

Generalized intuitionistic fuzzy multiplicative interactive geometric operators and their application to multiple criteria decision making

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Abstract In this paper, group decision making methods based on intuitionistic fuzzy multiplicative preference relations has been developed. For it, firstly some new operational laws on intuitionistic multiplicative numbers have been defined and then by using these operations some new intuitionistic fuzzy multiplicative interactive weighted geometric, intuitionistic fuzzy multiplicative interactive ordered weighted geometric and intuitionistic fuzzy multiplicative interactive hybrid weighted geometric operators have been developed. Some desirable properties of these operators, such as idempotency, boundedness, monotonicity etc., are studied in the paper. The major advantage of the proposed operators as compared to existing ones are that it consider the proper interaction between the membership and non-membership functions and proposed operators are more pessimistic than existing ones. Furthermore, these operators are applied to decision making problems in which experts provide theory preference relation by intuitionistic fuzzy multiplicative intuitionistic fuzzy environment to show the validity, practicality and effectiveness of the new approach. Finally, a systematic comparison between the existing work and the proposed work has been given.

Keywords Multi-criteria decision making · Intuitionistic multiplicative preference set · Aggregation operator · Intuitionistic multiplicative numbers

1 Introduction

Multiple criteria group decision making problems are the important parts of modern decision theory due to the rapid development of economic and social uncertainties. Today's, decision maker wants to attain more than one goal in selecting the course of action while simultaneously satisfying the constraints. But due to the complexity of management environments and decision problems themselves, decision makers may provide their ratings or judgments to some certain degree, but it is possible that they are not so sure about their judgments and hence in many decision making problems, crisp data are unavailable due to the fuzziness or vagueness of the data in the domain of the problem. To depict the decision making problem mathematically, the preference relation is proposed which stores the preference information of the decision maker with respect to a set of alternatives or criteria in a matrix. There are mainly three sorts of preference relations, which are fuzzy preference relation (FPR) [11], intuitionistic fuzzy preference relation (IFPR) [23] and multiplicative preference relation (MPR) [12]. Xu [24] made a survey of different kinds of preference relation and discussed their properties. The FPR employs the 0–1 scale to express the decision maker's evaluation information provided by comparing each pair of objects, while, the MPR uses a ratio scale named 1/9–9 scale to measure the intensity of the pairwise comparisons of different objects (alternatives or attributes). All the elements in both the MPR and the FPR are single values, which can only be used to describe the intensities of preferences, but can't depict the degrees of non-preferences. To cope with such situation, fuzzy set theory [28] and their extensions, namely, intuitionistic fuzzy set [2], interval-valued fuzzy set [1] has been widely used for handling the uncertainties and vagueness of the

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data. Intuitionistic fuzzy set (IFS) theory [2] is one of the most permissible extensions of the fuzzy set theory and has been widely used in multi-criteria decision making (MCDM). In practical applications, when evaluating some candidate alternatives, the decision makers may not be able to express their preferences accurately due to the fact that they may not grasp sufficient knowledge of the alternatives. In such cases, the decision makers may express the decision makers' preference information in intuitionistic fuzzy values (IFVs) [or called intuitionistic fuzzy numbers (IFNs)] which are composed of a membership degree, a non-membership degree and a hesitancy (or indeterminacy) degree.

As IFPR are much easier to handle the fuzzy decision information up to desired degree of accuracy, so some researchers have applied IFPR theory to the field of decision making for aggregating the different preferences using weighted and ordered weighted operators. For instance, Deshrijver and Kerre [3] have constructed a generalized union and a generalized intersection of IFSs from a general t -norm and t -conorm. Xu and Xia [21] studied the induced generalized aggregation operators under intuitionistic fuzzy environments in which some new induced generalized intuitionistic fuzzy Choquet integral operators and induced generalized intuitionistic fuzzy Dempster–Shafer operators have developed. Xu [22] proposed the intuitionistic fuzzy weighted averaging (IFWA) operator, ordered IFWA (IFOWA) operator and the intuitionistic hybrid aggregation (IFHA) operator. Xu and Yager [25] proposed some geometric aggregation operators such as the intuitionistic fuzzy weighted geometric (IFWG) operator, intuitionistic fuzzy ordered weighted geometric (IFOGA) operator, and intuitionistic fuzzy hybrid geometric (IFHG) operator. Wei [18] proposed some induced geometric aggregation operators with intuitionistic fuzzy information. Zhao et al. [29] combined Xu and Yager's operators to develop some generalized aggregation operators, such as the generalized intuitionistic fuzzy weighted averaging (GIFWA) operator, generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator. Wang and Liu [14] presented some geometric operators under the intuitionistic fuzzy environment using Einstein operators. Wei and Zhao [19] investigated some multiple attribute group decision making problems in which both the attribute weights and the expert weights are taken in the form of intuitionistic fuzzy values and developed the induced intuitionistic fuzzy correlated averaging and induced intuitionistic fuzzy correlated geometric operators. Wang and Liu [13] developed some intuitionistic fuzzy aggregation operators such as intuitionistic fuzzy Einstein

weighted averaging (IFEWA) operator and the intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) operator to aggregate intuitionistic fuzzy values with the help of Einstein operations. Liu [7] presented an multi-criteria decision making method based on Hamacher aggregation operators. He et al. [5] proposed some new geometric operations on intuitionistic fuzzy sets, namely generalized intuitionistic fuzzy weighted geometric interaction averaging (GIFWGIA) operator, the generalized intuitionistic fuzzy ordered weighted geometric interaction averaging (GIFOWGIA) operator and the generalized intuitionistic fuzzy hybrid geometric interaction averaging (GIFHGIA) operator. Liu et al. [8–10] presented an approach for MCDM based on intuitionistic uncertain linguistic weighted Bonferroni ordered weighted average, Heronian mean Operator and interval grey uncertain linguistic variable generalized hybrid averaging operator, respectively. Wang et al. [15–17] classified the fuzzy application based on maximum fuzzy entropy, maximum ambiguity and maximum fuzziness. Garg et al. [4] proposed entropy based multi-criteria decision making method under the fuzzy environment and by unknown attribute weights. Yu [26] presented a decision-making method for aggregating the alternative under intuitionistic fuzzy environment and then applied to the assessment of typhoon disaster in Zhejiang province, China.

Based on the above works, it has been analyzed that the above operators have several drawbacks. For instance, if we take the different IFNs as $\alpha_1 = (1, 0)$, $\alpha_2 = (0, 1)$, $\alpha_3 = (0, 1)$ and $\alpha_4 = (0, 1)$ and their corresponding weight vectors be $\omega = (0.25, 0.25, 0.25, 0.25)^T$ then by using IFWA operator [22] and IFEWA [13] operators we get the aggregated IFN as $IFWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0)$ and $IFEWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0)$. Therefore, it gives an inconsistent and unable to rank the different IFNs on the respective scales. This issue has been resolved by defining the experts preferences on the different scale named as 1/9–9 instead of the 0–1 and deal the situation by expressing a multiplicative preference relation $S = (\alpha_{ij})_{m \times n}$ with the condition that $\alpha_{ij}\alpha_{ji} = 1$ and $\frac{1}{9} \leq \alpha_{ij} \leq 9$ where α_{ij} indicates the degree that the alternative x_i is preferred to x_j and is asymmetrical distribution around 1. Moreover, in IFPR it was assumed that the grades are distributed uniformly and symmetrically, but in real life, there exist the problems where the grades assign corresponding to the variables are not uniformly and symmetrical distributed. For overcoming these drawbacks, intuitionistic multiplicative number (IMN) is preferable which is based on unbalanced scale and asymmetric about 1. As lots of work has been done about the interval fuzzy preference relations,

interval multiplicative preference relations and the intuitionistic fuzzy preference relation. Apart from these, a less work has been investigated on the intuitionistic fuzzy multiplicative preference relation. To the best of my knowledge, Xia et al. [20] introduced the concept of multiplicative intuitionistic fuzzy preference relations and define some operators for aggregating the intuitionistic multiplication information in the decision making process. Yu et al. [27] extended their ideas to the interval-valued multiplicative intuitionistic preference information and its aggregation techniques. Liao and Xu [6] presented an approach related to the multiplicative consistent of IFPR, which is based on the membership and nonmembership degrees of the intuitionistic fuzzy judgments. But, it has been concluded from their studies that whenever their proposed aggregated operational laws have been used for aggregating the different IMNs then the resultant intuitionistic multiplicative numbers will not give the right decision to the system analyst. These shortcoming has been highlighted in the present manuscript. Also, in the existing operational laws, the interaction between the membership and non-membership degree are mutually exclusive and hence degree of non-membership functions does not play any effects on the degree of membership functions. Thus, there is a need to improve their corresponding basic operational laws so that the interaction between the membership and non-membership will take part during the aggregation process simultaneously.

Therefore, the main objective of this manuscript is to present some generalized aggregated intuitionistic fuzzy multiplicative geometric aggregated operators for aggregating the different intuitionistic multiplicative numbers (IMNs). For it, firstly some new operational laws on intuitionistic multiplicative sets by proper considering the interaction between the membership and non-membership functions has been developed and hence based on it an some intuitionistic fuzzy multiplicative interactive weighted geometric (IFMIWG), intuitionistic fuzzy multiplicative interactive ordered weighted geometric (IFMIOGW) and intuitionistic fuzzy multiplicative interactive hybrid weighted geometric (IFMIHWG) operators have been proposed. Some desirable properties of these operators are also investigated. Furthermore, a series of some generalized aggregation operator has been developed which includes a generalized intuitionistic fuzzy multiplicative interactive weighted geometric (GIFMIWG) operators, GIFMIOGW operator and GIFMIHWG operators, which are more practical for an geometric aggregation operator. By comparison with the existing method, it is concluded that the method

proposed in this paper is a good complement and more pessimistic than the existing works on IMNs.

