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ORIGINAL ARTICLE

General relation-based variable precision rough fuzzy set

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Abstract In order to effectively handle the real-valued data sets in practice, it is valuable from theoretical and practical aspects to combine fuzzy rough set and variable precision rough set so that a powerful tool can be developed. That is, the model of fuzzy variable precision rough set, which not only can handle numerical data but also is less sensitive to misclassification and perturbation,In this paper, we propose a new variable precision rough fuzzy set by introducing the variable precision parameter to generalized rough fuzzy set, i.e., the variable precision rough fuzzy set based on general relation. We, respectively, define the variable precision rough lower and upper approximations of any fuzzy set and it level set with variable precision parameter by constructive approach. Also, we present the properties of the proposed model in detail. Meanwhile, we establish the relationship between the variable precision rough approximation of a fuzzy set and the rough approximation of the level set for a fuzzy set. Furthermore, we give a new approach to uncertainty measure for variable precision rough fuzzy set established in this paper in order to overcome the limitations of the traditional methods. Finally, some numerical example are used to illuminate the validity of the conclusions given in this paper.

Keywords Rough set · General relations · Uncertainty measure · Variable precision rough fuzzy set

1 Introduction

Rough set theory, as one kind of generalization of the notions of classical set theory, was proposed to deal with uncertainty and indiscernibility [1–3]. One of the main advantages of rough set theory is that it does not need any preliminary or additional information about data, such as probability distribution in statistics, basic probability assignment in the Dempster–Shafer theory, or grade of membership or the value of possibility in fuzzy set theory [1]. The generalizations of rough set model considered with respect to various generalized binary relations are main research topics of rough set theory [4–7].

The standard rough set model is a qualitative model that defines three regions for approximating a subset of a universe of objects based on an equivalence relation on the universe. A lack of consideration of the degree of overlap between an equivalence class and the set motivates many researchers to study quantitative rough set models [8]. There are two important approaches to establish the quantitative rough set model: probabilistic approach and parameterized approach. Probabilistic approach to rough set was firstly proposed by Yao [8, 9] and a non-parameterized definition way named as decision-theoretic rough set was also defined in recently years [10–13]. The variable precision rough set (VPRS) model, as one of parameterized approaches to Pawlak rough set, proposed by Ziarko was the other quantitative rough set model [14, 15]. In the very

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first paper of Ziarko [14], he used a set inclusion function to define approximations. Also, only one parameter was used. Later on, Ziarko reformulated the theory by using probabilistic terms. So, variable precision rough set also is one of probabilistic rough set models. Furthermore, an improved model of Ziarko's variable precision rough set was given by Katzberg and Ziarko [16, 17]. As a generalization, VPRS model was introduced to handle databases with noise (i.e. the wrong or missing information, such as error, misclassification or missing values). However, VPRS cannot effectively handle numerical (real number) data and is sensitive to perturbation of data (i.e. the small change between observed values and true values). Then it inspired to develop other new rough set models which could overcome these difficulties. To handle databases with non-symbolic values, fuzzy rough set (FRS) model has been introduced by combining fuzzy sets and rough sets. Dubois and Prade [18, 19] and Nakamura [20] were among the first who showed that the basic idea of rough set given in the form of lower and upper approximations can be extended in order to approximate fuzzy set [21] defined in terms of membership functions. Subsequently, Daniel et al. gives a systematic study of the fuzzy rough set theory by using the fuzzy logic [22]. It can handle more complex databases with numerical data. However, FRS is sensitive to noise and perturbation of data [23]. It is valuable from theoretical and practical viewpoints to combine FRS and VPRS so that a powerful tool which can handle numerical data and is less-sensitive to noise and perturbation of data can be developed.

In the past years, there are many researches on the combination of FRS and VPRS and also propose some generalized models [17, 24, 25]. Chen et al. proposed a model of fuzzy variable precision rough set (FVPRS) by introducing the variable precision parameter to fuzzy rough set and also discussed the attribute reduction based on this new model [24, 25]. As is well known, uncertainty processing plays a key role in relation-based learning systems [26–29]. It is found that, in comparison with the variable precision rough fuzzy set, the modeling of fuzziness and roughness can significantly improves the performance of a learning system [30–33]. By the notions of the fuzzy inclusion set and α-inclusion error based on the residual implicators, the variable precision fuzzy rough set was defined by using the extended version of the variable precision rough set model [34]. Meantime, they developed a decision model with fuzzy attribute based on the proposed model [35]. Subsequently, Ren and Zhang discussed the properties for the Alicja's model in detail [36]. Based on the idea of Ziarko, Huang and Zhang define the lower and upper approximations of a cut set of any fuzzy set with variable precision parameter [37, 38]. A variable precision fuzzy rough set based on the power inclusion degree of fuzzy sets was also defined by Xu et al. and the variable precision fuzzy rough set can carry the computing on fuzzy rough set by fuzzy sets operating properties [39].

In view of the existing results with the variable precision fuzzy rough set, we present the rough approximation of a fuzzy set with variable precision parameter in generalized approximation space, i.e., the variable precision rough fuzzy set model with general relations. Then we discuss the relation between the model we established and other existing models and also present the main properties in detail.

The remainder of this paper is organized as follows. Section 2 gives some preliminaries such as rough fuzzy set and variable precision rough set. Section 3 presents the model of variable precision rough fuzzy set based on general relation by approximating a fuzzy set and its level set, respectively. At the same time, the relationship among of the proposed variable precision rough lower and upper approximations were given. Also, we discuss the basic properties for the proposed model in detail. Section 4 presents a new approach to uncertainty measure for variable precision rough fuzzy set based on general binary relation. Finally, we draw a conclusion and point out the future work in Sect. 5.

2 Review of rough set models

In this section, we briefly review the concept of rough set theory as well as their extension forms.

2.1 Pawlak rough set

Let U be a non-empty finite universe. R be an equivalence relation of $U \times U$. The equivalence relation R induces a partition of U, denoted by $[x]_R$ or $[x]$ and $U/R = \{[x] | x \in U\}$ stands for the equivalence classes of x. Then (U, R) be the Pawlak approximation space.

