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ORIGINAL ARTICLE

Global exponential stability of uncertain memristor-based recurrent neural networks with mixed time delays

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Abstract

The global exponential stability of the equilibrium point for uncertain memristor-based recurrent neural networks is studied in this paper. The memristor-based recurrent neural networks considered in this paper are based on a realistic memristor model, and can be considered as the extension of some existing memristor-based recurrent neural networks. By virtue of homomorphic theory, it is proved that the uncertain memristor-based recurrent neural networks have a unique equilibrium point under some mild assumptions. Moreover, the unique equilibrium point is proved to be globally exponentially stable by constructing a suitable Lyapunov functional. Finally, the obtained results are applied to determine the dynamical behaviors and circuit design of the memristor-based recurrent neural networks by some numerical examples.

Keywords Memristor-based recurrent neural network · Global robust exponential stability · Homomorphic theory · Lyapunov functional

1 Introduction

In 1971, Professor Chua [1] theoretically predicted the existence of a new two-terminal circuit element called the memristor (a contraction for memory and resistor). Chua believed that memristor has every right to be the fourth fundamental passive circuit element. However, until 2008, the Williams group built the first solid-state memristor, which was modeled as a thin semiconductor film (TiO_2) sandwiched between two metal contacts [2]. Because of the important memory feature, the memristor has generated unprecedented worldwide interest since its potential applications in next generation computers and powerful brain-like neural computers (see [3, 4]).

In recent years, various recurrent neural networks have been proposed and their dynamic behaviors are studied extensively since their wide applications in pattern recognition, image processing, associative memory, neurodynamic optimization problems and so on (see [5–15]). Meanwhile, more and more researchers observe that time delays are unavoidable and can influence greatly the dynamical behaviors of neural networks (see [16–20]). In general, these delays include discrete delays, time-varying delays, distributed delays and so on. However, the conventional recurrent neural network's connection weights are implemented by resistors, and do not have any memory property. The memristor works like a biological synapse, with its conductance varying with experience, or with the current flowing through it over time [21, 22]. Compared with the resistor, the memristor is more suitably used as synapse in neural networks since its nanoscale size, automatic information storage, and nonvolatile characteristic with respect to long periods of power-down [23]. This special behavior can be applied in artificial neural networks, i.e., memristor-based neural network (see [21]). Memristor-based neural networks have proven as a promising architecture in neuromorphic systems for the non-volatility, high-density, and unique memristive

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characteristic. There exist several different mathematical models for the memristor-based neural networks. For example, in 2010, Hu and Wang [24] proposed a mathematical model for the memristor-based neural networks and studied its global uniform asymptotic stability by a constructing proper Lyapunov functional. In [25], combining with the typical current–voltage characteristics of memristor, Wu et.al introduced a simple model of the memristor-based recurrent neural networks.

It is well known that the applications of neural networks rely heavily on the dynamical behaviors of the networks, such as stability, periodic oscillatory, chaos, and so on. Meanwhile, the analysis of dynamical behaviors of the memristor-based neural networks has been found useful to address a number of interesting engineering applications and therefore have been studied extensively (see [22, 26–32]). Based on a realistic memristor model and differential inclusion theory, authors in [22] studied the convergence and attractivity of memristor-based cellular neural networks with time delays. Wu et al. in [28] introduced some Lagrange stability criteria dependent on the network parameters for the Lagrange stability of the memristor-based recurrent neural networks with discrete and distributed delays. The paper [33] presented some new theoretical results on the invariance and attractivity of memristor-based cellular neural networks with time-varying delays. In [34], Wu et.al designed a simple memristor-based neural network model. Based on the fuzzy theory and Lyapunov method, they studied the problem of global exponential synchronization of a class of memristor-based recurrent neural networks with time-varying delays. In [35], the global asymptotic stability and synchronization of a class of fractional-order memristor-based delayed neural networks were investigated.

Meanwhile, the estimation errors are unavoidable for the numerical values of the neural network parameters including the neuron fire rate and the weight coefficients depending on certain resistance and capacitance. Moreover, some other external disturbances such as noise are also unavoidable. It should be noted that the uncertainty may change the stability of the neural network. On the other hand, transmission delay is also unavoidable when signals are communicated among neurons, and the transmission delay may lead to some undesired complex dynamical behaviors. In general, the transmission delay includes discrete delays, time-varying delays and distributed delays and so on. For example, by exploiting all possible information in mixed time delays, two discrete-time mixed delay neural networks were studied separately in [36, 37]. So, it is reasonable to study the dynamical behaviors of the uncertain memristor-based neural networks with time delays. Recently, more and more literatures focus on the stability of uncertain neural networks with mixed time delays (see [36–39]). For example, in [38], author studied the global asymptotic robust stability of delayed neural networks with norm-bounded uncertainties. The problems of robust stability analysis and robust controller designing of an uncertain memristive neural networks were studied in [40]. Reference [41] was concerned with the global robust synchronization of multiple memristive neural networks with nonidentical uncertain parameters.

However, as far as we know, there are very few literatures concerning on the stability of the uncertain memristor-based recurrent neural networks with time-varying delays and distributed delays. Motivated by the above works, we will study the existence and global exponential stability of the equilibrium point for a class of the uncertain memristor-based recurrent neural networks with time-varying delays and distributed delays. The neural network considered in this paper can be considered as an extension of the neural network in [34]. The structure of this paper is outlined as follows. In Sect. 2, we introduce the memristor-based recurrent neural network model and some related preliminaries. In Sect. 3, we prove the existence and global exponential stability of the equilibrium point for a class of the uncertain memristor-based recurrent neural networks. In Sect. 4, we present several numerical simulations to show the effectiveness of our results. Finally, the main conclusions drawn in the paper are summarized.

Notation Given the vector $x=(x_1,x_2,\dots,x_n)^T$, where the superscript T is the transpose operator, we let $\|x\| := (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$. $\mathbb{R}$ is the set of real numbers. Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and define $\|A\| = \sqrt{\lambda_M(A^TA)}$, where $\lambda_M(A)$ stands for the operation of taking the maximum eigenvalue of A. $I\in\mathbb{R}^{n\times n}$ is the $n\times n$ identity matrix. For a real symmetric matrix A, $A<0(>0)$ means that A is negative (positive) definite.

2 Neural network model and preliminaries

As shown in [34], the memristor-based recurrent neural network can be implemented by very large scale of integration circuits with memristors (see Fig. 1). By Kirchoff's current law, the following memristor-based recurrent neural network was introduced in [34],

$$\begin{aligned} C_i\dot{x}_i(t) = &-\left[\sum_{j=1}^{n}\left(\frac{1}{R_{f_{ij}}}+\frac{1}{R_{g_{ij}}}\right)+W_i(x_i(t))\right]x_i(t) \\ &+\sum_{j=1}^{n}\frac{\text{sign}_{ij}}{R_{f_{ij}}}f_j(x_j(t)) \\ &+\sum_{j=1}^{n}\frac{\text{sign}_{ij}}{R_{g_{ij}}}f_j(x_j(t-\tau_j(t)))+I_i, \end{aligned} \tag{1}$$

where f_j is the activation function, $\tau_j(t)$ is the time-varying delay, and $x_i(t)$ is the voltage of the capacitor C_i. $R_{f_{ij}}$ is the