In order to do so, the remainder of this paper is set out as follows. Some basic definition related to the intuitionistic multiplicative preference relations and shortcoming of the existing work are given in the next section. In Sect. 3, we developed some intuitionistic fuzzy multiplicative interactive weighted operator such as the intuitionistic fuzzy multiplicative interactive weighted geometric (IFMIWG) operator, intuitionistic fuzzy multiplicative interactive ordered weighted geometric (IFMIOGW) operator and intuitionistic fuzzy multiplicative interactive hybrid weighted geometric (IFMIHWG) operators in which given arguments are intuitionistic multiplicative values and study some desired properties of these operators. The generalized version of these operators are described in Sect. 4. In Sect. 5, we proposed a method for solving the multi-criteria decision making problems using these aggregation operators. In Sect. 6, some illustrative examples are pointed out. Finally, some concrete conclusion about the paper has been summarized and give some remarks in Sect. 7.

2 Intuitionistic multiplicative preference relations

In this section, basic concepts of intuitionistic multiplicative set (IMS) and intuitionistic multiplicative preference relation has been discussed.

2.1 Intuitionistic multiplicative set (IMS)

Let X be a fixed or universal set then an IMS is defined as [20]

$$D = \{ \langle x, \mu_D(x), \nu_D(x) \rangle | x \in X \} \quad (1)$$

which assigns to each element x a membership information $\mu_D(x)$ and a non-membership information $\nu_D(x)$, with the conditions $\frac{1}{9} \leq \mu_D(x), \nu_D(x) \leq 9, \mu_D(x)\nu_D(x) \leq 1, \forall x \in X$. For convenience, let the pair $(\mu_D(x), \nu_D(x))$ be an IMN and M be the set of all IMNs. Let α_1 and α_2 be two IMNs and denote the partial order as $\alpha_1 \geq \alpha_2$ if and only if $\mu_{\alpha_1} \geq \mu_{\alpha_2}$ and $\nu_{\alpha_1} \leq \nu_{\alpha_2}$. Especially, $\alpha_1 = \alpha_2$ if and only if $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$. The top and bottom element of $9_P = (9, 1/9)$ and $1/9_P = (1/9, 9)$ respectively.

Definition 1 In order to compare for any two IMNs, Xia et al. [20] define the score and accuracy function as $S(\alpha) = \frac{\mu}{\nu}$ and $H(\alpha) = \mu \cdot \nu$ respectively for an IMN $\alpha = \langle \mu, \nu \rangle$.

Thus, based on these score function S and accuracy function H , an order relation between two IMNs $\alpha = \langle \mu_1, \nu_1 \rangle$ and $\beta = \langle \mu_2, \nu_2 \rangle$, are defined as follows.

1. If $S(\alpha) < S(\beta)$, then $\alpha \prec \beta$;
2. If $S(\alpha) > S(\beta)$, then $\alpha \succ \beta$;
3. If $S(\alpha) = S(\beta)$,
 - If $H(\alpha) < H(\beta)$, then $\alpha \prec \beta$.
 - If $H(\alpha) > H(\beta)$, then $\alpha \succ \beta$.
 - If $H(\alpha) = H(\beta)$, then α and β represent the same information, denoted by $\alpha = \beta$.

As for the IMNs, Xia et al. [20] defined the operations for three IMNs $\alpha = \langle \mu, \nu \rangle$, $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$, $\lambda > 0$ be a real number, as follows

$$\begin{aligned} \bullet \alpha_1 \oplus \alpha_2 &= \left(\frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\nu_1\nu_2}{(2+\nu_1)(2+\nu_2)-\nu_1\nu_2} \right) \\ \bullet \alpha_1 \otimes \alpha_2 &= \left(\frac{2\mu_1\mu_2}{(2+\mu_1)(2+\mu_2)-\mu_1\mu_2}, \frac{(1+2\nu_1)(1+2\nu_2)-1}{2} \right) \\ \bullet \lambda\alpha &= \left(\frac{(1+2\mu)^\lambda-1}{2}, \frac{2\nu^\lambda}{(2+\nu)^\lambda-\nu^\lambda} \right) \\ \bullet \alpha^\lambda &= \left(\frac{2\mu^\lambda}{(2+\mu)^\lambda-\mu^\lambda}, \frac{(1+2\nu)^\lambda-1}{2} \right) \end{aligned}$$

Based on these operations, Xia et al. [20] gave intuitionistic multiplicative weighted geometric (IMWG) operator for the family of IMNs $(\alpha_1, \alpha_2, \dots, \alpha_n)$ corresponding to $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\alpha_i (i = 1, 2, \dots, n)$ and $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$ as follows.

$$\begin{aligned} IMWG(\alpha_1, \alpha_2, \dots, \alpha_n) &= \omega_1\alpha_1 \otimes \omega_2\alpha_2 \otimes \dots \otimes \omega_n\alpha_n \\ &= \left(\frac{2 \prod_{i=1}^n \mu_i^{\omega_i}}{\prod_{i=1}^n (2 + \mu_i)^{\omega_i} - \prod_{i=1}^n \mu_i^{\omega_i}}, \frac{\prod_{i=1}^n (1 + 2\nu_i)^{\omega_i} - 1}{2} \right). \end{aligned} \tag{2}$$

Especially, if $\omega = (1/n, 1/n, \dots, 1/n)$, then the IMWG operator reduces to the intuitionistic multiplicative averaging (IMG) operator $IMG(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigotimes_{i=1}^n \alpha_i^{1/n}$.

2.2 Shortcoming of the existing operator

From the above operational law on IMNs and their corresponding IMWG operator, it has been observed that the membership function of the IMWG operator is independent of the degree of non-membership functions and hence does not give the accurate results or an undesirable feature of the operator. Also the pair of interaction between the membership and non-membership does not take into account while defining their operational laws. Therefore, the

existing IMWG operators do not give the sufficient information in the phase of aggregation process. For example,

Example 1 Let $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the collection of IMNs where $\alpha_1 = \langle 2/3, 1/2 \rangle$, $\alpha_2 = \langle 3, 1/5 \rangle$, $\alpha_3 = \langle 1/4, 1/3 \rangle$ and $\alpha_4 = \langle 1/6, 4 \rangle$ be four IMNs, $\omega = (0.3, 0.2, 0.1, 0.4)$ is the standardized weight vector of the four IMNs. By using the IMWG operator, the aggregate IMN is calculated as $IMWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle 0.4137, 1.1687 \rangle$. On the other hand, if we take $B = (\beta_1, \beta_2, \beta_3, \beta_4)$ where $\beta_1 = \langle 1/4, 3 \rangle$, $\beta_2 = \langle 1/6, 5 \rangle$, $\beta_3 = \langle 7, 1/9 \rangle$ and $\beta_4 = \langle 0.6407, 0.1780 \rangle$ be four IMNs corresponding to same weight set then we get $IMWG(\beta_1, \beta_2, \beta_3, \beta_4) = \langle 0.4137, 1.1687 \rangle$. Thus, the score functions corresponding to these IMNs are same and hence it cannot rank the alternatives. Therefore, it is difficult to choose the best alternatives among the existing ones by using the IMWG operator.

Example 2 Let $C = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ where $\gamma_1 = \langle 2/3, 1/4 \rangle$, $\gamma_2 = \langle 3, 1/7 \rangle$, $\gamma_3 = \langle 1/4, 3 \rangle$ and $\gamma_4 = \langle 1/6, 4 \rangle$ be four IMNs, $\omega = (0.3, 0.2, 0.1, 0.4)$ is the standardized weight vector of the four IMNs then $IMWG(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \langle 0.4137, 1.2371 \rangle$. Thus it has been observed from the observation that degree of membership of $IMWG(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is same as that of degree of membership value of $IMWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ i.e. 0.4137. Hence, the effect of change of ν_i is independent on μ_{IMWG} and therefore it is inconsistent to rank the alternative up to desired degree. In other words, it does not consider the interaction between the membership function and non-membership function of different IMNs.

Therefore, it has been concluded that the existing IMWG operator is invalid to rank the alternative and hence there is a necessary to pay more attention on this issue and to need other measuring functions. For this, a new feature of operational laws has been introduced here by considering the proper interaction between the membership functions and non-membership functions of different IMNs.

3 Intuitionistic fuzzy multiplicative interactive weighted operators

3.1 Improved operational laws on intuitionistic multiplicative numbers

Definition 2 Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle$ and $\alpha = \langle \mu, \nu \rangle$ be three IMNs and $\lambda > 0$ be a real number then the new operations on these IMNs are defined as follows.

1. $\alpha_1 \oplus \alpha_2 = \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\{1-(1-\mu_1v_1)(1-\mu_2v_2)\}}{(1+2\mu_1)(1+2\mu_2)-1} \right\rangle$
2. $\alpha_1 \otimes \alpha_2 = \left\langle \frac{2\{1-(1-\mu_1v_1)(1-\mu_2v_2)\}}{(1+2v_1)(1+2v_2)-1}, \frac{(1+2v_1)(1+2v_2)-1}{2} \right\rangle$
3. $\lambda\alpha = \left\langle \frac{(1+2\mu)^\lambda-1}{2}, \frac{2\{1-(1-\mu v)^\lambda\}}{(1+2\mu)^\lambda-1} \right\rangle$
4. $\alpha^\lambda = \left\langle \frac{2\{1-(1-\mu v)^\lambda\}}{(1+2v)^\lambda-1}, \frac{(1+2v)^\lambda-1}{2} \right\rangle$.

From $\alpha_1 \oplus \alpha_2$ it has been obtained that the membership function of $\alpha_1 \oplus \alpha_2$ does not contain the pair of μ_1, v_2 and v_1, μ_2 while the non-membership function contains $\mu_1 \cdot v_2$ and $v_1 \cdot \mu_2$. Thus, the influence of membership function is greater than the influence on non-membership function, which means that that attitude of decision maker is optimistic. Similarly, the geometric meaning of new multiplication operator $\alpha_1 \otimes \alpha_2$ has been obtained and found that influence of non-membership function is greater than that of membership functions. This is to say, the attitude of decision maker is pessimistic. We extend these operations to the n IMNs, $\alpha_1, \alpha_2, \dots, \alpha_n$ and get the following definition.

