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ORIGINAL ARTICLE

An information fusion technology for triadic decision contexts

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Abstract In this paper, the notion of a projected context is proposed to explore a novel algorithm of computing triadic concepts of a triadic context, and a triadic decision context is defined by combining triadic contexts. Then a rule acquisition method is presented for triadic decision contexts. It can be considered as an information fusion technology for decision-making analysis of multi-source data if the data under each condition is viewed as a single-source data. Moreover, a knowledge reduction framework is established to simplify knowledge discovery. Finally, discernibility matrix and Boolean function are constructed to compute all reducts, which is beneficial to the acquisition of compact rules from a triadic decision context.

Keywords Triadic concept analysis · Triadic context · Triadic decision context · Rule acquisition · Information fusion

1 Introduction

Formal concept analysis [1] mainly discusses how to obtain binary concepts and their hierarchy from a given binary relation between objects and attributes. Now, this theory has been applied to a variety of fields such as knowledge discovery, data mining and software engineering [2–8]. However, in the real world, the relation between objects and attribute is often established under certain conditions. In order to widen the application scope, Lehmann and Wille [9] extended classical formal concept analysis into triadic concept analysis. More studies on triadic concept analysis can be found in [10–15].

Formal decision context was proposed by Zhang and Qiu [16] in 2005 for decision-making analysis of the data, and in recent years it has been studied by many scholars. For example, Shao [17] and Qu et al. [18] discussed rule acquisition in formal decision contexts. Wei et al. [19] put forward two types of attribute reduction methods for formal decision contexts and Hong et al. [27] gave another one. Wang and Zhang [20] divided formal decision contexts into two categories: consistent and inconsistent ones, and developed attribute reduction approaches for consistent decision contexts. Based on granular computing, Wu et al. [21] defined a new kind of consistent decision contexts. Shao et al [22] also investigated the issue of attribute reduction in consistent decision contexts from the viewpoint of rule acquisition. Considering that an inconsistent decision context appears more often than a consistent one, Li et al. [23] presented an attribute reduction technology for inconsistent decision contexts. Since computing a minimal reduct of a formal decision context is computationally expensive, Li et al. [24] designed a heuristic attribute reduction method for formal decision contexts. For making better decision analysis, Li et al. [25] gave a rule acquisition oriented attribute reduction approach for general formal decision contexts. Also, it deserves to be mentioned that the issue of object reduction was discussed in [26] for formal decision contexts. Except the classical formal decision contexts, rule acquisition and attribute

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reduction were investigated in generalized decision contexts such as incomplete [28], fuzzy [29, 30] and real ones [31–33].

According to the above discussion, although some researchers investigated rule acquisition and attribute (object) reduction in classical, incomplete, fuzzy and real decision contexts, little work has been done on these topics for triadic decision contexts. Motivated by this problem, the current study focuses on the issues of rule acquisition and attribute reduction in triadic decision contexts, which can be regarded as an information fusion technology for decision-making analysis since a triadic decision context can be viewed as multi-source data if one treats the data under each condition as a single-source data.

The rest of this paper is organized as follows. Section 2 briefly recalls some basic notions related to triadic context, triadic concept and triadic decision context. Section 3 discusses how to derive implication rules from a triadic decision context and puts forward a rule acquisition method. Section 4 constructs discernibility matrix and Boolean function to compute all reducts of a triadic decision context. The paper is then concluded with a summary.

2 Triadic concept analysis

In this section, we briefly introduce some basic notions about triadic concept analysis to facilitate our subsequent discussion.

Definition 1 [9] A triadic context is a quadruple (U, A, C, I), where U is a nonempty and finite set of objects, A is a nonempty and finite set of attributes, C is a nonempty and finite set of conditions, and I is a ternary relation between U, A and C (i.e., $I \subseteq U \times A \times C$). Here, $(x, a, c) \in I$ means that the object x owns the attribute a under the condition c.

For a triadic context (U, A, C, I), we can build three binary contexts $(X \times A, C, I^{(1)})$, $(U, B \times C, I^{(2)})$ and $(U, A \times H, I^{(3)})$ by fixing the nonempty object set $X \subseteq U$, nonempty attribute set $B \subseteq A$ and nonempty condition set $H \subseteq C$, respectively. Note that here X, B and H are all treated as a whole. To be more concrete, for any $X \times \{a\} \subseteq X \times A$ and $c \in C$, the relationship $(X \times \{a\}, c) \in I^{(1)}$ means that for every object $x \in X$, there is $(x, a, c) \in I$. Similarly, for any $x \in U$ and $B \times \{c\} \subseteq B \times C$, the relationship $(x, B \times \{c\}) \in I^{(2)}$ means that for every attribute $a \in B$, there is $(x, a, c) \in I$; for any $x \in U$ and $\{a\} \times H \subseteq A \times H$, the relationship $(x, \{a\} \times H) \in I^{(3)}$ means that for every condition $c \in H$, there is $(x, a, c) \in I$.

