0) & (0.7, 0.2, 0.1) \\ (0.2, 0.7, 0.1) & (0.5, 0.1, 0.4) & (0.3, 0.1, 0.6) \\ (0.4, 0.6, 0.0) & (0.4, 0.3, 0.0) & (0.5, 0.5, 0.0) \end{bmatrix}.$$

We now use the developed measure given by Eq. (10) to derive the attribute weighting vector w, with which to aggregate all these opinions above to form an overall evaluation for each alternative.

Step 1. Aggregate all of the individual opinions, $R^{(k)} = \left(r_{ij}^{(k)}\right)_{7\times 3} (k = 1, 2, 3)$, into a group one, $R = \left(r_{ij}\right)_{7\times 3}$, by using the intuitionistic fuzzy weighted averaging (IFWA) operator [30], where

$$r_{ij} = IFWA_\lambda\left(r_{ij}^{(1)}, r_{ij}^{(2)}, r_{ij}^{(3)}\right) = \left(1 - \prod_{k=1}^{3}\left(1 - \mu_{ij}^{(k)}\right)^{\lambda_k}, \prod_{k=1}^{3}\left(\nu_{ij}^{(k)}\right)^{\lambda_k}, \prod_{k=1}^{3}\left(1 - \mu_{ij}^{(k)}\right)^{\lambda_k} - \prod_{k=1}^{3}\left(\nu_{ij}^{(k)}\right)^{\lambda_k}\right).$$

Thus the group opinion is shown as

$$R = \begin{bmatrix} (0.809, 0.100, 0.091) & (0.877, 0.123, 0.000) & (0.829, 0.123, 0.048) \\ (0.829, 0.171, 0.000) & (0.572, 0.162, 0.265) & (0.633, 0.162, 0.204) \\ (0.462, 0.468, 0.070) & (0.745, 0.171, 0.084) & (0.710, 0.123, 0.166) \\ (0.868, 0.100, 0.032) & (0.784, 0.132, 0.084) & (0.829, 0.171, 0.000) \\ (0.572, 0.191, 0.237) & (0.754, 0.226, 0.020) & (0.619, 0.162, 0.219) \\ (0.295, 0.638, 0.067) & (0.447, 0.230, 0.323) & (0.396, 0.263, 0.342) \\ (0.405, 0.409, 0.186) & (0.432, 0.369, 0.199) & (0.532, 0.468, 0.000) \end{bmatrix}.$$

Step 2. Determine the individual knowledge weights on attributes with R for each d_k. Calculate first the amount of knowledge associated with c_j for each d_k on the basis of his/her personality traits, i.e., α_k, by using Eq. (10). After normalization of these amounts of knowledge for each expert, the individual weighting vectors of attributes pertaining to R, denoted by $w^{(k)} = (w_1^{(k)}, w_2^{(k)}, w_3^{(k)})^T (k = 1, 2, 3)$, are obtained and shown as

$$w^{(1)} = (0.337, 0.335, 0.328)^T, \quad w^{(2)} = (0.332, 0.336, 0.332)^T,$$
$$w^{(3)} = (0.335, 0.335, 0.330)^T.$$

It is clear from above that for the same decision matrix R, different attitudes may lead to different logic on the importance weight of attributes.

Step 3. Derive the global weights on attributes, $w = (w_1, w_2, w_3)^T$, from the composition of λ with $w^{(k)}$ $(k = 1, 2, 3)$, that is,

$$w = \begin{bmatrix} w_1^{(1)} & w_1^{(2)} & w_1^{(3)} \\ w_2^{(1)} & w_2^{(2)} & w_2^{(3)} \\ w_3^{(1)} & w_3^{(2)} & w_3^{(3)} \end{bmatrix} \circ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = (0.335, 0.335, 0.330)^T.$$

Step 4. With w and R, make a group assessment r_i on each alternative $a_i (i = 1, 2, \dots, 7)$ by using the IFWA operator again, where

$$r_i = IFWA_w\left(r_{i1}, r_{i2}, r_{i3}\right) = \left(1 - \prod_{j=1}^{3}\left(1 - \mu_{ij}\right)^{w_j}, \prod_{j=1}^{3}\left(\nu_{ij}\right)^{w_j}, \prod_{j=1}^{3}\left(1 - \mu_{ij}\right)^{w_j} - \prod_{j=1}^{3}\left(\nu_{ij}\right)^{w_j}\right), \quad i = 1, 2, \dots, 7.$$