Definition 3 Let $\alpha = \langle \mu, v \rangle$, $\alpha_i = \langle \mu_i, v_i \rangle$, ($i = 1, 2, \dots, n$) be the collection of IMNs and $\lambda > 0$ be a real number then

1. $\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_n = \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_i) - 1}{2}, \frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)\}}{\prod_{i=1}^n (1 + 2\mu_i) - 1} \right\rangle$
2. $\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n = \left\langle \frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)\}}{\prod_{i=1}^n (1 + 2v_i) - 1}, \frac{\prod_{i=1}^n (1 + 2v_i) - 1}{2} \right\rangle$
3. $\lambda\alpha = \left\langle \frac{(1+2\mu)^\lambda-1}{2}, \frac{2\{1-(1-\mu v)^\lambda\}}{(1+2\mu)^\lambda-1} \right\rangle$
4. $\alpha^\lambda = \left\langle \frac{2\{1-(1-\mu v)^\lambda\}}{(1+2v)^\lambda-1}, \frac{(1+2v)^\lambda-1}{2} \right\rangle$.

3.2 Intuitionistic fuzzy multiplicative interactive weighted geometric (IFMIWG) operator

Definition 4 Let $\alpha_i = \langle \mu_i, v_i \rangle$, ($i = 1, 2, \dots, n$) be the collection of IMNs, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight vector of α_i ($i = 1, 2, \dots, n$) with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, and let $IFMIWG : \Omega^n \rightarrow \Omega$, if

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \dots \otimes \alpha_n^{\omega_n}$$

where Ω is the set of all IMNs then IFMIWG is called the intuitionistic fuzzy multiplicative interactive weighted averaging operator.

Theorem 1 Let $\alpha_i = \langle \mu_i, v_i \rangle$, ($i = 1, 2, \dots, n$) be the collection of IMNs, then

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1}{2} \right\rangle. \tag{3}$$

Proof We prove this theorem by induction on n .

When $n = 1, \omega_1 = 1$, we have

$$IFMIWG(\alpha_1) = \alpha_1^{\omega_1} = \langle \mu_1, v_1 \rangle = \left\langle \frac{2\{1 - (1 - \mu_1 v_1)^1\}}{(1 + 2v_1)^1 - 1}, \frac{(1 + 2v_1)^1 - 1}{2} \right\rangle.$$

Thus, Eq. (3) hold for $n = 1$. Assume that the Eq. (3) holds for $n = k$, i.e.,

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_k) = \left\langle \frac{2\{1 - \prod_{i=1}^k (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^k (1 + 2v_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^k (1 + 2v_i)^{\omega_i} - 1}{2} \right\rangle.$$

Then, when $n = k + 1$ by the operational laws in Definition 5, we have

$$\begin{aligned} IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_{k+1}) &= \bigotimes_{i=1}^{k+1} \alpha_i^{\omega_i} \\ &= IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_k) \otimes \alpha_{k+1}^{\omega_{k+1}} \\ &= \left\langle \frac{2\{1 - \prod_{i=1}^k (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^k (1 + 2v_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^k (1 + 2v_i)^{\omega_i} - 1}{2} \right\rangle \\ &\quad \otimes \left\langle \frac{2\{1 - (1 - \mu_{k+1} v_{k+1})^{\omega_{k+1}}\}}{(1 + 2v_{k+1})^{\omega_{k+1}} - 1}, \frac{(1 + 2v_{k+1})^{\omega_{k+1}} - 1}{2} \right\rangle \\ &= \left\langle \frac{2\{1 - \prod_{i=1}^{k+1} (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^{k+1} (1 + 2v_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^{k+1} (1 + 2v_i)^{\omega_i} - 1}{2} \right\rangle. \end{aligned}$$

Thus, result is true for $n = k + 1$ and hence by principle of mathematical induction, result is true for all $n \in N$.

Lemma 1 Let $\alpha_i = \langle \mu_i, v_i \rangle, \omega_i > 0$ for $i = 1, 2, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$, then

$$\prod_{i=1}^n \alpha_i^{\omega_i} \leq \sum_{i=1}^n \omega_i \alpha_i$$

with equality holds if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_n$.

Corollary 1 *The IMWG and IFMIWG operators have the following relation:*

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \geq IMWG(\alpha_1, \alpha_2, \dots, \alpha_n)$$

where $\alpha_i (i = 1, 2, \dots, n)$ be a collections of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of α_i such that $\omega_i \in [0, 1], i = 1, 2, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$.

Proof Since

$$\frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1} \geq \frac{2 \prod_{i=1}^n \mu_i^{\omega_i}}{\prod_{i=1}^n (\mu_i + 2)^{\omega_i} - \prod_{i=1}^n \mu_i^{\omega_i}}$$

where equality holds if and only if $\mu_1 = \mu_2 = \dots = \mu_n$ and $v_1 = v_2 = \dots = v_n$.

Let $IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \langle \mu_\alpha^p, v_\alpha^p \rangle = \alpha^p$ and $IMWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \langle \mu_\alpha, v_\alpha \rangle = \alpha$, then $\mu_\alpha^p \geq \mu_\alpha$ and $v_\alpha^p = v_\alpha$. Thus,

$$S(\alpha^p) = \frac{\mu_\alpha^p}{v_\alpha^p} \geq \frac{\mu_\alpha}{v_\alpha} = S(\alpha).$$

If $S(\alpha^p) > S(\alpha)$ then by Definition 1, for every ω , we have

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) > IMWG(\alpha_1, \alpha_2, \dots, \alpha_n).$$

If $S(\alpha^p) = S(\alpha)$ i.e. $\frac{\mu_\alpha^p}{v_\alpha^p} = \frac{\mu_\alpha}{v_\alpha}$ then by the condition $v_\alpha^p = v_\alpha$, we have $\mu_\alpha^p = \mu_\alpha$, thus the accuracy function $H(\alpha^p) = \mu_\alpha^p v_\alpha^p = \mu_\alpha v_\alpha = H(\alpha)$. Thus in this case, from the Definition 1, it follows that

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) = IMWG(\alpha_1, \alpha_2, \dots, \alpha_n).$$

Hence,

$$IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \geq IMWG(\alpha_1, \alpha_2, \dots, \alpha_n)$$

where that equality holds if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_n$.

From the improved operational laws on IMS and the proposed aggregation operator, the following points are summarized.

1. From $\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n$, it has been obtained that the non-membership function of it does not contain the pairs of $\mu_i \cdot v_j$ and $\mu_j \cdot v_i$ for $i \neq j$ while the membership functions contain these pairs. Thus the influence of the non-membership function is greater than the membership function., which means that the attitude of the decision maker is pessimistic.
2. Also, it has been observed that the degree of membership function of the proposed aggregated operator is greater than the degree of the membership of the existing operator [20].
3. From Corollary 1, it has been concluded that the score value computed from the IFMIWG operator is greater than the existing IMWG operator [20].

Thus, it has been concluded that the proposed IFMIWG operator shows the decision maker’s more pessimistic attitude than the existing IMWG operator in the aggregation process.

Example 3 If we apply the proposed IFMIWG operator on the different IMNs defined in Example 2 then the corresponding aggregated IMNs are obtained as

$$IFMIWG(A) = \langle 0.4527, 1.1687 \rangle; \quad S(A) = 0.3874$$

$$IFMIWG(B) = \langle 0.5323, 1.1687 \rangle; \quad S(B) = 0.4555.$$

Thus, based on the definition of score function, given in Definition 1, it has been concluded that alternative B is better than A. Moreover, it is clear that $IFMIWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > IMWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $IFMIWG(\beta_1, \beta_2, \beta_3, \beta_4) > IMWG(\beta_1, \beta_2, \beta_3, \beta_4)$.

Example 4 If we apply the proposed new operations on the Example 3 and compute that

$$IFMIWG(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \langle 0.4245, 1.2371 \rangle.$$

Thus, there is a significant impact of degree of non-membership on the degree of membership functions and hence it has been concluded that the interaction between membership function and non-membership functions of different IMNs.

Theorem 2 *If $\alpha_i = \langle \mu_i, v_i \rangle \in IMNs, i = 1, 2, \dots, n$, then the aggregated value by using the IFMIWG operator is also an intuitionistic multiplicative number i.e. $IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \in IMNs$.*

Proof Since $\alpha_i = \langle \mu_i, v_i \rangle \in IMNs, i = 1, 2, \dots, n$, by definition of IMNs, we have

$$\frac{1}{9} \leq \mu_i, v_i \leq 9 \quad \text{and} \quad \mu_i v_i \leq 1.$$

$$\text{As, } \frac{1}{9} \leq \frac{\prod_{i=1}^n (1+2v_i)^{\omega_i} - 1}{2} \leq 9; \quad \frac{1}{9} \leq \frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^n (1+2v_i)^{\omega_i} - 1} \leq 9 \text{ and}$$

$$\left(\frac{\prod_{i=1}^n (1+2v_i)^{\omega_i} - 1}{2} \right) \left(\frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^n (1+2v_i)^{\omega_i} - 1} \right) = 1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i} \leq 1$$

Thus, $IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \in IMNs$.

Property 1 *Let $A_i = \langle \mu_{A_i}, v_{A_i} \rangle (i = 1, 2, \dots, n)$ be the collection of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weighted vector satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. If $A_i = A_0 = \langle \mu_{A_0}, v_{A_0} \rangle$ for all i , then*

$$IFMIWG(A_1, A_2, \dots, A_n) = A_0.$$

This property is called idempotency.

Proof Since $A_i = A_0 = \langle \mu_{A_0}, v_{A_0} \rangle (i = 1, 2, \dots, n)$ and $\sum_{i=1}^n \omega_i = 1$, so by Theorem 1, we have

$$IFMIWG(A_1, A_2, \dots, A_n) = \left\langle \frac{2\{1 - \prod_{i=1}^n (1 - \mu_{A_0} v_{A_0})^{\omega_i}\}}{\prod_{i=1}^n (1 + 2v_{A_0})^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2v_{A_0})^{\omega_i} - 1}{2} \right\rangle = \left\langle \frac{2\{1 - (1 - \mu_{A_0} v_{A_0})^{\sum_{i=1}^n \omega_i}\}}{(1 + 2v_{A_0})^{\sum_{i=1}^n \omega_i} - 1}, \frac{(1 + 2v_{A_0})^{\sum_{i=1}^n \omega_i} - 1}{2} \right\rangle = \langle \mu_{A_0}, v_{A_0} \rangle = A_0.$$

Property 2 Let $A_i = \langle \mu_{A_i}, v_{A_i} \rangle (i = 1, 2, \dots, n)$ be a collection of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weighted vector satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Let $A^- = (\min_i(\mu_{A_i}), \min_i(v_{A_i}))$ and $A^+ = (\max_i(\mu_{A_i}), \max_i(v_{A_i}))$ then

$$A^- \leq IFMIWG(A_1, A_2, \dots, A_n) \leq A^+.$$

This property is called boundedness.

