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ORIGINAL ARTICLE

Multiple attribute group decision making based on generalized power aggregation operators under interval-valued dual hesitant fuzzy linguistic environment

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Abstract This paper aims to investigate the type of fuzzy multiple attribute group decision making (MAGDM) where arguments being aggregated are allowed to support each other. In order to enable decision makers to express their preferences more comprehensively, we firstly put forward a hybrid tool, an interval-valued dual hesitant fuzzy linguistic set (IVDHFLS), which employs interval-valued hesitant membership and nonmembership degrees to assess linguistic terms. Basic operational laws for IVDHFLS are discussed, also a distance measure is designed to overcome irrationality in traditional methodology for hesitant fuzzy sets, i.e., artificially adding values to mismatching membership or nonmembership degrees. We next develop fundamental generalized power average aggregation operators for IVDHFLS, including power average operator, power geometric average operator, power ordered weighted average operator and power ordered weighted geometric average operator. Desirable properties and special cases of these aggregation operators are further analyzed. Furthermore, based on the generalized operators above, we construct two approaches for MAGDM with mutually supportive arguments being aggregated under interval-valued dual hesitant fuzzy linguistic environments. Finally, case studies are conducted to verify effectiveness and practicality of the developed approaches.

Keywords Multiple attribute group decision making · Interval-valued dual hesitant fuzzy linguistic set · Generalized aggregation operators · Power aggregation operators

1 Introduction

Due to increasing complexity in socioeconomic environments and fuzziness in human cognition, single decision maker is usually incompetent in considering all relevant aspects of complicated decision making problems arising in different systems, therefore, group decision making (GDM) becomes common activity and analytics in modern technological society [3, 33, 39]. The purpose of GDM is to find most desirable solution(s) from finite alternatives by a group of experts assessing on a set of criteria [48, 68]. As an important part of GDM theories, multiple attribute group decision making (MAGDM) has been widely studied under different uncertain environments, for which fuzzy set theory [64] and its extensions have been shown as effective quantitative fuzzy tools for decision modelling with uncertainty, such as interval-valued fuzzy set [38], intuitionistic fuzzy set (IFS) [1], interval-valued intuitionistic fuzzy set [2]. However, these fuzzy theories may not suitable for situations where decision makers are hesitant and irresolute in establishing membership degrees because of several possible values, so Torra [35], Torra and Narukawa [37] defined hesitant fuzzy set (HFS) to allow membership degrees of an element to be a set of values. HFS are highly useful in handling complex decision situations where decision makers hesitate when providing their

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preferences [17, 44]. Recently, in view of that nonmembership degrees play the same important role as membership degrees in describing vague preferences, Zhu et al. [72] developed dual hesitant fuzzy set (DHFS) to denote membership and nonmembership degrees of an element to a given concept by two sets of crisp values. Then Ju et al. [15] and Farhadinia [8] extended DHFS to interval-valued environments. Evidently, DHFS is more practical in MAGDM as it provides an effective and flexible way to assign values for each element in the domains [8, 71].

With respect to those complex MAGDM problems that cannot be well-defined for quantification, Zadeh [65] suggested linguistic variables for expressing qualitative fuzzy preferences. Being capable of enhancing classical decision models [22, 44], linguistic variables have been studied in depth [4, 7, 26, 27, 29, 49, 50] and applied into many fields [5, 9, 14, 43]. Especially, noticing inadequacy of single linguistic term to reflect decision uncertainty completely, some novel expression forms have been developed to accommodate membership and nonmembership degrees of an element to a particular linguistic term. Such as, based on IFS, Wang and Li [42] introduced the intuitionistic linguistic set, Liu and Jin [20] studied intuitionistic uncertain linguistic set, Liu [19] further proposed interval-valued intuitionistic uncertain linguistic set to incorporate advantages and flexibility of both interval values and uncertain linguistic variables. In order to tackle more complex decision situations where exists decision hesitancy among several possible values, based on HFS, Lin et al. [18] developed hesitant fuzzy linguistic set whose elements hold the structure comprising of a certain linguistic term and a set of possible crisp membership degrees. Wang et al. [44] studied interval-valued hesitant fuzzy linguistic set to integrate advantages of both linguistic variables and interval-valued hesitant fuzzy elements. Meng et al. [23] presented linguistic hesitant fuzzy set to consider possible membership degrees of each possible linguistic term. Unfortunately, above HFS based fuzzy linguistic tools can only permit possible values for membership degrees.

In practice, for MAGDM problems under ill-defined circumstances, decision makers may want to state their preferences by certain linguistic variable with not only a set of possible membership degrees but also a set of nonmembership degrees. Till now, to our best knowledge, only Yang and Ju [62] carried out forerunning work based on DHFS to investigate dual hesitant fuzzy linguistic set (DHFLS), but only allowing exact numbers to collect possible membership and nonmembership degrees associated with a specified linguistic term. However, as noted in [30, 52, 54], due to time pressure, lack of knowledge or data, and limited expertise about complicated problems, decision makers are usually only willing or able to use interval values to express their preferences. Therefore, this paper extends DHFLS into interval-valued environments to propose another desirable hybrid tool, i.e., interval-valued dual hesitant fuzzy linguistic set (IVDHFLS). IVDHFLS takes a compound structure $\langle x, s, \tilde{h}(x), \tilde{g}(x)\rangle$ to denote its elements. For the evaluated object x, s provides certain linguistic term, $\tilde{h}(x)$ and $\tilde{g}(x)$ represent two interval-valued fuzzy sets to reflect possible membership and nonmembership degrees to s. When $\tilde{h}(x)$ and $\tilde{g}(x)$ reduce to exact sets, IVDHFLS becomes DHFLS. Obviously, IVDHFLS retains advantages of linguistic variables and DHFS in depicting fuzzy properties of an complex object, also holds the flexibility of interval numbers when assigning possible membership and nonmembership degrees.

During the procedures of MAGDM, aggregation operators play an indispensable role in aggregating individual evaluations into collective ones [12]. Based on traditional aggregation operators [6, 58, 61], such as operators WA, WG, OWA, OWG, and etc., numerous extended aggregation operators have been introduced to support decision making, such as, intuitionistic fuzzy aggregation operators [63, 70], hesitant fuzzy aggregation operators [24, 45, 46], linguistic aggregation operators [26, 27, 29, 49, 50], induced aggregation operators [25, 28] and generalized aggregation operators [27, 29, 66, 69]. However, most of extant aggregation operators do not sufficiently consider supportive correlations among arguments, so Yager [59] developed power average (PA) operator and power ordered weighted average (POWA) operator, in which weighting vectors depend on input arguments and allow arguments being aggregated to support each other. Then Xu and Yager [57] developed the power geometric average (PG) operator and the power ordered weighted average (POWGA) operator. Following their work, power average aggregation operators have been further extended to accommodate MAGDM under different uncertain environments. Such as, the linguistic power aggregation operators by Zhou and Chen [67], the power aggregation operators by Xu [55] under intuitionistic fuzzy and interval-valued intuitionistic fuzzy decision making environments, the power aggregation operators by Wan [40] under trapezoidal intuitionistic fuzzy decision making environments, and generalized argument-dependent power operators by Zhou et al. [68] to accommodate intuitionistic fuzzy preferences. In this paper, we continue to focus on investigating effective approaches for fuzzy MADGM with arguments being aggregated to support each other, but in which decision preferences take the form of IVDHFLS. To do so, firstly, a novel distance measure for IVDHFLS is put forward to overcome irrationality in traditional methodology for HFS, i.e., artificially complementing mismatching membership or nonmembership degrees [32, 56]. Then, fundamental generalized power aggregation operators are developed for

IVDHFLS, including weighted generalized interval-valued dual hesitant fuzzy linguistic power average (WGIVDHFLPA) operator, weighted generalized interval-valued dual hesitant fuzzy linguistic power geometric average (WGIVDHFLPGA) operator, generalized interval-valued dual hesitant fuzzy linguistic power ordered weighted average (GIVDHFLPOWA) operator, and generalized interval-valued dual hesitant fuzzy linguistic power ordered weighted geometric average (GIVDHFL-POWGA) operator. Their desirable properties and special cases are also discussed. Furthermore, by utilizing the developed generalized power operators, two effective approaches are constructed for MAGDM with mutually supportive arguments being aggregated under interval-valued dual hesitant fuzzy linguistic environments.

The remainder of this paper is organized as follows. Section 2 briefly reviews concepts of linguistic variables and interval-valued dual hesitant fuzzy set. In Sect. 3, we define the interval-valued dual hesitant fuzzy linguistic set (IVDHFLS), for which operational rules and a new distance measure are studied. Section 4 investigates fundamental generalized power aggregation operators for IVDHFLS, their properties and special cases. In Sect. 5, two MAGDM approaches are constructed in details. Additionally, to verify effectiveness and practicality of the approaches, illustrative case studies are carried out in Sect. 6. Finally, conclusions and further research are given in Sect. 7.

2 Preliminaries

2.1 Linguistic variables

Suppose $S = \{s_\alpha | \alpha = 0, 1, \ldots, l_\alpha - 1\}$ be a finite and totally ordered discrete linguistic label set with odd cardinality, such as 3, 7 and 9, where s_a is called a linguistic variable that represents a possible value in S, l_α is the cardinality.

Example 1 A linguistic set of nine terms, S, can be defined as:

$$S = \{s_0 = none; s_1 = verylow; s_2 = low; s_3 = almostmedium; s_4 = medium; s_5 = almosthigh; s_6 = high; s_7 = veryhigh; s_8 = perfect\}.$$

For the linguistic term set S, it is usually required that there exist the following characteristics:

1. The set S is ordered: $s_\alpha \geq s_\beta$ if $\alpha \geq \beta$;
2. Negative operator: $neg(s_\alpha) = s_\beta$ such that $\beta = g - 1 - \alpha$;
3. Max operator: $max(s_\alpha, s_\beta) = s_\beta$ if $s_\alpha \leq s_\beta$;
4. Min operator: $min(s_\alpha, s_\beta) = s_\beta$ if $s_\alpha \geq s_\beta$.

In the process of aggregating linguistic information, however, some results may not exactly match any linguistic terms presented in S. To preserve all given information, Herrera et al. [10] extended discrete term set S to the continuous term set $\bar{S} = \{s_\alpha | \alpha \in [0, l_\alpha - 1]\}$. Elements in $\bar{S}$ are called virtual linguistic terms and s_α is called a continuous linguistic variable [49]. Obviously, linguistic symbolic computational model based on continuous term set $\bar{S}$ is simple and convenient to use in decision making; most importantly, it can avoid information loss. Therefore, in the following, we adopt the continuous term set $\bar{S}$ to denote linguistic information.

Consider any two continuous linguistic variables: s_α and s_β, $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, Xu [49] defined some operational laws:

1. $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}$;
2. $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}$;
3. $s_\alpha / s_\beta = s_{\alpha/\beta}$ if $\beta \neq 0$;
4. $\lambda s_\alpha = s_{\lambda\alpha}$;
5. $s_\alpha^\lambda = s_{\alpha^\lambda}$;
6. $\lambda(s_\alpha \oplus s_\beta) = \lambda s_\alpha \oplus \lambda s_\beta$;
7. $\lambda_1 s_\alpha \oplus \lambda_2 s_\alpha = (\lambda_1 + \lambda_2) s_\alpha$;
8. $(s_\alpha \otimes s_\beta)^\lambda = s_\alpha^\lambda \otimes s_\beta^\lambda$;
9. $s_\alpha^{\lambda_1} \otimes s_\alpha^{\lambda_2} = s_\alpha^{\lambda_1 + \lambda_2}$.

To simplify representation and facilitate calculation, Xu [51] also presented the following functions to transform linguistic terms into corresponding term indices.

Definition 2.1 [51] Let $\bar{S} = \{s_\alpha | \alpha \in [0, l_\alpha - 1]\}$ be a continuous linguistic term set, where l_α is the cardinality; $s_\alpha \in \bar{S}$ is a continuous linguistic variable, denoting a virtual linguistic term. Then the corresponding term index can be derived by the function I as follows,

$$I : \bar{S} \rightarrow [0, l_\alpha - 1],$$

$$I(s_\alpha) = \frac{\alpha}{l_\alpha - 1}. \tag{1}$$

Sometimes, we need to map different linguistic term sets with different cardinalities into the linguistic term set with same cardinality. The following transformation function provide an effective way to map different linguistic term sets into the one with largest cardinality.

Definition 2.2 [51] Given two continuous linguistic term sets: $\bar{S}_1 = \{s_\alpha | \alpha \in [0, l_\alpha - 1]\}$ and $\bar{S}_2 = \{s_\beta | \beta \in [0, l_\beta - 1]\}$, where l_α and l_β are their cardinalities; $s_\alpha \in \bar{S}_1$ and $s_\beta \in \bar{S}_2$ are continuous linguistic variables. Then the transformation function f between s_α and s_β can be defined as follows,

$$f : \bar{S}_1 \rightarrow \bar{S}_2$$

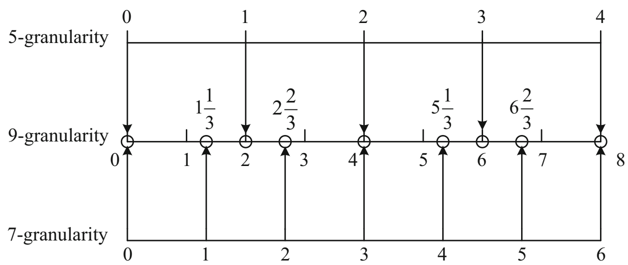

Fig. 1 Mapping 5-granularity and 7-granularity linguistic term sets into 9-granularity linguistic term set

$$\beta = f(\alpha) = \alpha \frac{l_\beta - 1}{l_\alpha - 1}; \quad (2)$$

$$f^{-1}: \bar{S}_2 \to \bar{S}_1$$

$$\alpha = f^{-1}(\beta) = \beta \frac{l_\alpha - 1}{l_\beta - 1}. \quad (3)$$

The above transformation function can be explained clearly in Fig. 1. Taken 5-granularity and 7-granularity linguistic term sets for examples, according to Eq. (2), we can map them into a 9-granularity linguistic term set as shown in Fig. 1.

2.2 Interval-valued dual hesitant fuzzy set (IVDHFS)

Torra [35] and Torra and Narukawa [37] proposed hesitant fuzzy set (HFS) to handle situations in which several values are possible for defining a membership function to a fuzzy set.

Definition 2.3 [35, 37] Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0, 1], which can be represented as the following symbol,

$$E = \{\langle x, h_E(x)\rangle | x \in X\}, \quad (4)$$

where $h_E(x)$ is a set of values in [0, 1], denoting possible membership degrees of the element $x \in X$ to the set E.

For convenience, Xu and Xia [56] named $h = h_E(x)$ as a hesitant fuzzy element (HFE) and H as the set of all HFEs.

Whereas, HFS accommodates only membership degrees of an element to a given set without considering nonmembership degrees. To overcome this limitation, Zhu et al. [72] proposed the dual hesitant fuzzy set (DHFS).

Definition 2.4 [72] Let X be a fixed set, a dual hesitant fuzzy set (DHFS) on X is represented as

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\}, \quad (5)$$

where $h(x) = \bigcup_{\mu \in h(x)} \{\mu\}$ and $g(x) = \bigcup_{v \in g(x)} \{v\}$ are two sets of crisp values in [0, 1], denoting possible membership degrees and possible nonmembership degrees of the element $x \in X$ to the set D, respectively. μ and v satisfy conditions: $\mu, v \in [0,1]$ and $0 \le \mu^+ + v^+ \le 1$, where $\mu \in h(x), v \in g(x), \mu^+ \in h^+(x) = \bigcup_{\mu \in h(x)} \max\{\mu\}$ and $v^+ \in g^+(x) = \bigcup_{v \in g(x)} \max\{v\}$ for all $x \in X$.

However, precise membership degrees and nonmembership degrees of an element to a set are sometimes hard to be specified, to overcome this barrier, Ju et al. [15] defined the concept of interval-valued dual hesitant fuzzy set (IVDHFS).

Definition 2.5 [15] Let X be a fixed set, then we can define an interval-valued dual hesitant fuzzy set (IVDHFS) on X as:

$$\tilde{D} = \{\langle x, \tilde{h}(x), \tilde{g}(x)\rangle | x \in X\}, \quad (6)$$

where $\tilde{h}(x) = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}(x)} \{\tilde{\mu}\} = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}(x)} \{[\mu^L, \mu^U]\}$ and $\tilde{g}(x) = \bigcup_{[v^L, v^U] \in \tilde{g}(x)} \{\tilde{v}\} = \bigcup_{[v^L, v^U] \in \tilde{g}(x)} \{[v^L, v^U]\}$ are two sets of interval values in [0, 1], denoting possible membership degrees and nonmembership degrees of the element $x \in X$ to set $\tilde{D}$, respectively; $\tilde{\mu}, \tilde{v} \in [0,1]$, $0 \le (\mu^U)^+ + (v^U)^+ \le 1$, where $(\mu^U)^+ \in \tilde{h}^+(x) = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}(x)} \max\{\mu^U\}$ and $(v^U)^+ \in \tilde{g}^+(x) = \bigcup_{[v^L, v^U] \in \tilde{g}(x)} \max\{v^U\}$ for all $x \in X$.

Here, we denote $\tilde{d} = \{\tilde{h}, \tilde{g}\}$ as an interval-valued dual hesitant fuzzy element (IVDHFE) and $\tilde{D}$ as the set of all IVDHFEs. Some operational rules for IVDHFEs were defined as follows.

Definition 2.6 [15] Given three IVDHFEs represented by $\tilde{d} = \{\tilde{h}, \tilde{g}\}, \tilde{d}_1 = \{\tilde{h}_1, \tilde{g}_1\}, \tilde{d}_2 = \{\tilde{h}_2, \tilde{g}_2\}$,, then basic operations between them can be described as

1. $\tilde{d}^\lambda = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}, [v^L, v^U] \in \tilde{g}} \{\{[(\mu^L)^\lambda, \quad (\mu^U)^\lambda]\}, \{[1-(1-v^L)^\lambda, 1-(1-v^U)^\lambda]\}\}, \lambda > 0;$
2. $\lambda \tilde{d} = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}, [v^L, v^U] \in \tilde{g}} \{\{[1-(1-\mu^L)^\lambda, \quad 1-(1-\mu^U)^\lambda]\}, \{[(v^L)^\lambda, (v^U)^\lambda]\}\}, \lambda > 0;$
3. $\tilde{d}_1 \oplus \tilde{d}_2 = \bigcup_{[\mu_1^L, \mu_1^U] \in \tilde{h}_1, [\mu_2^L, \mu_2^U] \in \tilde{h}_2, [v_1^L, v_1^U] \in \tilde{g}_1,} \quad [v_2^L, v_2^U] \in \tilde{g}_2 \{\{[\mu_1^L + \mu_2^L - \mu_1^L \mu_2^L, \mu_1^U + \mu_2^U - \mu_1^U \mu_2^U]\}, \{[v_1^L v_2^L, v_1^U v_2^U]\}\};$
4. $\tilde{d}_1 \otimes \tilde{d}_2 = \bigcup_{[\mu_1^L,} \quad \mu_1^U] \in \tilde{h}_1, [\mu_2^L, \mu_2^U] \in \tilde{h}_2, [v_1^L, v_1^U] \in \tilde{g}_1, [v_2^L, v_2^U] \in \tilde{g}_2 \{\{[\mu_1^L \mu_2^L, \mu_1^U \mu_2^U]\}, \{[v_1^L + v_2^L - v_1^L v_2^L, v_1^U + v_2^U - v_1^U v_2^U]\}\}.$

Definition 2.7 [15] Let $\tilde{d}_1 = \{\tilde{h}_1, \tilde{g}_1\}$ and $\tilde{d}_2 = \{\tilde{h}_2, \tilde{g}_2\}$ be any two IVDHFEs, then basic operational rules between them can be defined as:

1. $\tilde{d}_1 \oplus \tilde{d}_2 = \tilde{d}_2 \oplus \tilde{d}_1;$
2. $\tilde{d}_1 \otimes \tilde{d}_2 = \tilde{d}_2 \otimes \tilde{d}_1;$

3. $\lambda(\tilde{d}_1 \oplus \tilde{d}_2) = \lambda\tilde{d}_1 \oplus \lambda\tilde{d}_2, \lambda \geq 0$;
4. $\tilde{d}_1^{\lambda} \otimes \tilde{d}_2^{\lambda} = (\tilde{d}_2 \otimes \tilde{d}_1)^{\lambda}, \lambda \geq 0$.

Ju et al. [15] introduced a score function $S(\tilde{d})$ to calculate score of $\tilde{d}$, and also defined an accuracy function $P(\tilde{d})$ of $\tilde{d}$ to evaluate accuracy degree of $\tilde{d}$, where

$$S(\tilde{d}) = \frac{1}{2}\left(\frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^L - \frac{1}{l(\tilde{g})}\sum_{[\nu^L,\nu^U]\in\tilde{g}}\nu^L + \frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^U - \frac{1}{l(\tilde{g})}\sum_{[\nu^L,\nu^U]\in\tilde{g}}\nu^U\right), \tag{7}$$

$$P(\tilde{d}) = \frac{1}{2}\left(\frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^L + \frac{1}{l(\tilde{g})}\sum_{[\nu^L,\nu^U]\in\tilde{g}}\nu^L + \frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^U + \frac{1}{l(\tilde{g})}\sum_{[\nu^L,\nu^U]\in\tilde{g}}\nu^U\right), \tag{8}$$

where $l(\tilde{h})$ and $l(\tilde{g})$ are numbers of interval values in $\tilde{h}$ and $\tilde{g}$, respectively. The larger the score $S(\tilde{d})$, the larger the accuracy $P(\tilde{d})$, the greater the IVDHFE $\tilde{d}$. Then, ordering relation between $\tilde{d}_1 = \{\tilde{h}_1, \tilde{g}_1\}$ and $\tilde{d}_2 = \{\tilde{h}_2, \tilde{g}_2\}$ can be described as follows.

If $S(\tilde{d}_1) < S(\tilde{d}_2)$, then $\tilde{d}_1 < \tilde{d}_2$.

If $S(\tilde{d}_1) = S(\tilde{d}_2)$, then

1. If $P(\tilde{d}_1) = P(\tilde{d}_2)$, then $\tilde{d}_1 = \tilde{d}_2$;
2. If $P(\tilde{d}_1) < P(\tilde{d}_2)$, then $\tilde{d}_1 < \tilde{d}_2$.

2.3 Power aggregation operators

In this section, we briefly review some fundamental power aggregation operators in literatures.

Definition 2.8 [59] A power average (PA) operator of dimension n is a mapping: $R^n \to R$, according to following formula:

$$PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^n (1 + T(a_i))a_i}{\sum_{i=1}^n (1 + T(a_i))}, \tag{9}$$

where

$$T(a_i) = \sum_{j=1, j\neq i}^n Sup(a_i, a_j). \tag{10}$$

$Sup(a_i, a_j)$ is the support function to calculate the degree that a_i receives support from a_j. $Sup(a_i, a_j)$ satisfies following three properties:

1. $Sup(a_i, a_j) \in [0, 1]$;
2. $Sup(a_i, a_j) = Sup(a_j, a_i)$;
3. $Sup(a_i, a_j) \geq Sup(a_k, a_s)$, if $|a_i - a_j| \leq |a_k - a_s|$.

PA operator is a non-linear weighted average aggregation operator, allowing arguments to support each other in the aggregation process. Corresponding weight $(1 + T(a_i))/\sum_{i=1}^n (1 + T(a_i))$ of argument a_i depends on all input arguments $a_i (i = 1, 2, \ldots, n)$. The closer two values are, the more they support each other.

Yager [59] also defined the power ordered weighted average (POWA) operator as follows.

Definition 2.9 [59] A power ordered weighted average (POWA) operator of dimension n is a mapping: $R^n \to R$, according to the following formula:

$$POWA_w(a_1, a_2, \ldots, a_n) = \sum_{i=1}^n w_i a_{\sigma(i)}, \tag{11}$$

where

$$w_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right), \quad R_i = \sum_{j=1}^i V_{\sigma(j)},$$
$$TV = \sum_{i=1}^n V_{\sigma(i)}, \quad V_{\sigma(i)} = 1 + T(a_{\sigma(i)}),$$

$$T(a_{\sigma(i)}) = \sum_{j=1, j\neq i}^n Sup(a_{\sigma(i)}, a_{\sigma(j)}). \tag{12}$$

Here, $a_{\sigma(i)}$ denotes the i th largest arguments $a_j (j = 1, 2, \ldots, n)$; g: $[0, 1] \to [0, 1]$ is a basic BUM function [60] which satisfies: $g(0) = 0, g(1) = 1, g(x) \geq g(y)$ if $x > y$. $Sup(a_{\sigma(i)}, a_{\sigma(j)})$ indicates to what degree the jth largest argument supports the ith largest argument; $T(a_{\sigma(i)})$ denotes the support that the ith largest argument receives from all the other arguments.

Xu and Yager [57] further proposed the power geometric (PG) operator and the power ordered weighted geometric (POWG) operator.