For any $X \subseteq U$, its lower and upper approximations are defined as follows [1–3]:

$$\underline{R}(X) = \{x \in U | [x] \subseteq X\} = \cup\{[x] | [x] \subseteq X\},$$
$$\overline{R}(X) = \{x \in U | [x] \cap X \neq \emptyset\} = \cup\{[x] | [x] \cap X \neq \emptyset\}.$$

The lower approximation $\underline{R}X$ is the union of all elementary sets which are the subset of X, and the upper approximation $\overline{R}X$ is the union of all elementary sets which have a non-empty intersection with X. The positive, boundary and negative regions of X can be defined as follows [4, 15, 40]:

$$pos(X) = \underline{R}(X), \quad bn(X) = \overline{R}(X) - \underline{R}(X), \quad neg(X) = U - \overline{R}(X).$$

The positive region $pos(X)$ consists of all objects that are definitely contained in the set X. The negative region

$neg(X)$ consists of all objects that are definitely not contained in the set X. The boundary region $bn(X)$ consists of all objects that may be contained in X. Since approximations are from equivalence classes, inclusion into the boundary region reflects uncertainty about the classification of object.

2.2 Variable precision rough set

In this part, we review the probabilistic formulations of rough sets: the variable precision rough set. In probabilistic approaches to rough set model [8], the classification knowledge is assumed to be supplemented with the probabilistic knowledge. The probabilistic knowledge reflects the relation occurrence frequencies of sets. It is normally assumed that all subsets $X \subseteq U$ under consideration are measurable by a probabilistic measure function P with $o < P(X) < 1$. That is to say, they are likely to occur but their occurrence is not certain.

Let U be a non-empty set and R be an equivalence relation on U. P is the probabilistic measure defined on the σ-algebra of measure subsets of U. For any subsets $X \subseteq U$ and the precision control parameter $\beta \in (0.5, 1]$, the lower and upper approximation of X about the approximation space (U, R) are defined as follows, respectively.

$$\underline{R}_\beta X = \cup\{[x]_R | P(X|[x]_R) \geq \beta, x \in U\},$$
$$\overline{R}_\beta X = \cup\{[x]_R | P(X|[x]_R) > 1 - \beta, x \in U\}.$$

Similar to the classical Pawlak rough set, the positive region, boundary region and negative region of the target set X are defined by

$$pos_\beta(X) = \underline{R}_\beta X, \quad bn_\beta(X) = \overline{R}_\beta X - \underline{R}_\beta X,$$
$$neg_\beta(X) = \cup\{[x]_R | P(\sim X|[x]_R) \geq \beta, x \in U\}.$$

where $\sim$ stands for the complementary of sets.

It is easy to known that the variable precision rough set would be degenerated to the classical Pawlak rough set when the precision parameter $\beta = 1$.

3 Variable precision rough fuzzy set model based on general relation

From the above analysis, we know that there has a solid necessity to approximate a fuzzy concept in probabilistic approximation space or discuss the theory of probabilistic rough set in fuzzy environment for the management decision-making in practice. With the objective of bringing together existing studies on probabilistic rough set approximations in fuzzy environment, we discuss the approximation of a fuzzy concept of the universe of discourse on the probabilistic approximation space in this section. That is, we will establish the probabilistic rough fuzzy set model [41]. Similar to the existing probabilistic rough set models [8–10, 15, 42], we also present several generalized forms for the proposed model.

3.1 Variable precision rough approximation of a fuzzy set

The philosophy of the variable precision rough set is to introduce a parameter $\alpha \in (0.5, 1]$ and a majority inclusion relation defined on the equivalence classes of universe. Then the lower and upper approximations are given by confining the domain of the parameter α. On the other hand, as a generalization of variable precision rough sets, we also may consider the set-inclusion function named as inclusion degree which used by Skowron and Stepaniuk [40]. Here we use the conditional probabilistic of a fuzzy event in order to keep the consistency with other generalizations in the existing papers.

First of all, we present the concept of general binary relations R and the generalized approximation space (U, R).

Let U be a nonempty finite set. For any $x \in U$, a subset $n(x)$ is called a neighborhood of x. A mapping $n : U \to P(U)$ (where $P(U)$ denotes all crisp subset) is called a neighborhood operator. For any $X \subseteq U$, denote $n(X) = \cup_{x \in X} n(x)$. Then, $n(x)$ is called the neighborhood of X. Based on the neighborhood operator, one can easily obtain an general binary relations R.

For $x, y \in U$, if xRy, then R is called general binary relations of U. That is, $(x, y) \in R$, x is called the predecessor of y. Meanwhile, y is called the successor of x. Denote

$$R_s(x) = \{y \in U | xRy\}, \quad R_p(x) = \{y \in U | yRx\}.$$

Then $R_s(x)$ and $R_p(x)$ are called the successor and predecessor neighborhoods of x.

Furthermore, the binary relation and the neighborhood operators R_s, R_p can be determined one by one. i.e., $xRy \Leftrightarrow x \in R_p(x) \Leftrightarrow y \in R_s(x)$ [34]. Also, we suppose the binary relation satisfy the property of serial for any element of universe in this paper.

Definition 3.1 [43] Let U be a nonempty finite universe. R is a general binary relation on U. We call (U, R) the generalized approximation space.

In the following, we present the lower and upper approximations of any fuzzy set on the generalized approximation space with variable precision parameter.

As is well known, the key concept is conditional probability between the target set and the successor or predecessor neighborhoods of x in variable precision rough set.

So, we firstly define the definition of conditional probability of any fuzzy event in probabilistic space.

Definition 3.2 [44] Let $U = \{x_1, x_2, \ldots, x_n, \ldots\}$. Denote $P(x_n)(n = 1, 2, \ldots)$ be the probability of x_n and satisfy $P(x_n) \geq 0$, $\sum_{n=1}^{\infty} P(x_n) = 1$. For any $A \in F(U)$(*where* $F(U)$ *denotes all the fuzzy subsets of* U), the probability of the fuzzy event A is defined as follows:

$$P(A) = \sum_{n=1}^{\infty} A(x_n) P(x_n)$$

where $A(x_n)$ stands for the membership function of fuzzy set A.

If the probabilistic space is continuous, then the probability of fuzzy event A is defined as follows:

$$P(A) = \int_U A(x) dP = E(A(x)).$$

Here dP is the *Lebesgue–Stieltjes* integral [45].

By this definition and the concept of conditional probability of classical measure theory, we define the conditional probability of a fuzzy event given the description of a crisp set as follows.

Definition 3.3 Let U be a non-empty finite universe, R be a general binary relation of U. Denote $U/R_s = \{R_s(x) | x \in U\}$ and P the probabilistic measure. For any $A \in F(U)$ and $x \in U$. $P(A|R_s(x))$ is called the conditional probability of fuzzy event A given the description $R_s(x)$. Define

$$P(A|R_s(x)) = \frac{\sum_{y \in R_s(x)} A(y)}{|R_s(x)|} = \frac{|A(y)|}{|R_s(x)|}$$

where $|\bullet|$ stands for the cardinality of a crisp set and $|A(y)|$ stands for the cardinality of a fuzzy set A.