Group assessments on the agro-ecological regions can then be shown as

$$r_1 = (0.841, 0.115, 0.044),\ r_2 = (0.701, 0.165, 0.134),$$
$$r_3 = (0.659, 0.215, 0.126),$$
$$r_4 = (0.830, 0.131, 0.039),\ r_5 = (0.658, 0.192, 0.150),$$
$$r_6 = (0.382, 0.338, 0.280),\ r_7 = (0.459, 0.413, 0.128).$$

Step 5. Evaluate the values of $r_i (i = 1, 2, \dots, 7)$ by using the following method [7]

$$Z_{IFV}(r_i) = \left(1 - \frac{1}{2}\pi_{r_i}\right)\left(\mu_{r_i} + \frac{1}{2}\pi_{r_i}\right), \quad i = 1, 2, \dots, 7,$$

where the larger the value of $Z_{IFV}(r_i) \in [0, 1]$, the better the IFV r_i. We then have

$$Z_{IFV}\left(r_1\right) = 0.844,\ Z_{IFV}\left(r_2\right) = 0.716,\ Z_{IFV}\left(r_3\right) = 0.676,$$
$$Z_{IFV}\left(r_4\right) = 0.833,$$
$$Z_{IFV}\left(r_5\right) = 0.678,\ Z_{IFV}\left(r_6\right) = 0.449,\ Z_{IFV}\left(r_7\right) = 0.489.$$

Step 6. Finally, rank all agro-ecological regions with respect to the set of attributes in terms of the values of $Z_{IFV}\left(r_i\right)(i = 1, 2, \dots, 7)$, i.e.,

$$a_1 \succ a_4 \succ a_2 \succ a_5 \succ a_3 \succ a_7 \succ a_6.$$

This is exactly the same as the ranking list by Xu and Yager [31].

7 Conclusions

A measure of knowledge should not be viewed simply as a dual measure of entropy in the context of IFSs as there is no natural logic between them with the introduction of hesitancy. We point out in this paper that there are at least two facets of knowledge associated with an IFS, one of which is the information content while the other is related to the information clarity. We further argue that the more information content and the greater information clarity an IFS has, the larger amount of knowledge it will carry. With this understanding, we develop a novel axiomatic definition of knowledge measure for IFSs, and an attitudinal-based measuring model in the context of IFSs. Experimental results show that the knowledge measure provided in this manner is characterized by the excellent robustness and high flexibility, and can better capture the unique features of an IFS by comparison with other ones. This actually provides us with a feasible analytical tool for practical applications in more complex situation in which different attitudes of users have to be considered. At least, the developed measure could help us tackle some special problems that are really difficult to handle by using entropy alone in real-world applications with IFSs. In this sense, it could be extensively considered as a safe and effective alternative for fuzzy entropy in many fields such as soft computing, pattern recognition, information theory, operations management, decision making under uncertainty, etc. Future research is required to consider the personal preference between the information content and the information clarity in the developed model.

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References

1. Atanassov K (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96
2. Atanassov K (1989) More on intuitionistic fuzzy sets. Fuzzy Sets Syst 33:37–46
3. Atanassov K (1994) New operations defined over the intuitionistic fuzzy sets. Fuzzy Sets Syst 61:137–142
4. Atanassov K (2012) On intuitionistic fuzzy sets theory. Springer, Berlin
5. Atanassov K, Gargov G (1989) Interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31:343–349
6. Das S, Dutta B, Guha D (2016) Weight computation of criteria in a decision-making problem by knowledge measure with intuitionistic fuzzy set and interval-valued intuitionistic fuzzy set. Soft Comput 20:3421–3442