Definition 2.10 [57] A power geometric (PG) operator of dimension n is a mapping: $R^n \to R$, according to the following formula:

$$PG(a_1, a_2, \ldots, a_n) = \prod_{i=1}^n a_i^{\frac{1+T(a_i)}{\sum_{i=1}^n (1+T(a_i))}}, \tag{13}$$

where $T(a_i)$ satisfies Eq. (10).

Definition 2.11 [57] A power ordered weighted geometric (POWG) operator of dimension n is a mapping: $R^n \to R$, according to the following formula:

$$POWG(a_1, a_2, \ldots, a_n) = \prod_{i=1}^n a_{\sigma(i)}^{w_i}, \tag{14}$$

where w_i satisfies Eq. (12), and $a_{\sigma(i)}$ is the ith largest among the arguments $a_j (j = 1, 2, \ldots, n)$.

By combining PA operator with generalized mean operator, Zhou et al. [68] defined the following weighted generalized power average (WGPA) operator, and generalized power ordered weighted average (GPOWA) operator.

Definition 2.12 [68] Given parameter A weighted generalized power average (WGPA) operator of dimension n is a mapping: $R^n \to R$, according to the following formula:

$$WGPA(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{i=1}^{n} \omega_i (1 + T(a_i)) a_i^{\lambda}}{\sum_{i=1}^{n} \omega_i (1 + T(a_i))} \right)^{1/\lambda}, \tag{15}$$

where

$$T(a_i) = \sum_{j=1, j \neq i}^{n} \omega_j Sup(a_i, a_j). \tag{16}$$

Here,$\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1; Sup(a_i, a_j)$ is the support that a_i receives from a_j; $\lambda \in (-\infty, +\infty)$ and $\lambda \neq 0$.

Definition 2.13 [68] A generalized power ordered weighted average (GPOWA) operator of dimension n is a mapping: $R^n \to R$, defined by following formula:

$$GPOWA_{w,\lambda}(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} w_i a_{\sigma(i)}^{\lambda} \right)^{1/\lambda}, \tag{17}$$

where w_i satisfied Eq. (12); $\lambda \in (-\infty, +\infty)$ and $\lambda \neq 0$.

3 Interval-valued dual hesitant fuzzy linguistic set (IVDHFLS)

For better tackling complex MAGDM problems, dual hesitant fuzzy linguistic sets (DHFLS) [62] managed to represent decision maker's preferences by certain linguistic term with possible membership degrees and nonmembership degrees to indicate decision maker's hesitancy. While usually under circumstances of high complexity, due to time pressure and knowledge limitations, decision makers are only willing or able to provide their preferences with interval values [30, 52, 54]. To accommodate this kind of situations, inspired by DHFLS [62], here we define the concept of interval-valued dual hesitant fuzzy linguistic set (IVDHFLS), and study its basic operational rules; then, we propose an effective distance measure for IVDHFLS to overcome irrationality in conventional method.

3.1 Definition of IVDHFLS

Definition 3.1 Let X be a fixed set, $\bar{S}$ be a finite and continuous linguistic term set, then an interval-valued dual hesitant fuzzy linguistic set (IVDHFLS) SD on X is defined as

$$SD = \left\{ \langle x, s_{\alpha(x)}, \tilde{h}(x), \tilde{g}(x) \rangle | x \in X \right\}, \tag{18}$$

where $s_{\alpha(x)}$ is selected from predefined $\bar{S}$ to represent decision maker's judgment to an object x being evaluated; $\tilde{h}(x) = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}(x)} \{\tilde{\mu}\} = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}(x)} \{[\mu^L, \mu^U]\}$ is a set of closed interval values belonging to $[0, 1]$ for denoting possible membership degrees to which x belongs to $s_{\alpha(x)}$; $\tilde{g}(x) = \bigcup_{[v^L, v^U] \in \tilde{g}(x)} \{\tilde{v}\} = \bigcup_{[v^L, v^U] \in \tilde{g}(x)} \{[v^L, v^U]\}$ is a set of closed interval values belonging to $[0, 1]$ for denoting possible nonmembership degrees to which x belongs to $s_{\alpha(x)}$, that is, possible degrees to which x does not belong to $s_{\alpha(x)}$. $\tilde{h}(x)$ and $\tilde{g}(x)$ satisfy: $\tilde{\mu}, \tilde{v} \in [0, 1], 0 \leq (\mu^U)^+ + (v^U)^+ \leq 1$, where $(\mu^U)^+ \in \tilde{h}^+(x) = \bigcup_{[\mu^L, \mu^U] \in \tilde{h}(x)} \max\{\mu^U\}$ and $(v^U)^+ \in \tilde{g}^+(x) = \bigcup_{[v^L, v^U] \in \tilde{g}(x)} \max\{v^U\}$ for $\forall x$.

When $X = \{x_1, x_2, \ldots, x_n\}$ has only one element, SD reduces to $(s_\alpha, \tilde{h}, \tilde{g})$. For convenience, $sd = (s_\alpha, \tilde{h}, \tilde{g})$ is called an interval-valued dual hesitant fuzzy linguistic number (IVDHFLN), and IVDHFLNs represent all elements in IVDHFLS.

3.2 Operational rules for IVDHFLS

Definition 3.2 Given three IVDHFLNs: $sd = (s_\alpha, \tilde{h}, \tilde{g})$, $sd_1 = (s_{\alpha_1}, \tilde{h}_1, \tilde{g}_1)$ and $sd_2 = (s_{\alpha_2}, \tilde{h}_2, \tilde{g}_2), \lambda \in [0, 1]$, basic operations on these sets are defined by

1. $\lambda sd = \bigcup_{(s_\alpha, \tilde{h}, \tilde{g}) \in sd} (s_{\lambda\alpha}, \bigcup_{[\mu^L, \mu^U] \in \tilde{h}, [v^L, v^U] \in \tilde{g}} \{\{[1 - (1 - \mu^L)^\lambda, 1 - (1 - \mu^U)^\lambda]\}, \{[(v^L)^\lambda, (v^U)^\lambda]\}\}) = \bigcup_{(s_\alpha, \tilde{h}, \tilde{g}) \in sd} \left(s_{\lambda\alpha}, \bigcup_{[\mu^L, \mu^U] \in \tilde{h},} [v^L, v^U] \in \tilde{g} \{\{[1 - (1 - \mu^L)^\lambda, 1 - (1 - \mu^U)^\lambda]\}, \{[(v^L)^\lambda, (v^U)^\lambda]\}\}\right);$
2. $sd^\lambda = \bigcup_{(s_\alpha, \tilde{h}, \tilde{g}) \in sd} \left(s_{\alpha^\lambda}, \bigcup_{[\mu^L, \mu^U] \in \tilde{h}, [v^L, v^U] \in \tilde{g}} \{\{[(\mu^L)^\lambda, (\mu^U)^\lambda]\}, \{[1 - (1 - v^L)^\lambda, 1 - (1 - v^U)^\lambda]\}\}\right) = \bigcup_{(s_\alpha, \tilde{h}, \tilde{g}) \in sd} \left(s_{\alpha^\lambda}, \bigcup_{[\mu^L, \mu^U] \in \tilde{h}, [v^L, v^U] \in \tilde{g}} \{\{[(\mu^L)^\lambda, (\mu^U)^\lambda]\}, \{[1 - (1 - v^L)^\lambda, 1 - (1 - v^U)^\lambda]\}\}\right);$
3. $sd_1 \oplus sd_2 = \bigcup_{(s_{\alpha_1}, \tilde{h}_1, \tilde{g}_1)} \in sd_1, (s_{\alpha_2}, \tilde{h}_2, \tilde{g}_2) \in sd_2 (s_{\alpha_1 + \alpha_2}, \bigcup_{[\mu_1^L, \mu_1^U] \in \tilde{h}_1, [\mu_2^L, \mu_2^U] \in \tilde{h}_2, [v_1^L, v_1^U] \in \tilde{g}_1, [v_2^L, v_2^U] \in \tilde{g}_2} \{\{[\mu_1^L + \mu_2^L - \mu_1^L \mu_2^L, \mu_1^U + \mu_2^U - \mu_1^U \mu_2^U]\}, \{[v_1^L v_2^L, v_1^U v_2^U]\}\});$

4. $sd_1 \otimes sd_2 = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2}(s_{\alpha_1\alpha_2}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\{\{[\mu_1^L\mu_2^L, \mu_1^U\mu_2^U]\}, \{[v_1^L+v_2^L-v_1^Lv_2^L, v_1^U+v_2^U-v_1^Uv_2^U]\}\})$.

It can be easily proven that all results given above are also IVDHFLNs, then we can have the following theorem.

Theorem 3.1 *Let* $sd = (s_\alpha, \tilde{h}, \tilde{g}), sd_1 = (s_{\alpha_1}, \tilde{h}_1, \tilde{g}_1)$ *and* $sd_2 = (s_{\alpha_2}, \tilde{h}_2, \tilde{g}_2)$ *be any three IVDHFLNs, then following properties are true*:

1. $sd_1 \oplus sd_2 = sd_2 \oplus sd_1$;
2. $sd_1 \otimes sd_2 = sd_2 \otimes sd_1$;
3. $\lambda(sd_1 \oplus sd_2) = \lambda sd_1 \oplus \lambda sd_2, \lambda \in [0,1]$;
4. $sd_1^\lambda \otimes sd_2^\lambda = (sd_1 \otimes sd_2)^\lambda, \lambda \in [0,1]$;
5. $\lambda_1 sd \oplus \lambda_2 sd = (\lambda_1+\lambda_2)sd, \lambda_1, \lambda_2 \in [0,1]$;
6. $sd^{\lambda_1} \otimes sd^{\lambda_2} = sd^{\lambda_1+\lambda_2}, \lambda_1, \lambda_2 \in [0,1]$.

Proof See Appendix 1. □

In order to compare two IVDHFLNs, we next define score function and accuracy function, based on which a comparison method for two IVDHFLNs is presented.

Definition 3.3 Let $sd = (s_\alpha, \tilde{h}, \tilde{g})$ be an IVDHFLN, then score function $S(sd)$ can be denoted as following

$$S(sd) = I(s_\alpha) \times \frac{1}{2}\left(\frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^L - \frac{1}{l(\tilde{g})}\sum_{[v^L,v^U]\in\tilde{g}}v^L + \frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^U - \frac{1}{l(\tilde{g})}\sum_{[v^L,v^U]\in\tilde{g}}v^U\right), \quad (19)$$

where I is the function in Definition 2.1, $l(\tilde{h})$ and $l(\tilde{g})$ are the numbers of interval values in $\tilde{h}$ and $\tilde{g}$, respectively.

Definition 3.4 Let $sd = (s_\alpha, \tilde{h}, \tilde{g})$ be an IVDHFLN, then accuracy function $P(sd)$ can be denoted as following

$$P(sd) = I(s_\alpha) \times \frac{1}{2}\left(\frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^L + \frac{1}{l(\tilde{g})}\sum_{[v^L,v^U]\in\tilde{g}}v^L + \frac{1}{l(\tilde{h})}\sum_{[\mu^L,\mu^U]\in\tilde{h}}\mu^U + \frac{1}{l(\tilde{g})}\sum_{[v^L,v^U]\in\tilde{g}}v^U\right), \quad (20)$$

where I is the function in Definition 2.1, $l(\tilde{h})$ and $l(\tilde{g})$ are the numbers of interval values in $\tilde{h}$ and $\tilde{g}$, respectively.

Definition 3.5 Let any two IVDHFLNs $sd_1 = (s_{\alpha_1}, \tilde{h}_1, \tilde{g}_1)$ and $sd_2 = (s_{\alpha_2}, \tilde{h}_2, \tilde{g}_2)$, then

1. If $S(sd_1) < S(sd_2)$, then $sd_1 < sd_2$.
2. If $S(sd) = S(sd)$, then
   (a) If $P(sd_1) = P(sd_2)$, then $sd_1 = sd_2$;
   (b) If $P(sd_1) < P(sd_2)$, then $sd_1 < sd_2$.

3.3 An improved distance measure for IVDHFLS

When it comes to calculating distance between hesitant fuzzy elements in HFS and its extensions, the first problem is that lengths of membership set or nonmembership set in hesitant fuzzy elements could be unequal. Normally for this kind of situations, the complementing method [32, 56] is suggested to construct distance measures, that is, to make the lengths equal by adding values into membership set or nonmembership set with shorter length. While artificially adding some values will inevitably cause information distortion to some extent. Therefore, without inserting any artificial values, here we define a novel normalized Euclidean distance in following Definition 3.6 for IVDHFLS. It is worth noticing that Definition 3.6 is capable of calculating distance between any two IVDHFLNs using linguistic term sets with different cardinalities.

Definition 3.6 Given two IVDHFLNs $sd_1 = (s_{\vartheta_1}, \tilde{h}_1, \tilde{g}_1)$ and $sd_2 = (s_{\vartheta_2}, \tilde{h}_2, \tilde{g}_2)$. Let $l_{\tilde{h}_1}, l_{\tilde{h}_2}, l_{\tilde{g}_1}$ and $l_{\tilde{g}_2}$ be the lengths of $\tilde{h}_1, \tilde{h}_2, \tilde{g}_1$ and $\tilde{g}_2$ respectively, that is, denote the number of elements in $\tilde{h}_1, \tilde{h}_2, \tilde{g}_1$ and $\tilde{g}_2$. Suppose $\bar{S}_1 = \{s_{\vartheta_1} | \vartheta_1 \in [0, g_1 - 1]\}$ and $\bar{S}_2 = \{s_{\vartheta_2} | \vartheta_2 \in [0, g_2 - 1]\}$ be two sets of continuous linguistic terms, where $g_1 < g_2$. Then the normalized Euclidean distance for IVDHFLNs can be defined as

$$d(sd_1, sd_2) = \left(\frac{1}{2}\left(\frac{1}{l_{\tilde{h}_1}l_{\tilde{h}_2}}\sum_{j=1}^{l_{\tilde{h}_1}}\sum_{k=1}^{l_{\tilde{h}_2}}\left(\left|\frac{f(\alpha_1)}{g_2-1}\mu_{\tilde{h}_1}^{L_j} - \frac{\alpha_2}{g_2-1}\mu_{\tilde{h}_2}^{L_k}\right|^2 + \left|\frac{f(\alpha_1)}{g_2-1}\mu_{\tilde{h}_1}^{U_j} - \frac{\alpha_2}{g_2-1}\mu_{\tilde{h}_2}^{U_k}\right|^2\right) + \frac{1}{l_{\tilde{g}_1}l_{\tilde{g}_2}}\sum_{j=1}^{l_{\tilde{g}_1}}\sum_{k=1}^{l_{\tilde{g}_2}}\left(\left|\frac{f(\alpha_1)}{g_2-1}v_{\tilde{g}_1}^{L_j} - \frac{\alpha_2}{g_2-1}v_{\tilde{g}_2}^{L_k}\right|^2 + \left|\frac{f(\alpha_1)}{g_2-1}v_{\tilde{g}_1}^{U_j} - \frac{\alpha_2}{g_2-1}v_{\tilde{g}_2}^{U_k}\right|^2\right)\right)\right)^{1/2} \quad (21)$$

Especially, if and only if $\vartheta_1 = \vartheta_2, \tilde{h}_1 = \tilde{h}_2$ and $\tilde{g}_1 = \tilde{g}_2$, we call sd_1 and sd_2 are perfectly consistent, and the distance between sd_1 and sd_2 is equal to 0.

Theorem 3.2 *The above interval-valued dual hesitant fuzzy linguistic Euclidean distance satisfies following fundamental properties*:

1. $0 \le d(sd_1, sd_2) \le 1$;
2. $d(sd_1, sd_2) = 0$ *if and only if* sd_1 *and* sd_2 *are perfectly consistent*;
3. $d(sd_1, sd_2) = d(sd_2, sd_1)$.

4 Generalized power aggregation operators for IVDHFLS

In this section, we carry out extension study of traditional power average operators to accommodate interval-valued dual hesitant fuzzy linguistic information. A series of generalized power aggregation operators for IVDHFLS are proposed, their desirable properties and special cases are also discussed.

4.1 Weighted generalized interval-valued dual hesitant fuzzy linguistic power aggregation operators

Definition 4.1 For a collection of IVDHFLNs $sd_j(j = 1, 2, \ldots, n)$, a weighted generalized interval-valued dual hesitant fuzzy linguistic power average (WGIVDHFLPA) operator is a mapping $S^n \to S$:

$$WGIVDHFLPA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_n) = \left(\frac{\oplus_{j=1}^{n}(\omega_j(1 + T(sd_j))sd_j^{\lambda})}{\sum_{i=1}^{n}\omega_i(1 + T(sd_i))}\right)^{1/\lambda}, \quad (22)$$

where

$$T(sd_j) = \sum_{k=1, k\neq j}^{n} \omega_k Sup(sd_j, sd_k) \quad (23)$$

$\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector for $sd_j(j = 1, 2, \ldots, n)$ with conditions $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$ Parameter $\lambda \in (0, +\infty)$ $Sup(sd_j, sd_k)$ is the support function to calculate the degree that sd_j receives from sd_k $Sup(sd_j, sd_k)$ satisfies following three properties:

1. $Sup(sd_j, sd_k) \in [0, 1]$;
2. $Sup(sd_j, sd_k) = Sup(sd_k, sd_j)$;
3. $Sup(sd_j, sd_k) \geq Sup(sd_i, sd_s)$ if $d(sd_j, sd_k) < d(sd_i, sd_s)$ where d is a distance measure between two IVDHFLNs.

Theorem 4.1 *Let $sd_j = (s_\alpha, \tilde{h}_j, \tilde{g}_j)$ be a collection of IVDHFLNs, then aggregation results from Definition 4.1 are still IVDHFLNs, and we have*

$$\begin{aligned} &WGIVDHFLPA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_n) = \bigcup_{(s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\in sd_j}\Bigg(s_{\left(\sum_{j=1}^{n}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}(\alpha_j)^\lambda\right)^{1/\lambda}}, \\ &\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j}\Bigg(\Bigg\{\Bigg[\Bigg(1-\prod_{j=1}^{n}(1-(\mu_j^L)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_j(1+T(sd_i))}}\Bigg)^{1/\lambda}, \Bigg(1-\prod_{j=1}^{n}(1-(\mu_j^U)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\Bigg)^{1/\lambda}\Bigg]\Bigg\}, \\ &\Bigg\{\Bigg[1-\Bigg(1-\prod_{j=1}^{n}(1-(1-v_j^L)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\Bigg)^{1/\lambda}, 1-\Bigg(1-\prod_{j=1}^{n}(1-(1-v_j^U)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\Bigg)^{1/\lambda}\Bigg]\Bigg\}\Bigg)\Bigg) \end{aligned} \quad (24)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector for $sd_j(j = 1, 2, \ldots, n)$. And ω satisfies $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

Proof Theorem 4.1 can be proved by mathematical induction method. See Appendix 2. □

Theorem 4.2 *Given a collection of IVDHFLNs $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)(j = 1, 2, \ldots, n)$. λ is a parameter and $\lambda > 0$. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is weighting vector for $sd_j, \omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. If $\omega = (1/n, 1/n, \ldots, 1/n)^T$, operator WGIVDHFLPA reduces to following generalized interval-valued dual hesitant fuzzy linguistic power average (GIVDHFLPA) operator:*

$$GIVDHFLPA_{\lambda}(sd_1, sd_2, \ldots, sd_n) = \left(\frac{\overset{n}{\underset{j=1}{\oplus}}(1+T(sd_j))sd_j^{\lambda}}{\sum_{i=1}^{n}(1+T(sd_i))}\right)^{1/\lambda} = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\left(\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(\alpha_j)^{\lambda}\right)^{1/\lambda}},\right.$$
$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[\nu_j^L,\nu_j^U]\in\tilde{g}_j}\left\{\left\{\left[\left(1-\prod_{j=1}^{n}(1-(\mu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda},\left(1-\prod_{j=1}^{n}(1-(\mu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right. \tag{25}$$
$$\left.\left.\left\{\left[1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right)$$

Besides, along with variations of λ and ω, the WGIVDHFLPA operator reduces to several other special cases, which are listed in Theorem 4.3.

Theorem 4.3 *Given a collection of IVDHFLNs $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)(j = 1, 2, \ldots, n)$. λ is a parameter and $\lambda > 0$. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is weighting vector for $sd_j, \omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$. Then*

1. *If $\lambda = 1$ and $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then WGIVDHFLPA reduces to the interval-valued dual hesitant fuzzy linguistic power average (IVDHFLPA) operator*:

$$IVDHFLPA(sd_1, sd_2, \ldots, sd_n) = \frac{\overset{n}{\underset{j=1}{\oplus}}(1+T(sd_j))sd_j}{\sum_{i=1}^{n}(1+T(sd_i))} = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}\alpha_j},\right.$$
$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[\nu_j^L,\nu_j^U]\in\tilde{g}_j}\left\{\left\{\left[1-\prod_{j=1}^{n}(1-\mu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},1-\prod_{j=1}^{n}(1-\mu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right]\right\}\right. . \tag{26}$$
$$\left.\left.\left\{\left[\prod_{j=1}^{n}(\nu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},\prod_{j=1}^{n}(\nu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right]\right\}\right\}\right)$$

2. *If $\lambda = 1, \omega = (1/n, 1/n, \ldots, 1/n)^T$ and $Sup(sd_i, sd_j) = k$ (i.e., a constant) for all $i \neq j$, then WGIVDHFLPA reduces to the interval-valued dual hesitant fuzzy linguistic average (IVDHFLA) operator*:

$$IVDHFLA(sd_1, sd_2, \ldots, sd_n) = \frac{1}{n}\overset{n}{\underset{j=1}{\oplus}}sd_j = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\frac{1}{n}\sum_{j=1}^{n}\alpha_j},\right.$$
$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[\nu_j^L,\nu_j^U]\in\tilde{g}_j}\left\{\left\{\left[1-\prod_{j=1}^{n}(1-\mu_j^L)^{\frac{1}{n}},1\right.\right.\right. \tag{27}$$
$$\left.\left.\left.\left.-\prod_{j=1}^{n}(1-\mu_j^U)^{\frac{1}{n}}\right]\right\},\left\{\left[\prod_{j=1}^{n}(\nu_j^L)^{\frac{1}{n}},\prod_{j=1}^{n}(\nu_j^U)^{\frac{1}{n}}\right]\right\}\right\}\right).$$

Definition 4.2 For a collection of IVDHFLNs: $sd_j(j = 1, 2, \ldots, n)$, a weighted generalized interval-valued dual hesitant fuzzy linguistic power geometric average (WGIVDHFLPGA) operator is a mapping $S^n \to S$:

$$WGIVDHFLPGA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{j=1}{\otimes}}(\lambda sd_j)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right), \tag{28}$$

where $T(sd_j)$ is the same as Eq. (23). $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector for $sd_j, \omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. λ is a parameter and $\lambda \in (0, +\infty)$.

Theorem 4.4 *Let $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\,(j = 1, 2, \ldots, n)$ be a collection of IVDHFLNs, then aggregation results from Definition 4.2 are still IVDHFLNs, and we have*

$$WGIVDHFLPGA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_n) = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j} \left(s_{\frac{1}{\lambda}\prod_{j=1}^{n}(\lambda\alpha_j)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}}, \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j} \right.$$
$$\left\{\left\{\left[1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right. \tag{29}$$
$$\left.\left.\left\{\left[\left(1-\prod_{j=1}^{n}(1-(v_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}(1-(v_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right),$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector for $sd_j, \omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof Theorem 4.4 can be proved by mathematical induction method. See Appendix 3. □

Theorem 4.5 *For a collection of IVDHFLNs: $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\,(j = 1, 2, \ldots, n)$. λ is a parameter and $\lambda > 0$. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector for $sd_j, \omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. If $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then WGIVDHFLPGA reduces to following generalized interval-valued dual hesitant fuzzy linguistic power geometric average (GIVDHFLPGA) operator,*

$$GIVDHFLPGA_{\lambda}(sd_1, sd_2, \ldots, sd_n) = \frac{1}{\lambda}\left(\bigotimes_{j=1}^{n}(\lambda sd_j)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)$$
$$= \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j} \left(s_{\frac{1}{\lambda}\prod_{j=1}^{n}(\lambda\alpha_j)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}}, \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j} \right.$$
$$\left\{\left\{\left[1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right. \tag{30}$$
$$\left.\left.\left\{\left[\left(1-\prod_{j=1}^{n}(1-(v_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}(1-(v_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right).$$

Theorem 4.6 *For a collection of IVDHFLNs: $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\,(j = 1, 2, \ldots, n)$. λ is a parameter and $\lambda > 0$.*

$\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector for sd_j, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, Then

1. *If $\lambda = 1$ and $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then WGIVDHFLPGA reduces to the interval-valued dual hesitant fuzzy linguistic power geometric average (IVDHFLPGA) operator:*

$$IVDHFLPGA(sd_1, sd_2, \ldots, sd_n) = \overset{n}{\underset{j=1}{\otimes}} sd_j^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}$$

$$= \bigcup_{(s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j) \in sd_j} \left(s_{\prod_{j=1}^{n} \alpha_j^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}}, \bigcup_{[\mu_j^L, \mu_j^U] \in \tilde{h}_j, [v_j^L, v_j^U] \in \tilde{g}_j} \left\{ \left\{ \left[\prod_{j=1}^{n} (\mu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n} 1+T(sd_i)}}, \prod_{j=1}^{n} (\mu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \right] \right\}, \left\{ \left[\prod_{j=1}^{n} (v_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}, \prod_{j=1}^{n} (v_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \right] \right\} \right\} \right). \quad (31)$$

2. *If $\lambda = 1, \omega = (1/n, 1/n, \ldots, 1/n)^T, Sup(sd_i, sd_j) = k$ (i.e., a constant) for all $i \neq j$, then WGIVDHFLPGA reduces to the interval-valued dual hesitant fuzzy linguistic geometric average (IVDHFLGA) operator:*

$$IVDHFLGA(sd_1, sd_2, \ldots, sd_n) = \overset{n}{\underset{j=1}{\otimes}} sd_j^{\frac{1}{n}} = \bigcup_{(s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j) \in sd_j} \left(s_{\prod_{j=1}^{n} \alpha_j^{\frac{1}{n}}}, \bigcup_{[\mu_j^L, \mu_j^U] \in \tilde{h}_j, [v_j^L, v_j^U] \in \tilde{g}_j} \left\{ \left\{ \left[\prod_{j=1}^{n} (\mu_j^L)^{\frac{1}{n}}, \prod_{j=1}^{n} (\mu_j^U)^{\frac{1}{n}} \right] \right\}, \left\{ \left[\prod_{j=1}^{n} (v_j^L)^{\frac{1}{n}}, \prod_{j=1}^{n} (v_j^U)^{\frac{1}{n}} \right] \right\} \right\} \right). \quad (32)$$

Now, we can analyze some properties of operators developed above.