The $P(A|R_s(x))$ also can be understand the probability of an object $x \in U$ random selected belongs to the fuzzy concept A given the description $R_s(x)$.

Remark 3.1 The conditional probability of a fuzzy event based on general binary relation given in Definition 3.3 is a direct generalization of the conditional probability of a crisp set in the fuzzy environment. Especially, the $P(A|R_s(x))$ will degenerate the form in Ref. [46] when R is an equivalence relation over universe of discourse. Further, Ref. [47], Sarkar proposes a rough-fuzzy membership function for any two fuzzy sets of the universe of discourse as:

$$\mu_{\tilde{A}}(x) = \frac{|\tilde{A} \cap \tilde{B}|}{|\tilde{B}|}, \quad \text{for any } \tilde{A}, \tilde{B} \in F(U),\ x \in U.$$

So, the conditional probability $P(\tilde{A}|R_s(x))$ also can be regarded as the rough-fuzzy membership function between a fuzzy set $\tilde{A}$ and a crisp set $R_s(x) \subseteq U$.

By this definition, the following properties are clear.

Proposition 3.1 *Let* U *be a non-empty finite universe,* R *be an equivalence relation of* U. P *is the probabilistic measure. Then the following conclusions hold.*

1. $0 \leq P(A|R_s(x)) \leq 1$,
2. If $A, B \in F(U)$ *and* $A \subseteq B$, *then* $P(A|R_s(x)) \leq P(B|R_s(x))$,
3. $P(A^c|R_s(x)) = 1 - P(A|R_s(x))$ (*where* A^c *stands for the complementary set of* A).

In the following, we give the variable precision rough approximations of a fuzzy set in generalized approximation space.

Let (U, R) be a generalized approximation space. For any $A \in F(U)$, $\alpha \in (0.5, 1]$ and $x \in U$. $R_s(x)$ is the successor neighborhood of $x \in U$. P is the probabilistic measure defined on the $\sigma-$algebra of measure subsets of U. The lower and upper approximations of fuzzy set A on (U, R) with variable parameter α are, respectively, defined as follows:

$$\underline{R}_\alpha(A)(x) = \min\{A(y) | P(A|R_s(x)) \geq \alpha, y \in R_s(x)\}, \quad x \in U,$$
$$\overline{R}_\alpha(A)(x) = \max\{A(y) | P(A|R_s(x)) > 1 - \alpha, y \in R_s(x)\}, \quad x \in U.$$

Obviously, $\underline{R}_\alpha(A)$ and $\overline{R}_\alpha(A)$ are two binary operators from $F(U) \longrightarrow F(U)$.

In general, if $\underline{R}_\alpha(A) = \overline{R}_\alpha(A)$, then we call A definable fuzzy set on (U, R). Otherwise, A is called rough fuzzy set based on general binary relation.

By the definition variable precision rough fuzzy set based on general binary relation, it is easy to verify the following properties for the binary operators $\underline{R}_\alpha$ and $\overline{R}_\alpha$.

Theorem 3.1 *Let* (U, R) *be a generalized approximation space.* $\underline{R}_\alpha$ *and* $\overline{R}_\alpha$ *are the binary operators from* $F(U) \longrightarrow F(U)$. *Then*

1. $\underline{R}_\alpha(\emptyset) = \overline{R}_\alpha(\emptyset) = \emptyset, \quad \underline{R}_\alpha(U) = \overline{R}_\alpha(U) = U$,
2. $\underline{R}_\alpha(A) \subseteq \overline{R}_\alpha(A)$,
3. $\underline{R}_\alpha(A) = (\overline{R}_{1-\alpha}(A^c))^c, \quad \overline{R}_\alpha(A) = (\underline{R}_\alpha(A^c))^c$,
4. $\overline{R}_\alpha(A \cup B) \supseteq \overline{R}_\alpha(A) \cup \overline{R}_\alpha(B)$, $\underline{R}_\alpha(A \cap B) \subseteq \underline{R}_\alpha(A) \cap \underline{R}_\alpha(B)$,
5. $\overline{R}_\alpha(A \cap B) \subseteq \overline{R}_\alpha(A) \cap \overline{R}_\alpha(B)$, $\underline{R}_\alpha(A \cup B) \supseteq \underline{R}_\alpha(A) \cup \underline{R}_\alpha(B)$.

Proof It can be easily verified by the definition.

Remark 3.2 In general, the following relation may not satisfy but it holds in the other existed rough fuzzy set model [18, 19, 22]:

$$\underline{R}_\alpha(A) \subseteq A \subseteq \overline{R}_\alpha(A)$$

Similar to the Pawlak rough set, we also define the uncertainty measure of probabilistic rough fuzzy set as the way of the Pawlak rough set in the following:

We call $\rho_\alpha(A) = \frac{|\underline{R}_\alpha(A)|}{|\overline{R}_\alpha(A)|}$ the accuracy of approximation for fuzzy set A in generalized approximation space.

Moreover, the approximated quality of lower and upper approximations are, respectively, define as follows:

$$\underline{q}(A) = \frac{|\underline{R}_\alpha(A)|}{|U|} = P(\underline{R}_\alpha(A)), \quad \overline{q}(A) = \frac{|\overline{R}_\alpha(A)|}{|U|} = P(\overline{R}_\alpha(A)).$$

Furthermore, the relationship between the accuracy and quality of approximation can be expressed as follows:

$$\rho_\alpha(A) = \frac{\underline{q}(A)}{\overline{q}(A)}.$$

Then, we call $\sigma_\alpha(A) = 1 - \rho_\alpha(A)$ the roughness for fuzzy set $\tilde{A}$ in generalized approximation space.

Actually, there are the similar properties for the accuracy of approximation and roughness of the variable precision rough fuzzy set based on general binary relation and also can establish the relationship between the accuracy of approximation and roughness like the existing rough set models [39].

In order to illuminate the above results for the variable rough fuzzy set model based on general binary relation, we present a numerical example as follows.