Proof As $\max_i(v_{A_i}) \leq v_{A_i} \leq \min_i(v_{A_i})$

$$\Rightarrow \frac{\prod_{i=1}^n (1 + 2 \max_i(v_{A_i}))^{\omega_i} - 1}{2} \leq \frac{\prod_{i=1}^n (1 + 2v_{A_i})^{\omega_i} - 1}{2} \leq \frac{\prod_{i=1}^n (1 + 2 \min_i(v_{A_i}))^{\omega_i} - 1}{2} \Rightarrow \max_i(v_{A_i}) \leq v_{IFMIWG(A_1, \dots, A_n)} \leq \min_i(v_{A_i}). \tag{4}$$

Also, $\min_i(\mu_{A_i}) \leq \mu_i \leq \max_i(\mu_{A_i})$

$$\Rightarrow \min_i(\mu_{A_i}) \max_i(v_{A_i}) \leq \mu_i v_i \leq \max_i(\mu_{A_i}) \min_i(v_{A_i}) \Rightarrow 1 - \max_i(\mu_{A_i}) \min_i(v_{A_i}) \leq 1 - \mu_i v_i \leq 1 - \min_i(\mu_{A_i}) \max_i(v_{A_i})$$

$$\Rightarrow \left(1 - \max_i(\mu_{A_i}) \min_i(v_{A_i})\right)^{\sum_i \omega_i} \leq \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i} \leq \left(1 - \min_i(\mu_{A_i}) \max_i(v_{A_i})\right)^{\sum_i \omega_i}$$

$$\Rightarrow \min_i(\mu_{A_i}) \max_i(v_{A_i}) \leq 1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i} \leq \max_i(\mu_{A_i}) \min_i(v_{A_i}) \Rightarrow \frac{\min_i(\mu_{A_i}) \max_i(v_{A_i})}{\min_i(v_{A_i})} \leq \frac{2\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\}}{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1} \leq \frac{\max_i(\mu_{A_i}) \min_i(v_{A_i})}{\max_i(v_{A_i})} \Rightarrow \frac{\min_i(\mu_{A_i}) \max_i(v_{A_i})}{\min_i(v_{A_i})} \leq \mu_{IFMIWG(A_1, \dots, A_n)} \leq \frac{\max_i(\mu_{A_i}) \min_i(v_{A_i})}{\max_i(v_{A_i})} \Rightarrow \min_i(\mu_{A_i}) \leq \mu_{IFMIWG(A_1, \dots, A_n)} \leq \max_i(\mu_{A_i}). \tag{5}$$

Let $IFMIWG(A_1, A_2, \dots, A_n) = \alpha = \langle \mu_\alpha, v_\alpha \rangle$ then Eqs. (4) and (5) are transformed into the following forms, respectively

$$\min_i(\mu_{A_i}) \leq \mu_\alpha \leq \max_i(\mu_{A_i}); \quad \max_i(v_{A_i}) \leq v_\alpha \leq \min_i(v_{A_i}).$$

Take $A^- = (\min_i(\mu_{A_i}), \min_i(v_{A_i}))$ and $A^+ = (\max_i(\mu_{A_i}), \max_i(v_{A_i}))$. Thus, $S(A) \leq S(A^+)$ and $S(A) \geq S(A^-)$ and hence by order relation between two IMNs, we have

$$A^- \leq IFMIWG(A_1, A_2, \dots, A_n) \leq A^+.$$

Property 3 Let $A_i = \langle \mu_{A_i}, v_{A_i} \rangle (i = 1, 2, \dots, n)$ and $B_i = \langle \mu_{B_i}, v_{B_i} \rangle (i = 1, 2, \dots, n)$ be two collection of intuitionistic multiplicative numbers and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weighted vector satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. When $A_i \leq B_i$ for all i , then

$$IFMIWG(A_1, A_2, \dots, A_n) \leq IFMIWG(B_1, B_2, \dots, B_n)$$

This property is called monotonicity.

Proof The monotonicity of the IFMIWG operator can be obtained by a similar proving method.

Property 4 Let $\alpha_i = \langle \mu_i, v_i \rangle (i = 1, 2, \dots, n)$ be a collection of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. If $\beta = \langle \mu_\beta, v_\beta \rangle$ is an IMN, then

$$IFMIWG(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \dots, \alpha_n \oplus \beta) = IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta$$

This property is called shift-invariance.

Proof As $\alpha_i, \beta \in$ IMNs, so

$$\alpha_i \otimes \beta = \left\langle \frac{2[1 - (1 - \mu_i v_i)(1 - \mu_\beta v_\beta)]}{(1 + 2v_i)(1 + 2v_\beta) - 1}, \frac{(1 + 2v_i)(1 + 2v_\beta) - 1}{2} \right\rangle.$$

Therefore,

$$IFMIWG(\alpha_1 \otimes \beta, \alpha_2 \otimes \beta, \dots, \alpha_n \otimes \beta) = \left\langle \frac{2[1 - \prod_{i=1}^n \{(1 - \mu_i v_i)(1 - \mu_\beta v_\beta)\}^{\omega_i}]}{\prod_{i=1}^n \{(1 + 2v_i)(1 + 2v_\beta)\}^{\omega_i} - 1}, \frac{\prod_{i=1}^n \{(1 + 2v_i)(1 + 2v_\beta)\}^{\omega_i} - 1}{2} \right\rangle = \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i} (1 - \mu_\beta v_\beta)]}{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} (1 + 2v_\beta) - 1}, \frac{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} (1 + 2v_\beta) - 1}{2} \right\rangle = \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}]}{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1}{2} \right\rangle \otimes \langle \mu_\beta, v_\beta \rangle = IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta.$$

Hence, $IFMIWG(\alpha_1 \otimes \beta, \alpha_2 \otimes \beta, \dots, \alpha_n \otimes \beta) = IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \otimes \beta$

Property 5 Let $\alpha_i = \langle \mu_i, \nu_i \rangle (i = 1, 2, \dots, n)$ be a collection of IMNs and let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of them such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. If $\beta > 0$, $\beta = \langle \mu_\beta, \nu_\beta \rangle$ is an IMN, then

$$IFMIWG(\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n) = \beta IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n)$$

This property is called homogeneity.

Proof Since $\alpha_i = \langle \mu_i, \nu_i \rangle \in$ IMNs for $i = 1, 2, \dots, n$. Therefore, for $\beta > 0$, we have

$$\beta\alpha_i = \left\langle \frac{2[1 - (1 - \mu_i \nu_i)^\beta]}{(1 + 2\nu_i)^\beta - 1}, \frac{(1 + 2\nu_i)^\beta - 1}{2} \right\rangle.$$

Therefore,

$$\begin{aligned} IFMIWG(\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n) &= \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\beta\omega_i}]}{\prod_{i=1}^n \left(1 + 2\frac{(1+2\nu_i)^\beta - 1}{2}\right)^{\omega_i}} - 1, \frac{\prod_{i=1}^n \left(1 + 2\frac{(1+2\nu_i)^\beta - 1}{2}\right)^{\omega_i} - 1}{2} \right\rangle \\ &= \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\beta\omega_i}]}{\prod_{i=1}^n [(1 + 2\nu_i)^\beta]^{\omega_i} - 1}, \frac{\prod_{i=1}^n [(1 + 2\nu_i)^\beta]^{\omega_i} - 1}{2} \right\rangle \\ &= \left\langle \frac{2[1 - (\prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i})^\beta]}{(\prod_{i=1}^n (1 + 2\nu_i)^{\omega_i})^\beta - 1}, \frac{(\prod_{i=1}^n (1 + 2\nu_i)^{\omega_i})^\beta - 1}{2} \right\rangle \\ &= \beta \left\langle \frac{2\{1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i}\}}{\prod_{i=1}^n (1 + 2\nu_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2\nu_i)^{\omega_i} - 1}{2} \right\rangle \\ &= \beta IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n). \end{aligned}$$

Hence, $IFMIWG(\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n) = \beta IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Property 6 If $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle (i = 1, 2, \dots, n)$ be two collections of IMNs then

$$\begin{aligned} IFMIWG(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \dots, \alpha_n \oplus \beta_n) \\ = IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus IFMIWG(\beta_1, \beta_2, \dots, \beta_n). \end{aligned}$$

Proof As $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle (i = 1, 2, \dots, n)$ be two collections of IMNs, then

$$\alpha_i \otimes \beta_i = \left\langle \frac{2[1 - (1 - \mu_{\alpha_i} \nu_{\alpha_i})(1 - \mu_{\beta_i} \nu_{\beta_i})]}{(1 + 2\nu_{\alpha_i})(1 + 2\nu_{\beta_i}) - 1}, \frac{(1 + 2\nu_{\alpha_i})(1 + 2\nu_{\beta_i}) - 1}{2} \right\rangle.$$