Theorem 4.7 *GIVDHFLPA operator and GIVDHFLPGA operator hold following properties:*

1. *Commutativity: Let $(sd_1^*, sd_2^*, \ldots, sd_n^*)$ be any permutation of $(sd_1, sd_2, \ldots, sd_n)$, then*

$$GIVDHFLPA_\lambda(sd_1^*, sd_2^*, \ldots, sd_n^*) = GIVDHFLPA_\lambda(sd_1, sd_2, \ldots, sd_n),$$
$$GIVDHFLPGA_\lambda(sd_1^*, sd_2^*, \ldots, sd_n^*) = GIVDHFLPGA_\lambda(sd_1, sd_2, \ldots, sd_n).$$

2. *Idempotency: Let $sd_j = sd$, for all $j = 1, 2, \ldots, n$, then*

$$GIVDHFLPA_\lambda(sd_1, sd_2, \ldots, sd_n) = sd,$$
$$GIVDHFLPGA_\lambda(sd_1, sd_2, \ldots, sd_n) = sd.$$

3. *Boundedness: The GIVDHFLPA operator and the GIVDHFLPGA operator lie between the max and min operators,*

$$sd^- \leq GIVDHFLPA_\lambda(sd_1, sd_2, \ldots, sd_n) \leq sd^+,$$
$$sd^- \leq GIVDHFLPGA_\lambda(sd_1, sd_2, \ldots, sd_n) \leq sd^+.$$

Proof See Appendix 4. □

Similar to GIVDHFLPA and GIVDHFLPGA, operators IVDHFLPA, IVDHFLA, IVDHFLPGA and IVDHFLGA also hold properties of commutativity, idempotency and boundedness.

Clearly, operators WGIVDHFLPA and WGIVDHFLPGA are idempotent and bounded, but they do not hold commutativity. Take WGIVDHFLPA for example, if $(sd_1^*, sd_2^*, \ldots, sd_n^*)$ is any permutation of $(sd_1, sd_2, \ldots, sd_n)$, then $T(sd_j^*) = \sum_{k=1,k\neq j}^{n} \omega_k Sup(sd_j^*, sd_k^*)$. Since $(T(sd_1^*), T(sd_2^*), \ldots, T(sd_n^*))$ may not be the permutation of $(T(sd_1), T(sd_2), \ldots, T(sd_n))$, then

$$WGIVDHFLPA_\lambda(sd_1, sd_2, \ldots, sd_n) = WGIVDHFLPA_\lambda(sd_1^*, sd_2^*, \ldots, sd_n^*)$$

generally does not hold. And it is the same with operator WGIVDHFLPGA.

Lemma 4.1 [36] *Assume that $x_j > 0, \lambda_j > 0, j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \lambda_j = 1$, then*

$$\prod_{j=1}^{n} x_j^{\lambda_j} \le \sum_{j=1}^{n} \lambda_j x_j,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 4.8 *For a collection of IVDHFLNs* $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n), \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector for* $sd_j, \omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1$, *then*

$$IVDHFLPGA_{\omega}(sd_1, sd_2, \ldots, sd_n) \le IVDHFLPA_{\omega}(sd_1, sd_2, \ldots, sd_n).$$

Proof See Appendix 5. □

Theorem 4.8 shows that values obtained by IVDHFLPGA operator are not bigger than the ones obtained by the IVDHFLPA operator. Considering variations of parameter λ, we also can derive the following Theorems 4.9 and 4.10.

Theorem 4.9 *For a collection of IVDHFLNs*: $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n), \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector for* $sd_j, \omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1, \lambda > 0$, *then*

$$IVDHFLPGA_{\omega}(sd_1, sd_2, \ldots, sd_n) \le GIVDHFLPA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_n).$$

Proof See Appendix 6. □

Theorem 4.10 *For a collection of IVDHFLNs*: $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j), \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector for* $sd_j, \omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1, \lambda > 0 j = 1, 2, \ldots, n$, *then*

$$GIVDHFLPGA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_n) \le IVDHFLPA_{\omega}(sd_1, sd_2, \ldots, sd_n).$$

Proof See Appendix 7. □

Theorems 4.9 and 4.10 give us the result that values obtained by IVDHFLPGA operator are not bigger than the ones obtained by IVDHFLPA operator, no matter how parameter λ changes.

4.2 Generalized interval-valued dual hesitant fuzzy linguistic power ordered weighted aggregation operators

In the above developed operators WGIVDHFLPA and WGIVDHFLPGA, weighting vectors only depend upon the mutually supportive input arguments themselves. However, in many group decision making problems, we need to rearrange all given arguments in descending (or ascending) order, then weight the ordered positions of input arguments so as to relieve influence of unfair arguments on decision results by assigning low weights to those “false” or “biased” ones. Therefore, based on ideas of OWA operator [58] and power average aggregation operator [59], in this section, we study fundamental generalized power ordered weighted operators for IVDHFLS.

Definition 4.3 For a collection of IVDHFLNs $sd_j (j = 1, 2, \ldots, n), sd_{\sigma(j)}$ be the jth largest of them, parameter $\lambda \in (0, +\infty)$. Then a generalized interval-valued dual hesitant fuzzy linguistic power ordered weighted averaging (GIVDHFLPOWA) operator is a mapping $S^n \to S$:

$$GIVDHFLPOWA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n) = \left(\mathop{\oplus}_{j=1}^{n} (w_j sd_{\sigma(j)}^{\lambda}) \right)^{1/\lambda}, \tag{33}$$

where

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right)$$
$$\text{with } w_j \in [0, 1] \text{ and } \sum_{j=1}^{n} w_j = 1,\ R_j = \sum_{k=1}^{j} V_{\sigma(k)}.$$

$$TV = \sum_{j=1}^{n} V_{\sigma(j)}, \quad V_{\sigma(j)} = 1 + T(sd_{\sigma(j)}),$$
$$T(sd_{\sigma(j)}) = \sum_{k=1, k \ne j}^{n} Sup(sd_{\sigma(j)}, sd_{\sigma(k)}). \tag{34}$$

In Eq. (34), $T(sd_{\sigma(j)})$ denotes support that jth largest argument receives from all the other arguments, $Sup(sd_{\sigma(j)}, sd_{\sigma(k)})$ indicates support for $sd_{\sigma(j)}$ from $sd_{\sigma(k)}$. $g : [0, 1] \to [0, 1]$ is a basic BUM function [60] which satisfies $g(0) = 0, g(1) = 1$ and $g(x) \ge g(y)$ if $x > y$.

Theorem 4.11 *Let* $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n)$ *be a collection of IVDHFLNs, aggregation results yielded from Definition* 4.3 *are still IVDHFLNs, and we have*

$$GIVDHFLPOWA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n) = \bigcup_{\left(s_{\alpha_{\sigma(j)}}, \tilde{h}_{\sigma(j)}, \tilde{g}_{\sigma(j)}\right) \in sd_{\sigma(j)}} \left(s_{\left(\sum_{j=1}^{n} w_j (\alpha_{\sigma(j)})^{\lambda}\right)^{1/\lambda}}, \right.$$
$$\bigcup_{[\mu^L_{\sigma(j)}, \mu^U_{\sigma(j)}] \in \tilde{h}_{\sigma(j)}, [v^L_{\sigma(j)}, v^U_{\sigma(j)}] \in \tilde{g}_{\sigma(j)}} \left\{ \left\{ \left[\left(1 - \prod_{j=1}^{n} (1 - (\mu^L_{\sigma(j)})^{\lambda})^{w_j}\right)^{1/\lambda}, \left(1 - \prod_{j=1}^{n} (1 - (\mu^U_{\sigma(j)})^{\lambda})^{w_j}\right)^{1/\lambda} \right] \right\}, \right. \tag{35}$$
$$\left. \left. \left\{ \left[1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - v^L_{\sigma(j)})^{\lambda})^{w_j}\right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - v^U_{\sigma(j)})^{\lambda})^{w_j}\right)^{1/\lambda} \right] \right\} \right\} \right)$$

where w_j satisfies Eq. (34) and $sd_{\sigma(j)}$ be the jth largest of $sd_j (j = 1, 2, \ldots, n)$. λ is a parameter and $\lambda \in (0, +\infty)$.

Proof Similar to proof of Theorem 4.1, Theorem 4.11 can also be proved by mathematical induction method, detailed proof steps are omitted here for conciseness.

Theorem 4.12 *For a collection of IVDHFLNs: $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n)$, then we have*

1. *If $\lambda = 1$, then GIVDHFLPOWA operator reduces to the following interval-valued dual hesitant fuzzy linguistic power ordered weighted averaging (IVDHFLPOWA) operator:*

$$IVDHFLPOWA_w(sd_1, sd_2, \ldots, sd_n) = \mathop{\oplus}_{j=1}^{n} w_j sd_{\sigma(j)}$$
$$= \bigcup_{(s_{\alpha_{\sigma(j)}}, \tilde{h}_{\sigma(j)}, \tilde{g}_{\sigma(j)}) \in sd_{\sigma(j)}} \left(s_{\sum_{j=1}^{n} w_j \alpha_{\sigma(j)}}, \bigcup_{[\mu^L_{\sigma(j)}, \mu^U_{\sigma(j)}] \in \tilde{h}_{\sigma(j)}, [v^L_{\sigma(j)}, v^U_{\sigma(j)}] \in \tilde{g}_{\sigma(j)}} \right.$$
$$\left\{ \left\{ \left[1 - \prod_{j=1}^{n} (1 - \mu^L_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \mu^U_{\sigma(j)})^{w_j} \right] \right\}, \right.$$
$$\left. \left. \left\{ \left[\prod_{j=1}^{n} (v^L_{\sigma(j)})^{w_j}, \prod_{j=1}^{n} (v^U_{\sigma(j)})^{w_j} \right] \right\} \right\} \right). \tag{36}$$

2. *If $\lambda = 1$ and $w = (1/n, 1/n, \ldots, 1/n)^T$, then GIVD HFLPOWA reduces to the following interval-valued dual hesitant fuzzy linguistic average (IVDHFLA) operator:*

$$IVDHFLA(sd_1, sd_2, \ldots, sd_n) = \mathop{\oplus}_{j=1}^{n} \frac{1}{n} sd_j$$
$$= \bigcup_{(s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j) \in sd_j} \left(s_{\sum_{j=1}^{n} \frac{1}{n} \alpha_j}, \bigcup_{[\mu^L_j, \mu^U_j] \in \tilde{h}_j, [v^L_j, v^U_j] \in \tilde{g}_j} \right.$$
$$\left\{ \left\{ \left[1 - \prod_{j=1}^{n} (1 - \mu^L_j)^{\frac{1}{n}}, 1 - \prod_{j=1}^{n} (1 - \mu^U_j)^{\frac{1}{n}} \right] \right\}, \right. \tag{37}$$
$$\left. \left. \left\{ \left[\prod_{j=1}^{n} (v^L_j)^{\frac{1}{n}}, \prod_{j=1}^{n} (v^U_j)^{\frac{1}{n}} \right] \right\} \right\} \right).$$

Definition 4.4 For a collection of IVDHFLNs $sd_j (j = 1, 2, \ldots, n)$, let $sd_{\sigma(j)}$ be the jth largest of them, λ is a parameter and $\lambda \in (0, +\infty)$. A generalized interval-valued dual hesitant fuzzy linguistic power ordered weighted geometric average (GIVDHFLPOWGA) operator is a mapping $S^n \to S$:

$$GIVDHFLPOWGA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n) = \frac{1}{\lambda} \left(\mathop{\otimes}_{j=1}^{n} (\lambda sd_{\sigma(j)})^{w_j} \right), \tag{38}$$

in which w_j satisfies Eq. (34).

Theorem 4.13 *Let $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)$ be a collection of IVDHFLNs, then aggregation results from Definition 4.4 are still IVDHFLNs, and we can have*

$$GIVDHFLPOWGA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n) = \bigcup_{\left(s_{\alpha_{\sigma(j)}}, \tilde{h}_{\sigma(j)}, \tilde{g}_{\sigma(j)}\right) \in sd_{\sigma(j)}} \left(s_{\frac{1}{\lambda}\prod_{j=1}^{n}(\lambda\alpha_{\sigma(j)})^{\omega_j}}, \right.$$
$$\cup_{[\mu^L_{\sigma(j)},\mu^U_{\sigma(j)}]\in\tilde{h}_{\sigma(j)},[v^L_{\sigma(j)},v^U_{\sigma(j)}]\in\tilde{g}_{\sigma(j)}} \left\{ \left\{ \left[1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - \mu^L_{\sigma(j)})^{\lambda})^{w_j} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - \mu^U_{\sigma(j)})^{\lambda})^{w_j} \right)^{1/\lambda} \right] \right\}, \right. \tag{39}$$
$$\left. \left. \left\{ \left[\left(1 - \prod_{j=1}^{n} (1 - (v^L_{\sigma(j)})^{\lambda})^{\omega_j} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^{n} (1 - (v^U_{\sigma(j)})^{\lambda})^{\omega_j} \right)^{1/\lambda} \right] \right\} \right\} \right)$$

where w_j satisfies Eq. (34), $sd_{\sigma(j)}$ is the jth largest of $sd_j (j = 1, 2, \ldots, n)$, λ is a parameter and $\lambda \in (0, +\infty)$.

Proof Similar to proof of Theorem 4.4, Theorem 4.13 can be proved by mathematical induction method, and proof steps are omitted here.

Theorem 4.14 *For a collection of IVDHFLNs $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n)$, then*

1. *If $\lambda = 1$, then operator GIVDHFLPOWGA reduces to interval-valued dual hesitant fuzzy linguistic power ordered weighted geometric average (IVDHFLPOWGA) operator, where*

$$IVDHFLPOWGA_w(sd_1, sd_2, \ldots, sd_n) = \bigcup_{(s_{\alpha_{\sigma(j)}}, \tilde{h}_{\sigma(j)}, \tilde{g}_{\sigma(j)}) \in sd_{\sigma(j)}} \left(s_{\prod_{j=1}^{n} \alpha_{\sigma(j)}^{w_j}}, \right.$$
$$\left. \bigcup_{[\mu^L_{\sigma(j)},\mu^U_{\sigma(j)}]\in\tilde{h}_{\sigma(j)},[v^L_{\sigma(j)},v^U_{\sigma(j)}]\in\tilde{g}_{\sigma(j)}} \left\{ \left\{ \left[\prod_{j=1}^{n} (\mu^L_{\sigma(j)})^{w_j}, \prod_{j=1}^{n} (\mu^U_{\sigma(j)})^{w_j} \right] \right\}, \left\{ \left[1 - \prod_{j=1}^{n} (1 - v^L_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n} (1 - v^U_{\sigma(j)})^{w_j} \right] \right\} \right\} \right). \tag{40}$$

2. *If $\lambda = 1, w = (1/n, 1/n, \ldots, 1/n)^T$, then operator GIVDHFLPOWGA reduces to interval-valued dual hesitant fuzzy linguistic geometric average (IVDHFLGA) operator, where*

$$IVDHFLGA(sd_1, sd_2, \ldots, sd_n) = \overset{n}{\underset{j=1}{\otimes}} (sd_j)^{\frac{1}{n}} = \bigcup_{(s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j) \in sd_j} \left(s_{\prod_{j=1}^{n} (\alpha_j)^{\frac{1}{n}}}, \right.$$
$$\left. \bigcup_{[\mu^L_j,\mu^U_j]\in\tilde{h}_j,[v^L_j,v^U_j]\in\tilde{g}_j} \left\{ \left\{ \left[\prod_{j=1}^{n} (\mu^L_j)^{\frac{1}{n}}, \prod_{j=1}^{n} (\mu^U_j)^{\frac{1}{n}} \right] \right\}, \left\{ \left[1 - \prod_{j=1}^{n} (1 - v^L_j)^{\frac{1}{n}}, 1 - \prod_{j=1}^{n} (1 - v^U_j)^{\frac{1}{n}} \right] \right\} \right\} \right). \tag{41}$$

In resemblance to proof of Theorem 4.7, operators IVHFLPOWA and IVHFLPOWGA can be proved holding properties of commutativity, idempotency and boundedness. Besides, similar to Theorems 4.8, 4.9 and 4.10, we can also conclude the theorems as follows.

Theorem 4.15 *For a collection of IVDHFLNs: $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n), \lambda > 0$, then,*

$$GIVDHFLPOWGA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n) \le GIVDHFLPOWA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n).$$

Theorem 4.16 *Let $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)\ (j = 1, 2, \ldots, n)$ be a collection of IVDHFLNs, $\lambda > 0$, then,*

$$IVDHFLPOWGA_w(sd_1, sd_2, \ldots, sd_n) \leq GIVDHFLPOWA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n).$$

Theorem 4.17 *Let* $sd_j = (s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j)$ $(j = 1, 2, \ldots, n)$ *be a collection of IVDHFLNs*, $\lambda > 0$, *then*,

$$GIVDHFLPOWGA_{w,\lambda}(sd_1, sd_2, \ldots, sd_n) \leq IVDHFLPOWA_w(sd_1, sd_2, \ldots, sd_n).$$

5 Approaches for MAGDM with interval-valued dual hesitant fuzzy linguistic information

In this section, we apply afore-developed generalized power aggregation operators to construct effective approaches for MAGDM under interval-valued dual hesitant fuzzy linguistic environments. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a discrete set of alternatives, $G = \{G_1, G_2, \ldots, G_m\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weighting vector for attribute vector G, where $\omega_j \geq 0, j = 1, 2, \ldots, m$ and $\sum_{j=1}^{m} \omega_j = 1$. $D = \{d_1, d_2, \ldots, d_t\}$ denotes the set of decision makers, $\eta = (\eta_1, \eta_2, \ldots, \eta_t)$ represents the weighting vector for experts, with $\eta_k \geq 0$, and $\sum_{k=1}^{t} \eta_k = 1$, $k = 1, 2, \ldots, t$. Suppose that $R^k = (r_{ij}^k)_{n\times m}$ is the decision matrix, where $r_{ij}^k = (s_{\alpha_{ij}}^k, \tilde{h}_{ij}^k, \tilde{g}_{ij}^k)$ takes the form of IVDHFLN, given by decision maker d_k for alternative A_i with respect to attribute G_j.

Then, depending on actual decision situations where whether weighting information for decision makers and attributes can be determined in advance, in the following, we propose MAGDM approaches based on the developed generalized power aggregation operators for two basic cases: (1) case I, where expert weights and attribute weights are known; (2) case II, where expert weights and attribute weights are totally unknown.

5.1 Approach for MAGDM with known expert weights and attribute weights (case I)

Aiming at actual decision situations in which weighting vectors for decision makers and attributes can be obtained in advance, we apply WGIVDHFLPA and WGIVDHFLPGA operators to construct the following Approach I to resolve MAGDM under interval-valued dual hesitant fuzzy linguistic environments.

5.1.1 Approach I: MAGDM with known expert weights and attribute weights

Step I-1 According to Eqs. (2) or (3), transform decision matrices $R^k = (r_{ij}^k)_{n\times m}$ given with linguistic term sets of different cardinalities into the decision matrices $\widehat{R}^k = (\widehat{r}_{ij}^k)_{n\times m}$ denoted with same linguistic term set, where $\widehat{r}_{ij}^k = (\widehat{s}_{\alpha_{ij}}^k, \tilde{h}_{ij}^k, \tilde{g}_{ij}^k)$. Normally, we transform linguistic term sets of different cardinalities into the one with largest cardinality.

Step I-2 Calculate support degrees by the following function:

$$Sup(\widehat{r}_{ij}^k, \widehat{r}_{ij}^l) = 1 - d(\widehat{r}_{ij}^k, \widehat{r}_{ij}^l), \quad k, l = 1, 2, \ldots, t; k \neq l;\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m, \tag{42}$$

which satisfies the conditions for support functions in Definition 2.8. $d(\widehat{r}_{ij}^k, \widehat{r}_{ij}^l)$ is calculated by normalized Euclidean distance measure defined in Eq. (21).

Step I-3 Calculate the support degree $T(\widehat{r}_{ij}^k)$ that IVDHFLN $\widehat{r}_{ij}^k$ receives from other IVDHFLNs $\widehat{r}_{ij}^l (l = 1, 2, \ldots, t; l \neq k)$, where

$$T(\widehat{r}_{ij}^k) = \sum_{l=1, l\neq k}^{t} \eta_l Sup(\widehat{r}_{ij}^k, \widehat{r}_{ij}^l). \tag{43}$$

Step I-4 Utilize weights $\eta_k (k = 1, 2, \ldots, t)$ for decision makers d_k to calculate weights ξ_{ij}^k associated with the IVDHFLN r_{ij}^k,

$$\xi_{ij}^k = \frac{\eta_k(1 + T(\widehat{r}_{ij}^k))}{\sum_{k=1}^{t} \eta_k(1 + T(\widehat{r}_{ij}^k))}, \quad k = 1, 2, \ldots, t, \tag{44}$$

where $\xi_{ij}^k \geq 0$ and $\sum_{k=1}^{t} \xi_{ij}^k = 1$.

Step I-5 Aggregate all individual decision matrices $\widehat{R}^k = (\widehat{r}_{ij}^k)_{n\times m} (k = 1, 2, \ldots, t)$ into group decision matrix $R = (r_{ij})_{n\times m}$ by use of WGIVDHFLPA operator, or into group decision matrix $R^G = (r_{ij}^G)_{n\times m}$ by use of WGIVDHFLPGA operator, where

$$r_{ij} = WGIVDHFLPA_{\xi,\lambda}(\widehat{r}_{ij}^{1}, \widehat{r}_{ij}^{2}, \ldots, \widehat{r}_{ij}^{t}) = \bigcup_{\left(\widehat{s}_{\widehat{\alpha}_{ij}^{k}}, \tilde{h}_{ij}^{k}, g_{ij}^{k}\right)\in \widehat{r}_{ij}^{k}} \left(\widehat{s}_{\left(\sum_{k=1}^{t} \xi_{ij}^{k} (\widehat{\alpha}_{ij}^{k})^{\lambda}\right)^{1/\lambda}}, \right.$$
$$\bigcup_{[\mu_{ij}^{Lk},\mu_{ij}^{Uk}]\in\tilde{h}_{ij}^{k},[v_{ij}^{Lk},v_{ij}^{Uk}]\in\tilde{g}_{ij}^{k}} \left(\left\{ \left[\left(1 - \prod_{k=1}^{t} (1 - (\mu_{ij}^{Lk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda}, \left(1 - \prod_{k=1}^{t} (1 - (\mu_{ij}^{Uk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda} \right] \right\}, \right. \quad (45)$$
$$\left. \left. \left\{ \left[1 - \left(1 - \prod_{k=1}^{t} (1 - (1 - v_{ij}^{Lk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda}, 1 - \left(1 - \prod_{k=1}^{t} (1 - (1 - v_{ij}^{Uk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda} \right] \right\} \right) \right)$$

or

$$r_{ij}^{G} = WGIVDHFLPGA_{\xi,\lambda}(\widehat{r}_{ij}^{1}, \widehat{r}_{ij}^{2}, \ldots, \widehat{r}_{ij}^{t}) = \bigcup_{(\widehat{s}_{\widehat{\alpha}_{ij}^{k}}, \tilde{h}_{ij}^{k}, g_{ij}^{k})\in \widehat{r}_{ij}^{k}} \left(\widehat{s}_{\frac{1}{\lambda}\prod_{k=1}^{t} (\lambda \widehat{\alpha}_{ij}^{k})^{\xi_{ij}^{k}}}, \right.$$
$$\bigcup_{[\mu_{ij}^{Lk},\mu_{ij}^{Uk}]\in\tilde{h}_{ij}^{k},[v_{ij}^{Lk},v_{ij}^{Uk}]\in\tilde{g}_{ij}^{k}} \left\{ \left\{ \left[1 - \left(1 - \prod_{k=1}^{t} (1 - (1 - \mu_{ij}^{Lk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda}, 1 - \left(1 - \prod_{k=1}^{t} (1 - (1 - \mu_{ij}^{Uk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda} \right] \right\}, \right. \quad (46)$$
$$\left. \left. \left\{ \left[\left(1 - \prod_{k=1}^{t} (1 - (v_{ij}^{Lk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda}, \left(1 - \prod_{k=1}^{t} (1 - (v_{ij}^{Uk})^{\lambda})^{\xi_{ij}^{k}} \right)^{1/\lambda} \right] \right\} \right\} \right).$$

Step I-6 Calculate support degrees $Sup(r_{ij}, r_{ip})$ and $Sup(r_{ij}^{G}, r_{ip}^{G})$ by following functions:

$$Sup(r_{ij}, r_{ip}) = 1 - d(r_{ij}, r_{ip}), \quad i = 1, 2, \ldots, n;\ j, p = 1, 2, \ldots, m;\ j \neq p, \quad (47)$$

$$Sup(r_{ij}^{G}, r_{ip}^{G}) = 1 - d(r_{ij}^{G}, r_{ip}^{G}), \quad i = 1, 2, \ldots, n;\ j, p = 1, 2, \ldots, m;\ j \neq p,, \quad (48)$$

which satisfy the conditions for support functions in Definition 4.1. Here, $d(r_{ij}, r_{ip})$ and $d(r_{ij}^{G}, r_{ip}^{G})$ are calculated according to Eq. (21).