Hereinafter, we say that $(X \times A, C, I^{(1)})$, $(U, B \times C, I^{(2)})$ and $(U, A \times H, I^{(3)})$ are projected contexts of the triadic context (U, A, C, I) by fixing the nonempty object set $X \subseteq U$, nonempty attribute set $B \subseteq A$ and nonempty condition set $H \subseteq C$, respectively. Similar to other binary contexts, we can define concept-forming operators for the projected contexts $(X \times A, C, I^{(1)})$, $(U, B \times C, I^{(2)})$ and $(U, A \times H, I^{(3)})$.

Definition 2 [15] Let $(X \times A, C, I^{(1)})$ be a projected context of (U, A, C, I) with $X \subseteq U$. For $B \subseteq A$ and $H \subseteq C$, we define

$$B^{X^{(1)}} = \{c \in C \mid \forall a \in B, (X \times \{a\}, c) \in I^{(1)}\},$$
$$H^{X^{(1)}} = \{a \in A \mid \forall c \in H, (X \times \{a\}, c) \in I^{(1)}\},$$

where $X^{(1)}$ is a mapping from the power set 2^A to the power set 2^C, and it is also used as a mapping from 2^C to 2^A when there is no confusion.

Definition 3 [15] Let $(U, B \times C, I^{(2)})$ be a projected context of (U, A, C, I) with $B \subseteq A$. For $X \subseteq U$ and $H \subseteq C$, we define

$$X^{B^{(2)}} = \{c \in C \mid \forall x \in X, (x, B \times \{c\}) \in I^{(2)}\},$$
$$H^{B^{(2)}} = \{x \in U \mid \forall c \in H, (x, B \times \{c\}) \in I^{(2)}\},$$

where $B^{(2)}$ is a mapping from the power set 2^U to the power set 2^C, and it is also used as a mapping from 2^C to 2^U when there is no confusion.

Definition 4 [15] Let $(U, A \times H, I^{(3)})$ be a projected context of (U, A, C, I) with $H \subseteq C$. For $X \subseteq U$ and $B \subseteq A$, we define

$$X^{H^{(3)}} = \{a \in A \mid \forall x \in X, (x, \{a\} \times H) \in I^{(3)}\},$$
$$B^{H^{(3)}} = \{x \in U \mid \forall a \in B, (x, \{a\} \times H) \in I^{(3)}\},$$

where $H^{(3)}$ is a mapping from the power set 2^U to the power set 2^A, and it is also used as a mapping from 2^A to 2^U when there is no confusion.

Definition 5 [15] Let (U, A, C, I) be a triadic context. For $X \subseteq U$, $B \subseteq A$ and $H \subseteq C$, the ordered pair (X, B, H) is called a triadic concept of (U, A, C, I) if $B = H^{X^{(1)}}$, $H = B^{X^{(1)}}$, $X = H^{B^{(2)}}$, $H = X^{B^{(2)}}$, $X = B^{H^{(3)}}$ and $B = X^{H^{(3)}}$. Here X, B and H are termed as the extent, intent and modus of the triadic concept (X, B, H), respectively.

In other words, the concept-forming operators for projected contexts are employed in Definition 5 to define triadic concepts. Such a way of formalizing a triadic concept is equivalent to the one given in [9]. For convenience, we denote the set of all triadic concepts of (U, A, C, I) by $\mathfrak{B}(U, A, C, I)$.

Proposition 1 *Let (U, A, C, I) be a triadic context. For $X, X_1, X_2 \subseteq U$, $B, B_1, B_2 \subseteq A$ and $H \subseteq C$, the following conclusions hold:*

(i) $X_1 \subseteq X_2 \Rightarrow X_2^{H^{(3)}} \subseteq X_1^{H^{(3)}}$;

(ii) $B_1 \subseteq B_2 \Rightarrow B_2^{H^{(3)}} \subseteq B_1^{H^{(3)}}$;

(iii) $X \subseteq X^{H^{(3)}H^{(3)}}, B \subseteq B^{H^{(3)}H^{(3)}}$.