Therefore,

$$\begin{aligned} IFMIWG(\alpha_1 \otimes \beta_1, \alpha_2 \otimes \beta_2, \dots, \alpha_n \otimes \beta_n) &= \left\langle \frac{2[1 - \prod_{i=1}^n \{(1 - \mu_{\alpha_i} \nu_{\alpha_i})(1 - \mu_{\beta_i} \nu_{\beta_i})\}^{\omega_i}]}{\prod_{i=1}^n \left(1 + 2\frac{(1+2\nu_{\alpha_i})(1+2\nu_{\beta_i})-1}{2}\right)^{\omega_i}} - 1, \frac{\prod_{i=1}^n \left(1 + 2\frac{(1+2\nu_{\alpha_i})(1+2\nu_{\beta_i})-1}{2}\right)^{\omega_i} - 1}{2} \right\rangle \\ &= \left\langle \frac{2[1 - \prod_{i=1}^n \{(1 - \mu_{\alpha_i} \nu_{\alpha_i})(1 - \mu_{\beta_i} \nu_{\beta_i})\}^{\omega_i}]}{\prod_{i=1}^n \{(1 + 2\nu_{\alpha_i})(1 + 2\nu_{\beta_i})\}^{\omega_i} - 1}, \frac{\prod_{i=1}^n \{(1 + 2\nu_{\alpha_i})(1 + 2\nu_{\beta_i})\}^{\omega_i} - 1}{2} \right\rangle \\ &= \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_{\alpha_i} \nu_{\alpha_i})^{\omega_i} \prod_{i=1}^n (1 - \mu_{\beta_i} \nu_{\beta_i})^{\omega_i}]}{\prod_{i=1}^n (1 + 2\nu_{\alpha_i})^{\omega_i} \prod_{i=1}^n (1 + 2\nu_{\beta_i})^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2\nu_{\alpha_i})^{\omega_i} \prod_{i=1}^n (1 + 2\nu_{\beta_i})^{\omega_i} - 1}{2} \right\rangle \\ &= \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_{\alpha_i} \nu_{\alpha_i})^{\omega_i}]}{\prod_{i=1}^n (1 + 2\nu_{\alpha_i})^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2\nu_{\alpha_i})^{\omega_i} - 1}{2} \right\rangle \\ &\quad \otimes \left\langle \frac{2[1 - \prod_{i=1}^n (1 - \mu_{\beta_i} \nu_{\beta_i})^{\omega_i}]}{\prod_{i=1}^n (1 + 2\nu_{\beta_i})^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2\nu_{\beta_i})^{\omega_i} - 1}{2} \right\rangle \\ &= IFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus IFMIWG(\beta_1, \beta_2, \dots, \beta_n). \end{aligned}$$

Hence, $IFMIWG(\alpha_1 \otimes \beta_1, \dots, \alpha_n \otimes \beta_n) = IFMIWG(\alpha_1, \dots, \alpha_n) \otimes IFMIWG(\beta_1, \dots, \beta_n)$.

3.3 Intuitionistic fuzzy multiplicative interactive ordered weighted geometric (IFMIOWG) operator

In this section, we intend to take the idea of OWA into IFMIWG operator and propose a new operator called intuitionistic fuzzy weighted multiplicative interactive ordered weighted geometric (IFMIOWG) operator. In the following, we first introduce the concept of IFMIOWG operator and then illustrate it with a numerical example.

Definition 5 Suppose there is a family of IMNs $(\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i = \langle \mu_i, \nu_i \rangle$ for $i = 1, 2, \dots, n$ then the IFMIOWG operator is defined as follows.

$$IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_{\delta(1)}^{\omega_1} \otimes \dots \otimes \alpha_{\delta(n)}^{\omega_n}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$ and $\delta: (1, 2, \dots, n) \rightarrow (1, 2, \dots, n)$, IMN $\alpha_{\delta(i)}$ is the i th largest of IMN α_i .

Theorem 3 Let $\alpha_i = \langle \mu_i, \nu_i \rangle (i = 1, 2, \dots, n)$ be the collection of intuitionistic multiplicative numbers, then based on IFMIOWG operator, the aggregated IMN can be expressed as

$$IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{2\{1 - \prod_{i=1}^n (1 - \mu_{\delta(i)} \nu_{\delta(i)}^{\omega_i})\}}{\prod_{i=1}^n (1 + 2\nu_{\delta(i)}^{\omega_i}) - 1}, \frac{\prod_{i=1}^n (1 + 2\nu_{\delta(i)}^{\omega_i}) - 1}{2} \right\rangle.$$

Proof The proof of this theorem is similar to that of Theorem 1 and hence it is omitted here.

Corollary 2 The IFMIOWG operator and IFMOWG operator have the following relation

$$IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) \geq IFMOWG(\alpha_1, \alpha_2, \dots, \alpha_n).$$

Proof Proof is similar to that of Corollary 1 and hence it is omitted here.

Property 7 Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle (i = 1, 2, \dots, n)$ be a collection of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weighting vector of the IFMIOWG operator, $\omega_i \in [0, 1], i = 1, 2, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$ then we have the following properties.

1. *Idempotency:* if all $\alpha_i, (i = 1, 2, \dots, n)$ are equal i.e., $\alpha_i = \alpha$ for all i , then

$$IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha.$$

2. *Boundedness*

$$\alpha_{\min} \leq IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha_{\max}$$

where $\alpha_{\min} = \min\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\alpha_{\max} = \max\{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

3. *Monotonicity:* let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle, (j, i = 1, 2, \dots, n)$ be two collections of IMNs and $\alpha_i \leq \beta_i$ then for every weight vector ω , we have

$$IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) \leq IFMIOWG(\beta_1, \beta_2, \dots, \beta_n).$$

4. *shift-invariance:* let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ be collections of IMNs and $\beta = \langle \mu_{\beta}, \nu_{\beta} \rangle$ be an IMN then

$$IFMIOWG(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta \otimes \dots \otimes \alpha_n \oplus \beta) = IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta.$$

5. *Homogeneity:* let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle (i = 1, 2, \dots, n)$ be a collection of IMNs and $\beta > 0$ be a real number, then

$$IFMIOWG(\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n) = \beta IFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n).$$

Proof The proof of this is similar to that of IFMIWG operator properties.

Example 5 If we apply IFMIOWG operator to aggregate the different IMNs as given in Example 2, we get $IFMIOWG(\alpha_1, \dots, \alpha_4) = \langle 0.4976, 1.1103 \rangle$ and $IFMIOWG(\beta_1, \dots, \beta_4) = \langle 0.5708, 1.2891 \rangle$. On the other hand, for Example 3 we get $IFMIOWG(\gamma_1, \dots, \gamma_4) = \langle 0.4483, 1.2106 \rangle$.

3.4 Intuitionistic fuzzy multiplicative interactive hybrid weighted geometric (IFMIHWG) Operator

Here in the above studies, the IFMIWG and IFMIOWG operators have been studied. Now, in the following, we introduce the intuitionistic fuzzy multiplicative interactive hybrid weighted geometric (IFMIHWG) operator that combines the advantage of both IFMIWG and IFMIOWG operators.

Definition 6 Suppose there is a family of IMNs, $\alpha_i = \langle \mu_i, \nu_i \rangle (i = 1, 2, \dots, n)$ then the IFMIHWG operator is defined as follows.

$$IFMIHWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \check{\alpha}_{\sigma(1)}^{\omega_1} \otimes \dots \otimes \check{\alpha}_{\sigma(n)}^{\omega_n}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the associated standardized weight vector of IFMIHWG operator satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. $\check{\alpha}_{\sigma(i)}$ is the i th largest of the weighted IMNs $\check{\alpha}_i (\check{\alpha}_i = \alpha_i^{n\omega_i}, i = 1, 2, \dots, n)$, n is the number of IMNs and $w = (w_1, w_2, \dots, w_n)^T$ is the standard weight vector of $\alpha_i (i = 1, 2, \dots, n)$.

From the Definition 11, it has been concluded that

- The IFMIHWG operator first weights the given arguments, and the reorders the weighted arguments in descending order and weights these ordered arguments by the IFMIHG weights, and finally aggregates all the weighted arguments into a collective one.
- The IFMIHWG operator generalizes both the IFMIWG and IFMIOWG operators, and reflects the importance degrees of both the given arguments and their ordered positions. For instance, if $w = (1/n, 1/n, \dots, 1/n)$ then IFMIOWG operator is a special case of the IFMIHWG operator. On the other hand, if $\omega = (1/n, 1/n, \dots, 1/n)$ then the IFMIWG is a special case of the IFMIHWG operator.

Based on the proposed improved operational rules of the IMNs, we can derive the result shown in Theorem 4.

Theorem 4 Suppose that there is a family of IMNs $(\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i = (\mu_i, \nu_i) \in IMNs (i = 1, 2, \dots, n)$ based on the IFMIHWG operator, then the aggregated IMN can be expressed as

$$IFMIHWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{2\{1 - \prod_{i=1}^n (1 - \dot{\mu}_{\sigma(i)} \dot{v}_{\sigma(i)})^{\omega_i}\}}{\prod_{i=1}^n (1 + 2\dot{v}_{\sigma(i)})^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2\dot{v}_{\sigma(i)})^{\omega_i} - 1}{2} \right\rangle.$$

Proof The proof is similar to Theorem 1, so it is omitted here.

Example 6 Let $\alpha_1 = \langle 1/3, 2 \rangle$, $\alpha_2 = \langle 1/7, 3 \rangle$, $\alpha_3 = \langle 4, 1/5 \rangle$ and $\alpha_4 = \langle 6, 1/7 \rangle$ be four IMNs and $w = (0.12, 0.27, 0.24, 0.31)^T$ be the standardized weight vector of the four IMNs, and $\omega = (0.3, 0.2, 0.1, 0.4)^T$ is the associated weighted vector of the IFMIHWG operator. Then, $\dot{\alpha}_i = \alpha_i^{w_i}$, ($i = 1, 2, 3, 4$) becomes $\dot{\alpha}_1 = \langle 0.7034, 0.5826 \rangle$, $\dot{\alpha}_2 = \langle 0.1264, 3.5896 \rangle$, $\dot{\alpha}_3 = \langle 4.1266, 0.1906 \rangle$ and $\dot{\alpha}_4 = \langle 4.9799, 0.1828 \rangle$. Thus $S(\dot{\alpha}_4)S(\dot{\alpha}_3) > S(\dot{\alpha}_1) > S(\dot{\alpha}_2)$. Hence, $\dot{\alpha}_{\sigma(1)} = \alpha_4$; $\dot{\alpha}_{\sigma(2)} = \alpha_3$; $\dot{\alpha}_{\sigma(3)} = \alpha_1$; $\dot{\alpha}_{\sigma(4)} = \alpha_2$. Now, in order to aggregate these IMNs by IFMIHWG operator corresponding to weight vector ω , the aggregated IMN becomes $\langle 0.7284, 0.9753 \rangle$.

multiplicative interactive weighted geometric (GIFMIWG) operator is a mapping $GIFMIWG : \Omega^n \rightarrow \Omega$ such that

$$GIFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{\lambda} \otimes_{i=1}^n (\lambda \alpha_i)^{\omega_i}$$

where λ is a real number greater than zero, $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector of α_i such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$.