Step I-7 Calculate support degree $T(r_{ij})$ that IVDHFLN r_{ij} receives from other IVDHFLNs $r_{ip}(p = 1, 2, \ldots, m; p \neq j)$, or $T(r_{ij}^{G})$ of IVDHFLN r_{ij}^{G} by $r_{ip}^{G}(p = 1, 2, \ldots, m; p \neq j)$, where

$$T(r_{ij}) = \sum_{p=1, p\neq j}^{m} \omega_p Sup(r_{ij}, r_{ip}), \quad (49)$$

$$T(r_{ij}^{G}) = \sum_{p=1, p\neq j}^{m} \omega_p Sup(r_{ij}^{G}, r_{ip}^{G}). \quad (50)$$

Step I-8 Calculate weighting vector $w_{ij}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ associated with r_{ij}, and w_{ij}^{G} associated with r_{ij}^{G}:

$$w_{ij} = \frac{\omega_j(1 + T(r_{ij}))}{\sum_{j=1}^{m} \omega_j(1 + T(r_{ij}))}, \quad (51)$$

$$w_{ij}^{G} = \frac{\omega_j(1 + T^{G}(r_{ij}^{G}))}{\sum_{j=1}^{m} \omega_j(1 + T^{G}(r_{ij}^{G}))}. \quad (52)$$

Step I-9 Use WGIVDHFLPA operator or WGIVDHFLPG operator to aggregate all evaluation values $r_{ij}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ or r_{ij}^{G} into overall evaluation values $r_i(i = 1, 2, \ldots, n)$ or r_i^{G} corresponding to each alternative $A_i(i = 1, 2, \ldots, n)$:

$$r_i = WGIVDHFLPA_{w,\lambda}(r_{i1}, r_{i2}, \ldots, r_{im}) = \bigcup_{(s_{\alpha_{ij}}, \tilde{h}_{ij}, g_{ij}) \in r_{ij}} \Bigg(s_{\left(\sum_{j=1}^{m} w_{ij}(\alpha_{ij})^{\lambda}\right)^{1/\lambda}}, \cup_{[\mu_{ij}^L, \mu_{ij}^U] \in \tilde{h}_{ij}, [v_{ij}^L, v_{ij}^U] \in \tilde{g}_{ij}} \left(\left\{ \left[\left(1 - \prod_{j=1}^{m} (1 - (\mu_{ij}^L)^{\lambda})^{w_{ij}} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^{m} (1 - (\mu_{ij}^U)^{\lambda})^{w_{ij}} \right)^{1/\lambda} \right] \right\}, \left\{ \left[1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - v_{ij}^L)^{\lambda})^{w_{ij}} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - v_{ij}^U)^{\lambda})^{w_{ij}} \right)^{1/\lambda} \right] \right\} \right) \Bigg) \tag{53}$$

or

$$r_i^G = WGIVDHFLPGA_{w,\lambda}(r_{i1}^G, r_{i2}^G, \ldots, r_{im}^G) = \bigcup_{\left(s_{\alpha_{ij}}^G, \tilde{h}_{ij}^G, g_{ij}^G\right) \in r_{ij}^G} \Bigg(s_{\frac{1}{\lambda}\prod_{j=1}^{m} (\lambda \alpha_{ij}^G)^{w_{ij}^G}}, \cup_{[\mu_{ij}^{LG}, \mu_{ij}^{UG}] \in \tilde{h}_{ij}^G, [v_{ij}^{LG}, v_{ij}^{UG}] \in \tilde{g}_{ij}^G} \left\{ \left\{ \left[1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - \mu_{ij}^{LG})^{\lambda})^{w_{ij}^G} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - \mu_{ij}^{UG})^{\lambda})^{w_{ij}^G} \right)^{1/\lambda} \right] \right\}, \left\{ \left[\left(1 - \prod_{j=1}^{m} (1 - (v_{ij}^{LG})^{\lambda})^{w_{ij}^G} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^{m} (1 - (v_{ij}^{UG})^{\lambda})^{w_{ij}^G} \right)^{1/\lambda} \right] \right\} \right\} \Bigg) \tag{54}$$

Step I-10 Calculate scores $s(r_i)$ or $s(r_i^G)$ for the overall interval-valued dual hesitant fuzzy linguistic numbers $r_i (i = 1, 2, \ldots, n)$ or r_i^G of alternatives $A_i (i = 1, 2, \ldots, n)$ by Definition 3.5.

Step I-11 Rank all alternatives $A_i (i = 1, 2, \ldots, n)$ and select the best one(s) in accordance with the ranking order of $r_i (i = 1, 2, \ldots, n)$ or $r_i^G (i = 1, 2, \ldots, n)$.

5.2 Approach for MAGDM with unknown expert weights and attribute weights (case II)

To tackle more uncertain decision situations where weighting information for decision makers and attributes are totally unknown, we choose GIVDHFLPOWA or GIVDHFLPOWGA operators to develop following Approach II for MAGDM under interval-valued dual hesitant fuzzy linguistic environments.

5.2.1 *Approach II: MAGDM with unknown expert weights and attribute weights*

Step II-1 Same as step I-1.

Step II-2 Calculate support degrees according to following function

$$Sup(\widehat{r}_{ij}^{\sigma(k)}, \widehat{r}_{ij}^{\sigma(l)}) = 1 - d(\widehat{r}_{ij}^{\sigma(k)}, \widehat{r}_{ij}^{\sigma(l)}), \quad k, l = 1, 2, \ldots, t; k \neq l; \ i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m, \tag{55}$$

which satisfies conditions for support functions given in Definition 2.8. Here, $d(\widehat{r}_{ij}^{\sigma(k)}, \widehat{r}_{ij}^{\sigma(l)})$ is calculated by normalized Euclidean distance measure in Eq. (21).

Step II-3 Calculate support degree $T(\widehat{r}_{ij}^{\sigma(k)})$ that the kth largest IVDHFLN $\widehat{r}_{ij}^{\sigma(k)}$ receives from other IVDHFLNs $\widehat{r}_{ij}^{\sigma(l)} (l = 1, 2, \ldots, t; l \neq k)$, where

$$T(\widehat{r}_{ij}^{\sigma(k)}) = \sum_{l=1,l\neq k}^{t} Sup(\widehat{r}_{ij}^{\sigma(k)}, \widehat{r}_{ij}^{\sigma(l)}). \tag{56}$$

Step II-4 Utilize Eq. (34) to calculate weights η_{ij}^{k} associated with the kth largest IVDHFLN $\widehat{r}_{ij}^{k}$, we have $\eta_{ij}^{k} = g(\frac{R^k}{TV}) - g(\frac{R^{k-1}}{TV})$ with $\eta_{ij}^{k} \in [0,1]$ and $\sum_{k=1}^{t}\eta_{ij}^{k} = 1$, g is the BUM function [60], $R^k = \sum_{l=1}^{k} V_{ij}^{\sigma(l)}, TV_{ij} = \sum_{l=1}^{t} V_{ij}^{\sigma(l)}, V_{ij}^{\sigma(l)} = 1 + T(\widehat{r}_{ij}^{\sigma(l)})$.

Step II-5 Aggregate all individual decision matrices $\widehat{R}^{k} = (\widehat{r}_{ij}^{k})_{n\times m}(k = 1,2,\ldots,t)$ into group decision matrix $R = (r_{ij})_{n\times m}$ or $R^G = (r_{ij}^{G})_{n\times m}$ by GIVDHFLPOWA or GIVDHFLPOWG operator, where

$$r_{ij} = GIVDHFLPOWA_{\lambda}(\widehat{r}_{ij}^{1}, \widehat{r}_{ij}^{2}, \ldots, \widehat{r}_{ij}^{t}) = \bigcup_{\left(s_{\widehat{\alpha}_{ij}^{\sigma(k)}}, \tilde{h}_{ij}^{\sigma(k)}, g_{ij}^{\sigma(k)}\right) \in \widehat{r}_{ij}^{\sigma(k)}} \Bigg(\widehat{s}_{\left(\sum_{k=1}^{t}\eta_{ij}^{k}(\widehat{\alpha}_{ij}^{\sigma(k)})^{\lambda}\right)^{1/\lambda}}, \bigcup_{[\mu_{ij}^{L\sigma(k)},\mu_{ij}^{U\sigma(k)}]\, in \tilde{h}_{ij}^{\sigma(k)}, [v_{ij}^{L\sigma(k)}, v_{ij}^{U\sigma(k)}] \in \tilde{g}_{ij}^{\sigma(k)}} \Bigg(\Bigg\{ \Bigg[\left(1 - \prod_{k=1}^{t}(1-(\mu_{ij}^{L\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda}, \left(1 - \prod_{k=1}^{t}(1-(\mu_{ij}^{U\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda} \Bigg] \Bigg\}, \Bigg\{ \Bigg[1 - \left(1 - \prod_{k=1}^{t}(1-(1-v_{ij}^{L\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda}, 1 - \left(1 - \prod_{k=1}^{t}(1-(1-v_{ij}^{U\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda} \Bigg] \Bigg\} \Bigg) \Bigg) \tag{57}$$

or

$$r_{ij}^{G} = GIVDHFLPOWGA_{\lambda}(\widehat{r}_{ij}^{1}, \widehat{r}_{ij}^{2}, \ldots, \widehat{r}_{ij}^{t}) = \bigcup_{\left(s_{\widehat{\alpha}_{ij}^{\sigma(k)}}, \tilde{h}_{ij}^{\sigma(k)}, g_{ij}^{\sigma(k)}\right) \in \widehat{r}_{ij}^{\sigma(k)}} \Bigg(\widehat{s}_{\frac{1}{\lambda}\prod_{k=1}^{t}(\lambda\widehat{\alpha}_{ij}^{\sigma(k)})^{\eta_{ij}^{k}}}, \bigcup_{[\mu_{ij}^{L\sigma(k)},\mu_{ij}^{U\sigma(k)}] \in \tilde{h}_{ij}^{\sigma(k)}, [v_{ij}^{L\sigma(k)}, v_{ij}^{U\sigma(k)}] \in \tilde{g}_{ij}^{\sigma(k)}} \Bigg\{ \Bigg\{ \Bigg[1 - \left(1 - \prod_{k=1}^{t}(1-(1-\mu_{ij}^{L\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda}, 1 - \left(1 - \prod_{k=1}^{t}(1-(1-\mu_{ij}^{U\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda} \Bigg] \Bigg\}, \Bigg\{ \Bigg[\left(1 - \prod_{k=1}^{t}(1-(v_{ij}^{L\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda}, \left(1 - \prod_{k=1}^{t}(1-(v_{ij}^{U\sigma(k)})^{\lambda})^{\eta_{ij}^{k}}\right)^{1/\lambda} \Bigg] \Bigg\} \Bigg\} \Bigg). \tag{58}$$

Step II-6 Calculate support degrees $Sup(r_{i\sigma(j)}, r_{i\sigma(p)})$ or $Sup(r_{i\sigma(j)}^{G}, r_{i\sigma(p)}^{G})$ according to following functions:

$$Sup(r_{i\sigma(j)}, r_{i\sigma(p)}) = 1 - d(r_{i\sigma(j)}, r_{i\sigma(p)}), \quad i = 1,2,\ldots,n;\ j,p = 1,2,\ldots,m;\ j \neq p, \tag{59}$$

$$Sup(r_{i\sigma(j)}^{G}, r_{i\sigma(p)}^{G}) = 1 - d(r_{i\sigma(j)}^{G}, r_{i\sigma(p)}^{G}), \quad i = 1,2,\ldots,n;\ j,p = 1,2,\ldots,m;\ j \neq p, \tag{60}$$

which satisfy conditions for support functions in Definition 4.1. Here, $d(r_{ij}, r_{ip})$ and $d(r_{ij}^{G}, r_{ip}^{G})$ are calculated by Eq. (21).

Step II-7 Calculate support degree $T(r_{i\sigma(j)})$ that the jth largest IVDHFLN $r_{i\sigma(j)}$ receives from other IVDHFLNs $r_{i\sigma(p)}(p = 1,2,\ldots,m; p \neq j)$:

$$T(r_{i\sigma(j)}) = \sum_{p=1,p\neq j}^{m} Sup(r_{i\sigma(j)}, r_{i\sigma(p)}); \tag{61}$$

or support degree $T(r_{i\sigma(j)}^{G})$ that the jth largest IVDHFLN $r_{i\sigma(j)}^{G}$ receives from IVDHFLNs $r_{i\sigma(p)}^{G}(p = 1,2,\ldots,m; p \neq j)$, where

$$T(r^G_{i\sigma(j)}) = \sum_{p=1,p\neq j}^{m} Sup(r^G_{i\sigma(j)}, r^G_{i\sigma(p)}). \tag{62}$$

Step II-8 Utilize Eq. (34) to calculate weights $w_{ij}(i = 1,2,\ldots,n; j = 1,2,\ldots,m)$ or w^G_{ij}:

$w_{ij} = g(\frac{R_{ij}}{TV}) - g(\frac{R_{i(j-1)}}{TV})$ with $w_{ij} \in [0,1]$ and $\sum_{j=1}^{m} w_{ij} = 1$, g is the BUM function [60],

$$R_{ij} = \sum_{p=1}^{j} V_{i\sigma(p)}, \quad TV = \sum_{j=1}^{m} V_{i\sigma(j)}, \quad V_{i\sigma(j)} = 1 + T(r_{i\sigma(j)});$$

or $w^G_{ij} = g(\frac{R^G_{ij}}{TV^G}) - g(\frac{R^G_{i(j-1)}}{TV^G})$ with $w^G_{ij} \in [0,1]$ and $\sum_{j=1}^{m} w^G_{ij} = 1$, g is the BUM function [60], $R^G_{ij} = \sum_{p=1}^{j} V^G_{i\sigma(p)}, TV^G = \sum_{j=1}^{m} V^G_{i\sigma(j)}, V^G_{i\sigma(j)} = 1 + T(r^G_{i\sigma(j)})$.

Step II-9 Utilize WGIVDHFLPOWA operator or WGIVDHFLPOWG operator to aggregate $r_{ij}(i = 1,2,\ldots,n; j = 1,2,\ldots,m)$ or r^G_{ij} into overall values $r_i(i = 1,2,\ldots,n)$ or r^G_i related to each alternative A_i, where

Step II-10 Calculate scores $s(r_i)$ $(i = 1,2,\ldots,n)$ or $s(r^G_i)$ for r_i $(i = 1,2,\ldots,n)$ or r^G_i related to alternatives A_i $(i = 1,2,\ldots,n)$ according to Definition 3.5;

Step II-11 Rank all the alternatives A_i $(i = 1,2,\ldots,n)$ and select the best one(s) in accordance with the ranking of r_i $(i = 1,2,\ldots,n)$ and r^G_i.

For more clarity, Fig. 2 shows flowcharts of the above-proposed IVDHFLS-based approaches for MAGDM under the situations of cases I and II.

5.3 Advantages of proposed approaches

Based on IVDHFLS and its generalized power operators, the above-proposed approaches can effectively tackle MAGDM problems where arguments being aggregated are mutually supportive [41, 47, 55]. The main advantages of proposed approaches can be outlined as follows.

1. The newly defined expression form of IVDHFLS holds advantages of linguistic variables and DHFS when

$$r_i = WGIVDHFLPOWA_{w,\lambda}(r_{i1}, r_{i2}, \ldots, r_{im}) = \bigcup_{\left(s_{\alpha_{i\sigma(j)}}, \tilde{h}_{i\sigma(j)}, g_{i\sigma(j)}\right) \in r_{i\sigma(j)}} \Bigg(s_{\left(\sum_{j=1}^{m} w_{ij}(\alpha_{i\sigma(j)})^{\lambda}\right)^{1/\lambda}},$$
$$\bigcup_{[\mu^L_{i\sigma(j)}, \mu^U_{i\sigma(j)}] \in \tilde{h}_{i\sigma(j)}, [v^L_{i\sigma(j)}, v^U_{i\sigma(j)}] \in \tilde{g}_{i\sigma(j)}} \left(\left\{ \left[\left(1 - \prod_{j=1}^{m} (1 - (\mu^L_{i\sigma(j)})^{\lambda})^{w_{ij}}\right)^{1/\lambda}, \left(1 - \prod_{j=1}^{m} (1 - (\mu^U_{i\sigma(j)})^{\lambda})^{w_{ij}}\right)^{1/\lambda} \right] \right\}, \right. \tag{63}$$
$$\left. \left\{ \left[1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - v^L_{i\sigma(j)})^{\lambda})^{w_{ij}}\right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - v^U_{i\sigma(j)})^{\lambda})^{w_{ij}}\right)^{1/\lambda} \right] \right\} \right) \Bigg)$$

or

$$r^G_i = WGIVDHFLPOWGA_{w,\lambda}(r^G_{i1}, r^G_{i2}, \ldots, r^G_{im})$$
$$= \bigcup_{\left(s_{\alpha^G_{i\sigma(j)}}, \tilde{h}^G_{i\sigma(j)}, g^G_{i\sigma(j)}\right) \in r^G_{i\sigma(j)}} \Bigg(s_{\frac{1}{\lambda}\prod_{j=1}^{n} (\lambda\alpha^G_{i\sigma(j)})^{w^G_{ij}}}, \bigcup_{[\mu^{LG}_{i\sigma(j)}, \mu^{UG}_{i\sigma(j)}] \in \tilde{h}^G_{i\sigma(j)}, [v^{LG}_{i\sigma(j)}, v^{UG}_{i\sigma(j)}] \in \tilde{g}^G_{i\sigma(j)}}$$
$$\left\{ \left\{ \left[1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - \mu^{LG}_{i\sigma(j)})^{\lambda})^{w^G_{ij}}\right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{m} (1 - (1 - \mu^{UG}_{i\sigma(j)})^{\lambda})^{w^G_{ij}}\right)^{1/\lambda} \right] \right\}, \right. \tag{64}$$
$$\left. \left\{ \left[\left(1 - \prod_{j=1}^{m} (1 - (v^{LG}_{i\sigma(j)})^{\lambda})^{w^G_{ij}}\right)^{1/\lambda}, \left(1 - \prod_{j=1}^{m} (1 - (v^{UG}_{i\sigma(j)})^{\lambda})^{w^G_{ij}}\right)^{1/\lambda} \right] \right\} \right\} \Bigg)$$

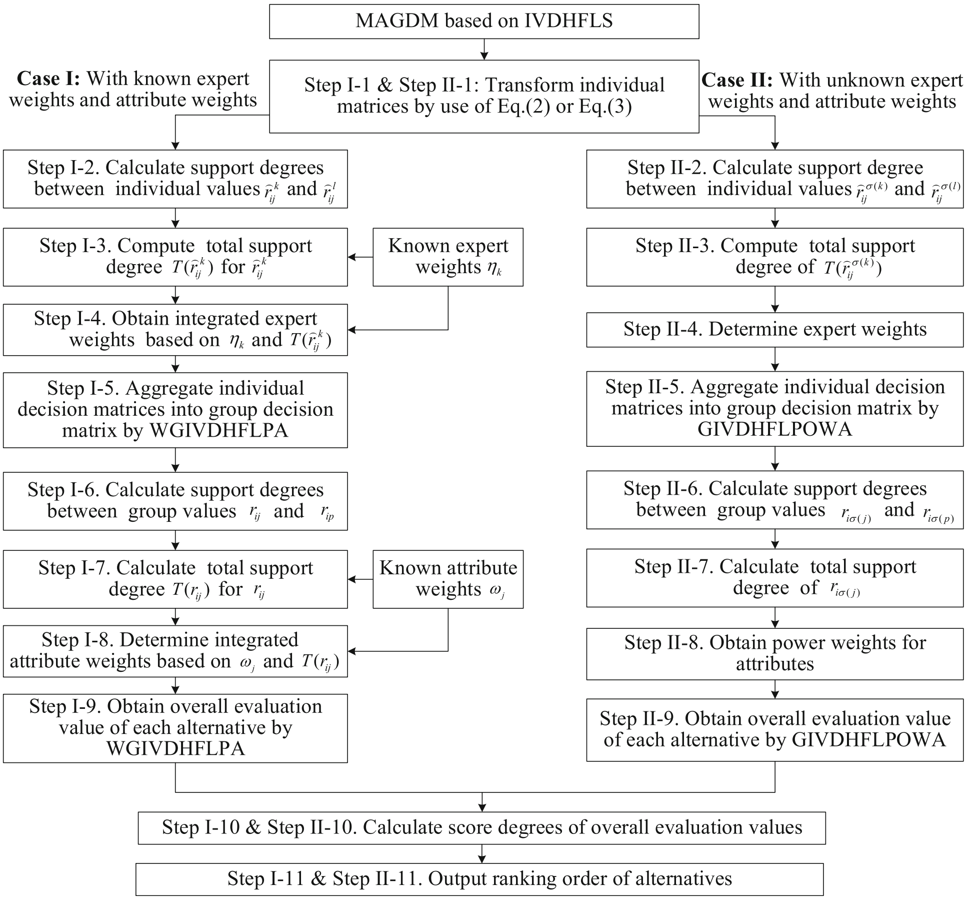

Fig. 2 Flowcharts of proposed IVDHFLS-based approaches for MAGDM

depicting decision hesitancy in ill-defined problems. IVDHFLS is specially suitable for complex decision situations where experts are willing to or only capable of indicating their hesitancy with interval values rather than crisp ones on both membership and nonmembership degrees. On the other hand, by setting upper bounds and lower bounds as equal in $\tilde{h}(x)$ and $\tilde{g}(x)$, IVDHFLS flexibly reduces to accommodate such decision scenarios where membership or nonmembership degrees can be assigned exactly with crisp values.

2. For calculating distances between IVDHFLNs, the improved distance measure defined in Eq. (21) manages to help proposed MAGDM approaches avoid the information distortion caused by conventional complementing method [32, 56].
3. As for MAGDM case I with known expert weights and attribute weights, approach I derives synthesized expert weights (using Eq. (44)) and attribute weights (using Eqs. (51) or (52)) by taking into account known weighting information and supportive correlations among arguments simultaneously, so that approach I manages to consider characteristics of problems in case I more comprehensively. While for more complex MAGDM in case II without weighting information (i.e., expert weights and attribute weights are totally unknown), approach II utilizes BUM function [60] to objectively derive appropriate expert weights and attribute weights from supportive correlations among interval-valued dual hesitant fuzzy linguistic arguments. Overall, by employing IVDHFLS to depict decision preferences, the proposed approaches I and II can exploit decision information more adequately, so as to effectively resolve MAGDM with mutually supportive arguments being aggregated.

In what follows, application and comparative studies are carried out to verify our proposed MAGDM approaches.

6 Illustrative examples

6.1 Case study

Emergency management is a very important issue in social systems because frequently happening emergency events may cause disastrous consequences, especially in countries with high population density, such as China. One of the most important components in emergency management is to evaluate the emergency response capacity (ERC) of emergency departments or emergency solutions [16]. In this subsection, we apply approaches developed in Sect. 5 to evaluate emergency response capacity of emergency solutions.