Proof

(i) For any $a \in X_2^{H^{(3)}}$, we have $(x, \{a\} \times H) \in I^{(3)}$ for every $x \in X_2$. Since $X_1 \subseteq X_2$, it follows $(x, \{a\} \times H) \in I^{(3)}$ for every $x \in X_1$, which yields $a \in X_1^{H^{(3)}}$. Then, $X_2^{H^{(3)}} \subseteq X_1^{H^{(3)}}$ is at hand.

(ii) For any $x \in B_2^{H^{(3)}}$, we have $(x, \{a\} \times H) \in I^{(3)}$ for every $a \in B_2$. Since $B_1 \subseteq B_2$, it follows $(x, \{a\} \times H) \in I^{(3)}$ for every $a \in B_1$, which leads to $x \in B_1^{H^{(3)}}$. That is, $B_2^{H^{(3)}} \subseteq B_1^{H^{(3)}}$ is established.

(iii) For any $x \in X$, we have $(x, \{a\} \times H) \in I^{(3)}$ if $a \in X^{H^{(3)}}$, which implies $x \in X^{H^{(3)}H^{(3)}}$. As a result, $X \subseteq X^{H^{(3)}H^{(3)}}$. In a similar manner, we can prove $B \subseteq B^{H^{(3)}H^{(3)}}$. □

Similar to the case in binary contexts, the ordered pair $(X, B \times H)$ is called a concept of the projected context $(U, A \times H, I^{(3)})$ if $X = B^{H^{(3)}}$ and $B = X^{H^{(3)}}$. Then, we denote the set of all concepts of $(U, A \times H, I^{(3)})$ by $\underline{\mathfrak{B}}(U, A \times H, I^{(3)})$. Moreover, we define *meet*, *join* and partial order relation on $\underline{\mathfrak{B}}(U, A \times H, I^{(3)})$ as follows:

(i) $(X_1, B_1 \times H) \wedge (X_2, B_2 \times H)$
$= (X_1 \cap X_2, (B_1 \cup B_2)^{H^{(3)}H^{(3)}} \times H)$,

(ii) $(X_1, B_1 \times H) \vee (X_2, B_2 \times H)$
$= ((X_1 \cup X_2)^{H^{(3)}H^{(3)}}, (B_1 \cap B_2) \times H)$,

(iii) $(X_1, B_1 \times H) \preceq (X_2, B_2 \times H) \Leftrightarrow X_1 \subseteq X_2$.

Definition 6 [15] Let $(U, A \times H, I^{(3)})$ be a projected context of (U, A, C, I), $E \subseteq A$ and $I_E^{(3)} = I^{(3)} \cap (U \times (E \times H))$. Then, the binary context $(U, E \times H, I_E^{(3)})$ is called a sub-context of $(U, A \times H, I^{(3)})$.

Moreover, we define

$$X^{H_E^{(3)}} = \{a \in E \mid \forall x \in X, (x, \{a\} \times H) \in I_E^{(3)}\},$$
$$B^{H_E^{(3)}} = \{x \in U \mid \forall a \in B, (x, \{a\} \times H) \in I_E^{(3)}\}.$$

Then, $(X, B \times H)$ is called a concept of the sub-context $(U, E \times H, I_E^{(3)})$ if $X = B^{H_E^{(3)}}$ and $B = X^{H_E^{(3)}}$. Similarly, we denote the set of all concepts of $(U, E \times H, I_E^{(3)})$ by $\underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$ and

$$\mathfrak{U}(U, E \times H, I_E^{(3)}) = \{X | (X, B \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})\}.$$

Proposition 2 *Let $(U, A \times H, I^{(3)})$ be a projected context of (U, A, C, I) and $E \subseteq A$. For $X, X_1, X_2 \subseteq U$ and $B, B_1, B_2 \subseteq E$, the following conclusions hold:*

(i) $X^{H_E^{(3)}} = X^{H^{(3)}} \cap E$, $B^{H_E^{(3)}} = B^{H^{(3)}}$;

(ii) $X_1 \subseteq X_2 \Rightarrow X_2^{H_E^{(3)}} \subseteq X_1^{H_E^{(3)}}$;

(iii) $B_1 \subseteq B_2 \Rightarrow B_2^{H_E^{(3)}} \subseteq B_1^{H_E^{(3)}}$;

(iv) $X \subseteq X^{H_E^{(3)}H_E^{(3)}}$, $B \subseteq B^{H_E^{(3)}H_E^{(3)}}$;

(v) $(X^{H_E^{(3)}H_E^{(3)}}, X^{H_E^{(3)}} \times H), (B^{H_E^{(3)}}, B^{H_E^{(3)}H_E^{(3)}} \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$.