Especially,

- If $\lambda = 1$ then the GIFMIWG reduces to IFMIWG operator;
- If $\omega = (1/n, 1/n, \dots, 1/n)$ then GIFMIWG reduces to the generalized intuitionistic fuzzy weighted multiplicative averaging (GIFWMG) operator, $GIFWMG(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{\lambda} \otimes_{i=1}^n (\lambda \alpha_i)^{1/n}$.

Theorem 5 Let $\alpha_i = \langle \mu_i, v_i \rangle (i = 1, 2, \dots, n)$ be a family of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector of α_i such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, then based on GIFMIWG the aggregated IMN can be expressed as

4 Generalized intuitionistic fuzzy multiplicative interactive geometric operators

4.1 Generalized intuitionistic fuzzy multiplicative interactive weighted geometric (GIFMIWG) operator

Definition 7 Let $\alpha_i = \langle \mu_i, v_i \rangle (i = 1, 2, \dots, n)$ be a collection of IMNs and Ω be the set of all intuitionistic multiplicative numbers, then the generalized intuitionistic fuzzy

$$GIFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$= \left\{ \prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \right. \\ \times \left\langle \frac{-4 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i}}{2 \{ \prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \}^{1/\lambda}}, \right. \\ \left. \frac{2 \{ \prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \}^{1/\lambda} [1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}]}{\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}} \right\rangle \\ \left. - \left\{ 4 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i} \right\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \right\}$$

Proof As $\alpha_i = \langle \mu_i, v_i \rangle$ and $\lambda > 0$ be a real number, therefore

$$\begin{aligned} \lambda \alpha_i &= \left\langle \frac{(1 + 2\mu_i)^\lambda - 1}{2}, \frac{2[1 - (1 - \mu_i v_i)^\lambda]}{(1 + 2\mu_i)^\lambda - 1} \right\rangle \\ \Rightarrow (\lambda \alpha_i)^{\omega_i} &= \left\langle \frac{2[1 - \{(1 - \mu_i v_i)^\lambda\}^{\omega_i}]}{\left\{1 + 2 \left\{ \frac{2(1 - (1 - \mu_i v_i)^\lambda)}{(1 + 2\mu_i)^\lambda - 1} \right\}^{\omega_i} - 1\right\}}, \frac{\left\{1 + 2 \left\{ \frac{2(1 - (1 - \mu_i v_i)^\lambda)}{(1 + 2\mu_i)^\lambda - 1} \right\}^{\omega_i} - 1\right\}}{2} \right\rangle \\ &= \left\langle \frac{2\{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} [1 - (1 - \mu_i v_i)^{\lambda \omega_i}]}{\{(1 + 2\mu_i)^\lambda - 1 + 4\{1 - (1 - \mu_i v_i)^\lambda\}\}^{\omega_i} - \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}}, \right. \\ &\quad \left. \frac{\{(1 + 2\mu_i)^\lambda - 1 + 4\{1 - (1 - \mu_i v_i)^\lambda\}\}^{\omega_i} - \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}}{2\{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}} \right\rangle \\ &= \left\langle \frac{2\{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} [1 - (1 - \mu_i v_i)^{\lambda \omega_i}]}{\{(1 + 2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}}, \right. \\ &\quad \left. \frac{\{(1 + 2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}}{2\{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}} \right\rangle \\ \Rightarrow \bigotimes_{i=1}^n (\lambda \alpha_i)^{\omega_i} &= \left\langle \frac{2 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} [1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i}]}{\prod_{i=1}^n \{(1 + 2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}}, \right. \\ &\quad \left. \frac{\prod_{i=1}^n \{(1 + 2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}}{2 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}} \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{1}{\lambda} \otimes_{i=1}^n (\lambda \alpha_i)^{\omega_i} &= \left\langle \frac{\left(1 + 2 \frac{2 \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i} [1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i}]}{\prod_{i=1}^n \{(1+2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i}}}\right)^{1/\lambda} - 1}{2} \right. \\
 &\quad \left. \frac{2[1 - (\prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i})^{1/\lambda}]}{\left(1 + 2 \frac{2 \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i} [1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i}]}{\prod_{i=1}^n \{(1+2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i}}}\right)^{1/\lambda} - 1} \right\rangle \\
 &= \left\langle \frac{\left(\frac{\prod_{i=1}^n \{(1+2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i} + 4 \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i} [1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i}]}{\prod_{i=1}^n \{(1+2\mu_i)^\lambda + 3 - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1+2\mu_i)^\lambda - 1\}^{\omega_i}}}\right)^{1/\lambda} - 1}{2} \right. \\
 &\quad \left. \frac{2 \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)^{1/\lambda} [1 - (\prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i})^{1/\lambda}]}{\left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)} \right\rangle \\
 &\quad - 4 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i} \Big)^{1/\lambda} \\
 &\quad - \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)^{1/\lambda} \\
 &\quad \times \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right) \\
 &= \left\langle \frac{-4 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i} \Big)^{1/\lambda} - \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)^{1/\lambda}}{2 \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)^{1/\lambda}} \right. \\
 &\quad \left. \frac{2 \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)^{1/\lambda} \left[1 - \left(\prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i}\right)^{1/\lambda}\right]}{\left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda - 4(1 - \mu_i v_i)^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)} \right. \\
 &\quad \left. - 4 \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i v_i)^{\lambda \omega_i} \Big)^{1/\lambda} \right. \\
 &\quad \left. - \left(\prod_{i=1}^n \{3 + (1 + 2\mu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_i)^\lambda - 1\}^{\omega_i}\right)^{1/\lambda} \right.
 \end{aligned}$$

The parameter λ plays a regulatory role during the information aggregation process. When the parameter λ set to a special number then the GIFMIWG operator can be reduced.

For example, when $\lambda = 1$ then

$$\begin{aligned}
 &GIFMIWG(\alpha_1, \alpha_2, \dots, \alpha_n) \\
 &= \left\langle \frac{2 \left\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\right\}}{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2v_i)^{\omega_i} - 1}{2} \right\rangle.
 \end{aligned}$$

4.2 Generalized intuitionistic fuzzy multiplicative interactive ordered weighted geometric (GIFMIOWG) operator

Definition 8 Suppose there is a family of IMNs $(\alpha_1, \alpha_2, \dots, \alpha_n)$ then GIFMIOWG operator is defined as follows.

$$GIFMIOWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{\lambda} \otimes_{i=1}^n (\lambda \alpha_{\delta(i)})^{\omega_i}$$

where λ is the real number greater than zero, $\delta : (1, 2, \dots, n) \rightarrow (1, 2, \dots, n)$, IMN $\alpha_{\delta(i)}$ is the i th largest of IMN α_i .

Theorem 6 Let $\alpha_i = \langle \mu_i, \nu_i \rangle$ ($i = 1, 2, \dots, n$) be a family of IMNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector of α_i such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, then based on GIFMIOWG operator the aggregated IMN can be expressed as

$$\text{GIFMIOWG}(\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$= \left\{ \prod_{i=1}^n \{3 + (1 + 2\mu_{\delta(i)})^\lambda - 4(1 - \mu_{\delta(i)}\nu_{\delta(i)})^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right. \\ \times \left. \left\langle \frac{-4 \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_{\delta(i)}\nu_{\delta(i)})^{\lambda\omega_i}}{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\mu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}}, \right. \right. \\ \left. \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\mu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \left[1 - \prod_{i=1}^n (1 - \mu_{\delta(i)}\nu_{\delta(i)})^{\omega_i} \right]}{\left\{ \prod_{i=1}^n \{2 + (1 + 2\mu_{\delta(i)})^\lambda - 4(1 - \mu_{\delta(i)}\nu_{\delta(i)})^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}} \right. \\ \left. \left. - 4 \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_{\delta(i)}\nu_{\delta(i)})^{\lambda\omega_i} \right\}^{1/\lambda} \right. \\ \left. - \left\{ \prod_{i=1}^n \{3 + (1 + 2\mu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\mu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \right. \right\}.$$

Proof The proof is similar to Theorem 5, so it is omitted here.

When the parameter $\lambda = 1$, then

$$\text{GIFMIOWG}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{2 \{1 - \prod_{i=1}^n (1 - \mu_{\delta(i)}\nu_{\delta(i)})^{\omega_i}\}}{\prod_{i=1}^n (1 + 2\nu_{\delta(i)})^{\omega_i} - 1}, \frac{\prod_{i=1}^n (1 + 2\nu_{\delta(i)})^{\omega_i} - 1}{2} \right\rangle. \tag{6}$$

4.3 Generalized intuitionistic fuzzy multiplicative interaction hybrid weighted geometric (GIFMIHWG) operator

Definition 9 Let Ω be the set of all IMNs, the generalized intuitionistic fuzzy multiplicative interaction hybrid weighted geometric (GIFMIHWG) operator of dimension n is a mapping $\text{GIFMIHWG} : \Omega^n \rightarrow \Omega$ such that

$$\text{GIFMIHWG}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{\lambda} \otimes_{i=1}^n (\lambda \dot{\alpha}_{\sigma(i)})^{\omega_i}$$

where $\dot{\alpha}_{\sigma(i)}$ is the i th largest of the weighted IMNs $\dot{\alpha}_i$ ($\dot{\alpha}_i = \alpha_i^{w_i}, i = 1, 2, \dots, n$), n is the number of IMNs and $w = (w_1, w_2, \dots, w_n)^T$ is the standard weight vector of α_i .