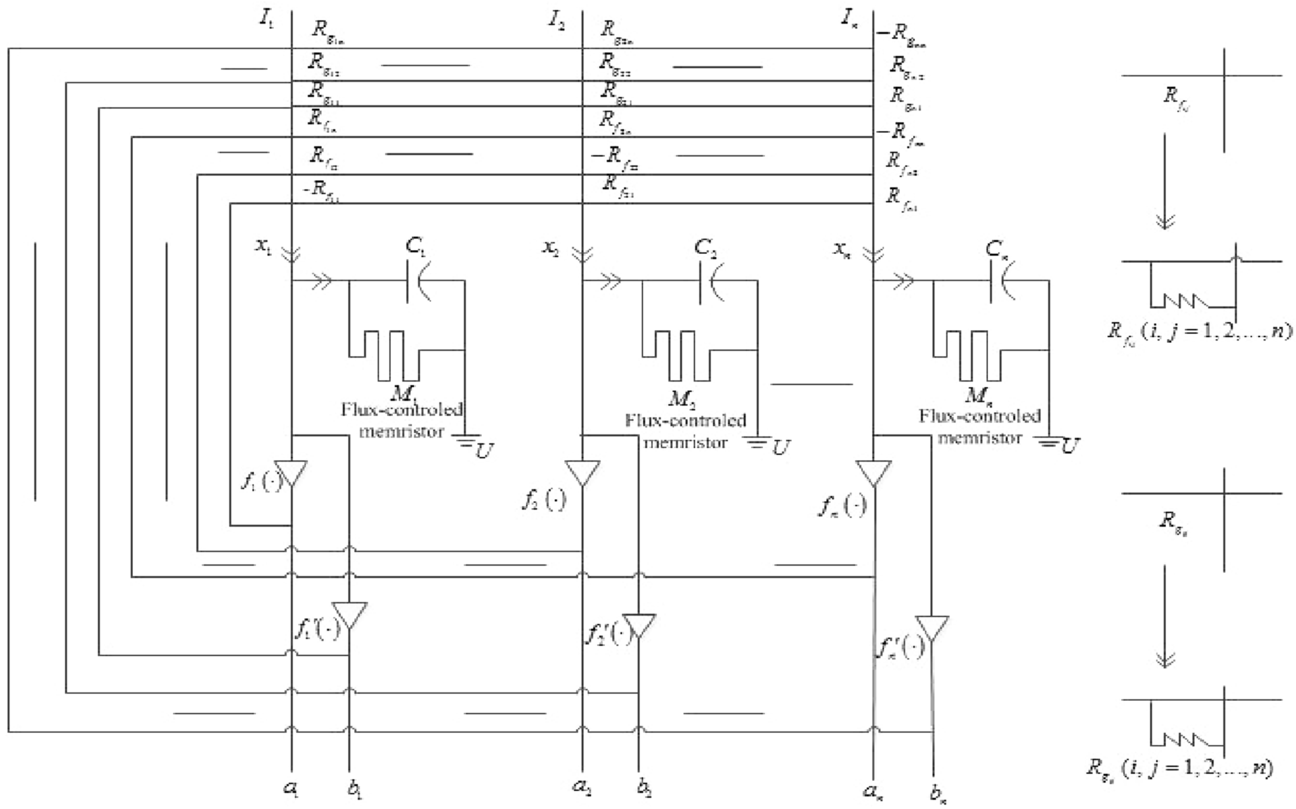

Fig. 1 Circuit of memristor-based recurrent neural network in [34]

resistor between the feedback function $f_j(x_j(t))$ and $x_i(t)$, and $R_{g_{ij}}$ is the resistor between the feedback function $f_j(x_j(t-\tau_j(t)))$ and $x_i(t)$. sign_{ij} is defined as

$$\text{sign}_{ij} = \begin{cases} 1, & \text{if} \quad i \neq j; \\ 0, & \text{if} \quad i = j. \end{cases} \tag{2}$$

W_i is the memductance of the i-th memristor satisfying

$$W_i(x_i(t)) = \begin{cases} W_i', & \text{if} \quad x_i(t) \leq 0; \\ W_i'', & \text{if} \quad x_i(t) > 0. \end{cases} \tag{3}$$

I_i is an external input or bias. Let

$$s(x_i(t)) := \begin{cases} -1, & \text{if} \quad x_i(t) \leq 0; \\ 1, & \text{if} \quad x_i(t) > 0, \end{cases} \tag{4}$$

and

$$m_i := \frac{W_i'' - W_i'}{2C_i}. \tag{5}$$

Then, by (3), we obtain that

$$\frac{W_i(x_i(t))}{C_i} = m_i s(x_i(t)) + \frac{W_i' + W_i''}{2C_i}.$$

For simplicity, we let

$$\begin{aligned} d_i &= \sum_{j=1}^{n} \left[\frac{1}{C_i R_{f_{ij}}} + \frac{1}{C_i R_{g_{ij}}} \right] + \frac{W_i' + W_i''}{2C_i}, \\ a_{ij} &= \frac{\text{sign}_{ij}}{C_i R_{f_{ij}}},\ b_{ij} = \frac{\text{sign}_{ij}}{C_i R_{g_{ij}}},\ U_i = \frac{I_i}{C_i}. \end{aligned} \tag{6}$$

It follows that $[\sum_{j=1}^{n}(\frac{1}{C_i R_{f_{ij}}} + \frac{1}{C_i R_{g_{ij}}}) + \frac{W_i(x_i)}{C_i}]x_i = [d_i + m_i s(x_i)]x_i = d_i x_i + m_i |x_i|$. Hence, the memristor-based neural network (1) can be simplified as follows,

$$\begin{aligned} \dot{x}_i(t) = &- d_i x_i(t) - m_i |x_i(t)| + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j(t))) + U_i, \end{aligned} \tag{7}$$

or,

$$\dot{x}(t) = -Dx(t) - M|x(t)| + Af(x(t)) + Bf(x(t-\tau(t))) + U, \tag{8}$$

where $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, $M = \text{diag}\{m_1, m_2, \ldots, m_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $|x(t)| = (|x_1(t)|, \ldots, |x_n(t)|)^T$, $\tau(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T$, $x(t-\tau(t)) = (x_1(t-\tau_1(t)), \ldots, x_n(t-\tau_n(t)))^T$, and $U = (U_1, U_2, \ldots, U_n)^T$.

Throughout the paper, we also need the following assumptions introduced in [34].

($\mathbf{A_1}$) For $i \in \{1, 2, \ldots, n\}$, the activation function f_i is Lipschitz continuous. That is, there exists $l_i > 0$ such that for all $r_1, r_2 \in \mathbb{R}$ with $r_1 \neq r_2$,

$$0 \leq \frac{f_i(r_1) - f_i(r_2)}{r_1 - r_2} \leq l_i. \tag{9}$$

Here, we let $L = \text{diag}\{l_1, l_2, \ldots, l_n\}$.

($\mathbf{A_2}$) For $i \in \{1, 2, \ldots, n\}$, $\tau_i(t)$ satisfies

$$0 \leq \tau_i(t) \leq \overline{\tau}_i, \quad \dot{\tau}_i(t) \leq 0. \tag{10}$$

Here, we let $\overline{\tau} = \max\{\overline{\tau}_1, \ldots, \overline{\tau}_n\}$.

Different from [34], we will study the following memristor-based neural network with time-varying delays and distributed delays,

$$\dot{x}_i(t) = -d_i x_i(t) - M_i|x_i(t)| + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{n} e_{ij} \int_{t-\mu_j}^{t} f_j(x_j(s))ds + U_i, \tag{11}$$

$i = 1, 2, \ldots, n$. Neural network (11) can be considered as an extension of neural network (7). e_{ij} and the time delay $\mu_j > 0$ are two constants, and $\int_{t-\mu_j}^{t} f_j(x_j(s))ds$ is the distributed delay.