7. Guo K (2014) Amount of information and attitudinal based method for ranking Atanassov's intuitionistic fuzzy values. IEEE Trans Fuzzy Syst 22:177–188
8. Guo K (2016) Knowledge measure for Atanassov's intuitionistic fuzzy sets. IEEE Trans Fuzzy Syst 24:1072–1078
9. Guo K (2014) Quantifiers induced by subjective expected value of sample information. IEEE Trans Cybern 44:1784–1794
10. Guo K (2016) Quantifiers induced by subjective expected value of sample information with Bernstein polynomials. Eur J Oper Res 254:226–235
11. Guo K, Li W (2012) An attitudinal-based method for constructing intuitionistic fuzzy information in hybrid MADM under uncertainty. Inf Sci 208:28–38
12. Guo K, Li W (2017) A unified framework for the key weights in MAGDM under uncertainty. Soft Comput 21:2251–2262
13. Guo K, Song Q (2014) On the entropy for Atanassov's intuitionistic fuzzy sets: An interpretation from the perspective of amount of knowledge. Appl Soft Comput 24:328–340
14. Guo K, Xu H (2018) Personalized quantifier by Bernstein polynomials combined with interpolation spline. Int J Intell Syst 33:1507–1533
15. Lei Y (2015) Technique for image de-noising based on non-subsampled shearlet transform and improved intuitionistic fuzzy entropy. Optik—Int J Light Electron Opt 126:446–453
16. Li JQ, Deng GN, Li HX, Zeng WY (2012) The relationship between similarity measure and entropy of intuitionistic fuzzy sets. Inf Sci 188:314–321
17. Liu XD, Zheng SH, Xiong FL (2005) Entropy and subsethood for general interval-valued intuitionistic fuzzy sets. Lect Notes Artif Intell 3613:42–52
18. Nguyen N (2016) A new interval-valued knowledge measure for interval-valued intuitionistic fuzzy sets and application in decision making. Expert Syst Appl 56:143–155
19. Nguyen N (2015) A new knowledge-based measure for intuitionistic fuzzy sets and its application in multiple attribute group decision making. Expert Syst Appl 42:8766–8774
20. Sun M, Liu J (2012) New entropy and similarity measures for interval-valued intuitionistic fuzzy sets. J Inf Comput Sci 9:5799–5806
21. Szmidt E, Kacprzyk J (2001) Entropy for intuitionistic fuzzy sets. Fuzzy Sets Syst 118:467–477
22. Szmidt E, Kacprzyk J, Bujnowski P (2014) How to measure the amount of knowledge conveyed by Atanassov's intuitionistic fuzzy sets. Inf Sci 257:276–285
23. Verma R, Sharma DB (2013) Exponential entropy on intuitionistic fuzzy sets. Kybernetika (Prague) 49:114–127
24. Wang R, Wang XZ, Kwong S et al (2017) Incorporating diversity and informativeness in multiple-instance active learning. IEEEE Trans Fuzzy Syst 25:1460–1475
25. Wang XZ, Wang R, Xu C (2018) Discovering the relationship between generalization and uncertainty by incorporating complexity of classification. IEEE Trans Cybern 48:703–715
26. Wang XZ, Xing HJ, Li Y et al (2015) A study on relationship between generalization abilities and fuzziness of base classifiers in ensemble learning. IEEE Trans Fuzzy Syst 23:1638–1654
27. Wang XZ, Zhang T, Wang R (2017) Non-iterative deep learning: incorporating restricted boltzmann machine into multilayer random weight neural networks. IEEE Trans Syst Man Cybern: Syst. https://doi.org/10.1109/TSMC.2017.2701419 (In press).
28. Wei C, Wang P, Zhang Y (2011) Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications. Inf Sci 181:4273–4286
29. Wei C, Zhang Y (2015) Entropy measure for interval-valued intuitionistic fuzzy sets and their application in group decision-making. Math Probl Eng. https://doi.org/10.1155/2015/563745 (Article ID: 563745, 13 pages)
30. Xu ZS (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15:1179–1187
31. Xu ZS, Yager RR (2008) Dynamic intuitionistic fuzzy multi-attribute decision making. Int J Approx Reason 48:246–262
32. Yager RR (2004) OWA aggregation over a continuous interval argument with applications to decision making. IEEE Trans Syst Man Cybern (Part B: Cybern) 34:1952–1963
33. Yager RR (1996) Quantifier guided aggregation using OWA operators. Int J Intell Syst 11:49–73
34. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
35. Zadeh LA (1965) Fuzzy sets and systems. In: Proceedings of the symposium on systems theory, Polytechnic Institute of Brooklyn, New York, pp 29–37
36. Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning-I. Inf Sci 8:199–249
37. Zeng WY, Li HX (2006) Relationship between similarity measure and entropy of interval-valued fuzzy sets. Fuzzy Sets Syst 157:1477–1484
38. Zhang QS, Jiang S (2008) A note on information entropy measures for vague sets and its applications. Inf Sci 178:4184–4191
39. Zhang QS, Jiang S, Jia B, Luo SH (2010) Some information measures for interval-valued intuitionistic fuzzy sets. Inf Sci 180:5130–5145
40. Zhang HY, Zhang WX, Mei CL (2009) Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure. Knowl-Based Syst 22:449–454