Theorem 7 Let $\alpha_i = \langle \mu_i, \nu_i \rangle$ ($i = 1, 2, \dots, n$) be a collection of IMNs, $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the normalized weight vector of α_i , then $\text{GIFMIHWG}(\alpha_1, \alpha_2, \dots, \alpha_n) \in \text{IMNs}$. Moreover,

$GIFMIHWG(\alpha_1, \alpha_2, \dots, \alpha_n)$

$$\begin{aligned}
 &= \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 4(1 - \dot{\mu}_{\sigma(i)}\dot{v}_{\sigma(i)})^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right. \\
 &\quad \left. - 4 \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \dot{\mu}_{\sigma(i)}\dot{v}_{\sigma(i)})^{\lambda\omega_i} \right\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}, \\
 &\times \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}}{\left\{ \prod_{i=1}^n \{2 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 4(1 - \dot{\mu}_{\sigma(i)}\dot{v}_{\sigma(i)})^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}} \\
 &\quad \left\langle \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} [1 - \prod_{i=1}^n (1 - \dot{\mu}_{\sigma(i)}\dot{v}_{\sigma(i)})^{\omega_i}]}{\left\{ \prod_{i=1}^n \{2 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 4(1 - \dot{\mu}_{\sigma(i)}\dot{v}_{\sigma(i)})^\lambda\}^{\omega_i} + 3 \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}} \right\rangle \\
 &\quad - 4 \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \dot{\mu}_{\sigma(i)}\dot{v}_{\sigma(i)})^{\lambda\omega_i} \Bigg\}^{1/\lambda} \\
 &\quad - \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{\mu}_{\sigma(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{\mu}_{\sigma(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}.
 \end{aligned}$$

Proof The proof is similar to Theorem 5, so it is omitted here.

5 An approach to multiple criteria decision making with intuitionistic multiplicative information

In this section, we shall investigate the multiple criteria decision making (MCDM) problems based on the GIFMIWG, GIFMIOWG, GIFMIHWG operator in which the criteria weights take the form of real numbers, criteria values take the form of intuitionistic multiplicative numbers. For a MCDM problem, assume that a set of option/alternatives $X = \{X_1, X_2, \dots, X_m\}$ to be considered under the set of criteria $G = \{G_1, G_2, \dots, G_n\}$, $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ is the corresponding weighting vector, satisfying $w_j \in [0, 1], \sum_{j=1}^n w_j = 1$. Let there be a prioritization between the criteria expressed by the linear ordering $G_1 \succ G_2 \succ \dots \succ G_n$ (indicating that the criteria G_i has a higher priority than G_j , if $i < j$), $D = (D_1, D_2, \dots, D_q)$ be the set of decision makers, and also, let there be a prioritization between the decision makers expressed by the linear ordering $D_1 \succ D_2 \succ \dots \succ D_n$, indicating that the decision maker D_η has a higher priority than D_ζ , if $\eta < \zeta$. Let $A^{(k)} = (\alpha_{ij}^k)_{m \times n}$ be the intuitionistic fuzzy multiplicative decision matrix, and α_{ij} be an attribute value provided by the decision maker $D^k \in D$, for the alternative $X_i \in X$ with respect to the criteria $G_j \in G$. The evaluated values of the alternatives $X_i (i = 1, 2, \dots, m)$ are represented by intuitionistic multiplicative

numbers $\alpha_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$, where μ_{ij} the degree that the alternative X_i satisfies the criteria G_j given by the decision maker, and ν_{ij} indicates the degree that the alternative X_i doesn't satisfy the criteria G_j given by the decision maker, $\frac{1}{9} \leq \mu_{ij}, \nu_{ij} \leq 9$ and $\mu_{ij}\nu_{ij} \leq 1$. Then, we have the following decision making method which consists of the following steps.

- Step 1 Utilize the GIFMIWG or GIFMIOWG or GIFMIHWG operators to aggregate all the individual intuitionistic fuzzy multiplicative matrix $R^{(k)} = (\alpha_{ij}^{(k)})_{m \times n} (k = 1, 2, \dots, K)$ into the collective intuitionistic fuzzy multiplicative matrix $R = (\alpha_{ij})_{m \times n}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$.
- Step 2 Aggregate all the values α_{ij} for each option $\alpha_i (i = 1, 2, \dots, n)$ by GIFMIWG, GIFMIOWG or GIFMIHWG operators to derive the overall preference value $\alpha_i (i = 1, 2, \dots, m)$ of the alternative X_i .
- Step 3 *Compute score values* Calculate the scores of the overall collective overall values $\alpha_i, i = 1, 2, \dots, m$ by using the score function. If there is no difference between two scores $S(\alpha_i)$ and $S(\alpha_j)$ then we need to calculate the accuracy function $H(\alpha_i)$ and $H(\alpha_j)$ of the collective overall preference values α_i and α_j , respectively.
- Step 4 *Ranking the alternative* Rank all the alternatives $X_i (i = 1, 2, \dots, m)$ and select the most desirable alternative accordance with descending order of their score function and accuracy function.

Step 5 Do the sensitivity analysis on the parameter λ according to decision makers' preferences.

6 Numerical example

In order to demonstrate the applications of the development methodology, we will consider an example where the main task is to find the best professor for the School of Mathematics in a Thapar University, Patiala, India. The appointment is done by a committee of four decision makers, which take the responsibility for evaluating the candidates $X_i (i = 1, 2, 3, 4)$ with respect to the criteria $G_j (j = 1, 2, 3, 4)$ where (1) G_1 , the research capability, (2) G_2 , the past experience, (3) G_3 , subject knowledge, (4) G_4 , the teaching skill. They provided their evaluation values in terms of intuitionistic fuzzy multiplicative numbers and constructed the following four intuitionistic fuzzy decision matrices $D^{(q)} = (\alpha_{ij}^{(q)})_{m \times n}$, ($q = 1, 2, 3, 4$) as shown below:

$$D^1 = \begin{bmatrix} (1, 1) & (1/3, 1/4) & (3, 2/7) & (3/5, 2/3) \\ (1/4, 1/3) & (1, 1) & (3/4, 1/3) & (2, 1/3) \\ (2/7, 3) & (1/3, 3/4) & (1, 1) & (3, 1/4) \\ (2/3, 3/5) & (1/3, 2) & (1/4, 3) & (1, 1) \end{bmatrix},$$

$$D^2 = \begin{bmatrix} (1, 1) & (1/5, 4) & (3, 1/6) & (4, 1/7) \\ (4, 1/5) & (1, 1) & (1/8, 5) & (1/3, 1/2) \\ (1/6, 3) & (5, 1/8) & (1, 1) & (3, 1/4) \\ (1/7, 4) & (1/2, 1/3) & (1/4, 3) & (1, 1) \end{bmatrix}$$

$$D^3 = \begin{bmatrix} (1, 1) & (3, 1/5) & (1/3, 1) & (1/5, 2) \\ (1/5, 3) & (1, 1) & (1/4, 3) & (2/5, 1/2) \\ (1, 1/3) & (3, 1/4) & (1, 1) & (2, 3/7) \\ (2, 1/5) & (1/2, 2/5) & (3/7, 2) & (1, 1) \end{bmatrix},$$

$$D^4 = \begin{bmatrix} (1, 1) & (2/3, 1/3) & (3, 1/7) & (3/5, 1/2) \\ (1/3, 2/3) & (1, 1) & (1/7, 1/4) & (1/3, 1) \\ (1/7, 3) & (1/4, 1/7) & (1, 1) & (1, 2/3) \\ (1/2, 3/5) & (1, 1/3) & (2/3, 1) & (1, 1) \end{bmatrix}$$

Here, in the first decision matrix, D^1 , for example, the first preference is (1, 1) implies that when the first candidate X_1 compares with himself then the preference is (1, 1). On the other hand, the IMN (2/5, 1/2) indicates that the first professor argued that the degree of first candidate is priority to the second candidate is 2/5 while at the same time, he thinks the degree of first candidate is not a priority to the second candidate is 1/2. Similarly, the other observations have their meaning. Based on these preferences, the following steps are being executed for aggregating these different preferences by using GIFMIWG and GIFMIHWG operators corresponding to $\lambda = 0.8$ and $\omega = (0.2, 0.3, 0.1, 0.4)^T$ be the weight vector corresponding to

IMNs such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. The detailed calculation process is shown as follows.

6.1 By GIFMIWG operator

Step 1 To make use of GIFMIWG operator to aggregate $(\alpha_{1j}^k, \alpha_{2j}^k, \dots, \alpha_{ij}^k)$ and obtain the α_i^k , ($i, k = 1, 2, 3, 4$). The results corresponding to it have been given as

$$\alpha_1^1 = \langle 3.1461, 0.3179 \rangle, \alpha_2^1 = \langle 2.1182, 0.4721 \rangle,$$

$$\alpha_3^1 = \langle 1.6611, 0.6020 \rangle, \alpha_4^1 = \langle 0.7001, 1.4284 \rangle$$

$$\alpha_1^2 = \langle 1.1584, 0.8633 \rangle, \alpha_2^2 = \langle 1.0648, 0.9391 \rangle,$$

$$\alpha_3^2 = \langle 2.7847, 0.3591 \rangle, \alpha_4^2 = \langle 1.1242, 0.8896 \rangle$$

$$\alpha_1^3 = \langle 1.6112, 0.6207 \rangle, \alpha_2^3 = \langle 0.9512, 1.0513 \rangle,$$

$$\alpha_3^3 = \langle 2.6725, 0.3742 \rangle, \alpha_4^3 = \langle 1.5904, 0.6288 \rangle$$

$$\alpha_1^4 = \langle 3.5801, 0.2793 \rangle, \alpha_2^4 = \langle 1.5760, 0.6345 \rangle,$$

$$\alpha_3^4 = \langle 2.2319, 0.4480 \rangle, \alpha_4^4 = \langle 1.8565, 0.5386 \rangle.$$

Step 2 By using the GIFMIWG operator to aggregate the $\alpha_1^k, \alpha_2^k, \alpha_3^k, \alpha_4^k$ to IMNs α_i , the results were shown as follows

$$\alpha_1 = \langle 1.8294, 0.5466 \rangle, \alpha_2 = \langle 1.2406, 0.8061 \rangle,$$

$$\alpha_3 = \langle 2.4409, 0.4097 \rangle, \alpha_4 = \langle 1.3161, 0.7598 \rangle$$

Step 3 By definition of the score function given in Definition 1, respective scores values corresponding to α_i 's are

$$S(\alpha_1) = 3.3467, \quad S(\alpha_2) = 1.5391,$$

$$S(\alpha_3) = 5.9578, \quad S(\alpha_4) = 1.7322.$$

Step 4 Therefore, based on their score functions, it has been concluded that $S(\alpha_3) \succ S(\alpha_1) \succ S(\alpha_4) \succ S(\alpha_2)$. Thus the order of the four candidate is $X_3 \succ X_1 \succ X_4 \succ X_2$ and the best candidate for the job is X_3 .