The memristor-based neural network (11) can be rewritten as follows,

$$\dot{x}(t) = -Dx(t) - M|x(t)| + Af(x(t)) + Bf(x(t-\tau(t))) + E\int_{t-\mu}^{t} f(x(s))ds + U, \tag{12}$$

where $E = (e_{ij})_{n\times n}$. It is clear that the neural network (12) can be considered as an extension of the neural network in [34] (i.e., (7) in this paper).

Next, we introduce the assumptions about the connection weight matrices as follows.

($\mathbf{A_3}$) The parameters $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, $M = \text{diag}\{m_1, m_2, \ldots, m_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, and $E = (e_{ij})_{n\times n}$ are assumed to be intervalised as follows,

$$\begin{aligned} D_{\mathcal{I}} &:= \{D = \text{diag}\{d_i\} : 0 < \underline{d}_i \leq d_i \leq \overline{d}_i, \forall i = 1, \ldots, n\}, \\ M_{\mathcal{I}} &:= \{M = \text{diag}\{m_i\} : \underline{m}_i \leq m_i \leq \overline{m}_i, \forall i = 1, \ldots, n\}, \\ A_{\mathcal{I}} &:= \{A = (a_{ij})_{n\times n} : \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, \forall i,j = 1, \ldots, n\}, \\ B_{\mathcal{I}} &:= \{B = (b_{ij})_{n\times n} : \underline{b}_{ij} \leq b_{ij} \leq \overline{B}_{ij}, \forall i,j = 1, \ldots, n\}, \\ E_{\mathcal{I}} &:= \{E = (e_{ij})_{n\times n} : \underline{e}_{ij} \leq e_{ij} \leq \overline{E}_{ij}, \forall i,j = 1, \ldots, n\}. \end{aligned} \tag{13}$$

Here, we let $\underline{A} = (\underline{a}_{ij})_{n\times n}$, $\underline{B} = (\underline{b}_{ij})_{n\times n}$, $\underline{E} = (\underline{e}_{ij})_{n\times n}$, $\overline{A} = (\overline{a}_{ij})_{n\times n}$, $\overline{B} = (\overline{B}_{ij})_{n\times n}$, and $\overline{E} = (\overline{E}_{ij})_{n\times n}$. From the above assumption ($\mathbf{A_3}$), the diagonal matrix D is invertible.

Lemma 1 *If $H(x) \in C^0$ satisfies the following conditions*

(i) $H(x) \neq H(y)$ for all $x \neq y$,

(ii) $\|H(x)\| \to \infty$ as $\|x\| \to \infty$,

then, $H(x)$ is a homeomorphism of $\mathbb{R}^n$.

The following lemmas are necessary for proving the existence and global exponential stability of the equilibrium point for the uncertain memristor-based recurrent neural network (11).

Lemma 2 [42] *For $x \in \mathbb{R}^n$, $A \in A_{\mathcal{I}}$ and any positive diagonal matrix P, we have*

$$x^T(PA + A^TP)x \leq x^T(PA^* + A^{*T}P + \| PA_* + A_*^TP \| I)x,$$

where $A^* = \frac{1}{2}(\underline{A} + \overline{A})$ and $A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Lemma 3 [42] *For any $B \in B_{\mathcal{I}}$ and $E \in E_{\mathcal{I}}$, we have*

$$\| B \| \leq \mathrm{b}, \quad \| E \| \leq \varrho,$$

where

$$\mathrm{b} = \min\{\| B^* \| + \| B_* \|, \| \hat{B} \|, \sqrt{\| B^* \|^2 + \| B_* \|^2 + 2\| B_*^T \mid B^* \mid \|}\},$$

$$\varrho = \min\{\| E^* \| + \| E_* \|, \| \hat{E} \|, \sqrt{\| E^* \|^2 + \| E_* \|^2 + 2\| E_*^T \mid E^* \mid \|}\}.$$

$B^* = \frac{1}{2}(\underline{B} + \overline{B})$, $B_* = \frac{1}{2}(\overline{B} - \underline{B})$, $E^* = \frac{1}{2}(\underline{E} + \overline{E})$, $E_* = \frac{1}{2}(\overline{E} - \underline{E})$. $\hat{B} = (\hat{b}_{ij})_{n\times n}$ with $\hat{b}_{ij} = \max\{| \underline{b}_{ij} |, | \overline{B}_{ij} |\}$, and $\hat{E} = (\hat{e}_{ij})_{n\times n}$ with $\hat{e}_{ij} = \max\{| \underline{e}_{ij} |, | \overline{E}_{ij} |\}$.

Lemma 4 [43] *For any constant matrix* $\chi \in \mathbb{R}^{n\times n}, \chi = \chi^T$, *scalar* $v > 0$, *vector function* $F : [0, v] \to \mathbb{R}^n$ *such that the integrations concerned are well-defined, we have,*

$$v\int_0^v F^T(s)\chi F(s)ds \geq \Big(\int_0^v F(s)ds\Big)^T \chi \Big(\int_0^v F(s)ds\Big).$$

3 Main results

In this section, we will study the existence and global exponential stability of the equilibrium point for the uncertain memristor-based neural network (11). For simplicity, we let

$$\begin{aligned} &\widehat{M} = \mathrm{diag}\{\widehat{m}_i\}, \quad \widehat{m}_i := \max\{|\underline{m}_i|, |\overline{m}_i|\}, \\ &\mu = \mathrm{diag}\{\mu_1, \dots, \mu_n\}, \quad \mu_0 := \max\{\mu_1, \dots, \mu_n\}, \end{aligned} \tag{14}$$

where $\underline{m}_i, \overline{m}_i$ are from (13), and μ_i is from (11).

Theorem 1 *Under the assumptions* $(\mathbf{A_1})$,$(\mathbf{A_2})$, *and* $(\mathbf{A_3})$, *if the diagonal matrix* $\underline{D} - \widehat{M} > 0$ *and there exists a positive diagonal matrix* $P = \mathrm{diag}\{p_1, p_2, \dots, p_n\}$ *such that*

$$\begin{aligned} \Psi := &- 2P(\underline{D} - \widehat{M})L^{-1} + (PA^* + A^{*T}P \\ &+ \parallel PA_* + A_*^T P \parallel I) + 2\|P\|(\mathrm{b} + \varrho\mu_0)I < 0, \end{aligned} \tag{15}$$

then, the memristor-based neural network (11) *has a unique equilibrium point.*

Proof Let $H(x) = -Dx - M|x| + (A + B + E\mu)f(x) + U$. It is obvious that $H(x^*) = 0$ if and only if x^* is an equilibrium point of the memristor-based neural network (11). Next, based on Lemma 1, we will prove that $H(\cdot)$ a homeomorphism of $\mathbb{R}^n$, and then the memristor-based neural network (11) has a unique equilibrium point.

Step 1 : We first prove $H(\cdot)$ is injective, i.e., the hypothesis (i) in Lemma 1 holds. In fact, for any $x, y \in \mathbb{R}^n$ with $x \neq y$, we have

$$\begin{aligned} H(x) - H(y) = &- D(x - y) - M(|x| - |y|) \\ &+ (A + B + E\mu)(f(x) - f(y)). \end{aligned} \tag{16}$$

The proof of this step is divided into two following cases:

Case 1 : $f(x) - f(y) = 0$. If $f(x) - f(y) = 0$, then $H(x) - H(y) = -D(x - y) - M(|x| - |y|)$, and

$$\begin{aligned} &(x - y)^T(H(x) - H(y)) \\ &\quad = -(x - y)^T D(x - y) - (x - y)^T M(|x| - |y|) \\ &\quad \leq -(x - y)^T D(x - y) + (x - y)^T |M|(x - y) \\ &\quad = -(x - y)^T (D - |M|)(x - y) \\ &\quad \leq (x - y)^T (\widehat{M} - \underline{D})(x - y) \\ &\quad < 0. \end{aligned} \tag{17}$$

Hence, $H(x) - H(y) \neq 0$.