Example 1 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and R be an general binary relation over universe U. Then the successor neighborhood of $x \in U$ is $U/R_s = \{R_s(x_1) = \{x_1, x_2\}$, $R_s(x_2) = \{x_2, x_3\}, R_s(x_3) = \{x_3, x_4\}, R_s(x_4) = \{x_4, x_5\}$, $R_s(x_5) = \{x_1, x_3, x_5, x_6\}$, $R_s(x_6) = \{x_1, x_2, x_3, x_4\}\}$ be a covering of universe U with binary relation R. Let $A \in F(U)$ be a fuzzy set on U with the membership function

$$A = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.2}{x_3} + \frac{0.5}{x_4} + \frac{0.9}{x_5} + \frac{0.4}{x_6}.$$

Then we have following results by Definition 3.3:

$$P(A|R_s(x_1)) = \frac{A(x_1) + A(x_2)}{|R_s(x_1)|} = \frac{0.3 + 0.6}{2} = 0.45,$$

$$P(A|R_s(x_2)) = \frac{A(x_2) + A(x_3)}{|R_s(x_2)|} = \frac{0.6 + 0.2}{2} = 0.4,$$

$$P(A|R_s(x_3)) = \frac{A(x_3) + A(x_4)}{|R_s(x_3)|} = \frac{0.2 + 0.5}{2} = 0.35,$$

$$P(A|R_s(x_4)) = \frac{A(x_4) + A(x_5)}{|R_s(x_4)|} = \frac{0.5 + 0.9}{2} = 0.7,$$

$$P(A|R_s(x_5)) = \frac{A(x_1) + A(x_3) + A(x_5) + A(x_6)}{|R_s(x_5)|} = \frac{0.3 + 0.2 + 0.9 + 0.4}{4} = 0.45,$$

$$P(A|R_s(x_6)) = \frac{A(x_1) + A(x_2) + A(x_3) + A(x_4)}{|R_s(x_6)|} = \frac{0.3 + 0.6 + 0.2 + 0.5}{4} = 0.4.$$

Suppose $\alpha = 0.6$. Then, we obtain the lower and upper approximations of A about the generalized approximation space (U, R) as follows:

$$\begin{aligned}
\underline{R}_\alpha(A)(x_1) = \underline{R}_{0.6}(A)(x_1) &= \min\{A(y)|P(A|R_s(x_1)) \geq 0.6, \\ &\quad y \in R_s(x_1)\} = 0, \\
\underline{R}_\alpha(A)(x_2) = \underline{R}_{0.6}(A)(x_2) &= \min\{A(y)|P(A|R_s(x_2)) \geq 0.6, \\ &\quad y \in R_s(x_2)\} = 0, \\
\underline{R}_\alpha(A)(x_3) = \underline{R}_{0.6}(A)(x_3) &= \min\{A(y)|P(A|R_s(x_2)) \geq 0.6, \\ &\quad y \in R_s(x_3)\} = 0, \\
\underline{R}_\alpha(A)(x_4) = \underline{R}_{0.6}(A)(x_4) &= \min\{A(y)|P(A|R_s(x_2)) \geq 0.6, \\ &\quad y \in R_s(x_4)\} \\ &= \min\{0.5, 0.9\} = 0.5, \\
\underline{R}_\alpha(A)(x_5) = \underline{R}_{0.6}(A)(x_5) &= \min\{A(y)|P(A|R_s(x_5)) \geq 0.6, \\ &\quad y \in R_s(x_5)\} = 0, \\
\underline{R}_\alpha(A)(x_6) = \underline{R}_{0.6}(A)(x_6) &= \min\{A(y)|P(A|R_s(x_6)) \geq 0.6, \\ &\quad y \in R_s(x_6)\} = 0,
\end{aligned}$$

So, we have

$$\underline{R}_{0.6}(A) = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0.5}{x_4} + \frac{0}{x_5} + \frac{0}{x_6}.$$

$$\begin{aligned}
\overline{R}_\alpha(A)(x_1) &= \overline{R}_{(0.6)}(A)(x_1) = \max\{A(y)|P(A|R_s(x_1)) > 0.4, \\ &\quad y \in R_s(x_1)\} \\ &= \max\{0.3, 0.6\} = 0.6, \\
\overline{R}_\alpha(A)(x_2) &= \overline{R}_{(0.6)}(A)(x_2) = \max\{A(y)|P(A|R_s(x_2)) > 0.4, \\ &\quad y \in R_s(x_2)\} = 0, \\
\overline{R}_\alpha(A)(x_3) &= \overline{R}_{(0.6)}(A)(x_3) = \max\{A(y)|P(A|R_s(x_3)) > 0.4, \\ &\quad y \in R_s(x_3)\} = 0, \\
\overline{R}_\alpha(A)(x_4) &= \overline{R}_{(0.6)}(A)(x_4) = \max\{A(y)|P(A|R_s(x_4)) > 0.4, \\ &\quad y \in R_s(x_4)\} \\ &= \max\{0.5, 0.9\} = 0.9, \\
\overline{R}_\alpha(A)(x_5) &= \overline{R}_{(0.6)}(A)(x_5) = \max\{A(y)|P(A|R_s(x_5)) > 0.4, \\ &\quad y \in R_s(x_5)\} \\ &= \max\{0.2, 0.3, 0.4, 0.9\} = 0.9, \\
\overline{R}_\alpha(A)(x_6) &= \overline{R}_{(0.6)}(A)(x_6) = \max\{A(y)|P(A|R_s(x_6)) > 0.4, \\ &\quad y \in R_s(x_6)\} = 0,
\end{aligned}$$

So, we have

$$\overline{R}_{0.6}(A) = \frac{0.6}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0.9}{x_4} + \frac{0.9}{x_5} + \frac{0}{x_6}.$$

Meanwhile, it is easy to verify that $\underline{R}_{0.6}(A) \not\subseteq A \not\subseteq \overline{R}_{0.6}(A)$.

Furthermore, the accuracy and roughness of $\tilde{A}$ with probabilistic approximation space can be calculated as follows:

$$\rho_{0.6}(A) = \frac{|\underline{R}_{0.6}(A)|}{|\overline{R}_{0.6}(A)|} = 0.208, \quad \sigma_{0.6}(A) = 1 - \rho_{0.6}(A) = 0.792.$$

Also, the approximated quality of lower and upper approximations are, respectively, calculated as follows:

$$\underline{q}(A) = \frac{|\underline{R}_{0.6}(A)|}{|U|} = 0.08, \quad \overline{q}(A) = \frac{|\overline{R}_{0.6}(A)|}{|U|} = 0.4.$$

So, the validity of the basic concepts of variable precision rough fuzzy set model are tested by this numerical example.

3.2 Variable precision rough approximation the level set of a fuzzy set

In this section, we present the rough lower and upper approximations for the level set of a fuzzy set about the generalized approximation space.