To implement rescue to a urban fire happened in China, for effective response, local emergency department have to select the most appropriate alternative from prepared emergency solutions (adapted from Ju et al. [16]). Suppose there are three emergency solutions to select: $\{A_1, A_2, A_3\}$. Three decision making teams d_k $(k = 1, 2, 3)$, including employees team (d_1), external experts team (d_2), senior managers team (d_3), are organized to evaluate the emergency solutions under four attributes: emergency process capability (G_1), emergency forecasting capability (G_2), emergency support capability (G_3), after-disaster process capability (G_4). Due to uncertainty and time pressure, decision makers are usually hesitant and are more likely willing to express their preferences by use of linguistic terms. Then the fuzzy tool of IVDHFLS defined in this paper is provided to all teams to assess the alternatives. Then, three interval-valued dual hesitant fuzzy linguistic decision matrices, i.e., $R^k = (r_{ij}^k)_{3\times4} (k = 1, 2, 3)$, are collected and shown in Tables 1, 2 and 3. Suppose R^1 given by d_1 uses the nine-granularity linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, R^2 given by d_2 uses the five-granularity linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4\}$, and R^3 given by d_3 uses the eleven- granularity linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\}$. Subsequently, we use the developed decision making approaches to derive ranking order of the three emergency solutions. Decision steps are detailed as follows.

Case I Suppose that weighting information is known. The weighting vector for decision making teams is $\eta = (0.4, 0.3, 0.3)^T$, weighting vector for attributes is $\omega = (0.2, 0.3, 0.3, 0.2)^T$. For this case, we use Approach I to obtain ranking order of the emergency solutions.

Step I-1 Transform decision matrices into the ones by use of linguistic term set with largest cardinality (i.e., the eleven-granularity linguistic term set) according to Eq. (2). Therefore, $R^1 = (r_{ij}^1)_{3\times4}$ is mapped into $\widehat{R}^1 = (\widehat{r}_{ij}^1)_{3\times4}$ as shown in Table 4, $R^2 = (r_{ij}^2)_{3\times4}$ is mapped into $R^2 = (r_{ij}^2)_{3\times4}$ as shown in Table 5, $\widehat{R}^3 = R^3$.

Step I-2 Calculate support degrees $Sup(\widehat{r}_{ij}^k, \widehat{r}_{ij}^l)$ $(k, l, i = 1, 2, 3; j = 1, 2, 3, 4; k \neq l)$ by Eq. (42). For simplicity, we denote $Sup(\widehat{r}_{ij}^k, \widehat{r}_{ij}^l)$ by Sup^{kl} referring to support degrees between $\widehat{R}^k$ and $\widehat{R}^l$, then we have

Table 1 The interval-valued dual hesitant fuzzy linguistic decision matrix R^1 provided by d_1

	G_1	G_2	G_3	G_4
A_1	$(s_5, \{[0.3, 0.5]\}, \{[0.1, 0.2]\})$	$(s_1, \{[0.1, 0.4]\}, \{[0.2, 0.3], [0.3, 0.4]\})$	$(s_7, \{[0.2, 0.4]\}, \{[0.4, 0.5]\})$	$(s_4, \{[0.6, 0.7], [0.7, 0.8]\}, \{[0.1, 0.2]\})$
A_2	$(s_5, \{[0.4, 0.7]\}, \{[0.2, 0.3]\})$	$(s_2, \{[0.5, 0.6]\}, \{[0.1, 0.2]\})$	$(s_8, \{[0.2, 0.3]\}, \{[0.5, 0.6], [0.6, 0.7]\})$	$(s_6, \{[0.4, 0.5]\}, \{[0.2, 0.4]\})$
A_3	$(s_7, \{[0.6, 0.8]\}, \{[0.1, 0.2]\})$	$(s_4, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\})$	$(s_2, \{[0.5, 0.6], [0.7, 0.8]\}, \{[0.1, 0.2]\})$	$(s_5, \{[0.5, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\})$

Table 2 The interval-valued dual hesitant fuzzy linguistic decision matrix R^2 provided by d_2

	G_1	G_2	G_3	G_4
A_1	(s_2, {[0.1, 0.2], [0.2, 0.3]}, {[0.3, 0.4]})	(s_1, {[0.6, 0.7]}, {[0.1, 0.2], [0.2, 0.3]})	(s_4, {[0.3, 0.4], [0.4, 0.5]}, {[0.4, 0.5]})	(s_1, {[0.4, 0.7]}, {[0.2, 0.3]})
A_2	(s_3, {[0.7, 0.8]}, {[0.1, 0.2]})	(s_2, {[0.2, 0.3]}, {[0.5, 0.6]})	(s_1, {[0.6, 0.8]}, {[0.1, 0.2]})	(s_4, {[0.3, 0.5]}, {[0.3, 0.4]})
A_3	(s_4, {[0.4, 0.5]}, {[0.3, 0.4], [0.4, 0.5]})	(s_1, {[0.7, 0.8]}, {[0.1, 0.2]})	(s_3, {[0.2, 0.5]]}, {[0.3, 0.4]})	(s_2, {[0.3, 0.4]}, {[0.2, 0.4], [0.4, 0.5]})

Table 3 The interval-valued dual hesitant fuzzy linguistic decision matrix R^3 provided by d_3

	G_1	G_2	G_3	G_4
A_1	(s_7, {[0.2, 0.4]}, {[0.3, 0.4], [0.5, 0.6]})	(s_5, {[0.6, 0.7]}, {[0.1, 0.2], [0.2, 0.3]})	(s_8, {[0.1, 0.2]}, {[0.5, 0.6], [0.6, 0.7]})	(s_4, {[0.3, 0.5]}, {[0.1, 0.3], [0.3, 0.5]})
A_2	(s_2, {[0.4, 0.6], [0.6, 0.7]}, {[0.1, 0.3]})	(s_6, {[0.3, 0.5]}, {[0.1, 0.3], [0.4, 0.5]})	(s_3, {[0.4, 0.6]}, {[0.3, 0.4]})	(s_7, {[0.7, 0.8]}, {[0.1, 0.2]})
A_3	(s_9, {[0.7, 0.8]}, {[0.1, 0.2]]})	(s_5, {[0.5, 0.6]}, {[0.3, 0.4]})	(s_6, {[0.6, 0.7]]}, {[0.1, 0.2], [0.2, 0.3]})	(s_2, {[0.6, 0.7]}, {[0.1, 0.3]})

Table 4 The transformed interval-valued hesitant fuzzy linguistic decision matrix $\widehat{R}^1$

	G_1	G_2	G_3	G_4
A_1	$(s_{6.25}, \{[0.3, 0.5]\}, \{[0.1, 0.2]\})$	$(s_{1.25}, \{[0.1, 0.4]\}, \{[0.2, 0.3], [0.3, 0.4]\})$	$(s_{8.75}, \{[0.2, 0.4]\}, \{[0.4, 0.5]\})$	$(s_5, \{[0.6, 0.7], [0.7, 0.8]\}, \{[0.1, 0.2]\})$
A_2	$(s_{6.25}, \{[0.4, 0.7]\}, \{[0.2, 0.3]\})$	$(s_{2.5}, \{[0.5, 0.6]\}, \{[0.1, 0.2]\})$	$(s_{10}, \{[0.2, 0.3]\}, \{[0.5, 0.6], [0.6, 0.7]\})$	$(s_{7.5}, \{[0.4, 0.5]\}, \{[0.2, 0.4]\})$
A_3	$(s_{8.75}, \{[0.6, 0.8]\}, \{[0.1, 0.2]\})$	$(s_5, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\})$	$(s_{2.5}, \{[0.5, 0.6], [0.7, 0.8]\}, \{[0.1, 0.2]\})$	$(s_{6.25}, \{[0.5, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\})$

Table 5 The transformed interval-valued hesitant fuzzy linguistic decision matrix $\widehat{R}^2$

	G_1	G_2	G_3	G_4
A_1	$(s_5, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.3, 0.4]\})$	$(s_{2.5}, \{[0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\})$	$(s_{10}, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\})$	$(s_{2.5}, \{[0.4, 0.7]\}, \{[0.2, 0.3]\})$
A_2	$(s_{7.5}, \{[0.7, 0.8]\}, \{[0.1, 0.2]\})$	$(s_5, \{[0.2, 0.3]\}, \{[0.5, 0.6]\})$	$(s_{2.5}, \{[0.6, 0.8]\}, \{[0.1, 0.2]\})$	$(s_{10}, \{[0.3, 0.5]\}, \{[0.3, 0.4]\})$
A_3	$(s_{10}, \{[0.4, 0.5]\}, \{[0.3, 0.4], [0.4, 0.5]\})$	$(s_{2.5}, \{[0.7, 0.8]\}, \{[0.1, 0.2]\})$	$(s_{7.5}, \{[0.2, 0.5]]\}, \{[0.3, 0.4]\})$	$(s_5, \{[0.3, 0.4]\}, \{[0.2, 0.4], [0.4, 0.5]\})$

$$Sup^{12} = Sup^{21} = \begin{pmatrix} 0.8234 & 0.9077 & 0.8387 & 0.7849 \\ 0.7699 & 0.7615 & 0.4285 & 0.8449 \\ 0.6797 & 0.8096 & 0.7319 & 0.7812 \end{pmatrix},$$

$$Sup^{13} = Sup^{31} = \begin{pmatrix} 0.7644 & 0.7446 & 0.8222 & 0.798 \\ 0.7309 & 0.7891 & 0.4909 & 0.7738 \\ 0.9243 & 0.9065 & 0.7536 & 0.7333 \end{pmatrix},$$

$$Sup^{23} = Sup^{32} = \begin{pmatrix} 0.8016 & 0.8268 & 0.7109 & 0.9214 \\ 0.5455 & 0.8354 & 0.9278 & 0.717 \\ 0.6487 & 0.8361 & 0.7895 & 0.8391 \end{pmatrix}.$$

Step I-3 Calculate support degree $T(\widehat{r}_{ij}^k)$ of $\widehat{r}_{ij}^k$ by $\widehat{r}_{ij}^l(l = 1,2,3; l \neq k)$ using Eq. (43). We denote $T(\widehat{r}_{ij}^k)$ by T^k as follows,

$$T^1 = \begin{pmatrix} 0.4763 & 0.4957 & 0.4983 & 0.4749 \\ 0.4502 & 0.4652 & 0.2758 & 0.4856 \\ 0.4812 & 0.5148 & 0.4456 & 0.4543 \end{pmatrix},$$

$$T^2 = \begin{pmatrix} 0.5698 & 0.6111 & 0.5487 & 0.5904 \\ 0.4716 & 0.5552 & 0.4497 & 0.5531 \\ 0.4665 & 0.5747 & 0.5296 & 0.5642 \end{pmatrix},$$

$$T^3 = \begin{pmatrix} 0.5462 & 0.5459 & 0.5422 & 0.5956 \\ 0.456 & 0.5663 & 0.4747 & 0.5246 \\ 0.5643 & 0.6134 & 0.5383 & 0.545 \end{pmatrix}.$$

Step I-4 Utilize Eq. (44) to calculate the weights $\xi_{ij}^k(k = 1,2,3)$ associated with r_{ij}^k, we denote $(\xi_{ij}^k)_{3\times 4}$ by ξ^k as follows,

$$\xi^1 = \begin{pmatrix} 0.3871 & 0.3871 & 0.3926 & 0.3817 \\ 0.3978 & 0.3849 & 0.3678 & 0.3916 \\ 0.3945 & 0.3878 & 0.3859 & 0.3841 \end{pmatrix},$$

$$\xi^2 = \begin{pmatrix} 0.3087 & 0.3128 & 0.3044 & 0.3087 \\ 0.3027 & 0.3064 & 0.3134 & 0.307 \\ 0.293 & 0.3024 & 0.3062 & 0.3098 \end{pmatrix},$$

$$\xi^3 = \begin{pmatrix} 0.3041 & 0.3001 & 0.3031 & 0.3097 \\ 0.2995 & 0.3086 & 0.3188 & 0.3014 \\ 0.3125 & 0.3098 & 0.3079 & 0.306 \end{pmatrix}.$$

Step I-5 Suppose $\lambda = 2$, then aggregate individual decision matrices $\widehat{R}^k = (\widehat{r}_{ij}^k)_{3\times 4}(k = 1,2,3)$ into group decision matrix $R = (r_{ij})_{3\times 4}$ by WGIVDHFLPA operator as shown in Table 6, or $R^G = (r_{ij}^G)_{3\times 4}$ by WGIVDHFLPGA operator, as shown Table 7.

Step I-6. Calculate support degrees $Sup(r_{ij}, r_{ip})$ $(i = 1,2,3; j,p = 1,2,3,4; j \neq p)$ by Eq. (47), and $Sup(r_{ij}^G, r_{ip}^G)$ by Eq. (48). Here, we denote $Sup(r_{ij}, r_{ip})$ by Sup_{jp}, and $Sup(r_{ij}^G, r_{ip}^G)$ by Sup_{jp}^G, referring to support degrees between jth row and pth row in R and R^G, respectively. We have

$$Sup_{12} = Sup_{21} = \begin{pmatrix} 0.8998 \\ 0.8281 \\ 0.636 \end{pmatrix},$$

$$Sup_{13} = Sup_{31} = \begin{pmatrix} 0.7138 \\ 0.909 \\ 0.7014 \end{pmatrix},$$

$$Sup_{14} = Sup_{41} = \begin{pmatrix} 0.9098 \\ 0.8647 \\ 0.6511 \end{pmatrix};$$

$$Sup_{23} = Sup_{32} = \begin{pmatrix} 0.6199 \\ 0.8352 \\ 0.923 \end{pmatrix},$$

$$Sup_{24} = Sup_{42} = \begin{pmatrix} 0.9301 \\ 0.7178 \\ 0.9696 \end{pmatrix},$$

$$Sup_{34} = Sup_{43} = \begin{pmatrix} 0.6462 \\ 0.8715 \\ 0.9376 \end{pmatrix};$$

and

$$Sup_{12}^G = Sup_{21}^G = \begin{pmatrix} 0.8299 \\ 0.8577 \\ 0.6345 \end{pmatrix},$$

$$Sup_{13}^G = Sup_{31}^G = \begin{pmatrix} 0.7461 \\ 0.8586 \\ 0.6499 \end{pmatrix},$$

$$Sup_{14}^G = Sup_{41}^G = \begin{pmatrix} 0.8791 \\ 0.8261 \\ 0.6248 \end{pmatrix},$$

$$Sup_{23}^G = Sup_{32}^G = \begin{pmatrix} 0.585 \\ 0.9501 \\ 0.9477 \end{pmatrix},$$

$$Sup_{24}^G = Sup_{42}^G = \begin{pmatrix} 0.8889 \\ 0.7455 \\ 0.961 \end{pmatrix},$$

$$Sup_{34}^G = Sup_{43}^G = \begin{pmatrix} 0.6398 \\ 0.7793 \\ 0.9592 \end{pmatrix}.$$

Step I-7 Calculate support degree $T(r_{ij})$ of r_{ij} by $r_{ip}(p = 1,2,3,4; p \neq j)$ according to Eq. (49), and support degree $T^G(r_{ij})$ of r_{ij}^G by $r_{ip}^G(p = 1,2,3,4; p \neq j)$ according to Eq. (50). Here, we denote $(T(r_{ij}))_{3\times 4}$ and $(T^G(r_{ij}))_{3\times 4}$ by T and T^G, respectively. Then we have

Table 6 Group decision matrix R yielded by WGIVDHFLPA operator

	G_1	G_2	G_3	G_4
A_1	$(s_{6.1432}$, {[0.2252, 0.4027], [0.2443, 0.4194]}, {[0.1925, 0.3016], [0.2198, 0.334]})	$(s_{3.1721}$, {[0.4922, 0.6175]}, {[0.1301, 0.2332], [0.1602, 0.2634], [0.1617, 0.2647], [0.2, 0.3], [0.1506, 0.2583], [0.1859, 0.2925], [0.1876, 0.294], [0.2332, 0.3343]})	$(s_{8.9385}$, {[0.216, 0.354], [0.2638, 0.3937]}, {[0.4268, 0.5269], [0.4477, 0.5475]})	$(s_{4.053}$, {[0.4731, 0.6522], [0.5369, 0.7049]}, {[0.1233, 0.256], [0.1719, 0.2951]})
A_2	$(s_{5.8107}$, {[0.527, 0.7133], [0.5781, 0.7355]}, {[0.1311, 0.2645]})	$(s_{4.6017}$, {[0.3759, 0.5013]}, {[0.157, 0.307], [0.2391, 0.3561]})	$(s_{6.4504}$, {[0.4357, 0.6259]}, {[0.2459, 0.3602], [0.2582, 0.3741]})	$(s_{8.2156}$, {[0.5092, 0.6315]}, {[0.1817, 0.3204]})
A_3	$(s_{9.2096}$, {[0.5953, 0.7441]}, {[0.1361, 0.2424], [0.1463, 0.2559]})	$(s_{4.3966}$, {[0.5297, 0.6343], [0.5501, 0.6543]}, {[0.234, 0.3457]})	$(s_{5.5426}$, {[0.4792, 0.6112], [0.5798, 0.7059]}, {[0.1381, 0.2446], [0.1711, 0.2775]})	$(s_{4.8962}$, {[0.4913, 0.6362]}, {[0.1234, 0.278], [0.1496, 0.2949], [0.161, 0.3271], [0.1962, 0.3479]})

Table 7 Group decision matrix R^G yielded by WGIVDHFLPGA operator

	G_1	G_2	G_3	G_4
A_1	$(s_{6.037}$, {[0.1865, 0.3441], [0.2332, 0.3954]}, {[0.2446, 0.3397], [0.3369, 0.4308]})	$(s_{2.3537}$, {[0.2737, 0.5462]}, {[0.1474, 0.2443], [0.1753, 0.2732], [0.1763, 0.2744], [0.2, 0.3], [0.2044, 0.2973], [0.2249, 0.3207], [0.2257, 0.3217], [0.2443, 0.3432]})	$(s_{8.8713}$, {[0.1813, 0.32], [0.1958, 0.3398]}, {[0.434, 0.5341], [0.4759, 0.5769]})	$(s_{3.7682}$, {[0.4184, 0.622], [0.4363, 0.6461]}, {[0.1391, 0.2669], [0.2112, 0.3539]})
A_2	$(s_{4.6951}$, {[0.4633, 0.6885], [0.5229, 0.7259]}, {[0.1485, 0.2741]})	$(s_{4.0498}$, {[0.3157, 0.4488]}, {[0.3012, 0.4075], [0.3684, 0.463]})	$(s_{4.4118}$, {[0.3397, 0.4795]}, {[0.3603, 0.4555], [0.4232, 0.5205]})	$(s_{8.0239}$, {[0.4201, 0.5617]}, {[0.2164, 0.3543]})
A_3	$(s_{9.1796}$, {[0.5476, 0.6729]}, {[0.1848, 0.2772], [0.2378, 0.3271]})	$(s_{4.0545}$, {[0.4385, 0.5386], [0.4977, 0.5963]}, {[0.3083, 0.4037]})	$(s_{4.5825}$, {[0.3838, 0.5896], [0.4241, 0.6473]}, {[0.1877, 0.2801], [0.2105, 0.3058]})	$(s_{4.1146}$, {[0.4424, 0.571]}, {[0.1393, 0.3064], [0.2432, 0.3537], [0.1758, 0.3351], [0.2649, 0.378]})

$$T = \begin{pmatrix} 0.666 & 0.5519 & 0.458 & 0.6549 \\ 0.6941 & 0.5597 & 0.6067 & 0.6497 \\ 0.5314 & 0.598 & 0.6047 & 0.7024 \end{pmatrix},$$

$$T^G = \begin{pmatrix} 0.6486 & 0.5193 & 0.4527 & 0.6344 \\ 0.6801 & 0.6057 & 0.6126 & 0.6227 \\ 0.5103 & 0.6034 & 0.6061 & 0.701 \end{pmatrix}.$$

Step I-8 Use Eqs. (51) and (52) to calculate weighting vectors: $w_{ij}(i = 1,2,3; j = 1,2,3,4)$ and w_{ij}^G associated with r_{ij} and r_{ij}^G, respectively. Here, we denote $(w_{ij})_{3\times4}$ and $(w_{ij}^G)_{3\times4}$ by w and w^G. Then we have

$$w = \begin{pmatrix} 0.2126 & 0.2971 & 0.2791 & 0.2112 \\ 0.2093 & 0.2891 & 0.2978 & 0.2038 \\ 0.1905 & 0.2982 & 0.2995 & 0.2118 \end{pmatrix},$$

$$w^G = \begin{pmatrix} 0.213 & 0.2944 & 0.2815 & 0.2111 \\ 0.2067 & 0.2962 & 0.2975 & 0.1996 \\ 0.1882 & 0.2997 & 0.3002 & 0.2119 \end{pmatrix}.$$

Step I-9 Use WGIVDHFLPA operator or WGIVDHFLPGA operator to aggregate all the evaluations $r_{ij}(i = 1,2,3; j = 1,2,3,4)$ or $r_{ij}^G(i = 1,2,3; j = 1,2,3,4)$ to overall evaluation values $r_i(i = 1,2,3)$ or $r_i^G(i = 1,2,3)$ corresponding to each alternative A_i, where

$r_1 = (s_{6.0648}$, {[0.3849, 0.5333], [0.4045, 0.5516], [0.3923, 0.5397], [0.4114, 0.5576], [0.3871, 0.5356], [0.4065, 0.5538], [0.3945, 0.542], [0.4134, 0.5597]}, {[0.1906, 0.3099], [0.2049, 0.3197], [0.1926, 0.3125], [0.2072, 0.3224], [0.2032, 0.322], [0.2186, 0.3322], [0.2054, 0.3247], [0.221, 0.3351], [0.2037, 0.3225], [0.2192, 0.3328], [0.206, 0.3252], [0.2217, 0.3356], [0.2173, 0.3352], [0.234, 0.346], [0.2197, 0.3381], [0.2366, 0.349], [0.1994, 0.32], [0.2145, 0.3302], [0.2015, 0.3227], [0.2168, 0.333], [0.2126, 0.3326], [0.2289, 0.3433], [0.2149, 0.3354], [0.2314, 0.3463], [0.2132, 0.3331], [0.2295, 0.3439], [0.2155, 0.336], [0.232, 0.3468], [0.2275, 0.3464], [0.2451, 0.3577], [0.23, 0.3494], [0.2479, 0.3608], [0.196, 0.3167], [0.2108, 0.3267], [0.1981, 0.3193], [0.2131, 0.3295], [0.209, 0.3291], [0.2249, 0.3397], [0.2113, 0.3319], [0.2274, 0.3426], [0.2096, 0.3296], [0.2256, 0.3402], [0.2119, 0.3324], [0.2281, 0.3431], [0.2236, 0.3427], [0.2409, 0.3538], [0.2261, 0.3456], [0.2436, 0.3569], [0.205, 0.3271], [0.2207, 0.3376], [0.2073, 0.3299], [0.2231, 0.3404], [0.2187, 0.34], [0.2355, 0.3511], [0.2211, 0.3429], [0.2382, 0.3541], [0.2193, 0.3406], [0.2362, 0.3516], [0.2217, 0.3435], [0.2388, 0.3547], [0.2341, 0.3542], [0.2524, 0.3659], [0.2367, 0.3573], [0.2552, 0.3691]}),

$r_2 = (s_{6.2718}$, {[0.4585, 0.6195], [0.4728, 0.6265]}, {[0.1774, 0.3143], [0.1799, 0.3177], [0.2002, 0.3279], [0.2031, 0.3315]}),

$r_3 = (s_{6.0166}$, {[0.5221, 0.6536], [0.5516, 0.6812], [0.5284, 0.6593], [0.5574, 0.6863]}, {[0.1568, 0.2774], [0.1634, 0.2809], [0.166, 0.287], [0.173, 0.2906], [0.1672, 0.2883], [0.1743, 0.2919], [0.1771, 0.2983], [0.1847, 0.3021], [0.159, 0.2804], [0.1657, 0.2839], [0.1683, 0.2901], [0.1755, 0.2937], [0.1696, 0.2914], [0.1768, 0.2951], [0.1796, 0.3016], [0.1873, 0.3054]});

and

$r_1^G = (s_{4.6157}$, {[0.243, 0.4288], [0.2448, 0.4313], [0.2485, 0.4369], [0.2504, 0.4395], [0.2552, 0.4426], [0.2571, 0.4453], [0.2611, 0.4511], [0.2631, 0.4539]}, {[0.2814, 0.3782], [0.2902, 0.3922], [0.3026, 0.3994], [0.3107, 0.4124], [0.2858, 0.3835], [0.2945, 0.3972], [0.3066, 0.4043], [0.3146, 0.4171], [0.286, 0.3837], [0.2946, 0.3975], [0.3068, 0.4045], [0.3148, 0.4173], [0.2903, 0.3889], [0.2988, 0.4024], [0.3108, 0.4093], [0.3187, 0.4219], [0.2911, 0.3884], [0.2997, 0.4019], [0.3116, 0.4088], [0.3194, 0.4214], [0.2954, 0.3935], [0.3037, 0.4067], [0.3155, 0.4136], [0.3232, 0.426], [0.2956, 0.3937], [0.3039, 0.407], [0.3156, 0.4138], [0.3234, 0.4262], [0.2997, 0.3987], [0.308, 0.4118], [0.3195, 0.4185], [0.3271, 0.4307], [0.301, 0.3975], [0.3092, 0.4106], [0.3207, 0.4173], [0.3283, 0.4296], [0.3051, 0.4025], [0.3132, 0.4153], [0.3245, 0.422], [0.332, 0.434], [0.3053, 0.4027], [0.3133, 0.4155], [0.3246, 0.4222], [0.3321, 0.4342], [0.3093, 0.4075], [0.3172, 0.4202], [0.3284, 0.4267], [0.3357, 0.4386], [0.3101, 0.407], [0.318, 0.4197], [0.3291, 0.4262], [0.3365, 0.4381], [0.314, 0.4118], [0.3218, 0.4243], [0.3328, 0.4307], [0.34, 0.4424], [0.3142, 0.412], [0.322, 0.4245], [0.3329, 0.4309], [0.3402, 0.4426], [0.3181, 0.4167], [0.3257, 0.429], [0.3365, 0.4354], [0.3437, 0.4469]}),

$r_2^G = (s_{4.9095}$, {[0.3682, 0.5176], [0.3763, 0.5215]}, {[0.2834, 0.391], [0.31, 0.4167], [0.3066, 0.4095], [0.3311, 0.4337]}),

$r_3^G = (s_{4.9209}$, {[0.4384, 0.5825], [0.4524, 0.5983], [0.455, 0.6013], [0.4698, 0.6182]}, {[0.2238, 0.3279], [0.2416, 0.3379], [0.2291, 0.3338], [0.2464, 0.3435], [0.2298, 0.3346], [0.2471, 0.3443], [0.2348, 0.3403], [0.2518, 0.3498], [0.233, 0.3363], [0.2501, 0.3459], [0.238, 0.342], [0.2547, 0.3514], [0.2387, 0.3428], [0.2553, 0.3522], [0.2435, 0.3483], [0.2599,0.3575]}).