Case 2 : $f(x) - f(y) \neq 0$. In this case, multiplying both sides of (16) by $2(f(x) - f(y))^T P$, we have

$$\begin{aligned} &2(f(x) - f(y))^T P(H(x) - H(y)) \\ &\quad = -2(f(x) - f(y))^T PD(x - y) \\ &\qquad - 2(f(x) - f(y))^T PM(|x| - |y|) \\ &\qquad + 2(f(x) - f(y))^T P(A + B + E\mu)(f(x) - f(y)). \end{aligned} \tag{18}$$

By the assumption $(\mathbf{A_1})$ and the fact that P and M are two diagonal matrices, it is clear that

$$\begin{aligned} &-2(f(x) - f(y))^T PM(|x| - |y|) \\ &\quad \leq 2|f(x) - f(y)|^T P|M||x - y| \\ &\quad = 2(f(x) - f(y))^T P|M|(x - y) \\ &\quad \leq 2(f(x) - f(y))^T P\widehat{M}(x - y). \end{aligned} \tag{19}$$

Meanwhile,

$$\begin{aligned} &2(f(x) - f(y))^T P(A + B + E\mu)(f(x) - f(y)) \\ &\leq (f(x) - f(y))^T (PA + A^T P)(f(x) - f(y)) \\ &\quad + 2(f(x) - f(y))^T \|P\|(\|B\| + \|E\mu\|)(f(x) - f(y)). \end{aligned} \tag{20}$$

Substituting (20) into (18), we have

$$\begin{aligned} &2(f(x) - f(y))^T P(H(x) - H(y)) \\ &\leq -2(f(x) - f(y))^T P(\underline{D} - \widehat{M})L^{-1}(f(x) - f(y)) \\ &\quad + (f(x) - f(y))^T \big(PA + A^T P + 2\|P\|(\|B\| \\ &\quad + \|E\mu\|)I\big)(f(x) - f(y)) \\ &= (f(x) - f(y))^T \big[-2P(\underline{D} - \widehat{M})L^{-1} \\ &\quad + PA + A^T P + 2\|P\|(\|B\| + \|E\mu\|)I\big](f(x) - f(y)) \\ &\leq (f(x) - f(y))^T \Psi (f(x) - f(y)). \end{aligned} \tag{21}$$

Hence, by (21), considering the assumptions that Ψ is negative definite and $f(x) - f(y) \neq 0$, we obtain

$$2(f(x) - f(y))^T P(H(x) - H(y)) < 0, \tag{22}$$

which means $H(x) - H(y) \neq 0$.

Thus, from **Cases 1** and **2**, it follows that H is injective.

Step 2 : We next prove that the hypothesis (ii) in Lemma 1 holds, i.e., $\|H(x)\| \to \infty$ as $\|x\| \to \infty$. In fact, by the definition of H, we have

$$\begin{aligned}
&\|H(x)\| \\
&\quad = \| -Dx - M|x| + (A + B + E\mu)f(x) + U\| \\
&\quad \geq \| -Dx\| - \|M|x|\| - \|(A + B + E\mu)f(x)\| - \|U\| \\
&\quad \geq (\|\underline{D}\| - \|\widehat{M}\|)\|x\| - \|A + B + E\mu\|\|f(x)\| - \|U\|.
\end{aligned} \tag{23}$$

It is noted that $\|\underline{D}\| - \|\widehat{M}\| > 0$ since the diagonal matrix $\underline{D} - \widehat{M}$ is positive definite. On the other hand, letting $y = 0$ in (21), we have

$$\begin{aligned}
&2(f(x) - f(0))^T P(H(x) - H(0)) \\
&\quad \leq (f(x) - f(0))^T \Psi (f(x) - f(0)) \\
&\quad \leq \lambda_M(\Psi)\|f(x) - f(0)\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
&-2(f(x) - f(0))^T P(H(x) - H(0)) \\
&\quad \geq -\lambda_M(\Psi)\|f(x) - f(0)\|^2.
\end{aligned} \tag{24}$$

Then, by (24), it follows that $\|2P\|\|H(x) - H(0)\| \geq -\lambda_M(\Psi)\|f(x) - f(0)\|$, and consequently,

$$\begin{aligned}
\|H(x)\| &\geq \frac{-\|H(0)\| - \lambda_M(\Psi)\|f(x) - f(0)\|}{\|2P\|} \\
&\geq \frac{-\|H(0)\| - \lambda_M(\Psi)\|f(x)\| + \lambda_M(\Psi)\|f(0)\|}{\|2P\|}.
\end{aligned} \tag{25}$$

It is also noted that $-\lambda_M(\Psi) > 0$. Hence,

$$\begin{aligned}
\|H(x)\| \geq \max \Big\{ &(\|\underline{D}\| - \|\widehat{M}\|)\|x\| \\
&- \|A + B + E\mu\|\|f(x)\| - \|U\|, \\
&\times \big[-\|H(0)\| - \lambda_M(\Psi)\|f(x)\| \\
&+ \lambda_M(\Psi)\|f(0)\|\big] / \|2P\| \Big\}.
\end{aligned} \tag{26}$$

Thus, we obtain that $\|H(x)\| \to \infty$ as $\|x\| \to \infty$.

Then, by Lemma 1, $H(\cdot)$ is a homeomorphism of $\mathbb{R}^n$, and consequently the memristor-based neural network (11) has a unique equilibrium point. □

We next study the global exponential stability of the equilibrium point for the memristor-based neural network (11).

Theorem 2 *Under the assumptions in Theorem 1, the unique equilibrium point of the memristor-based neural network (11) is globally exponentially stable.*

Proof From Theorem 1, the memristor-based neural network (11) has a unique equilibrium point. Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ be the unique equilibrium point of the memristor-based neural network (11). Then, by the definition of the equilibrium point, we have

$$0 = -Dx^* - M|x^*| + Af(x^*) + Bf(x^*) + E\mu f(x^*) + U.$$

To simplify the proof, we make the following transformation

$$z = x - x^*. \tag{27}$$

Then, the memristor-based neural network (11) can be expressed equivalently as follows,

$$\begin{aligned}
\dot{z}(t) = &-Dz(t) - M(|z(t) + x^*| - |x^*|) \\
&+ Ag(z(t)) + Bg(z(t - \tau(t))) + E\int_{t-\mu}^{t} g(z(s))ds,
\end{aligned} \tag{28}$$

where $g(z(t)) = f(z(t) + x^*) - f(x^*)$ and $g(z(t - \tau(t))) = f(z(t - \tau(t)) + x^*) - f(x^*)$.

We consider the following Lyapunov function,

$$V(t, z) = V_1(t, z) + V_2(t, z) + V_3(t, z), \tag{29}$$

where

$$\begin{aligned}
V_1(t, z) =& e^{\delta t} z^T D^{-1} z, \\
V_2(t, z) =& 2\alpha e^{\delta t} \sum_{i=1}^{n} p_i \int_0^{z_i} g_i(s)ds, \\
V_3(t, z) =& (\alpha\gamma_1 + \beta_1) \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_i^2(z_i(s)) e^{\delta(s+\overline{\tau}_i)} ds \\
&+ (\alpha\gamma_2 + \beta_2) \sum_{i=1}^{n} \int_{-\mu_i}^{0} \int_{t+\theta}^{t} e^{\delta(s-\theta)} g_i^2(z_i(s)) ds d\theta.
\end{aligned} \tag{30}$$

Here, $\alpha, \beta_j, \gamma_j$, and δ are some positive constants to be determined, $j = 1, 2$.