Let (U, R) be a generalized approximation space. For any fuzzy set $A \in F(U)$ (where $F(U)$ denotes all fuzzy subsets of universes U) and variable parameter $\alpha \in (0.5, 1]$. $R_s(x)$ is the successor neighborhood of $x \in U$. P is the probabilistic measure defined on the σ-algebra of measure subsets of U. For any parameter $\lambda \in [0, 1]$, the lower and upper approximations of fuzzy set A on (U, R) with variable parameter α are, respectively, defined as follows:

$$\underline{R}_\alpha(A_\lambda) = \{x \in U | P(A_\lambda | R_s(x)) \geq \alpha\},$$
$$\overline{R}_\alpha(A_\lambda) = \{x \in U | P(A_\lambda | R_s(x)) > 1 - \alpha\}.$$

Remark 3.3 Here we use the probability form to define the lower and upper approximations of any fuzzy set with the generalized approximation space but not the majority inclusion relation which is used by all variable precision rough sets model. Actually, the majority inclusion relation is the conditional probability between any two sets from the computational point of view. So, the above definition also can be regarded as the probability description of rough fuzzy set model based on general relation or variable precision probabilistic rough fuzzy set based on general relation.

Like the existing rough set models, we also can present the positive region, negative region and boundary region for the fuzzy set with generalized approximation space as follows.

$$pos_\alpha(A_\lambda) = \underline{R}_\alpha(A_\lambda), \quad neg_\alpha(A_\lambda) = \sim \overline{R}_\alpha(A_\lambda),$$
$$bn_\alpha(A_\lambda) = \overline{R}_\alpha(A_\lambda) - \underline{R}_\alpha(A_\lambda).$$

From the above definition, it is easy to know that we actually give the rough approximations of the level-set for any fuzzy set on the generalized approximation space with variable precision parameter. So, it also is an extension or a general form of the variable precision rough set model based on general relations defined by Gong and Sun [43, 48].

In the following, we establish the relationships between the variable precision rough fuzzy sets and the existing rough sets in details.

Remark 3.4 If fuzzy set $A \in F(U)$ is a crisp set, then $\underline{R}_\alpha(A_\lambda)$ and $\overline{R}_\alpha(A_\lambda)$ degenerate to the variable precision rough set based on general relations [22]. Furthermore, the lower approximation $\underline{R}_\alpha(A_\lambda)$ and upper approximation $\overline{R}_\alpha(A_\lambda)$ will be degenerated to the Ziarko's variable precision rough set when the binary relation R of universe U is an equivalence relation.

Remark 3.5 If $\alpha = 1$, the variable precision rough fuzzy set model proposed in this paper will be degenerated to the fuzzy rough set based on level-set of fuzzy set [23].

Similarly, we also can define the lower and upper approximations of any fuzzy set $A \in F(U)$ with variable precision parameter $\alpha \in (0.5, 1]$ on the generalized approximation space (U, R) by using the successor neighborhoods $R_p(x)$ of any $x \in U$. Meanwhile, the similar results can also be obtained using Definition 3.2.

Similarly, we define the accuracy and roughness of the variable precision rough fuzzy set model as follows.

Let (U, R) be a generalized approximation space. For any $A \in F(U), \alpha \in (0.5, 1]$. $\underline{R}_\alpha(A_\lambda)$ and $\overline{R}_\alpha(A_\lambda)$ are the lower and upper approximations of the level set of A about (U, R). Then the accuracy of A about the generalized approximation space is defined as follows:

$$\rho_\alpha^\lambda(A) = \frac{|\underline{R}_\alpha(A_\lambda)|}{|\overline{R}_\alpha(A_\lambda)|}.$$

Furthermore, we call $\sigma_\alpha^\lambda(A) = 1 - \rho_\alpha^\lambda(A) = \frac{|bn_\alpha(A_\lambda)|}{|\overline{R}_\alpha(A_\lambda)|}$ the roughness of A about the generalized approximation space (U, R). Also, it is easy to know that $0 \leq \rho_\alpha^\lambda(A) \leq 1$ and $0 \leq \sigma_\alpha^\lambda(A) \leq 1$.

In the following, we present some properties for the variable precision rough fuzzy set based on the general relation established in Sect. 3.

According to the Definition 3.2, the lower and upper approximations satisfy the following properties.

Theorem 3.2 *Let (U, R) be a generalized approximation space. For any fuzzy set $A, B \in F(U)$, $\alpha, \beta \in (0.5, 1]$ and $\lambda \in [0, 1]$. Then the following relations hold for the lower and upper approximation operators.*

1. $\underline{R}_\alpha(A_\lambda) \subseteq \overline{R}_\alpha(A_\lambda)$.
2. $\underline{R}_\alpha(\emptyset_\lambda) = \underline{R}_\alpha(\emptyset_\lambda) = \emptyset, \quad \underline{R}_\alpha(U_\lambda) = \overline{R}_\alpha(U_\lambda) = U$.
3. $A \subseteq B \Longrightarrow \underline{R}_\alpha(A_\lambda) \subseteq \underline{R}_\alpha(B_\lambda), \quad \overline{R}_\alpha(A_\lambda) \subseteq \overline{R}_\alpha(B_\lambda)$,
4. $\alpha \geq \beta \Longrightarrow \underline{R}_\alpha(A_\lambda) \supseteq \underline{R}_\beta(A_\lambda), \quad \overline{R}_\alpha(A_\lambda) \subseteq \overline{R}_\beta(A_\lambda)$,

5. $\lambda_1, \lambda_2 \in [0,1], \lambda_1 \geq \lambda_2 \Longrightarrow \underline{R}_\alpha(A_{\lambda_1}) \supseteq \underline{R}_\alpha(A_{\lambda_2})$,
   $\overline{R}_\alpha(A_{\lambda_1}) \subseteq \overline{R}_\alpha(A_{\lambda_2})$,
6. $\underline{R}_\alpha(A_\lambda \cup B_\lambda) \supseteq \underline{R}_\alpha(A_\lambda) \cup \underline{R}_\alpha(B_\lambda)$,
   $\overline{R}_\alpha(A_\lambda \cap B_\lambda) \subseteq \overline{R}_\alpha(A_\lambda) \cap \overline{R}_\alpha(B_\lambda)$.
7. $\underline{R}_\alpha(\sim A_\lambda) = \sim \overline{R}_\alpha(A_\lambda), \quad \overline{R}_\alpha(\sim A_\lambda) = \sim \underline{R}_\alpha(A_\lambda)$.

Proof It can be easily verified by the definitions.

Remark 3.6 In general, the following relationships dose not hold since the binary relations R is not an equivalence relation over universe U.