Step I-10 Calculate scores $s(r_i)(i = 1, 2, 3)$ and $s(r_i^G)$ of the above overall interval-valued dual hesitant fuzzy linguistic values $r_i(i = 1, 2, 3)$ and r_i^G related to solutions $A_i(i = 1, 2, 3)$, we have

$$s(r_1) = 1.1739, \quad s(r_2) = 1.8052, \quad s(r_3) = 2.2481;$$
$$s(r_1^G) = -0.0786, \quad s(r_2^G) = 0.4204, \quad s(r_3^G) = 1.1529.$$

Step I-11 Rank all solutions $A_i(i = 1, 2, 3)$ in accordance with scores $s(r_i)(i = 1, 2, 3)$ and $s(r_i^G)(i = 1, 2, 3)$, then we can obtain the following ranking orders:

$A_3 \succ A_2 \succ A_1$ by WGIVDHFLPA operator;
$A_3 \succ A_2 \succ A_1$ by WGIVDHFLPGA operator.

Thus we can derive that the most desirable solution is A_3.

Case II Suppose weighting vectors for decision making teams and attributes are unknown, for this kind of situations, we can apply Approach II to select the most appropriate emergency solution.

Step II-1 See step I-1.

Step II-2 We denote $(\hat{r}_{ij}^{\sigma(k)})_{3\times4}$ by $\hat{R}^{\sigma(k)}$, where $\hat{r}_{ij}^{\sigma(k)}$ is kth largest IVDHFLN of all the IVDHFLNs $\hat{r}_{ij}^{k}(k = 1, 2, 3)$. $\hat{R}^{\sigma(k)}(k = 1, 2, 3)$ are shown in Tables 8, 9 and 10. Then calculate support degrees $Sup(\hat{r}_{ij}^{\sigma(k)}, \hat{r}_{ij}^{\sigma(l)})$ $(k, l, i = 1, 2, 3; j = 1, 2, 3, 4; k \neq l)$ by Eq. (55), referring to the support degrees between $\hat{R}^{\sigma(k)}$ and $\hat{R}^{\sigma(l)}$. We denote $Sup(\hat{r}_{ij}^{\sigma(k)}, \hat{r}_{ij}^{\sigma(l)})$ by Sup^{kl} for clarity, then

$$Sup^{12} = Sup^{21} = \begin{pmatrix} 0.8234 & 0.8268 & 0.8387 & 0.7849 \\ 0.7699 & 0.7891 & 0.9278 & 0.7738 \\ 0.9243 & 0.8361 & 0.7536 & 0.7333 \end{pmatrix},$$
$$Sup^{13} = Sup^{31} = \begin{pmatrix} 0.6273 & 0.6979 & 0.7109 & 0.798 \\ 0.5455 & 0.7615 & 0.4285 & 0.717 \\ 0.6487 & 0.8096 & 0.7895 & 0.7812 \end{pmatrix},$$
$$Sup^{23} = Sup^{32} = \begin{pmatrix} 0.6822 & 0.8671 & 0.8222 & 0.9214 \\ 0.7309 & 0.8354 & 0.4909 & 0.8449 \\ 0.6797 & 0.9065 & 0.7319 & 0.8391 \end{pmatrix}.$$

Table 8 The reordered interval-valued dual hesitant fuzzy linguistic decision matrix $\widehat{R}^{\sigma(1)}$

	G_1	G_2	G_3	G_4
A_1	$(s_{6.25}, \{[0.3, 0.5]\}, \{[0.1, 0.2]\})$	$(s_5, \{[0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\})$	$(s_{10}, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\})$	$(s_5, \{[0.6, 0.7], [0.7, 0.8]\}, \{[0.1, 0.2]\})$
A_2	$(s_{7.5}, \{[0.7, 0.8]\}, \{[0.1, 0.2]\})$	$(s_{2.5}, \{[0.5, 0.6]\}, \{[0.1, 0.2]\})$	$(s_{2.5}, \{[0.6, 0.8]\}, \{[0.1, 0.2]\})$	$(s_7, \{[0.7, 0.8]\}, \{[0.1, 0.2]\})$
A_3	$(s_9, \{[0.7, 0.8]\}, \{[0.1, 0.2]]\})$	$(s_{2.5}, \{[0.7, 0.8]\}, \{[0.1,0.2]\})$	$(s_6, \{[0.6, 0.7]]\}, \{[0.1, 0.2], [0.2, 0.3]\})$	$(s_{6.25}, \{[0.5, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\})$

Table 9 The reordered interval-valued dual hesitant fuzzy linguistic decision matrix $\widehat{R}^{\sigma(2)}$

	G_1	G_2	G_3	G_4
A_1	$(s_5, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.3, 0.4]\})\}$	$(s_{2.5}, \{[0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\})$	$(s_{8.75}, \{[0.2, 0.4]\}, \{[0.4, 0.5]\})$	$(s_{2.5}, \{[0.4, 0.7]\}, \{[0.2, 0.3]\})$
A_2	$(s_{6.25}, \{[0.4, 0.7]\}, \{[0.2, 0.3]\})$	$(s_6, \{[0.3, 0.5]\}, \{[0.1, 0.3], [0.4, 0.5]\})$	$(s_3, \{[0.4, 0.6]\}, \{[0.3, 0.4]\})$	$(s_{7.5}, \{[0.4, 0.5]\}, \{[0.2, 0.4]\})$
A_3	$(s_{8.75}, \{[0.6, 0.8]\}, \{[0.1, 0.2]\})$	$(s_5, \{[0.5, 0.6]\}, \{[0.3, 0.4]\})$	$(s_{2.5}, \{[0.5, 0.6], [0.7, 0.8]\}, \{[0.1, 0.2]\})$	$(s_2, \{[0.6, 0.7]\}, \{[0.1, 0.3]\})$

Table 10 The reordered interval-valued dual hesitant fuzzy linguistic decision matrix $\widehat{R}^{\sigma(3)}$

	G_1	G_2	G_3	G_4
A_1	$(s_7, \{[0.2, 0.4]\}, \{[0.3, 0.4], [0.5, 0.6]\})$	$(s_{1.25}, \{[0.1, 0.4]\}, \{[0.2, 0.3], [0.3, 0.4]\})$	$(s_8, \{[0.1, 0.2]\}, \{[0.5, 0.6], [0.6, 0.7]\})$	$(s_4, \{[0.3, 0.5]\}, \{[0.1, 0.3], [0.3, 0.5]\})$
A_2	$(s_2, \{[0.4, 0.6], [0.6, 0.7]\}, \{[0.1, 0.3]\})$	$(s_5, \{[0.2, 0.3]\}, \{[0.5, 0.6]\})$	$(s_{10}, \{[0.2, 0.3]\}, \{[0.5, 0.6], [0.6, 0.7]\})$	$(s_{10}, \{[0.3, 0.5]\}, \{[0.3, 0.4]\})$
A_3	$(s_{10}, \{[0.4, 0.5]\}, \{[0.3, 0.4], [0.4, 0.5]\})$	$(s_5, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5]\})$	$(s_{7.5}, \{[0.2, 0.5]\}, \{[0.3, 0.4]\})$	$(s_5, \{[0.3, 0.4]\}, \{[0.2, 0.4], [0.4, 0.5]\})$

Step II-3 Calculate support degree $T(\widehat{r}_{ij}^{\sigma(k)})$ of $\widehat{r}_{ij}^{\sigma(k)}$ by $\widehat{r}_{ij}^{\sigma(l)}(l = 1, 2, 3; l \neq k)$ according to Eq. (56). We denote $T(\widehat{r}_{ij}^{\sigma(k)})(k = 1, 2, 3)$ by T^k, then we have

$$T^1 = \begin{pmatrix} 1.4507 & 1.5247 & 1.5496 & 1.583 \\ 1.3153 & 1.5506 & 1.3563 & 1.4908 \\ 1.573 & 1.6457 & 1.5431 & 1.5145 \end{pmatrix},$$

$$T^2 = \begin{pmatrix} 1.5056 & 1.6939 & 1.6609 & 1.7064 \\ 1.5007 & 1.6245 & 1.4187 & 1.6187 \\ 1.604 & 1.7425 & 1.4855 & 1.5725 \end{pmatrix},$$

$$T^3 = \begin{pmatrix} 1.3095 & 1.565 & 1.5331 & 1.7194 \\ 1.2764 & 1.5969 & 0.9195 & 1.5619 \\ 1.3284 & 1.7161 & 1.5214 & 1.6203 \end{pmatrix}.$$

Step II-4 Let $g(x) = x^2$, then utilize Eq. (34) to calculate weights w_{ij}^k associated with the kth largest IVDHFN $\widehat{r}_{ij}^k$. We denote $(w_{ij}^k)_{3\times4}$ by $w^k(k = 1, 2, 3)$, then

$$w^1 = \begin{pmatrix} 0.1138 & 0.1052 & 0.1084 & 0.104 \\ 0.1066 & 0.1077 & 0.1239 & 0.1054 \\ 0.1175 & 0.1066 & 0.1135 & 0.1064 \end{pmatrix},$$

$$w^2 = \begin{pmatrix} 0.3516 & 0.3443 & 0.3444 & 0.3322 \\ 0.3545 & 0.3357 & 0.3849 & 0.3382 \\ 0.3583 & 0.3355 & 0.3302 & 0.3292 \end{pmatrix},$$

$$w^3 = \begin{pmatrix} 0.5347 & 0.5505 & 0.5472 & 0.5638 \\ 0.5389 & 0.5566 & 0.4912 & 0.5564 \\ 0.5242 & 0.558 & 0.5564 & 0.5644 \end{pmatrix}.$$

Step II-5 Let $\lambda = 2$, then use GIVDHFLPOWA operator to aggregate all individual decision matrices $R^k = (\widehat{r}_{ij}^k)_{3\times4}(k = 1, 2, 3)$ into group decision matrix $R = (r_{ij})_{3\times4}$, as shown in Table 11; or use GIVDHFLPOWGA operator to obtain group decision matrix $R^G = (r_{ij}^G)_{3\times4}$, as collected in Table 12.

Step II-6 Calculate support degrees $Sup(r_{i\sigma(j)}, r_{i\sigma(p)})(i = 1, 2, 3; j, p = 1, 2, 3, 4; j \neq p)$ by Eq. (59), and $Sup(r_{i\sigma(j)}^G, r_{i\sigma(p)}^G)$ by Eq. (60). We denote $Sup(r_{i\sigma(j)}, r_{i\sigma(p)})$ and $Sup(r_{i\sigma(j)}^G, r_{i\sigma(p)}^G)$ by Sup_{jp} and Sup_{jp}^G, referring to support degrees between jth row and pth row in R and R^G, respectively. Then we have

Table 11 The group decision matrix R obtained by GIVDHFLPOWA operator

	G_1	G_2	G_3	G_4
A_1	($s_{6.2795}$, {[0.1883, 0.3618], [0.2141, 0.3835]}, {[0.2623, 0.3671], [0.3368, 0.4461]})	($s_{2.3753}$, {[0.4316, 0.5734]}, {[0.1457, 0.2491], [0.1798, 0.2888], [0.1855, 0.287], [0.2304, 0.3345], [0.1567, 0.26], [0.1937, 0.3019], [0.2, 0.3], [0.2491, 0.3503]})	($s_{8.4987}$, {[0.1713, 0.3104], [0.1942, 0.3277]}, {[0.4505, 0.5505], [0.4927, 0.5922]})	($s_{3.701}$, {[0.3817, 0.6051], [0.407, 0.6234]}, {[0.1253, 0.2872], [0.2318, 0.3766]})
A_2	($s_{4.6903}$, {[0.4511, 0.6668], [0.5586, 0.7132]}, {[0.1273, 0.2868]})	($s_{5.1646}$, {[0.286, 0.423]}, {[0.2306, 0.4071], [0.3782, 0.4883]})	($s_{7.3049}$, {[0.3644, 0.54]}, {[0.3258, 0.4353], [0.3493, 0.46]})	($s_{8.9346}$, {[0.4083, 0.553]}, {[0.2309, 0.3694]})
A_3	($s_{9.4536}$, {[0.5305, 0.6863]}, {[0.1747, 0.2836], [0.1994, 0.3137]})	($s_{4.796}$, {[0.4452, 0.5472], [0.483, 0.5852]}, {[0.3079, 0.4148]})	($s_{6.1192}$, {[0.3936, 0.5646], [0.506, 0.6608]}, {[0.1809, 0.2899], [0.1961, 0.3041]})	($s_{4.4254}$, {[0.4539, 0.5693]}, {[0.1471, 0.3356], [0.2107, 0.3763], [0.1584, 0.3517], [0.2277, 0.395]})

Table 12 The group decision matrix R^G obtained by GIVDHFLPOWGA operator

	G_1	G_2	G_3	G_4
A_1	($s_{6.1394}$, {[0.1628, 0.3161], [0.2091, 0.3688]}, {[0.285, 0.3836], [0.4141, 0.5124]})	($s_{1.8361}$, {[0.2067, 0.4997]}, {[0.1632, 0.2605], [0.2342, 0.3287], [0.1921, 0.2913], [0.2546, 0.3526], [0.1726, 0.2703], [0.2407, 0.3362], [0.2, 0.3], [0.2605, 0.3595]})	($s_{8.4527}$, {[0.1418, 0.27], [0.1457, 0.2756]}, {[0.4588, 0.5589], [0.5255, 0.6267]})	($s_{3.5022}$, {[0.3513, 0.5711], [0.3551, 0.5764]}, {[0.1417, 0.2914], [0.256, 0.4236]})
A_2	($s_{3.4486}$, {[0.4207, 0.6487], [0.5197, 0.7088]}, {[0.144, 0.2912]})	($s_{4.9332}$, {[0.2503, 0.3775]}, {[0.3896, 0.4976], [0.4442, 0.543]})	($s_{5.2987}$, {[0.2922, 0.4241]}, {[0.4047, 0.5028], [0.4759,0.5759]})	($s_{8.7381}$, {[0.3561, 0.5202]}, {[0.2552, 0.3848]})
A_3	($s_{9.4156}$, {[0.4855, 0.6044]}, {[0.2297, 0.3239], [0.3028, 0.3956]})	($s_{4.644}$, {[0.3814, 0.4824], [0.453, 0.5526]}, {[0.3492, 0.4474]})	($s_{5.0879}$, {[0.2964, 0.5482], [0.3205, 0.5898]}, {[0.2353, 0.3298], [0.2422, 0.3378]})	($s_{3.7869}$, {[0.3886, 0.4961]}, {[0.1645, 0.3539], [0.3125, 0.4236], [0.1739, 0.3608], [0.3172, 0.4291]})

$$Sup_{12} = Sup_{21} = \begin{pmatrix} 0.9159 \\ 0.7699 \\ 0.7331 \end{pmatrix}, \quad Sup^G_{12} = Sup^G_{21} = \begin{pmatrix} 0.8864 \\ 0.7199 \\ 0.6714 \end{pmatrix},$$

$$Sup_{13} = Sup_{31} = \begin{pmatrix} 0.868 \\ 0.8015 \\ 0.6311 \end{pmatrix}, \quad Sup^G_{13} = Sup^G_{31} = \begin{pmatrix} 0.8448 \\ 0.8138 \\ 0.6088 \end{pmatrix},$$

$$Sup_{14} = Sup_{41} = \begin{pmatrix} 0.6485 \\ 0.8577 \\ 0.6621 \end{pmatrix}, \quad Sup^G_{14} = Sup^G_{41} = \begin{pmatrix} 0.6362 \\ 0.8368 \\ 0.6796 \end{pmatrix},$$

$$Sup_{23} = Sup_{32} = \begin{pmatrix} 0.8184 \\ 0.8964 \\ 0.8891 \end{pmatrix}, \quad Sup^G_{23} = Sup^G_{32} = \begin{pmatrix} 0.7746 \\ 0.7992 \\ 0.9198 \end{pmatrix},$$

$$Sup_{24} = Sup_{42} = \begin{pmatrix} 0.5998 \\ 0.7381 \\ 0.9076 \end{pmatrix}, \quad Sup^G_{24} = Sup^G_{42} = \begin{pmatrix} 0.5718 \\ 0.7611 \\ 0.9385 \end{pmatrix},$$

$$Sup_{34} = Sup_{43} = \begin{pmatrix} 0.7735 \\ 0.8208 \\ 0.9407 \end{pmatrix}; \quad Sup^G_{34} = Sup^G_{43} = \begin{pmatrix} 0.7794 \\ 0.9495 \\ 0.9137 \end{pmatrix}.$$

and

Step II-7 Calculate support degree $T(r_{ij})$ of r_{ij} by $r_{ip}(p = 1,2,3,4; p \neq j)$, and $T^G(r_{ij})$ of r_{ij}^G by $r_{ip}^G(p = 1,2,3,4; p \neq j)$. We denote $(T(r_{ij}))_{3\times4}$ by T, $(T^G(r_{ij}))_{3\times4}$ by T^G, where

$$T = \begin{pmatrix} 2.4324 & 2.3341 & 2.4599 & 2.0218 \\ 2.4291 & 2.4044 & 2.5187 & 2.4166 \\ 2.0263 & 2.5298 & 2.4609 & 2.5104 \end{pmatrix},$$

$$T^G = \begin{pmatrix} 2.3674 & 2.2328 & 2.3988 & 1.9874 \\ 2.3705 & 2.2802 & 2.5625 & 2.5474 \\ 1.9598 & 2.5297 & 2.4423 & 2.5318 \end{pmatrix}.$$

Step II-8 Use Eq. (34) to calculate weights $w_{ij}(i = 1,2,3; j = 1,2,3,4)$ and w_{ij}^G associated with r_{ij} and r_{ij}^G, respectively. Here, we denote $(w_{ij})_{3\times4}$ by w, $(w_{ij}^G)_{3\times4}$ by w^G, then

$$w = \begin{pmatrix} 0.0671 & 0.1937 & 0.335 & 0.4042 \\ 0.062 & 0.1843 & 0.319 & 0.4347 \\ 0.05 & 0.1848 & 0.3134 & 0.4517 \end{pmatrix},$$

$$w^G = \begin{pmatrix} 0.0672 & 0.1911 & 0.3345 & 0.4072 \\ 0.06 & 0.1736 & 0.3173 & 0.4491 \\ 0.0483 & 0.184 & 0.3118 & 0.4558 \end{pmatrix}.$$

Step II-9 For each solution $A_i(i = 1,2,3)$, use GIVDHFLPOWA operator to aggregate all evaluations $r_{ij}(i = 1,2,3; j = 1,2,3,4)$ into overall evaluation values $r_i(i = 1,2,3)$, or use GIVDHFLPOWGA operator to aggregate all evaluations r_{ij}^G into overall evaluation values r_i^G. Then we have

$r_1 = (s_{5.7855}$, {[0.3309, 0.5126], [0.3435, 0.5229], [0.3348, 0.5156], [0.3472, 0.5258], [0.3319, 0.5135], [0.3444, 0.5238], [0.3358, 0.5165], [0.3481, 0.5266]}, {[0.2021, 0.3463], [0.2619, 0.3879], [0.2069, 0.3463], [0.2683, 0.3879], [0.2108, 0.3571], [0.2736, 0.4005], [0.2158, 0.3571], [0.2805, 0.4005], [0.2121, 0.3567], [0.2754, 0.4], [0.2172, 0.3567], [0.2823, 0.4], [0.2213, 0.3679], [0.2879, 0.4132], [0.2266, 0.3679], [0.2952, 0.4132], [0.2051, 0.3494], [0.2659, 0.3915], [0.21, 0.3494], [0.2725, 0.3915], [0.2139, 0.3604], [0.2779, 0.4043], [0.219, 0.3604], [0.2848, 0.4043], [0.2153, 0.3599], [0.2797, 0.4038], [0.2204, 0.3599], [0.2867, 0.4038], [0.2246, 0.3713], [0.2925, 0.4172], [0.23, 0.3713], [0.2999, 0.4172], [0.2053, 0.3506], [0.2662, 0.3929], [0.2102, 0.3506], [0.2728, 0.3929], [0.2141, 0.3616], [0.2782, 0.4058], [0.2193, 0.3616], [0.2851, 0.4058], [0.2155, 0.3611], [0.28, 0.4052], [0.2206, 0.3611], [0.287, 0.4052], [0.2248, 0.3726], [0.2927, 0.4187], [0.2302, 0.3726], [0.3002, 0.4187], [0.2084, 0.3538], [0.2703, 0.3966], [0.2133, 0.3538], [0.277, 0.3966], [0.2173, 0.3649], [0.2825, 0.4096], [0.2225, 0.3649], [0.2896, 0.4096], [0.2187, 0.3645], [0.2844, 0.4091], [0.2239, 0.3645], [0.2915, 0.4091], [0.2282, 0.376], [0.2974, 0.4227], [0.2337, 0.376], [0.305, 0.4227]}),

$r_2 = (s_{7.616}$, {[0.3784, 0.5378], [0.3885, 0.5428]}, {[0.2471, 0.3893], [0.2523, 0.3958], [0.2699, 0.4018], [0.2756, 0.4086]}),

$r_3 = (s_{5.413}$, {[0.4392, 0.5712], [0.4738, 0.6043], [0.4467, 0.578], [0.4805, 0.6104]}, {[0.1806, 0.3298], [0.2128, 0.347], [0.1868, 0.3367], [0.2204, 0.3544], [0.1852, 0.3349], [0.2184, 0.3525], [0.1916, 0.342], [0.2262, 0.36], [0.1818, 0.3315], [0.2143, 0.3488], [0.1881, 0.3385], [0.2219, 0.3563], [0.1864, 0.3366], [0.2199, 0.3543], [0.1929, 0.3438], [0.2277, 0.362]});

and

$r_1^G = (s_{4.3165}$, {[0.2196, 0.4062], [0.2204, 0.4074], [0.2217, 0.4093], [0.2225, 0.4106], [0.2234, 0.4107], [0.2243, 0.4119], [0.2255, 0.4139], [0.2264, 0.4152]}, {[0.3058, 0.4115], [0.333, 0.4532], [0.3456, 0.4507], [0.3692, 0.4876], [0.314, 0.4199], [0.3404, 0.4605], [0.3527, 0.4581], [0.3757, 0.4941], [0.3088, 0.415], [0.3357, 0.4563], [0.3482, 0.4538], [0.3716, 0.4904], [0.3169, 0.4233], [0.343, 0.4635], [0.3552, 0.4611], [0.378, 0.4968], [0.3067, 0.4126], [0.3338, 0.4542], [0.3464, 0.4517], [0.3699, 0.4885], [0.3149, 0.4209], [0.3412, 0.4614], [0.3535, 0.459], [0.3764, 0.4949], [0.3097, 0.4161], [0.3365, 0.4572], [0.349, 0.4548], [0.3723, 0.4912], [0.3178, 0.4244], [0.3438, 0.4644], [0.356, 0.462], [0.3787, 0.4976], [0.3159, 0.4212], [0.3421, 0.4617], [0.3543, 0.4593], [0.3772, 0.4952], [0.3238, 0.4293], [0.3493, 0.4688], [0.3612, 0.4664], [0.3835, 0.5015], [0.3188, 0.4246], [0.3447, 0.4646], [0.3568, 0.4623], [0.3795, 0.4978], [0.3266, 0.4326], [0.3518, 0.4717], [0.3636, 0.4693], [0.3858, 0.5041], [0.3168, 0.4222], [0.3429, 0.4626], [0.3551, 0.4602], [0.3779, 0.496], [0.3246, 0.4303], [0.35, 0.4696], [0.3619, 0.4673], [0.3842, 0.5023], [0.3196, 0.4257], [0.3455, 0.4656], [0.3576, 0.4632], [0.3802, 0.4986], [0.3274, 0.4336], [0.3526, 0.4726], [0.3644, 0.4702], [0.3865, 0.5049]}),

$r_2^G = (s_{6.3849}$, {[0.3168, 0.4642], [0.3202, 0.4659]}, {[0.3321, 0.4436], [0.3633, 0.474], [0.3446, 0.4536], [0.3745, 0.4831]}),

$r_3^G = (s_{4.504}$, {[0.3586, 0.5133], [0.3679, 0.5243], [0.3695, 0.5264], [0.3792, 0.5378]}, {[0.235, 0.3654], [0.2951, 0.398], [0.238, 0.3685], [0.2975, 0.4008], [0.2372, 0.3677], [0.2968, 0.4], [0.2401, 0.3707], [0.2991, 0.4027], [0.239, 0.3688], [0.2983, 0.4011], [0.242, 0.3718], [0.3005, 0.4038], [0.2412, 0.371], [0.2999, 0.403], [0.2441, 0.374], [0.3022, 0.4057]}).