First, calculating the time derivative of $V_1(t,z)$ along the trajectories of the memristor-based neural network (28), we have

$$\begin{aligned}\frac{d}{dt}V_1(t,z(t))&=\delta e^{\delta t}z^T(t)D^{-1}z(t)+2e^{\delta t}z^T(t)D^{-1}\dot{z}(t)\\&=\delta e^{\delta t}z^T(t)D^{-1}z(t)-2e^{\delta t}z^T(t)z(t)\\&\quad-2e^{\delta t}z^T(t)D^{-1}M(|z(t)+x^*|-|x^*|)\\&\quad+2e^{\delta t}z^T(t)D^{-1}Ag(z(t))\\&\quad+2e^{\delta t}z^T(t)D^{-1}Bg(z(t-\tau(t)))\\&\quad+2e^{\delta t}z^T(t)D^{-1}E\int_{t-\mu}^{t}g(z(s))ds.\end{aligned}\tag{31}$$

Since D and M are all diagonal matrices, we have

$$\begin{aligned}&-2e^{\delta t}z^T(t)D^{-1}M(|z(t)+x^*|-|x^*|)\\&\quad\le 2e^{\delta t}|z(t)|^TD^{-1}|M|\cdot\big|(|z(t)+x^*|-|x^*|)\big|\\&\quad\le 2e^{\delta t}z^T(t)D^{-1}|M|z(t).\end{aligned}\tag{32}$$

Substituting (32) into (31), we obtain that

$$\begin{aligned}\frac{d}{dt}V_1(t,z(t))&\le e^{\delta t}z^T(t)(\delta D^{-1}-2I+2D^{-1}|M|)z(t)\\&\quad+2e^{\delta t}z^T(t)D^{-1}Ag(z(t))\\&\quad+2e^{\delta t}z^T(t)D^{-1}Bg(z(t-\tau(t)))\\&\quad+2e^{\delta t}z^T(t)D^{-1}E\int_{t-\mu}^{t}g(z(s))ds\\&\le e^{\delta t}z^T(t)\Big(\delta D^{-1}-2I+2D^{-1}|M|+\frac{3}{k}\Big)z(t)\\&\quad+ke^{\delta t}g^T(z(t))\|D^{-1}A\|^2g(z(t))\\&\quad+ke^{\delta t}g^T(z(t-\tau(t)))\|D^{-1}B\|^2g(z(t-\tau(t)))\\&\quad+ke^{\delta t}\Big[\int_{t-\mu}^{t}g(z(s))ds\Big]^T\|D^{-1}E\|^2\int_{t-\mu}^{t}g(z(s))ds,\end{aligned}\tag{33}$$

where k is a positive constant to be determined later.

Second, we calculate the time derivative of $V_2(t,z)$ along the trajectories of the memristor-based neural network (28) as follows,

$$\begin{aligned}\frac{d}{dt}V_2(t,z(t))&=2\delta\alpha e^{\delta t}\sum_{i=1}^{n}p_i\int_0^{z_i}g_i(s)ds\\&\quad+2\alpha e^{\delta t}g^T(z(t))P\dot{z}(t)\\&\le 2\delta\alpha e^{\delta t}g^T(z(t))Pz(t)-2\alpha e^{\delta t}g^T(z(t))PDz(t)\\&\quad-2\alpha e^{\delta t}g^T(z(t))PM(|z(t)+x^*|-|x^*|)\\&\quad+2\alpha e^{\delta t}g^T(z(t))PAg(z(t))\\&\quad+2\alpha e^{\delta t}g^T(z(t))PBg(z(t-\tau(t)))\\&\quad+2\alpha e^{\delta t}g^T(z(t))PE\int_{t-\mu}^{t}g(z(s))ds.\end{aligned}\tag{34}$$

Similarly to (32), and by the fact that g_i is nondecreasing with $g_i(0)=0$, we have

$$\begin{aligned}&-2\alpha e^{\delta t}g^T(z(t))PM(|z(t)+x^*|-|x^*|)\\&\quad\le 2\alpha e^{\delta t}g^T(z(t))P|M|z(t).\end{aligned}\tag{35}$$

Based on the assumption that $\underline{D}-\widehat{M}$ is positive definite, it can be obtained that the diagonal matrix $D-|M|$ is positive definite. Hence, we can choose a sufficient small constant $\delta>0$ such that

$$\delta I-D+|M|<0.\tag{36}$$

Meanwhile, by the assumption ($\mathbf{A_1}$) and the transformation (27), we have

$$g^T(z(t))z(t)\ge g^T(z(t))L^{-1}g(z(t)).\tag{37}$$

Thus, substituting (35) and (37) into (34), we obtain

$$\begin{aligned}
&\frac{d}{dt}V_2(t,z(t))\\
&\quad\le 2\alpha e^{\delta t}g^T(z(t))P(\delta I-D+|M|)z(t)\\
&\qquad+2\alpha e^{\delta t}g^T(z(t))PAg(z(t))\\
&\qquad+2\alpha e^{\delta t}g^T(z(t))PBg(z(t-\tau(t)))\\
&\qquad+2\alpha e^{\delta t}g^T(z(t))PE\int_{t-\mu}^{t}g(z(s))ds\\
&\quad\le \alpha e^{\delta t}g^T(z(t))\big[2P(\delta I-D+|M|)L^{-1}+PA+A^TP\\
&\qquad+\|PB\|I+\mu_0\|PE\|I\big]g(z(t))\\
&\qquad+\alpha e^{\delta t}g^T(z(t-\tau(t)))\|PB\|g(z(t-\tau(t)))\\
&\qquad+\alpha e^{\delta t}\Big[\int_{t-\mu}^{t}g(z(s))ds\Big]^T\mu_0^{-1}\|PE\|\int_{t-\mu}^{t}g(z(s))ds,
\end{aligned}\tag{38}$$

since $2g^T(z(t))PAg(z(t))=g^T(z(t))(PA+A^TP)g(z(t))$ and $2g^T(z(t))PBg(z(t-\tau(t)))\le g^T(z(t))\|PB\|g(z(t))+g^T(z(t-\tau(t)))\|PB\|g(z(t-\tau(t)))$.

Third, calculating the time derivative of $V_3(t,z)$ along the trajectories of the memristor-based neural network (28), we have

$$\begin{aligned}
&\frac{d}{dt}V_3(t,z(t))\\
&\quad=(\alpha\gamma_1+\beta_1)\sum_{i=1}^{n}e^{\delta(t+\overline{\tau}_i)}g_i^2(z_i(t))\\
&\qquad-(\alpha\gamma_1+\beta_1)\sum_{i=1}^{n}(1-\dot{\tau}_i(t))e^{\delta(t-\tau_i(t)+\overline{\tau}_i)}g_i^2(z_i(t-\tau_i(t)))\\
&\qquad+(\alpha\gamma_2+\beta_2)\sum_{i=1}^{n}g_i^2(z_i(t))\delta^{-1}e^{\delta t}(e^{\delta\mu_i}-1)\\
&\qquad-(\alpha\gamma_2+\beta_2)e^{\delta t}\sum_{i=1}^{n}\int_{t-\mu_i}^{t}g_i^2(z_i(s))ds\\
&\quad\le e^{\delta t}\big[(\alpha\gamma_1+\beta_1)e^{\delta\overline{\tau}}\\
&\qquad+(\alpha\gamma_2+\beta_2)\delta^{-1}(e^{\delta\mu_0}-1)\big]g^T(z(t))g(z(t))\\
&\qquad-(\alpha\gamma_1+\beta_1)e^{\delta t}g^T(z(t-\tau(t)))g(z(t-\tau(t)))\\
&\qquad-(\alpha\gamma_2+\beta_2)e^{\delta t}\sum_{i=1}^{n}\int_{t-\mu_i}^{t}g_i^2(z_i(s))ds.
\end{aligned}\tag{39}$$