1. $\underline{R}_\alpha(A_\lambda) = \cup\{R_s(x) | P(A_\lambda | R_s(x)) \leq \alpha, x \in U\}$,
2. $\overline{R}_\alpha(A_\lambda) = \cup\{R_s(x) | P(A_\lambda | R_s(x)) < 1 - \alpha, x \in U\}$.

By the above definition of the rough lower and upper approximations for the level set of a fuzzy set about the generalized approximation space, the following results also are clear.

Theorem 3.3 *Let* (U, R) *be a generalized approximation space. For any fuzzy set* $A \in F(U)$, $\alpha \in (0.5, 1]$ *and* $\lambda \in [0,1]$. *Then the following relationship holds for any* $x \in U$.

$$pos_\alpha(A_\lambda) = neg_\alpha(\sim A_\lambda) \quad where \ \sim A_\lambda = U - A_\lambda.$$

Proof By the definition of the position region of A, we have that

$$\begin{aligned}
pos_\alpha(A_\lambda) &= \underline{R}_\alpha(A_\lambda) = \{x \in U | P(A_\lambda | R_s(x)) \geq \alpha\} \\
&= \{x \in U | P(A_\lambda | R_s(x)) \geq \alpha\} \\
&= \left\{ x \in U | \frac{|A_\lambda \cap R_s(x)|}{|R_s(x)|} \geq \alpha \right\} \\
&= \left\{ x \in U | 1 - \frac{|A_\lambda \cap R_s(x)|}{|R_s(x)|} \leq 1 - \alpha \right\} \\
&= \left\{ x \in U | \frac{|(U - A_\lambda) \cap R_s(x)|}{|R_s(x)|} \leq 1 - \alpha \right\} \\
&= \left\{ x \in U | \frac{|(\sim A_\lambda) \cap R_s(x)|}{|R_s(x)|} \leq 1 - \alpha \right\} \\
&= \{x \in U | P(\sim A_\lambda | R_s(x)) \leq 1 - \alpha\} \\
&= \sim \overline{R}_\alpha(\sim A_\alpha) = neg_\alpha(\sim A_\lambda).
\end{aligned}$$

So, we prove the equation hold.

Theorem 3.4 *Let* (U, R) *be a generalized approximation space. For any fuzzy set* $A \in F(U)$, $\alpha \in (0.5, 1]$ *and* $\lambda \in [0,1]$. *Then the following relationships hold for any* $x \in U$.

1. $\bigcup_{\lambda \in [0,1]} \lambda(\underline{R}_\alpha(A_\lambda))(x) = \bigvee_{\lambda \in [0,1]} \{\lambda | P(A_\lambda | R_s(x)) \geq \alpha\}$,
2. $\bigcup_{\lambda \in [0,1]} \lambda(\overline{R}_\alpha(A_\lambda))(x) = \bigvee_{\lambda \in [0,1]} \{\lambda | P(A_\lambda | R_s(x)) > 1 - \alpha\}$.

Proof It is easy to prove similarly by using the decomposition theorem of classical fuzzy set theory.

3.3 The relationship between the variable precision approximations of a fuzzy set and its level set

In this section, we will establish the relationship between the variable precision rough approximation of a fuzzy set and the level set for the fuzzy set.

From the Theorem 3.2, for any $\alpha \in (0.5, 1]$ and $A \in F(U)$, $\lambda_1, \lambda_2 \in [0,1]$ and $\lambda_1 \geq \lambda_2$, the following relations hold:

$$\underline{R}_\alpha(A_{\lambda_1}) \subseteq \underline{R}_\alpha(A_{\lambda_2}), \quad \overline{R}_\alpha(A_{\lambda_1}) \subseteq \overline{R}_\alpha(A_{\lambda_2}).$$

Therefore, for any $\alpha \in (0.5, 1]$, it is easy to know that the family of the set $\{\underline{R}_\alpha(A_\lambda) | \lambda \in [0,1]\}$ and $\{\overline{R}_\alpha(A_\lambda) | \lambda \in [0,1]\}$ are two nested sets over the universe U.

By the Theorem 3.4, we present two symbols as follows:

$$\begin{aligned}
\underline{R}'_\alpha(A)(x) &= \bigvee_{\lambda \in [0,1]} \{\lambda | P(A_\lambda | R_s(x)) \geq \alpha\} \\
&= \bigvee_{\lambda \in [0,1]} \{\lambda | x \in \underline{R}_\alpha(A_\lambda)\}, \quad x \in U,
\end{aligned} \tag{1}$$

$$\begin{aligned}
\overline{R}'_\alpha(A)(x) &= \bigvee_{\lambda \in [0,1]} \{\lambda | P(A_\lambda | R_s(x)) > 1 - \alpha\} \\
&= \bigvee_{\lambda \in [0,1]} \{\lambda | x \in \overline{R}_\alpha(A_\lambda)\}, \quad x \in U,
\end{aligned} \tag{2}$$

As the former mentioned, both the family of the set $\{\underline{R}_\alpha(A_\lambda) | \lambda \in [0,1]\}$ and $\{\overline{R}_\alpha(A_\lambda) | \lambda \in [0,1]\}$ are two nested sets over the universe U. So, the set $\underline{R}'_\alpha(A)$ and $\overline{R}'_\alpha(A)$ are two fuzzy sets on the universe U. That is, we can obtain two fuzzy sets by using the variable precision rough lower approximation and upper approximation of the level set for a fuzzy set about the generalized approximation space (U, R).

Based on the above analysis, the following conclusion show the relationship between the variable precision rough approximation of a fuzzy set and the level set for the fuzzy set.

Theorem 3.5 *Let* (U, R) *be a generalized approximation space. For any fuzzy set* $A \in F(U)$, $\alpha \in (0.5, 1]$ *and* $\lambda \in [0,1]$. *Then the following relationship holds for any* $x \in U$.

$$\underline{R}_\alpha(A) = \underline{R}'_\alpha(A), \quad \overline{R}_\alpha(A) = \overline{R}'_\alpha(A).$$

Proof Denote $k_1 = \underline{R}_\alpha(A)(x), x \in U$, $k_2 = \underline{R}'_\alpha(A)(x)$, $x \in U$.

For any $\alpha \in (0.5, 1]$, suppose $\lambda \in [0,1]$ satisfy $x \in \underline{R}_\alpha(A_\lambda)$. Then, for any $y \in A_\lambda \cap R_s(x)$, there must be $A(y) \geq \lambda$. So, there is $\min_{y \in R_s(x) \cap A_\lambda} \geq \lambda$. Then we prove $k_1 \geq \lambda$. Therefore, $k_1 \geq k_2$.