Step II-10 Calculate scores $s(r_i)(i = 1,2,3)$ and $s(r_i^G)$ of the above overall interval-valued dual hesitant fuzzy linguistic values $r_i(i = 1,2,3)$ and r_i^G, respectively. Then we have

$s(r_1) = 0.6546, \quad s(r_2) = 1.0041, \quad s(r_3) = 1.3587;$
$s(r_1^G) = -0.3781, \quad s(r_2^G) = -0.1075, \quad s(r_3^G) = 0.539.$

Step II-11 Rank alternatives $A_i (i = 1, 2, 3)$ in accordance with the scores $s(r_i)(i = 1, 2, 3)$ or $s(r_i^G)(i = 1, 2, 3)$, we have

$A_3 \succ A_2 \succ A_1$ by GIVDHFLPOWA operator;
$A_3 \succ A_2 \succ A_1$ by GIVDHFLPGOWA operator.

Thus, A_3 is identified as the most desirable solution.

$$IVDHFLWA(sd_1, sd_2, \ldots, sd_n) = \overset{n}{\underset{j=1}{\otimes}} sd_j^{\omega_j} = \bigcup_{(s_{\alpha_j}, \tilde{h}_j, \tilde{g}_j) \in sd_j} \Bigg(s_{\prod_{j=1}^{n} \alpha_j^{\omega_j}}, \bigcup_{[\mu_j^L, \mu_j^U] \in \tilde{h}_j, [v_j^L, v_j^U] \in \tilde{g}_j} \left\{ \left\{ \left[1 - \prod_{j=1}^{n} (1 - \mu_j^L)^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \mu_j^U)^{\omega_j} \right] \right\}, \left\{ \left[\prod_{j=1}^{n} (v_j^L)^{\omega_j}, \prod_{j=1}^{n} (v_j^L)^{\omega_j} \right] \right\} \right\} \Bigg). \tag{65}$$

6.2 Comparative studies

In this section, we conduct comparative studies to demonstrate the feasibility and effectiveness of our proposed approach.

6.2.1 Comparison with conventional aggregation-operator-based MAGDM approach with known expert weights and attribute weights

Weighted averaging (WA) operator [31], as an effective aggregation operator, has been widely employed to construct conventional aggregation-operator-based models to accommodate decision making under different environments [53]. To compare with traditional aggregation-operator-based MAGDM approach, in the following, we firstly utilize WA operator to define the interval-valued dual hesitant fuzzy linguistic weighted averaging (IVDHFLWA) operator. Based on IVDHFLWA, we then construct an aggregation-operator-based MAGDM approach as detailed in approach III. Subsequently, we apply approach III to solve the same case in Sect. 6.1 with known attribute weights and expert weights.

Definition 6.1 For a collection of IVDHFLNs $sd_j (j = 1, 2, \ldots, n)$, an interval-valued dual hesitant fuzzy linguistic weighted averaging (IVDHFLWA) operator is a mapping $S^n \to S$, where

Suppose $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the known attribute weighting vector, $\eta = (\eta_1, \eta_2, \ldots, \eta_t)$ be the known expert weighting vector. $R^k = (r_{ij}^k)_{n \times m} (k = 1, 2, \ldots, t)$ denotes the decision matrices given by decision maker d_k for alternative A_i with respect to attribute G_j, $r_{ij}^k = (s_{\alpha_{ij}}^k, \tilde{h}_{ij}^k, \tilde{g}_{ij}^k)$ takes the form of IVDHFLNs. Then, based on IVDHFLWA operator, we here put forward the following Approach III to support MAGDM under interval-valued dual hesitant fuzzy linguistic environments.

6.2.1.1 Approach III: MAGDM based on IVDHFLWA operator *Step III-1* See step I-1.
Step III-2 Obtain group decision matrix.

Aggregate all individual decision matrices $R^k = (\overset{\frown}{r}{}_{ij}^k)_{n \times m} (k = 1, 2, \ldots, t)$ into group decision matrix $R = (r_{ij})_{n \times m}$ by use of IVDHFLWA operator, where

$$r_{ij} = IVDHFLWA(\overset{\frown}{r}{}_{ij}^1, \overset{\frown}{r}{}_{ij}^2, \ldots, \overset{\frown}{r}{}_{ij}^t) = \bigcup_{(\overset{\frown}{s}{}^k_{\alpha_{ij}}, \tilde{h}_{ij}^k, g_{ij}^k) \in \overset{\frown}{r}{}_{ij}^k} \Bigg(\overset{\frown}{s}_{\sum_{k=1}^{t} \eta_{ij}^k \overset{\frown}{\alpha}{}_{ij}^k}, \bigcup_{[\mu_{ij}^{Lk}, \mu_{ij}^{Uk}] \in \tilde{h}_{ij}^k, [v_{ij}^{Lk}, v_{ij}^{Uk}] \in \tilde{g}_{ij}^k} \left(\left\{ \left[1 - \prod_{k=1}^{t} (1 - \mu_{ij}^{Lk})^{\eta_{ij}^k}, 1 - \prod_{k=1}^{t} (1 - \mu_{ij}^{Uk})^{\eta_{ij}^k} \right] \right\}, \left\{ \left[\prod_{k=1}^{t} (v_{ij}^{Lk})^{\eta_{ij}^k}, \prod_{k=1}^{t} (v_{ij}^{Uk})^{\eta_{ij}^k} \right] \right\} \right) \Bigg). \tag{66}$$

Step III-3 Calculate overall evaluation values of alternatives.

Aggregate all evaluation values $r_{ij}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ into overall evaluation values r_i corresponding to each alternative A_i by use of IVDHFLWA operator, we have

$$r_i = IVDHFLWA(r_{i1}, r_{i2}, \ldots, r_{im}) = \bigcup_{(s_{\alpha_{ij}}, \tilde{h}_{ij}, g_{ij}) \in r_{ij}} \left(\widehat{s}_{\sum_{k=1}^{t} \eta_{ij}^{k} \overset{\frown}{\alpha}_{ij}^{k}}, \bigcup_{[\mu_{ij}^{L}, \mu_{ij}^{U}] \in \tilde{h}_{ij}, [v_{ij}^{L}, v_{ij}^{U}] \in \tilde{g}_{ij}} \right.$$
$$\left. \left(\left\{ \left[1 - \prod_{j=1}^{m} (1 - \mu_{ij}^{L})^{\omega_j}, 1 - \prod_{j=1}^{m} (1 - \mu_{ij}^{U})^{\omega_j} \right] \right\}, \left\{ \left[\prod_{j=1}^{m} (v_{ij}^{L})^{\omega_j}, \prod_{j=1}^{m} (v_{ij}^{U})^{\omega_j} \right] \right\} \right) \right). \tag{67}$$

Step III-4 Calculate scores of alternatives.

Calculate scores $s(r_i)(i = 1, 2, \ldots, n)$ of the above overall interval-valued dual hesitant fuzzy linguistic values $r_i(i = 1, 2, \ldots, n)$ of alternatives A_i by use of Definition 3.5.

Step III-5 Rank all alternatives $A_i(i = 1, 2, \ldots, n)$ in accordance with the ranking of r_i, then and select the most appropriate one(s).

After applying approach III to solve the case in Sect. 6.1, we obtain scores of the solutions as: $s(r_1) = 0.8252, s(r_2) = 1.4463$ and $s(r_3) = 1.9927$. Thus ranking order of the solutions can be derived as $A_3 \succ A_2 \succ A_1$. As can be seen, both approach I and approach III can significantly differentiate the three solutions and identify A_3 as the best option. Although the derived ranking orders are the same, there are differences between the two comparing approaches. Approach III is a conventional decision approach based on the assumption that attributes and preferences of decision makers are independent between one another. While for real decision making problems, there is always some kind of correlations among attributes or argument values [13, 21, 34]. When the arguments being aggregated support or reinforce each other [47], approach I, based on power average operator, can capture the correlations among arguments so as to provide more versatility in the information aggregation process. And by considering the correlations, the proposed approach I can utilize decision information more adequately in supporting MAGDM with mutually supportive arguments.

6.2.2 *Comparison with conventional TOPSIS-based MAGDM approach with unknown expert weights and attribute weights*

In this subsection, we focus on comparative studies in the more complex situations where attribute weights and expert weights cannot be determined in advance. Here we adopt the TOPSIS [11] method for comparison, because of its straight mathematical calculation but robustness in supporting multiple attribute decision making. We here firstly put forward the following Approach IV by extending conventional TOPSIS to accommodate group decision making under interval-valued dual hesitant fuzzy linguistic environments. Then we apply Approach IV to solve the same case in Sect. 6.1 when attribute weights and expert weights are totally unknown.

6.2.2.1 Approach IV: TOPSIS-based MAGDM with interval-valued dual hesitant fuzzy linguistic information *Step IV-1* See step I-1.

Step IV-2 Obtain group decision matrix.

In conventional TOPSIS-based methods for group decision making, expert weights are treated as undifferentiated when expert weights are unknown. Therefore, expert weights are configured as equal to $1/t$, where t is the number of experts. Then we can aggregate all individual decision matrices $R^k = (\overset{\frown}{r}_{ij}^{k})_{n \times m}(k = 1, 2, \ldots, t)$ into group decision matrix $R = (r_{ij})_{n \times m}$ by IVDHFLA operator, where

$$r_{ij} = IVDHFLA(\widehat{r}_{ij}^{1}, \widehat{r}_{ij}^{2}, \ldots, \widehat{r}_{ij}^{t}) = \bigcup_{(\widehat{s}_{\widehat{\alpha}_{ij}^{k}}, \tilde{h}_{ij}^{k}, g_{ij}^{k}) \in \widehat{r}_{ij}^{k}} \left(\widehat{s}_{\frac{1}{t}\sum_{k=1}^{t} \widehat{\alpha}_{ij}^{k}}, \bigcup_{[\mu_{ij}^{Lk}, \mu_{ij}^{Uk}] \in \tilde{h}_{ij}^{k}, [v_{ij}^{Lk}, v_{ij}^{Uk}] \in \tilde{g}_{ij}^{k}} \left(\left\{ \left[1 - \prod_{k=1}^{t} (1 - \mu_{ij}^{Lk})^{\frac{1}{t}}, 1 - \prod_{k=1}^{t} (1 - \mu_{ij}^{Uk})^{\frac{1}{t}} \right] \right\}, \left\{ \left[\prod_{k=1}^{t} (v_{ij}^{Lk})^{\frac{1}{t}}, \prod_{k=1}^{t} (v_{ij}^{Uk})^{\frac{1}{t}} \right] \right\} \right) \right). \tag{68}$$

Step IV-3 Calculate the separating measure from positive and negative ideal solutions.

Determine positive ideal solution (PIS) $X^{+} = (r_1^{+}, r_2^{+}, \ldots, r_i^{+}, \ldots, r_n^{+})$ and negative ideal solution (NIS) $X^{-} = (r_1^{-}, r_2^{-}, \ldots, r_i^{-}, \ldots, r_n^{-})$, where

$r_i^{+} = (\{1\}, \{0\}), r_i^{-} = (\{0\}, \{1\})$.

Then we can calculate the separating measure from the PIS and NIS for each alternative according to the distance measure introduced in Definition 3.6:

$$d_i^{+} = \sum_{j=1}^{m} d(r_{ij}, r_i^{+}), \quad d_i^{-} = \sum_{j=1}^{m} d(r_{ij}, r_i^{-}). \tag{69}$$

Step IV-4 Calculate the relative closeness to the ideal solution.

The relative closeness of the alternative A_i with respect to X^{+} is defined as c_i:

$$c_i = \frac{d_i^{-}}{d_i^{-} + d_i^{+}}. \tag{70}$$

Step IV-5 Ranking the alternatives according to the descending order of c_i.

After applying approach IV to solve the case in Sect. 6.1, we obtain the relative closeness of all three alternatives as: $c_1 = 0.5562, c_2 = 0.6274$ and $c_3 = 0.6690$. Thus the ranking order can be derived as $A_3 \succ A_2 \succ A_1$. Although approach IV can identify the same final ranking result as that by approach II, assumption for undifferentiated experts and neglect of attribute weights obviously make approach IV lack of exploiting decision information. While the proposed approach II can objectively determine expert weights and attribute weights reasonably by modelling supportive relationship among arguments. Therefore, when resolving MAGDM problems in which arguments are mutually supportive and weighing information (i.e., expert weights and attribute weights) is totally unknown, the proposed Approach II can derive decision results more reasonably and objectively.

6.2.3 *Comparative study with approach based on dual hesitant fuzzy linguistic set*

In order to further inspect feasibility and effectiveness of proposed group decision making model, this subsection presents comparative study with decision making approach proposed in Yang and Ju [62] most recently under dual hesitant fuzzy linguistic decision situations.

We take the same case problem used in Yang and Ju [62] for discussion. That is, suppose that an investment company wants to invest money in the most appropriate project. There is a panel with four alternative companies: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must make a decision according to the following three attributes: (1) C_1 is the risk factor; (2) C_2 is the growth factor; (3) C_3 is the environmental impact factor.

In Yang and Ju [62], they defined this investment problem as a single-person decision making problem with dual hesitant fuzzy linguistic information. $\omega = (0.35, 0.25, 0.40)^{T}$ is the known weighting vector for attributes. Here for comparison, we firstly transform original dual hesitant fuzzy linguistic decision matrix in [62] into interval-valued dual hesitant fuzzy linguistic preference information. The crisp membership degrees and nonmembership degrees are mapped into interval values with equal upper and lower limits (e.g., 0.2 turns out to be [0.2, 0.2]). Then, we can obtain the interval-valued dual hesitant fuzzy linguistic decision matrix as shown in Table 13. Since attribute weighing vector is known, approach I can be applied to resolve this investment decision problem.

After similar computing steps of approach I detailed in Sect. 6.1, ranking order of solutions obtained by WGIVDHFLPA operator is $A_4 \succ A_1 \succ A_3 \succ A_2$ and ranking order by WGIVDHFLPGA operator is $A_4 \succ A_3 \succ A_1 \succ A_2$, which indicates the same best solution (A_4) and the worst solution (A_2) as those in the ranking order $A_4 \succ A_1 \succ A_3 \succ A_2$ reported in Yang and Ju [62]. Difference in ranking order of A_1 and A_3 can be ascribed to that the WGIVDHFLPA and WGIVDHFLPGA additionally consider support degrees among arguments being aggregated.

Table 13 Interval-valued dual hesitant fuzzy linguistic decision matrix

	C_1	C_2	C_3
A_1	$(s_3, \{[0.4, 0.4], [0.5, 0.5], [0.6, 0.6]\}, \{[0.3, 0.3], [0.4, 0.4]\})$	$(s_4, \{[0.3, 0.3], [0.5, 0.5]\}, \{[0.2, 0.2], [0.3, 0.3]\})$	$(s_4, \{[0.3, 0.3], [0.5, 0.5], [0.6, 0.6]\}, \{[0.1, 0.1], [0.2, 0.2]\})$
A_2	$(s_2, \{[0.4, 0.4], [0.5, 0.5]\}, \{[0.3, 0.3], [0.4, 0.4]\})$	$(s_5, \{[0.4, 0.4], [0.5, 0.5]\}, \{[0.4, 0.4], [0.5, 0.5]\})$	$(s_3, \{[0.2, 0.2], [0.5, 0.5], [0.6, 0.6]\}, \{[0.2, 0.2], [0.4, 0.4]\})$
A_3	$(s_4, \{[0.5,0.5], [0.7, 0.7]\}, \{[0.2, 0.2], [0.3, 0.3]\})$	$(s_3, \{[0.2, 0.2], [0.4, 0.4], [0.5, 0.5]\}, \{[0.3, 0.3], [0.4, 0.4]\})$	$(s_3, \{[0.4, 0.4], [0.5, 0.5], [0.7, 0.7]\}, \{[0.2, 0.2], [0.3, 0.3]\})$
A_4	$(s_4, \{[0.4, 0.4], [0.6, 0.6], [0.8, 0.8]\}, \{[0.1, 0.1], [0.2, 0.2]\})$	$(s_2, \{[0.5, 0.5], [0.6, 0.6]\}, \{[0.2, 0.2], [0.4, 0.4]\})$	$(s_5, \{[0.6, 0.6], [0.7, 0.7]\}, \{[0.1, 0.1], [0.3, 0.3]\})$

In general, when coping with practical decision making problems of high uncertainty, it is sometimes difficult for decision makers to provide their assessments using crisp numbers, while they are more willing to indicate their preference with certain linguistic term set, additionally, they may also have difficulty and hesitancy in giving crisp membership and nonmembership degrees to a linguistic term. Under these decision environments, the developed fuzzy tool of interval-valued dual hesitant fuzzy linguistic set (IVDHFLS) is suitable for decision makers to convey their preference information more comprehensively, and the generalized power aggregation operators for IVDHFLS manage to consider supportive correlations among arguments. Therefore, the proposed approaches I and II, based on IVDHFLS and its power aggregation operators, can serve as effective models for MAGDM with mutually supportive arguments being aggregated.

7 Conclusions

Based on a newly defined hybrid fuzzy tool of interval-valued dual hesitant fuzzy linguistic set (IVDHFLS), we have developed two effective approaches for MAGDM allowing arguments being aggregated to support each other. IVDHFLS manages to attain the flexibility of interval numbers in assigning membership and nonmembership degrees, as well as the advantages of linguistic variable and dual hesitant fuzzy set in depicting fuzzy properties of evaluated objects. For IVDHFLS, we have put forward a novel distance measure to overcome irrationality in traditional methodology, i.e. artificially adding values to mismatching membership or nonmembership degrees. Then to consider supportive correlations among arguments being aggregated, we have developed fundamental generalized power aggregation operators, including WGIVDHFLPA, WGIVDHFLPGA, GIVDHFLPA, GIVDHFLPGA, GIVDHFLPOWA and GIVDHFLPOWGA. Their desirable properties, i.e. commutativity, idempotency, boundedness and monotonicity, have been inspected. Subsequently, two approaches for MAGDM under interval-valued dual hesitant fuzzy linguistic environments have been constructed and verified by case studies.

In future research, empirical studies can be carried out on applying the developed approaches to different areas, such as supply chain management, operations management, and intelligent interactive systems. Decision support system should be further developed to facilitate resolving practical problems. When different types of correlations exist among attributes in complex MAGDM, suitable aggregation operators should be studied for IVDHFLS, such as operators based on Choquet integral. Other methods, such as social network analysis, could also be integrated to tackle situations of high complexity.

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Appendix 1: Proof of theorem 3.1

Obviously, rules (1) and (2) are correct. For rule (3), we have

$$\lambda(sd_1 \oplus sd_2) = \lambda \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{\alpha_1+\alpha_2}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left.\{\{[\mu_1^L + \mu_2^L - \mu_1^L\mu_2^L, \mu_1^U + \mu_2^U - \mu_1^U\mu_2^U]\}, \{[v_1^L v_2^L, v_1^U v_2^U]\}\}\right)$$

$$= \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{\lambda(\alpha_1+\alpha_2)}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left.\{\{[1-(1-(\mu_1^L+\mu_2^L-\mu_1^L\mu_2^L))^\lambda, 1-(1-(\mu_1^U+\mu_2^U-\mu_1^U\mu_2^U))^\lambda]\}, \{[(v_1^L v_2^L)^\lambda, (v_1^U v_2^U)^\lambda]\}\}\right\rangle\Big\},$$

$$\lambda sd_1 = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1} \left(s_{\lambda\alpha_1}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[v_1^L,v_1^U]\in\tilde{g}_1} \{\{[1-(1-\mu_1^L)^\lambda, 1-(1-\mu_1^U)^\lambda]\}, \{[(v_1^L)^\lambda, (v_1^U)^\lambda]\}\}\right)$$

$$\lambda sd_2 = \bigcup_{(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{\lambda\alpha_2}, \bigcup_{[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_2^L,v_2^U]\in\tilde{g}_2} \{\{[1-(1-\mu_2^L)^\lambda, 1-(1-\mu_2^U)^\lambda]\}, \{[(v_2^L)^\lambda, (v_2^U)^\lambda]\}\}\right)$$

$$\lambda sd_1 \oplus \lambda sd_2 = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{\lambda(\alpha_1+\alpha_2)}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left.\{\{[1-(1-(\mu_1^L+\mu_2^L-\mu_1^L\mu_2^L))^\lambda, 1-(1-(\mu_1^U+\mu_2^U-\mu_1^U\mu_2^U))^\lambda]\}, \{[(v_1^L v_2^L)^\lambda, (v_1^U v_2^U)^\lambda]\}\}\right)$$

$$= \lambda(sd_1 \oplus sd_2);$$

For rule (4),

$$sd_1^\lambda = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd} \left(s_{\alpha_1^\lambda}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[v_1^L,v_1^U]\in\tilde{g}_1} \{\{[(\mu_1^L)^\lambda, (\mu_1^U)^\lambda]\}, \{[1-(1-v_1^L)^\lambda, 1-(1-v_1^U)^\lambda]\}\}\right),$$

$$sd_2^\lambda = \bigcup_{(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{\alpha_2^\lambda}, \bigcup_{[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_2^L,v_2^U]\in\tilde{g}_2} \{\{[(\mu_2^L)^\lambda, (\mu_2^U)^\lambda]\}, \{[1-(1-v_2^L)^\lambda, 1-(1-v_2^U)^\lambda]\}\}\right),$$

$$sd_1 \otimes sd_2 = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{\alpha_1\alpha_2}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left.\{\{[\mu_1^L\mu_2^L, \mu_1^U\mu_2^U]\}, \{[v_1^L+v_2^L-v_1^L v_2^L, v_1^U+v_2^U-v_1^U v_2^U]\}\}\right),$$

$$sd_1^\lambda \otimes sd_2^\lambda = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2} \left(s_{(\alpha_1\alpha_2)^\lambda}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left.\{\{[(\mu_1^L\mu_2^L)^\lambda, (\mu_1^U\mu_2^U)^\lambda]\}, \{[1-(1-(v_1^L+v_2^L-v_1^L v_2^L))^\lambda, 1-(1-(v_1^U+v_2^U-v_1^U v_2^U))^\lambda]\}\}\right)$$

$$= (sd_1 \otimes sd_2)^\lambda;$$

For rule (5),

$$\lambda_1 sd = \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_{\lambda_1\alpha}, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[1-(1-\mu^L)^{\lambda_1}, 1-(1-\mu^U)^{\lambda_1}]\}, \{[(v^L)^{\lambda_1}, (v^U)^{\lambda_1}]\}\}\right),$$

$$\lambda_2 sd = \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_{\lambda_2\alpha}, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[1-(1-\mu^L)^{\lambda_2}, 1-(1-\mu^U)^{\lambda_2}]\}, \{[(v^L)^{\lambda_2}, (v^U)^{\lambda_2}]\}\}\right), \lambda_1 sd + \lambda_2 sd$$

$$= \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_{(\lambda_1+\lambda_2)\alpha}, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[1-(1-\mu^L)^{\lambda_1+\lambda_2}, 1-(1-\mu^U)^{\lambda_1+\lambda_2}]\}, \{[(v^L)^{\lambda_1+\lambda_2}, (v^U)^{\lambda_1+\lambda_2}]\}\}\right)$$

$$= (\lambda_1+\lambda_2)sd;$$

For rule (6)

$$sd^{\lambda_1} = \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_{\alpha^{\lambda_1}}, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[(\mu^L)^{\lambda_1}, (\mu^U)^{\lambda_1}]\}, \{[1-(1-v^L)^{\lambda_1}, 1-(1-v^U)^{\lambda_1}]\}\}\right),$$

$$sd^{\lambda_2} = \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_{\alpha^{\lambda_2}}, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[(\mu^L)^{\lambda_2}, (\mu^U)^{\lambda_2}]\}, \{[1-(1-v^L)^{\lambda_2}, 1-(1-v^U)^{\lambda_2}]\}\}\right),$$

$$sd^{\lambda_1}\otimes sd^{\lambda_2} = \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_{\alpha^{\lambda_1+\lambda_2}},\right.$$

$$\left.\bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[(\mu^L)^{\lambda_1+\lambda_2}, (\mu^U)^{\lambda_1+\lambda_2}]\}, \{[1-(1-v^L)^{\lambda_1+\lambda_2}, 1-(1-v^U)^{\lambda_1+\lambda_2}]\}\}\right) = sd^{\lambda_1+\lambda_2}$$