Meanwhile, according to Lemma 4, we have

$$\begin{aligned}
&-\mu_0\sum_{i=1}^{n}\int_{t-\mu_i}^{t}g_i^2(z_i(s))ds\\
&\quad\le-\Big[\int_{t-\mu}^{t}g(z(s))ds\Big]^T\int_{t-\mu}^{t}g(z(s))ds.
\end{aligned}$$

Hence, by (33), (38), and (39), the time derivative of $V(t,z)$ can be calculated as follows,

$$\begin{aligned}
&\frac{d}{dt}V(t,z(t))\\
&\quad=\frac{d}{dt}V_1(t,z(t))+\frac{d}{dt}V_2(t,z(t))+\frac{d}{dt}V_3(t,z(t))\\
&\quad\le e^{\delta t}z^T(t)\Big[\delta D^{-1}-2I+2D^{-1}|M|+\frac{3}{k}I\Big]z(t)\\
&\qquad+\alpha e^{\delta t}g^T(z(t))\Big[k\alpha^{-1}\|D^{-1}A\|^2\\
&\qquad+2P(\delta I-D+|M|)L^{-1}+PA+A^TP\\
&\qquad+\|PB\|I+\mu_0\|PE\|I+\alpha^{-1}\big[(\alpha\gamma_1+\beta_1)e^{\delta\overline{\tau}}\\
&\qquad+(\alpha\gamma_2+\beta_2)\delta^{-1}(e^{\delta\mu_0}-1)\big]I\Big]g(z(t))\\
&\qquad+e^{\delta t}g^T(z(t-\tau(t)))\Big[k\|D^{-1}B\|^2+\alpha\|PB\|\\
&\qquad-(\alpha\gamma_1+\beta_1)\Big]g(z(t-\tau(t)))\\
&\qquad+e^{\delta t}\Big[k\|D^{-1}E\|^2+\alpha\mu_0^{-1}\|PE\|-(\alpha\gamma_2\\
&\qquad+\beta_2)\mu_0^{-1}\Big]\Big[\int_{t-\mu}^{t}g(z(s))ds\Big]^T\int_{t-\mu}^{t}g(z(s))ds.
\end{aligned}\tag{40}$$

Next, we let

$$\begin{aligned}
&\gamma_1=\|PB\|,\ \ \beta_1=k\|D^{-1}B\|^2,\\
&\gamma_2=\|PE\|,\ \ \beta_2=k\mu_0\|D^{-1}E\|^2.
\end{aligned}\tag{41}$$

Then, under the assumption $(\mathbf{A_3})$, by Lemmas 2 and 3, we have

$$\begin{aligned}
&\frac{d}{dt}V(t,z(t))\\
&\quad\le e^{\delta t}z^T(t)\Big[\delta\underline{D}^{-1}-2I+2\underline{D}^{-1}|\widehat{M}|+\frac{3}{k}I\Big]z(t)\\
&\qquad+\alpha e^{\delta t}g^T(z(t))\Big[k\alpha^{-1}\|D^{-1}A\|^2I+2\delta P+\Psi\\
&\qquad-\|P\|(b+\varrho\mu_0)I+\alpha^{-1}\big[(\alpha\gamma_1+\beta_1)e^{\delta\overline{\tau}}\\
&\qquad+(\alpha\gamma_2+\beta_2)\delta^{-1}(e^{\delta\mu_0}-1)\big]I\Big]g(z(t)).
\end{aligned}\tag{42}$$

Here, Ψ is from (15). By Lemma 3 and the choices of γ_1 and β_1 in (41), we have

$$\begin{aligned} & -\|P\|b + \alpha^{-1}\left(\alpha\gamma_1 + \beta_1\right)e^{\delta\overline{\tau}} \\ & \quad = -\|P\|b + \|PB\| + \alpha^{-1}k\|D^{-1}B\|^2 e^{\delta\overline{\tau}} \\ & \quad \le \frac{k}{\alpha}\|D^{-1}B\|^2 e^{\delta\overline{\tau}}. \end{aligned} \tag{43}$$

Similarly, we also have

$$\begin{aligned} & -\|P\|\varrho\mu_0 + \alpha^{-1}\left(\alpha\gamma_2 + \beta_2\right)\delta^{-1}(e^{\delta\mu_0} - 1) \\ & \quad = -\|P\|\varrho\mu_0 + \|PE\|\delta^{-1}(e^{\delta\mu_0} - 1) \\ & \qquad + \alpha^{-1}k\mu_0\|D^{-1}E\|^2\delta^{-1}(e^{\delta\mu_0} - 1) \\ & \quad \le \|P\|\varrho[-\mu_0 + \delta^{-1}(e^{\delta\mu_0} - 1)] \\ & \qquad + \alpha^{-1}k\mu_0\|D^{-1}E\|^2\delta^{-1}(e^{\delta\mu_0} - 1). \end{aligned} \tag{44}$$

Then, letting $\alpha = k^2$ and $\delta = \frac{1}{k}$, and substituting (43) and (44) into (42), we obtain that

$$\begin{aligned} & \frac{d}{dt}V(t, z(t)) \\ & \quad \le e^{\delta t}z^T(t)\underline{D}^{-1}\left[-2(\underline{D} - |\widehat{M}|) + \left(\frac{1}{k}I + \frac{3}{k}\underline{D}\right)\right]z(t) \\ & \qquad + k^2 e^{\delta t}g^T(z(t))\Big[\Psi + 2k^{-1}P \\ & \qquad + \left[k^{-1}\|D^{-1}A\|^2 + k^{-1}\|D^{-1}B\|^2 e^{k^{-1}\overline{\tau}}\right. \\ & \qquad + \|P\|\varrho[-\mu_0 + k(e^{k^{-1}\mu_0} - 1)] \\ & \qquad + \mu_0\|D^{-1}E\|^2(e^{k^{-1}\mu_0} - 1)\big]I\Big]g(z(t)). \end{aligned} \tag{45}$$

The facts that

$$\begin{aligned} & \text{(i)}\ \ \Psi_1 := \frac{1}{k}I + \frac{3}{k}\underline{D} \to 0, \ \text{ as } k \to +\infty; \\ & \text{(ii)}\ \ \Psi_2 := \frac{\|D^{-1}A\|^2 + \|D^{-1}B\|^2 e^{k^{-1}\overline{\tau}}}{k} \\ & \qquad\qquad + \|P\|\varrho[k(e^{k^{-1}\mu_0} - 1) - \mu_0] \\ & \qquad\qquad + \mu_0\|D^{-1}E\|^2(e^{k^{-1}\mu_0} - 1) \to 0, \ \text{ as } k \to +\infty \end{aligned}$$

imply that we can choose a sufficiently large k such that both (36) and the following inequalities hold,

Table 1 Parameter values in (1)

i	1	2	3	4
Capacitor C_i	2	3	2	7
External input I_i	9	3	9.5	6
Memductance W_i' for $x_i(t) \le 0$	1	3	4	1
Memductance W_i'' for $x_i(t) > 0$	4	1.5	2	3.5

Table 2 Resistors between $f_j(x_j(t))$ and $x_i(t)$ in (1)

$R_{f_{ij}}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$j = 1$	1	3	1.5	2
$j = 2$	2	4	6	12
$j = 3$	1.8	2.6	3.5	4
$j = 4$	7	3	4	3

$$\begin{aligned} & \text{(I)}\ \ -2(\underline{D} - |\widehat{M}|) + \Psi_1 < 0; \\ & \text{(II)}\ \ \Psi + 2k^{-1}P + \Psi_2 I < 0. \end{aligned} \tag{46}$$