Conversely, for any $\lambda > k_2$, by the definition of $k_2 = \underline{R}'_\alpha(A)(x)$, there is $A(y) < \lambda$ when $y \notin R_s(x) \cap A_\lambda$ and $y \in$

$R_s(x)$. Then we obtain that $\lambda > k_1$ by the definition of $k_1 = \underline{R}_\alpha(A)(x)$. Therefore, we can obtain that $k_2 \geq k_1$ since the relation $\lambda > k_2$ holds for any $\lambda \in [0,1]$.

So, we prove $\underline{R}_\alpha(A) = \underline{R}'_\alpha(A)$.

The second equation $\overline{R}_\alpha(A) = \overline{R}'_\alpha(A)$ can be easily proved as the same way of $\underline{R}_\alpha(A) = \underline{R}'_\alpha(A)$.

4 Uncertainty measure of variable precision rough fuzzy set

In Sect. 3, the accuracy and roughness are used to characterize uncertainty of a fuzzy set and approximation accuracy is employed to depict accuracy of a rough classification according to a general binary relation of universe. Although these measures are effective, they have some limitations when the lower and upper approximations of a fuzzy set with one level set is equal to that with another level set. To overcome these limitations, we address in this section the issue of uncertainty of a fuzzy set in the generalized approximation space.

Firstly, through an illustrative example, we real the limitations of the accuracy and roughness established in Sect. 3.2 for evaluating uncertainty of a fuzzy set and approximation accuracy of a rough classification according to a general binary relation.

Example 2 (Continued from Example 1) Let $\lambda = 0.5$, then we obtain $A_{0.5} = \{x_2, x_4, x_5\}$.

Taking $\alpha = 0.6$, then the lower and upper approximation of the 0.5−level set of fuzzy set A about the generalized approximation space (U, R), respectively, are as follows:

$$\underline{R}_{0.6}(A_{0.5}) = \{x_4\}, \quad \overline{R}_{0.6}(A_{0.5}) = \{x_1, x_2, x_3, x_4, x_6\},$$

there is

$$\rho^{\lambda}_{\alpha}(A) = \frac{|\underline{R}_\alpha(A_\lambda)|}{|\overline{R}_\alpha(A_\lambda)|} = \rho^{0.5}_{0.6}(A) = \frac{1}{5}, \quad \sigma^{0.5}_{0.6}(A) = 1 - \rho^{0.5}_{0.6}(A) = \frac{4}{5}.$$

Taking $\alpha = 0.7$, then the lower and upper approximation of the 0.5−level set of fuzzy set A about the generalized approximation space (U, R), respectively, are as follows:

$$\underline{R}_{0.7}(A_{0.5}) = \{x_5\}, \quad \overline{R}_{0.7}(A_{0.5}) = \{x_1, x_2, x_3, x_4, x_6\},$$

there is

$$\rho^{\lambda}_{\alpha}(A) = \frac{|\underline{R}_\alpha(A_\lambda)|}{|\overline{R}_\alpha(A_\lambda)|} = \rho^{0.5}_{0.7}(A) = \frac{1}{5}, \quad \sigma^{0.5}_{0.7}(A) = 1 - \rho^{0.5}_{0.7}(A) = \frac{4}{5}.$$

So,

$$\rho^{0.5}_{0.7}(A) = \rho^{0.5}_{0.6}(A) \quad \sigma^{0.5}_{0.7}(A) = \sigma^{0.5}_{0.6}(A).$$

Note that, in Example 2, there are two different values for the precision parameter with $\alpha = 0.6$ and $\alpha = 0.7$, but the same accuracy or roughness is obtained for the fuzzy set A about the generalized approximation space (U, R), respectively. Therefore, it is necessary to introduce more effective measure for the variable precision rough fuzzy set based on general binary relation.

In this section, we will propose a new uncertainty measure for the variable precision rough fuzzy set based on general binary relation by using the concept of the entropy of a fuzzy set on the generalized approximation space.

We first introduce the concept of entropy of a fuzzy set by an axiomatic approach.

Definition 4.1 [21, 49, 50] Let $e : F \to [0, +\infty)$ be a real function. If the following conditions are satisfied:

1. $e(D) = 0, \forall D \in P(U)$.
2. $e([\frac{1}{2}]_U) = \max_{A \in F(U)} e(A)$, (where $[\frac{1}{2}]_U$ stands for the fuzzy set over universe U and satisfies $[\frac{1}{2}]_U(x) = \frac{1}{2}$ for any $x \in U$).
3. For any $A, B \in F(U), x \in U$, if there is $B(x) \geq A(x)$ when $A(x) \geq \frac{1}{2}$ or $B(x) \leq A(x)$ when $A(x) \leq \frac{1}{2}$, then $e(A) \geq e(B)$.
4. $e(A^c) = e(A)$ for any $A \in F(U)$.

Then, we call e a entropy on the family of fuzzy set $F(U)$.

Let $U = \{x_1, x_2, \cdots, x_n\}$. Define

$$E(A) = \sum_{i=1}^{n} A(x_i)(1 - A(x_i)), \quad \forall A \in F(U).$$

Then, E is an entropy on the family of fuzzy set $F(U)$.

In the following, we present a new approach to uncertainty measure for variable precision rough fuzzy set based on general binary relation according to the conclusions of Theorem 3.5.

Definition 4.2 Let (U, R) be a generalized approximation space. For any fuzzy set $A \in F(U)$, $\alpha \in (0.5, 1]$. We call

$$E(\underline{R}'_\alpha(A)) = \sum_{x \in U} \underline{R}'_\alpha(A)(x)(1 - \underline{R}'_\alpha(A)(x))$$

and

$$E(\overline{R}'_\alpha(A)) = \sum_{x \in U} \underline{R}'_\alpha(A)(x)(1 - \underline{R}'_\alpha(A)(x))$$

the lower fuzzy entropy and upper fuzzy entropy of fuzzy set A about generalized approximation space (U, R) with precision parameter α.

Where $\underline{R}'_\alpha(A))$ *and* $\overline{R}'_\alpha(A)$ *were defined by the formula* (1) *and* (2) *in Sect.* 3.3.

It is east to know that $E(\underline{R}'_\alpha(A)) = \sum_{x\in U} \underline{R}_\alpha(A)(x)(1 - \underline{R}_\alpha(A)(x))$ and $E(\overline{R}'_\alpha(A)) = \sum_{x\in U} \underline{R}_\alpha(A)(x)(1 - \underline{R}_\alpha(A)(x))$ according to the conclusions of Theorem 3.5.