□

Appendix 2: Proof of Theorem 4.1

1. When $n = 1$, obviously, it is right.

$$WGIVDHFLPA(sd) = \left(s_\alpha, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[\mu_j^L, \mu_j^U]\}, \{[v_j^L, v_j^U]\}\}\right).$$

2. When $n = 2$,

$$sd_1^{\lambda} = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1}\left(s_{\alpha_1^{\lambda}}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[v_1^L,v_1^U]\in\tilde{g}_1}\left\{\left\{\left[(\mu_1^L)^{\lambda},(\mu_1^U)^{\lambda}\right]\right\},\left\{\left[1-(1-v_1^L)^{\lambda},1-(1-v_1^U)^{\lambda}\right]\right\}\right\}\right),$$

$$\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}sd_1^{\lambda} = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1}\left(s_{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}\alpha_1^{\lambda}},\right.$$

$$\bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[v_1^L,v_1^U]\in\tilde{g}_1}\left\{\left\{\left[1-(1-(\mu_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},1-(1-(\mu_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\},\right.$$

$$\left.\left.\left\{\left[(1-(1-v_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},(1-(1-v_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

$$sd_2^{\lambda} = \bigcup_{(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2}\left(s_{\alpha_2^{\lambda}}, \bigcup_{[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_2^L,v_2^U]\in\tilde{g}_2}\left\{\left\{\left[(\mu_2^L)^{\lambda},(\mu_2^U)^{\lambda}\right]\right\},\left\{\left[1-(1-v_2^L)^{\lambda},1-(1-v_2^U)^{\lambda}\right]\right\}\right\}\right),$$

$$\frac{\omega_2(1+T(sd_2)}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}sd_2^{\lambda} = \bigcup_{(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2}\left(s_{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}\alpha_2^{\lambda}},\right.$$

$$\bigcup_{[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_2^L,v_2^U]\in\tilde{g}_2}\left\{\left\{\left[1-(1-(\mu_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},1-(1-(\mu_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\},\right.$$

$$\left.\left.\left\{\left[(1-(1-v_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},(1-(1-v_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

$$\frac{\omega_1(1+T(sd_1)}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}sd_1^{\lambda}+\frac{\omega_2(1+T(sd_2)}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}sd_2^{\lambda} = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2}\left(s_{\sum_{j=1}^{2}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}(\alpha_j)^{\lambda}},\right.$$

$$\bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\left\{\left\{\left[1-(1-(\mu_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(\mu_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},\right.\right.\right.$$

$$\left.\left.1-(1-(\mu_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(\mu_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\},$$

$$\left\{\left[(1-(1-v_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(1-v_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},\right.\right.$$

$$\left.\left.\left.\left.(1-(1-v_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(1-v_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

$$\left(\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}sd_1^{\lambda}+\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}sd_2^{\lambda}\right)^{1/\lambda} = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\left(\sum_{j=1}^{2}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}(\alpha_j)^{\lambda}\right)^{1/\lambda}},\right.$$

$$\cup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j}\left\{\left\{\left[\left(1-\prod_{j=1}^{2}(1-(\mu_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda},\left(1-\prod_{j=1}^{2}(1-(\mu_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right.$$

$$\left.\left.\left\{\left[1-\left(1-\prod_{j=1}^{2}(1-(1-v_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{2}(1-(1-v_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right).$$

So when $n = 2$, Theorem 4.1 also is right.

3. Suppose when $n = k$, Theorem 4.1 is right, then we have

$$WGIVDHFLPA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_k) = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\left(\sum_{j=1}^{k}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}(\alpha_j)^{\lambda}\right)^{1/\lambda}},\right.$$
$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j}\left\{\left\{\left[\left(1-\prod_{j=1}^{k}(1-(\mu_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{k}(1-(\mu_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right.$$
$$\left.\left.\left\{\left[1-\left(1-\prod_{j=1}^{k}(1-(1-v_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{k}(1-(1-v_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right),$$

$$\overset{k}{\underset{j=1}{\oplus}}\omega_j sd_j^{\lambda} = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\sum_{j=1}^{k}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{k}\omega_i(1+T(sd_i))}(\alpha_j)^{\lambda}},\right.$$
$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j}\left\{\left\{\left[1-\prod_{j=1}^{k}(1-(\mu_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}, 1-\prod_{j=1}^{k}(1-(\mu_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right]\right\},\right.$$
$$\left.\left.\left\{\left[\prod_{j=1}^{k}(1-(1-v_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}, \prod_{j=1}^{k}(1-(1-v_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right).$$

Then when $n = k + 1$,

$$WGIVDHFLPA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_{k+1})$$
$$= \left(\left(\overset{k}{\underset{j=1}{\oplus}}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}sd_j^{\lambda}\right)\oplus\frac{\omega_{k+1}(1+T(sd_{k+1}))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}sd_{k+1}^{\lambda}\right)^{1/\lambda}$$
$$= \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\left(\sum_{j=1}^{k+1}\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}(\alpha_j)^{\lambda}\right)^{1/\lambda}},\right.$$
$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j}\left\{\left\{\left[\left(1-\prod_{j=1}^{k+1}(1-(\mu_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{k+1}(1-(\mu_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right.$$
$$\left.\left.\left\{\left[1-\left(1-\prod_{j=1}^{k+1}(1-(1-v_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{k+1}(1-(1-v_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right).$$

So, when $n = k + 1$, Theorem 4.1 is right too.

According to steps (1), (2), (3), we can conclude that Theorem 4.1 is right for all n. □

Appendix 3: Proof of Theorem 4.4

1. When $n = 1$, obviously, it is right.
2. When $n = 2$,

$$(\lambda sd_1)^{\frac{(\omega_1(1+T(sd_1)))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}} = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1}\left(s_{(\lambda\alpha_1)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}},\right.$$

$$\bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[v_1^L,v_1^U]\in\tilde{g}_1}\left\{\left\{\left[(1-(1-\mu_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},(1-(1-\mu_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\},\right.$$

$$\left.\left.\left\{\left[1-(1-(v_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},1-(1-(v_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

$$(\lambda sd_2)^{\frac{(\omega_2(1+T(sd_2)))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}} = \bigcup_{(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2}\left(s_{(\lambda\alpha_2)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}},\right.$$

$$\bigcup_{[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_2^L,v_2^U]\in\tilde{g}_2}\left\{\left\{\left[(1-(1-\mu_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},(1-(1-\mu_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\},\right.$$

$$\left.\left.\left\{\left[1-(1-(v_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},1-(1-(v_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

$$(\lambda sd_1)^{\omega_1}\otimes(\lambda sd_2)^{\omega_2} = \bigcup_{(s_{\alpha_1},\tilde{h}_1,\tilde{g}_1)\in sd_1,(s_{\alpha_2},\tilde{h}_2,\tilde{g}_2)\in sd_2}$$

$$\left(s_{(\lambda\alpha_1)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(\lambda\alpha_2)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}},\bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left\{\left\{\left[(1-(1-\mu_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(1-\mu_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},(1-(1-\mu_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(1-\mu_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\},\right.$$

$$\left.\left.\left\{\left[1-(1-(v_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(v_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}},1-(1-(v_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(v_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

$$\frac{1}{\lambda}((\lambda sd_1)^{\omega_1}\otimes(\lambda sd_2)^{\omega_2}) = \bigcup_{\langle\tilde{s}_{\vartheta_1},\tilde{h}_1,\tilde{g}_1\rangle\in sd_1,\langle\tilde{s}_{\vartheta_2},\tilde{h}_2,\tilde{g}_2\rangle\in sd_2}$$

$$\left(s_{\frac{1}{\lambda}\left((\lambda\alpha_1)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(\lambda\alpha_2)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right)},\bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2}\right.$$

$$\left\{\left\{\left[1-\left(1-(1-(1-\mu_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(1-\mu_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}},\right.\right.\right.$$

$$\left.\left.1-\left(1-(1-(1-\mu_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(1-\mu_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}}\right]\right\},$$

$$\left\{\left[\left(1-(1-(v_1^L)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(v_2^L)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}},\right.\right.$$

$$\left.\left.\left.\left.\left(1-(1-(v_1^U)^{\lambda})^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}(1-(v_2^U)^{\lambda})^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^{2}\omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}}\right]\right\}\right\}\right).$$

$$\frac{1}{\lambda}((\lambda sd_1)^{\omega_1} \otimes (\lambda sd_2)^{\omega_2})$$

$$= \bigcup_{\langle \tilde{s}_{\vartheta_1}, \tilde{h}_1, \tilde{g}_1 \rangle \in sd_1, \langle \tilde{s}_{\vartheta_2}, \tilde{h}_2, \tilde{g}_2 \rangle \in sd_2} \left(s_{\frac{1}{\lambda}\left((\lambda\alpha_1)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}(\lambda\alpha_2)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)}, \bigcup_{[\mu_1^L,\mu_1^U]\in\tilde{h}_1,[\mu_2^L,\mu_2^U]\in\tilde{h}_2,[v_1^L,v_1^U]\in\tilde{g}_1,[v_2^L,v_2^U]\in\tilde{g}_2} \right.$$

$$\left\{\left\{\left[1-\left(1-(1-(1-\mu_1^L)^\lambda)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}(1-(1-\mu_2^L)^\lambda)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}},\right.\right.\right.$$

$$\left.\left.1-\left(1-(1-(1-\mu_1^U)^\lambda)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}(1-(1-\mu_2^U)^\lambda)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}}\right]\right\},$$

$$\left\{\left[\left(1-(1-(v_1^L)^\lambda)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}(1-(v_2^L)^\lambda)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}},\right.\right.$$

$$\left.\left.\left.\left.\left(1-(1-(v_1^U)^\lambda)^{\frac{\omega_1(1+T(sd_1))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}(1-(v_2^U)^\lambda)^{\frac{\omega_2(1+T(sd_2))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{\frac{1}{\lambda}}\right]\right\}\right\}\right).$$

So when $n = 2$, Theorem 4.4 also is right.

3. Suppose when $n = k$, Theorem 4.4 is right, then we have

$$WGIVDHFLPGA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_k) = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}$$

$$\left(s_{\frac{1}{\lambda}\prod_{j=1}^k (\lambda\alpha_j)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}}, \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j} \right.$$

$$\left\{\left\{\left[1-\left(1-\prod_{j=1}^k(1-(1-\mu_j^L)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^k(1-(1-\mu_j^U)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right.$$

$$\left.\left.\left\{\left[\left(1-\prod_{j=1}^k(1-(v_j^L)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{1/\lambda}, \left(1-\prod_{j=1}^k(1-(v_j^U)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right),$$

$$\overset{k}{\underset{j=1}{\otimes}}(\lambda sd_j)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}} = \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j} \left(s_{\prod_{j=1}^k (\lambda\alpha_j)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}}, \right.$$

$$\bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j} \left\{\left\{\left[\prod_{j=1}^k(1-(1-\mu_j^L)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}, \prod_{j=1}^k(1-(1-\mu_j^U)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right]\right\},\right.$$

$$\left.\left.\left\{\left[1-\prod_{j=1}^k(1-(v_j^L)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}, 1-\prod_{j=1}^k(1-(v_j^U)^\lambda)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^2 \omega_i(1+T(sd_i))}}\right]\right\}\right\}\right),$$

then when $n = k + 1$,

$$WGIVDHFLPGA_{\omega,\lambda}(sd_1, sd_2, \ldots, sd_{k+1}) = \frac{1}{\lambda}\left(\left(\overset{k}{\underset{j=1}{\otimes}}(\lambda sd_j)^{\omega_j}\right) \otimes (\lambda sd_{k+1})^{\omega_{k+1}}\right)$$

$$= \bigcup_{(s_{\alpha_j},\tilde{h}_j,\tilde{g}_j)\in sd_j}\left(s_{\frac{1}{\lambda}\prod_{j=1}^{n}(\lambda\alpha_j)^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}}, \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j,[v_j^L,v_j^U]\in\tilde{g}_j}\right.$$

$$\left\{\left\{\left[1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\},\right.$$

$$\left.\left.\left\{\left[\left(1-\prod_{j=1}^{n}(1-(v_j^L)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}(1-(v_j^U)^{\lambda})^{\frac{\omega_j(1+T(sd_j))}{\sum_{i=1}^{n}\omega_i(1+T(sd_i))}}\right)^{1/\lambda}\right]\right\}\right\}\right).$$

So, when $n = k + 1$, Theorem 4.4 is right too.

According to steps (1), (2), (3), we can conclude that Theorem 4.4 is right for all n. □

Appendix 4: Proof of Theorem 4.7

1. Assume that $(sd_1^*, sd_2^*, \ldots, sd_n^*)$ is any permutation of $(sd_1, sd_2, \ldots, sd_n)$, then for each sd_j, there exists one and only one sd_k^* such that $sd_j = sd_k^*$ and vice versa. And, $T(sd_j) = T(sd_k^*)$. Thus, based on Theorems 4.2 and 4.5, we have

$$GIVDHFLPA_{\lambda}(sd_1, sd_2, \ldots, sd_n) = \left(\frac{\oplus_{j=1}^{n}(1+T(sd_j))sd_j^{\lambda}}{\sum_{i=1}^{n}(1+T(sd_i))}\right)^{1/\lambda} = \left(\frac{\oplus_{k=1}^{n}(1+T(sd_k^*))sd_k^{*\lambda}}{\sum_{i=1}^{n}(1+T(sd_i))}\right)^{1/\lambda}$$

$$= GIVDHFLPA_{\lambda}(sd_1^*, sd_2^*, \ldots, sd_n^*).$$

$$GIVDHFLPGA_{\lambda}(sd_1, sd_2, \ldots, sd_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{j=1}{\otimes}}(\lambda sd_j)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)$$

$$= \frac{1}{\lambda}\left(\overset{n}{\underset{k=1}{\otimes}}(\lambda sd_k^*)^{\frac{1+T(sd_k^*)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right) = GIVDHFLPGA_{\lambda}(sd_1^*, sd_2^*, \ldots, sd_n^*).$$

2. Since $sd_j = sd$ for all $j = 1, 2, \ldots, n$, thus

$$GIVDHFLPA_{\lambda}(sd_1, sd_2, \ldots, sd_n) = \bigcup_{(s_\alpha,\tilde{h},\tilde{g})\in sd}\left(s_\alpha, \bigcup_{[\mu^L,\mu^U]\in\tilde{h},[v^L,v^U]\in\tilde{g}}\{\{[\mu^L, \mu^U]\}, \{[v^L, v^U]\}\}\right)$$

$$= sd = GIVDHFLPGA_{\lambda}(sd_1, sd_2, \ldots, sd_n).$$

3. Let $sd^- = (s_\alpha^-, \tilde{h}^-, \tilde{g}^-)$, $sd^+ = \langle s_\alpha^+, \tilde{h}^+, \tilde{g}^+\rangle$, where $s_\alpha^- = \min_j(s_{\alpha_j})$, $s_\alpha^+ = \max_j(s_{\alpha_j})$,

$$\tilde{h}^- = \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j}\{[\mu^{L-}, \mu^{U-}]\} = \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j}\left\{\left[\min_{1\le j\le n}\mu_j^L, \min_{1\le j\le n}\mu_j^U\right]\right\},$$

$$\tilde{h}^+ = \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j}\{[\mu^{L+}, \mu^{U+}]\} = \bigcup_{[\mu_j^L,\mu_j^U]\in\tilde{h}_j}\left\{\left[\max_{1\le j\le n}\mu_j^L, \max_{1\le j\le n}\mu_j^U\right]\right\},$$

$$\tilde{g}^- = \bigcup_{[v_j^L,v_j^U]\in\tilde{g}_j}\{[v^{L-}, v^{U-}]\} = \bigcup_{[v_j^L,v_j^U]\in\tilde{g}_j}\left\{\left[\max_{1\le j\le n}v_j^L, \max_{1\le j\le n}v_j^U\right]\right\},$$

$$\tilde{g}^+ = \bigcup_{[v_j^L,v_j^U]\in\tilde{g}_j}\{[v^{L+}, v^{U+}]\} = \bigcup_{[v_j^L,v_j^U]\in\tilde{g}_j}\left\{\left[\min_{1\le j\le n}v_j^L, \min_{1\le j\le n}v_j^U\right]\right\},$$

for all $j = 1, 2, \ldots, n$, we have

$$\left(1-\prod_{j=1}^{n}(1-(\mu^{L+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu^{U+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$
$$\geq\left(1-\prod_{j=1}^{n}(1-(\mu_j^{L})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu_j^{U})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$
$$\geq\left(1-\prod_{j=1}^{n}(1-(\mu^{L-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu^{U-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda},$$

and mean while

$$\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{L-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)+\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{U-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)$$
$$\geq\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^{L})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)+\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^{U})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)$$
$$\geq\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{L+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)+\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{U+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right),$$

so we have

$$\left(1-\prod_{j=1}^{n}(1-(\mu_j^{L})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu_j^{U})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$
$$-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^{L})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^{U})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)$$
$$\leq\left(1-\prod_{j=1}^{n}(1-(\mu^{L+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu^{U+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$
$$-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{L+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{U+})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right).$$

And

$$\left(1-\prod_{j=1}^{n}(1-(\mu_j^{L})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu_j^{U})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$
$$-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^{L})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_j^{U})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)$$
$$\geq\left(1-\prod_{j=1}^{n}(1-(\mu^{L-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}+\left(1-\prod_{j=1}^{n}(1-(\mu^{U-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$
$$-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{L-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right)-\left(1-\left(1-\prod_{j=1}^{n}(1-(1-\nu^{U-})^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}\right).$$

Obviously, $max_j(s_{\alpha_j}) \geq s_{\left(\sum_{j=1}^{n}\omega_j(\alpha_j)^{\lambda}\right)^{1/\lambda}} \geq \min_j(s_{\alpha_j})$, then according to Definitions 3.3, 3.5 and Theorem 4.2, we have$sd^+ \geq GIVDHFLPA_{\lambda}(sd_1, sd_2, \ldots, sd_n) \geq sd^-$.

Similarly, according to Definitions 3.3, 3.5 and Theorem 4.5, we have

$sd^+ \geq GIVDHFLPGA_{\lambda}(sd_1, sd_2, \ldots, sd_n) \geq sd^-$,

which completes the proof. □

Appendix 5: Proof of Theorem 4.8

Based on Lemma 4.1, we have

$$\prod_{j=1}^{n}\alpha_j^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \leq \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}\alpha_j,$$

$$\prod_{j=1}^{n}(\mu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \leq \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}\mu_j^L = 1 - \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-\mu_j^L)$$
$$\leq 1 - \prod_{j=1}^{n}(1-\mu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},$$

$$\prod_{j=1}^{n}(\mu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \leq \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}\mu_j^U = 1 - \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-\mu_j^U)$$
$$\leq 1 - \prod_{j=1}^{n}(1-\mu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},$$

$$\prod_{j=1}^{n}(v_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \leq \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}v_j^L = 1 - \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-v_j^L)$$
$$\leq 1 - \prod_{j=1}^{n}(1-v_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},$$

$$\prod_{j=1}^{n}(v_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \leq \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}v_j^U = 1 - \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-v_j^U)$$
$$\leq 1 - \prod_{j=1}^{n}(1-v_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}.$$

By Definitions 3.3, 3.4 and 3.5, we have $\otimes_{j=1}^{n}\left(sd_j^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right) \leq \oplus_{j=1}^{n}\left(\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}sd_j\right)$, which completes the proof of Theorem 4.8. □

Appendix 6: Proof of Theorem 4.9

Based on Lemma 1, we have

$$\prod_{j=1}^{n} \alpha_j^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} = \left(\prod_{j=1}^{n} (\alpha_j^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \leq \left(\sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} \alpha_j^{\lambda}\right)^{1/\lambda},$$

$$\prod_{j=1}^{n} (\mu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} = \left(\prod_{j=1}^{n} ((\mu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$

$$\leq \left(\sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (\mu_j^L)^{\lambda}\right)^{1/\lambda} = \left(1 - \sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (1-(\mu_j^L)^{\lambda})\right)^{1/\lambda}$$

$$\leq \left(1 - \prod_{j=1}^{n} (1-(\mu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda},$$

$$\prod_{j=1}^{n} (\mu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} = \left(\prod_{j=1}^{n} ((\mu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \leq \left(\sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (\mu_j^U)^{\lambda}\right)^{1/\lambda}$$

$$= \left(1 - \sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (1-(\mu_j^U)^{\lambda})\right)^{1/\lambda} \leq \left(1 - \prod_{j=1}^{n} (1-(\mu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda},$$

$$1 - \left(1 - \prod_{j=1}^{n} (1-(1-\nu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \leq 1 - \left(1 - \sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (1-(1-\nu_j^L)^{\lambda})\right)^{1/\lambda}$$

$$= 1 - \left(\sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (1-\nu_j^L)^{\lambda}\right)^{1/\lambda} \leq 1 - \left(\prod_{j=1}^{n} (1-\nu_j^L)^{\frac{\lambda(1+T(sd_j))}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} = 1 - \prod_{j=1}^{n} (1-\nu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},$$

$$1 - \left(1 - \prod_{j=1}^{n} (1-(1-\nu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \leq 1 - \left(1 - \sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (1-(1-\nu_j^U)^{\lambda})\right)^{1/\lambda}$$

$$= 1 - \left(\sum_{j=1}^{n} \frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} (1-\nu_j^U)^{\lambda}\right)^{1/\lambda} \leq 1 - \left(\prod_{j=1}^{n} (1-\nu_j^U)^{\frac{\lambda(1+T(sd_j))}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} = 1 - \prod_{j=1}^{n} (1-\nu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}.$$

By Theorems 4.3 and 4.6, we have $\otimes_{j=1}^{n} (sd_j^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}) \leq \left(\oplus_{j=1}^{n} \left(\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))} sd_j^{\lambda}\right)\right)^{1/\lambda}$, which completes the proof of Theorem 4.9. □

Appendix 7: Proof of Theorem 4.10

$$\frac{1}{\lambda}\prod\nolimits_{j=1}^{n}(\lambda\alpha_j)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} \le \frac{1}{\lambda}\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(\lambda\alpha_j) = \sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}\alpha_j,$$

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \le 1-\left(1-\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-(1-\mu_j^L)^{\lambda})\right)^{1/\lambda}$$

$$=1-\left(\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-\mu_j^L)^{\lambda}\right)^{1/\lambda} \le 1-\left(\prod_{j=1}^{n}(1-\mu_j^L)^{\frac{\lambda(1+T(sd_j))}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}$$

$$=1-\prod_{j=1}^{n}(1-\mu_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},$$

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\mu_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \le 1-\left(1-\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-(1-\mu_j^U)^{\lambda})\right)^{1/\lambda}$$

$$=1-\left(\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-\mu_j^U)^{\lambda}\right)^{1/\lambda} \le 1-\left(\prod_{j=1}^{n}(1-\mu_j^U)^{\frac{\lambda(1+T(sd_j))}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} = 1-\prod_{j=1}^{n}(1-\mu_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}},$$

$$\prod_{j=1}^{n}(v_j^L)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} = \left(\prod_{j=1}^{n}((v_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \le \left(\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(v_j^L)^{\lambda}\right)^{1/\lambda}$$

$$=\left(1-\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-(v_j^L)^{\lambda})\right)^{1/\lambda} \le \left(1-\prod_{j=1}^{n}(1-(v_j^L)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda},$$

$$\prod_{j=1}^{n}(v_j^U)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}} = \left(\prod_{j=1}^{n}((v_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda} \le \left(\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(v_j^U)^{\lambda}\right)^{1/\lambda}$$

$$=\left(1-\sum_{j=1}^{n}\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}(1-(v_j^U)^{\lambda})\right)^{1/\lambda} \le \left(1-\prod_{j=1}^{n}(1-(v_j^U)^{\lambda})^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}}\right)^{1/\lambda}.$$

Thus, we have $\frac{1}{\lambda}\left(\otimes_{j=1}^{n}((\lambda sd_j)^{\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}})\right) \le \oplus_{j=1}^{n}\left(\frac{1+T(sd_j)}{\sum_{i=1}^{n}(1+T(sd_i))}sd_j\right)$, which completes the proof of Theorem 4.10. □

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