Hence, by (45), we have

$$\frac{d}{dt}V(t, z(t)) \le 0,$$

which means that

$$e^{\delta t}z^T(t)D^{-1}z(t) = V_1(t, z(t)) \le V(t, z(t)) \le V(0, z(0)).$$

More precisely,

$$\|x(t) - x^*\| = \|z(t)\| \le Me^{-\frac{\delta}{2}t},$$

where $M = [d_m V(0, z(0))]^{\frac{1}{2}}$ and $d_m = \max\{d_i : i = 1, \dots, n\}$. That is, the unique equilibrium point x^* of the memristor-based neural network (11) is globally exponentially stable. □

Remark 1 Recently, researchers propose several different mathematical models of the memristor-based neural networks, and study their dynamical behaviors extensively [24, 25, 33, 34]. These dynamical behaviors include the stability of equilibrium point, periodic solution, almost-periodic solution and synchronization and so on. For example, the global exponential synchronization and periodic solution of memristor-based neural network (11) were separately studied in [34, 44]. However, as far as we know, there are very few related conclusions about uncertain memristor-based recurrent neural network (11) with distributed delays.

Meanwhile, Theorems 1 and 2 can also be used to verify the global exponential stability of the equilibrium point not only for the memristor-based recurrent neural network in [34] but also for the general uncertain recurrent neural networks in [16, 18]. Hence, the conclusions in this paper can be considered as the generalization and improvement of the previous related works.

Corollary 1 *Under the assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$, *if the diagonal matrix* $D - \widehat{M} > 0$ *and there exists a positive diagonal matrix* $P = diag\{p_1, p_2, \ldots, p_n\}$ *such that*

$$\begin{aligned}\Psi :=& -2P(D-\widehat{M})L^{-1} + (PA + A^TP \\ &+ \| PA + A^TP \| I) + 2\|P\|\mathrm{b}I < 0,\end{aligned} \tag{47}$$

then, memristor-based neural network (1) *has a unique equilibrium point, which is globally exponentially stable.*

4 Numerical examples

In this section, we present some illustrative examples to show the effectiveness and application of the obtained results.

4.1 Analysis of dynamical behaviors of network (11)

First, we choose randomly the values of capacitor C_i, external input I_i, memductance W_i', W_i'', resistors $R_{f_{ij}}$, and $R_{g_{ij}}$ in (1) in Tables 1 and 2, and let $R_{f_{ij}} = R_{g_{ij}}$ for $i, j = 1, 2, 3, 4$.

Then, substituting the parameter values in Tables 1 and 2 into (11), we have

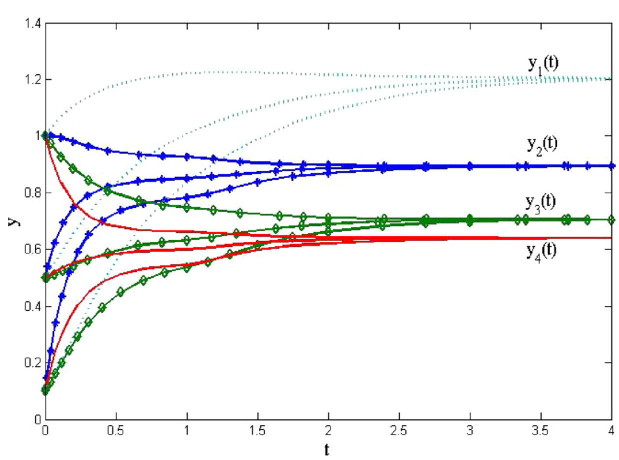

Fig. 2 Solution trajectories of the memristor-based neural network (11)

$$\begin{aligned}D =& \mathrm{diag}\{7.5, 4.0, 7.4759, 1.5884\},\\ M =& \mathrm{diag}\{0.75, -0.25, -0.5, 0.1786\},\\ U =& (4.5, 1.0, 2.75, 0.8571)^T,\end{aligned}$$

$$A = \begin{bmatrix} 0 & 0.1667 & 0.3333 & 0.25 \\ 0.1667 & 0 & 0.0556 & 0.0278 \\ 0.2278 & 0.1923 & 0 & 0.1250 \\ 0.0204 & 0.0476 & 0.0357 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0.1250 & 0.1111 & 0.2000 \\ 0.3333 & 0 & 0.0417 & 0.1667 \\ 0.1786 & 0.1087 & 0 & 0.2500 \\ 0.0286 & 0.1429 & 0.0476 & 0 \end{bmatrix}.$$

Additionally, the parameter e_{ij} in (11) are uniformly distributed pseudorandom numbers generated by *rand* in Matlab. In this numerical experiment,

$$E = \begin{bmatrix} 0.7094 & 0.6551 & 0.9597 & 0.7513 \\ 0.7547 & 0.1626 & 0.3404 & 0.2551 \\ 0.2760 & 0.1190 & 0.5853 & 0.5060 \\ 0.6797 & 0.4984 & 0.2238 & 0.6991 \end{bmatrix}.$$

The activation functions are given:

$$f_1(x) = f_2(x) = \frac{1}{2}(|x+1| - |x-1|). \tag{48}$$

Then, by (9), we have $L = \mathrm{diag}\{1, 1, 1, 1\}$. Let $\mu = \mathrm{diag}\{0.6, 0.6, 0.6, 0.6\}$ and $\tau_i(t) = 1$ for $i = 1, 2, 3, 4$ in (11).

Second, the assumption $(\mathbf{A_3})$ about the matrices D, M, A, B and E in (12) is shown as follows,

$$\begin{aligned}D_{\mathcal{I}} &:= \{D = \mathrm{diag}\{d_i\} : 0.9d_i \le d_i \le 1.1d_i, \forall i\},\\ M_{\mathcal{I}} &:= \{M = \mathrm{diag}\{m_i\} : 0.9m_i \le m_i \le 1.1m_i, \forall i\},\\ A_{\mathcal{I}} &:= \{A = (a_{ij})_{n\times n} : 0.9a_{ij} \le a_{ij} \le 1.1a_{ij}, \forall i,j\},\\ B_{\mathcal{I}} &:= \{B = (b_{ij})_{n\times n} : 0.9b_{ij} \le b_{ij} \le 1.1b_{ij}, \forall i,j\},\\ E_{\mathcal{I}} &:= \{E = (e_{ij})_{n\times n} : 0.9e_{ij} \le e_{ij} \le 1.1e_{ij}, \forall i,j\}.\end{aligned}$$

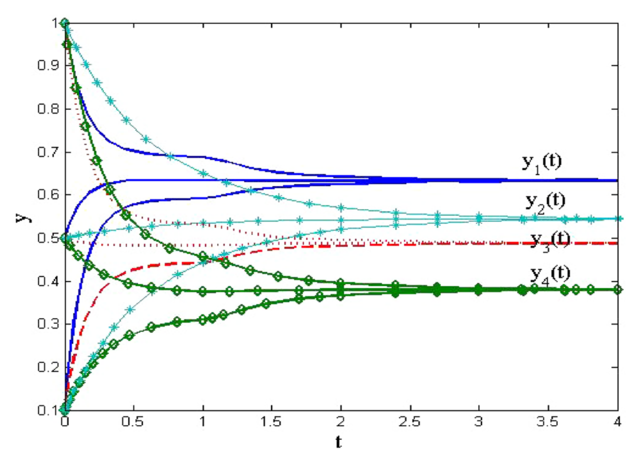

Fig. 3 Solution trajectories of the memristor-based neural network (1)

Furthermore, by Lemmas 2, 3, and (14), we can calculate A^*, A_*, $\widehat{M}$, $\varrho = 2.1649$, b = 0.5515, and $\mu_0 = 0.6$. It is clear that $\underline{D} - \widehat{M}$ is positive and there exists a positive definite diagonal matrix

$P = \text{diag}\{19.1338,\ 19.1338,\ 19.1338,\ 19.1338\}$

such that (15) holds. Thus, by Theorems 1 and 2, the unique equilibrium point of the memristor-based neural network (11) is globally exponentially stable. The initial values of the neural network (11) are set to be $(0.1, 0.1, 0.1, 0.1)^T$, $(0.5, 0.5, 0.5, 0.5)^T$, and $(1, 1, 1, 1)^T$, respectively. The solution trajectories of (11) are illustrated in Fig. 2.