Where $\underline{R}_\alpha(A))$ *and* $\overline{R}_\alpha(A)$ *were defined in Sect.* 3.1.

Definition 4.3 Let (U, R) be a generalized approximation space. For any fuzzy set $A \in F(U)$, $\alpha \in (0.5, 1])$ and $\lambda \in [0, 1]$. We call

$$Accuracy_\alpha(A) = \frac{E(\underline{R}'_\alpha(A))}{E(\overline{R}'_\alpha(A))} = \frac{\sum_{x\in U} \vee\{\lambda | x \in \underline{R}_\alpha(A_\lambda)\}(1 - \vee\{\lambda | x \in \underline{R}_\alpha(A_\lambda)\})}{\sum_{x\in U} \vee\{\lambda | x \in \overline{R}_\alpha(A_\lambda)(1 - \vee\{\lambda | x \in \overline{R}_\alpha(A_\lambda))\}}.$$

the accuracy of fuzzy set A about the generalized approximation space with precision parameter α. Furthermore, we call $Roughness_\alpha(A) = 1 - Accurcy_\alpha(A)$ the roughness of fuzzy set A about the generalized approximation space with precision parameter α.

From the Definition 4.3, the following propositions are clear.

Proposition 4.1 *Let* (U, R) *be a generalized approximation space.* $\forall A \in F(U)$, $\alpha \in (0.5, 1], \lambda \in [0, 1]$. *Then the following results are clear.*

1. $Accuracy_\alpha(A)$ *not increase with the decrease of precision parameter* α.
2. $Roughness_\alpha(A)$ *not decrease with the increase of precision parameter* α.

Proposition 4.2 *Let* (U, R) *be a generalized approximation space.* $\forall A \in F(U)$, $\alpha \in (0.5, 1], \lambda \in [0, 1]$. *Then the following results are clear.*

$$0 \le Accuracy_\alpha(A) \le 1, \quad 0 \le Roughness_\alpha(A) \le 1.$$

Proposition 4.3 *Let* (U, R) *be a generalized approximation space.* $\forall A \in F(U)$, $\alpha \in (0.5, 1], \lambda \in [0, 1]$. *If there is* $\underline{R}_\alpha(A_\lambda) = \overline{R}_\alpha(A_\lambda)$ *for any* $\lambda \in [0, 1]$. *Then*

$$Accuracy_\alpha(A) = 1, \quad and \quad Roughness_\alpha(A) = 0.$$

Corollary 4.1 *Let* (U, R) *be a generalized approximation space.* $\forall A \in F(U)$, $\alpha \in (0.5, 1], \lambda \in [0, 1]$. *If there exits one* $\lambda \in [0, 1]$ *satisfies* $\underline{R}_\alpha(A_\lambda) \neq \overline{R}_\alpha(A_\lambda)$. *Then*

$$Roughness_\alpha(A) \neq 0 \quad and \quad Accuracy_\alpha(A) \neq 1.$$

Example 3 (Continued from Example 1) Let $\alpha = 0.6$. By computing, we have that

$$\begin{aligned} Accuracy_{0.6}(A) =& \frac{E(\underline{R}'_{0.4}(A))}{E(\overline{R}'_{0.4}(A))} \\ =& \frac{0 \times 1 + 0 \times 1 + 0 \times 1 + 0.5 \times 0.5 + 0 \times 1 + 0 \times 1}{0.6 \times 0.4 + 0.6 \times 0.4 + 0.5 \times 0.5 + 0 \times 1 + 0.9 \times 0.1 + 0 \times 1} = 0.126. \\ Roughness_{0.6}(A) =& 1 - Accuracy_{0.6}(A) = 0.874. \end{aligned}$$

Let $\alpha = 0.7$. By computing, we have that

$$\begin{aligned} Accuracy_{0.7}(A) =& \frac{E(\underline{R}'_{0.7}(A))}{E(\overline{R}'_{0.7}(A))} \\ =& \frac{0 \times 1 + 0 \times 1 + 0 \times 1 + 0.5 \times 0.5 + 0 \times 1 + 0 \times 1}{0.6 \times 0.4 + 0.6 \times 0.4 + 0.5 \times 0.5 + 0.9 \times 0.1 + 0.9 \times 0.1 + 0.6 \times 0.4} = 0.097. \\ Roughness_{0.7}(A) =& 1 - Accuracy_{0.7}(A) = 0.903. \end{aligned}$$

So, $Accuracy_{0.6}(A) \neq Accuracy_{0.7}(A)$ and $Roughness_{0.6}(A) \neq Roughness_{0.7}(A)$.

Furthermore, there are $Accuracy_{0.6}(A) > Accuracy_{0.7}(A)$ and $Roughness_{0.6}(A) < Roughness_{0.7}(A)$.

This verifies the conclusions of Proposition 4.1 well.

5 Conclusion

In this paper, we propose a new rough set model named variable precision rough fuzzy set based on general relations. We define the lower and upper approximations of any fuzzy set and its level set on the generalized approximation space with variable precision parameter by using the conditional probability between the fuzzy set (level-set of the fuzzy set) and the neighborhood set of any element $x \in U$. In other words, we establish the variable precision probabilistic rough fuzzy set based on general relations or the probabilistic definition of variable precision rough fuzzy set model based on general relations. Meanwhile, we discuss the relationships between the established model and the exiting rough set models in detail. Also, we establish the relationship among of the variable precision rough approximation of a fuzzy set and its level set. The results show that the proposed model have extended the related rough set model and also included the existing models. Furthermore, we study the properties of the variable precision rough fuzzy set model based on general relations and we also compare the differences between the proposed model and the related models. Finally, we give new measurement for the accuracy and roughness of the variable precision rough approximation in generalized approximation space.

The proposed model in this paper gives a new perspective for investigating of the variable precision rough fuzzy set. Like the classical fuzzy rough set, the variable precision fuzzy rough set is also one of an important models both in theoretical and practical of the rough set theory. As far as the possible application of the proposed model, the binary relation over the universe of discourse and the idea of the definition for general relation-based variable precision rough fuzzy set can describe the characterization of uncertainty emergency decision-making problems of unconventional emergency events. Then, several interesting and valuable uncertainty decision-making models will be established for emergency decision-making by using the proposed approach in this paper. So, the future research will focus on the further discussion of the variable precision fuzzy rough set theory and its applications in the decision-making of unconventional emergency events with uncertainty.

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