On the other hand, if we aggregate these IMNs by the existing IMWG operator [20] then the aggregated IMNs becomes

$$\alpha_1 = \langle 0.7593, 0.6624 \rangle, \alpha_2 = \langle 0.4942, 0.9365 \rangle,$$

$$\alpha_3 = \langle 1.0277, 0.4798 \rangle, \alpha_4 = \langle 0.5896, 0.8914 \rangle$$

and hence their corresponding score values are

$$S(\alpha_1) = 1.1463, \quad S(\alpha_2) = 0.5277, \quad S(\alpha_3) = 2.1421,$$

$$S(\alpha_4) = 0.6614.$$

Therefore, $S(\alpha_3) \succ S(\alpha_1) \succ S(\alpha_4) \succ S(\alpha_2)$ and hence the best candidate for the job is X_3 which is same as that of the proposed aggregated operators.

6.2 By GIFMIHWG operator

Step 1 To make use of GIFMIHWG operator to aggregate $(\alpha_{1j}^k, \alpha_{2j}^k, \dots, \alpha_{lj}^k)$ and obtain the α_i^k . The results corresponding to it have been given as

$$\begin{aligned} \alpha_1^1 &= \langle 2.2556, 0.4433 \rangle, & \alpha_2^1 &= \langle 2.8430, 0.3517 \rangle, \\ \alpha_3^1 &= \langle 0.9232, 1.0832 \rangle, & \alpha_4^1 &= \langle 0.8512, 1.1748 \rangle \\ \alpha_1^2 &= \langle 1.8596, 0.5377 \rangle, & \alpha_2^2 &= \langle 0.9646, 1.0367 \rangle, \\ \alpha_3^2 &= \langle 1.0387, 0.9627 \rangle, & \alpha_4^2 &= \langle 0.4613, 2.1676 \rangle \\ \alpha_1^3 &= \langle 0.8480, 1.1793 \rangle, & \alpha_2^3 &= \langle 0.6653, 1.5030 \rangle, \\ \alpha_3^3 &= \langle 1.9806, 0.5049 \rangle, & \alpha_4^3 &= \langle 1.6003, 0.6249 \rangle \\ \alpha_1^4 &= \langle 2.4024, 0.4162 \rangle, & \alpha_2^4 &= \langle 2.3060, 0.4337 \rangle, \\ \alpha_3^4 &= \langle 0.8150, 1.2270 \rangle, & \alpha_4^4 &= \langle 1.3081, 0.7644 \rangle. \end{aligned}$$

Step 2 By using the GIFMIHWG operator to aggregate the $\alpha_1^k, \alpha_2^k, \alpha_3^k, \alpha_4^k$ to IMNs α_i , the results were shown as follows

$$\begin{aligned} \alpha_1 &= \langle 1.5224, 0.6568 \rangle, & \alpha_2 &= \langle 1.3358, 0.7486 \rangle, \\ \alpha_3 &= \langle 1.0521, 0.9505 \rangle, & \alpha_4 &= \langle 0.9464, 1.0566 \rangle. \end{aligned}$$

Step 3 By definition of the score function given in Definition 1, respective scores values corresponding to α_i 's are

$$\begin{aligned} S(\alpha_1) &= 2.3178, & S(\alpha_2) &= 1.7843, \\ S(\alpha_3) &= 1.1069, & S(\alpha_4) &= 0.8957. \end{aligned}$$

Step 4 Therefore, based on their score functions, it has been concluded that $S(\alpha_1) \succ S(\alpha_2) \succ S(\alpha_3) \succ S(\alpha_4)$. Thus the order of the four candidate is $X_1 \succ X_2 \succ X_3 \succ X_4$ and the best candidate for the job is X_1 .

On the other hand, if we apply IMWA operator to aggregated by using HWA operator then we get the aggregated IMN as

$$\begin{aligned} \alpha_1 &= \langle 0.6616, 0.7789 \rangle, & \alpha_2 &= \langle 0.2904, 0.9269 \rangle, \\ \alpha_3 &= \langle 0.5820, 1.0837 \rangle, & \alpha_4 &= \langle 0.5100, 1.2491 \rangle \end{aligned}$$

and hence order of the four candidate is $X_1 \succ X_3 \succ X_4 \succ X_2$. Therefore, the best candidate for the job is X_1 .

6.3 Sensitivity analysis

In order to investigate the variation trends of the scores and the rankings of the alternatives with the change of the values of the attitudinal character parameter λ from 0 to 10. The variation of the score values and their corresponding ranking of the alternatives obtained by the proposed operators, GIFMIWG, GIFMIOWG and GIFMIHWG for different values of λ are summarized in Table 1 which

Table 1 Effect of λ on the score values and ranking of the alternatives

	Score values ($\lambda = 0.5$)			Score values ($\lambda = 0.7$)			Score values ($\lambda = 1$)		
	GIFMIWG	GIFMIOWG	GIFMIHWG	GIFMIWG	GIFMIOWG	GIFMIHWG	GIFMIWG	GIFMIOWG	GIFMIHWG
X_1	12.1023	7.7095	6.8052	4.4614	3.5887	3.0614	2.0691	1.8542	1.5130
X_2	3.5198	6.4016	5.0295	1.9097	2.5962	2.3028	1.1053	1.1753	1.1428
X_3	17.8671	3.2211	2.6032	7.8277	1.7807	1.3759	3.9740	1.0745	0.7981
X_4	4.3896	1.7343	2.0762	2.1971	1.0431	1.1014	1.2139	0.6733	0.6490
Ranking	(3142)	(1234)	(1234)	(3142)	(1234)	(1234)	(3142)	(1234)	(1234)
	Score values ($\lambda = 2$)			Score values ($\lambda = 2.5$)			Score values ($\lambda = 3.5$)		
	GIFMIWG	GIFMIOWG	GIFMIHWG	GIFMIWG	GIFMIOWG	GIFMIHWG	GIFMIWG	GIFMIOWG	GIFMIHWG
X_1	0.6326	0.6695	0.5209	0.4564	0.5019	0.3927	0.2890	0.3307	0.2663
X_2	0.4655	0.3564	0.3277	0.3603	0.2625	0.2275	0.2436	0.1743	0.1340
X_3	1.4019	0.4750	0.3351	1.0025	0.3626	0.2513	0.5830	0.2407	0.1539
X_4	0.5266	0.3413	0.2940	0.4262	0.2849	0.2387	0.3147	0.2175	0.1751
Ranking	(3142)	(1324)	(1324)	(3142)	(1342)	(1342)	(3412)	(1342)	(1432)
	Score values ($\lambda = 5.5$)			Score values ($\lambda = 7.5$)			Score values ($\lambda = 10$)		
	GIFMIWG	GIFMIOWG	GIFMIHWG	GIFMIWG	GIFMIOWG	GIFMIHWG	GIFMIWG	GIFMIOWG	GIFMIHWG
X_1	0.1687	0.1958	0.1644	0.1786	0.1723	0.1196	0.0972	0.1114	0.0908
X_2	0.1437	0.1076	0.0741	0.1135	0.1073	0.0518	0.0758	0.0614	0.0388
X_3	0.2744	0.1420	0.0748	0.2067	0.1144	0.0461	0.1175	0.0772	0.0311
X_4	0.2083	0.1486	0.1104	0.1813	0.1270	0.1129	0.1183	0.0874	0.0541
Ranking	(3412)	(1432)	(1432)	(3412)	(1432)	(1423)	(4312)	(1432)	(1423)

shows to the decision makers that they can choose the values of it according to their preferences. If $\lambda = 1$ then the generalized intuitionistic multiplicative fuzzy weighted geometric interaction averaging operator reduces to the IFMIWG operator. Also, it has been highlighted that $\lambda = 1$ means that the attitude of decision makers is neutral and it is obtained that the overall score values of different alternatives are decreasing as the increase of λ . Thus, the management meaning of λ is that the decision makers' different preference had effects on the score values of alternatives, which leads to the different optimal alternative.

7 Conclusion

The objective of this paper is to present a robust geometric aggregation operators to fuse the different preference of the decision maker under the condition that the preferences corresponding to each alternatives are in the form of intuitionistic fuzzy multiplicative numbers. From the existing operators, it has been observed that interaction between membership and non-membership does not play any role and hence change of non-membership degree of IMNs does not effect on the degree of membership functions. Therefore, based on that it is unable to rank the alternative up to desired degree. For handling this issues, the new operational laws have been proposed and based on it, some generalized aggregation operations namely intuitionistic fuzzy multiplicative interactive weighted geometric (IFMIWG), intuitionistic fuzzy multiplicative interactive ordered weighted geometric (IFMIOWG) and intuitionistic fuzzy multiplicative interactive hybrid weighted geometric (IFMIHWG) operators have been developed. Also, some desirable properties of these operators such as idempotency, boundedness, monotonicity, shift variance, homogeneity etc., have been studied in details. All these operators have been investigated with a case study regarding the recruitment of an outstanding teacher in School of Mathematics, Thapar University, Patiala, India. From the comparative studies, it has been concluded that the decision making method proposed in this paper is more stable and pessimistic than the existing operational law given by Xia et al. [20]. Thus, it has been concluded from the aforementioned results that the proposed decision making method can be suitably utilized to solve the multiple and decision making problem. In our further research will focus on adopting this approach to some more complicated applications in the field of pattern recognition, fuzzy cluster analysis and uncertain programming.

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