4.2 Applications

In this section, we apply the proposed results to analysis the dynamic behaviors and design the circuit of memristor-based neural network in [34].

4.2.1 Analysis of the dynamical behaviors of (1)

We fix the values of parameters C_i, I_i, $R_{f_{ij}}$, $R_{g_{ij}}$, $\tau_j(t)$ for $i, j = 1, 2, 3, 4.$ in Section 4.1. It is clear that $D - \widehat{M}$ is positive and there exists a positive definite diagonal matrix

$P = \text{diag}\{0.1372, 0.1372, 0.1372, 0.1372\}$

such that (47) holds. Thus, by Corollary 1, the unique equilibrium point of the memristor-based neural network (1) is globally exponentially stable. The initial values of the neural network (1) are set to be $(0.1, 0.1, 0.1, 0.1)^T$, $(0.5, 0.5, 0.5, 0.5)^T$, and $(1, 1, 1, 1)^T$, respectively. The solution trajectories of (1) are illustrated in Fig. 3.

4.2.2 Design of network (1)

In this section, we apply the obtained results in Corollary 1 to design the circuit of memristor-based neural network with a unique globally exponentially stable equilibrium point. First, we fix the values of parameters C_i, I_i, $R_{f_{ij}}$, and $R_{g_{ij}}$ in (1). Then, we apply the obtained results to determine d_i and m_i in (1). Furthermore, we calculate the memductances W_i' and W_i'' of the $i-$th memristor, and finish the design of the circuit.

The design process of memristor-based neural network is described by three steps as follows.

Step 1 We fix a positive definite matrix P in (47). Based on the matrix inequality (47), we add the following two matrix inequalities

$$D - |\widehat{M}| > 0, \tag{49}$$

$$D - |\widehat{M}| < D \tag{50}$$

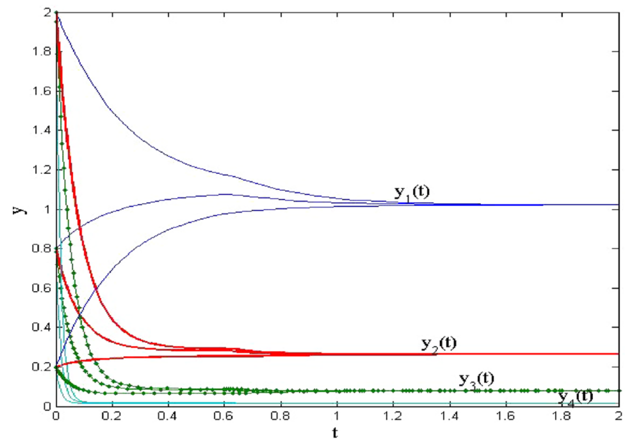

Fig. 4 Solution trajectories of the designed memristor-based neural network (1)

to solve the matrix $D - |\widehat{M}|$. Here, the conditions (49) and (50) guarantee that $D - |\widehat{M}|$ and $|\widehat{M}|$ are positive definite in both Theorem 1 and Corollary 1, respectively.

Step 2 By (5) and (6), if $m_i \geq 0$, then we have

$$d_i - |m_i| = \sum_{j=1}^{n} \left[\frac{1}{C_i R_{fij}} + \frac{1}{C_i R_{gij}} \right] + \frac{W_i'}{C_i}.$$

Furthermore, we calculate W_i' for $i \in \{1, 2, \cdots, n\}$. Meanwhile, the corresponding W_i'' can be assigned the arbitrary value satisfied $m_i = \frac{W_i'' - W_i'}{2C_i} > 0$ theoretically.

If $m_i < 0$, then we have

$$d_i - |m_i| = \sum_{j=1}^{n} \left[\frac{1}{C_i R_{fij}} + \frac{1}{C_i R_{gij}} \right] + \frac{W_i''}{C_i}.$$

Similarly, we obtain the W_i' and W_i'' in (1).

Step 3 Substituting W_i' and W_i'' into (5) and (6), we obtain d_i and m_i. That is, we complete the design of memristor-based neural network.

Now we set activation function $f_i(x)$ represented by (48), and the values of parameters C_i, I_i, $R_{f_{ij}}$, and $R_{g_{ij}}$ in (1) as same as in Tables 1 and 2. $\tau_i(t)$ are given as same as in Sect. 4.1. Consequently, we obtain the matrices U, A, B, and L as same as in Sect. 4.1. Next, we give a positive definite matrix

$P = \text{diag}\{1, 1, 1, 1\}.$

By Step 1, we have

$D - |\widehat{M}| = \text{diag}\{0.0143, 2.1977, 1.5685, 3.5865\}.$

By Step 2, letting $m_i > 0$ for $i = 1, 2, 3, 4$, we have $W_1' = -1.7109$, $W_2' = 0.1170$, $W_3' = -0.6748$, $W_4' = 4.8468$.

Moreover, fix $W_1'' = 1.2891$, $W_2'' = 3.1170$, $W_3'' = 2.3252$, and $W_4'' = 7.8468$. Then, by Step 3, we obtain

$$D = \text{diag}\{1.5143, 3.1977, 3.0685, 4.0150\},$$

$$\widehat{M} = \text{diag}\{1.5000, 1.0000, 1.5000, 0.4286\}.$$

The initial values of the neural network (1) are set to be $(0.2, 0.2, 0.2, 0.2)^T$, $(0.8, 0.8, 0.8, 0.8)^T$, and $(2, 2, 2, 2)^T$, respectively. We depict the solution trajectories of (1) in Fig. 4.

Remark 2 It should be noted that the obtained conclusions in this paper can also be applied to verify the global exponential stability of the equilibrium point for the general uncertain recurrent neural networks as follows,

$$\begin{aligned}\dot{x}_i(t) = &- d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j(t))) \\ &+ \sum_{j=1}^{n} e_{ij} \int_{t-\mu_j}^{t} f_j(x_j(s)) ds + U_i,\end{aligned}$$

which has been studied extensively (see [17, 38, 39]). However, the influence of the distributed delays were not considered in [17, 38, 39]. Hence, the obtained conclusions in this paper improve the previous related works.

5 Conclusion

The analysis of dynamical behaviors of the memristor-based neural networks is necessary when the engineering applications of such networks become more and more popular. In this paper, we study the existence and global exponential stability of the equilibrium point for a class of memristor-based recurrent neural networks. By virtue of homeomorphic theory, we prove that the memristor-based neural network has a unique equilibrium point. Furthermore, we prove that the unique equilibrium point is globally exponentially stable by constructing a suitable Lyapunov functional. From the circuit of memristor-based recurrent network, we present some conditions for the amplifiers, connection resistors between the amplifiers, the capacitors, and the memductances of memristor to guarantee the existence and global exponential stability of the equilibrium point of the circuit. Finally, some numerical examples are used to show the effectiveness of our main results. In the future, we will focus on the delay-distribution probability problem for memristor-based recurrent neural networks.

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