Proof (i) $X^{H_E^{(3)}} = \{a \in E \mid \forall x \in X, (x, \{a\} \times H) \in I_E^{(3)}\} = \{a \in A \mid \forall x \in X, (x, \{a\} \times H) \in I^{(3)}\} \cap E = X^{H^{(3)}} \cap E$. That is, $X^{H_E^{(3)}} = X^{H^{(3)}} \cap E$ is at hand. Moreover, $B^{H_E^{(3)}} = \{x \in U \mid \forall a \in B, (x, \{a\} \times H) \in I_E^{(3)}\} = \{x \in U \mid \forall a \in B, (x, \{a\} \times H) \in I^{(3)}\} = B^{H^{(3)}}$. That is, $B^{H_E^{(3)}} = B^{H^{(3)}}$ is established.

The proofs of (ii), (iii) and (iv) are similar to those of (i), (ii) and (iii) in Proposition 1. The remainder is to prove (v).

Note that $X \subseteq X^{H_E^{(3)}H_E^{(3)}}$ holds due to (iv), which leads to $X^{H_E^{(3)}H_E^{(3)}H_E^{(3)}} \subseteq X^{H_E^{(3)}}$. On the other hand, $X^{H_E^{(3)}} \subseteq X^{H_E^{(3)}H_E^{(3)}H_E^{(3)}}$ is true according to (iv). To sum up, it follows $X^{H_E^{(3)}} = (X^{H_E^{(3)}H_E^{(3)}})^{H_E^{(3)}}$. By combining it with $X^{H_E^{(3)}H_E^{(3)}} = (X^{H_E^{(3)}})^{H_E^{(3)}}$, we conclude $(X^{H_E^{(3)}H_E^{(3)}}, X^{H_E^{(3)}} \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$. Moreover, $(B^{H_E^{(3)}}, B^{H_E^{(3)}H_E^{(3)}} \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$ can be proved in a similar manner. □

Proposition 3 *Let $(U, A \times H, I^{(3)})$ be a projected context of (U, A, C, I) and $F \subseteq E \subseteq A$. Then $\mathfrak{U}(U, F \times H, I_F^{(3)}) \subseteq \mathfrak{U}(U, E \times H, I_E^{(3)})$.*

Proof For any $X \in \mathfrak{U}(U, F \times H, I_F^{(3)})$, there exists $B \subseteq A$ such that $(X, B \times H) \in \underline{\mathfrak{B}}(U, F \times H, I_F^{(3)})$. Let $S = E \backslash F$.

Case 1 If $X^{H_S^{(3)}} = \emptyset$, then $(X, B \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$. As a result, $X \in \mathfrak{U}(U, E \times H, I_E^{(3)})$.

Case 2 If $X^{H_S^{(3)}} \neq \emptyset$, we, without loss of generality, suppose $X^{H_S^{(3)}} = B'$. Then, $(X, (B \cup B') \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$. That is, $X \in \mathfrak{U}(U, E \times H, I_E^{(3)})$ still holds.

To sum up, $\mathfrak{U}(U, F \times H, I_F^{(3)}) \subseteq \mathfrak{U}(U, E \times H, I_E^{(3)})$ is at hand. □

Table 1 A triadic decision context $(U, A, C_A, I, D, C_D, J)$

	c_1					c_2					c_3			c_4		
	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5	h_1	h_2	h_3	h_1	h_2	h_3
x_1	1	1	0	1	1	0	1	1	1	0	0	1	0	0	1	1
x_2	1	1	1	0	0	1	0	0	0	1	1	0	1	1	0	1
x_3	0	0	0	1	0	1	0	1	1	0	1	0	1	1	0	0
x_4	1	1	1	0	0	0	1	0	1	0	1	1	0	0	0	1

Definition 7 A triadic decision context is a septuple $(U, A, C_A, I, D, C_D, J)$, where (U, A, C_A, I) and (U, D, C_D, J) with $A \cap D = \emptyset$ and $C_A \cap C_D = \emptyset$ are two triadic contexts. Here, A and D are called the conditional and decision attribute sets, respectively.

Example 1 Table 1 shows a triadic decision context $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$, where $U = \{x_1, x_2, x_3, x_4\}$ means four days: Monday, Tuesday, Wednesday, Thursday, $A = \{a_1, a_2, a_3, a_4, a_5\}$ means five kinds of ingredients: Potatoes, Tomatoes, Eggs, Milk, Pork, $C_A = \{c_1, c_2\}$ means two suppliers: Supermarket 1 and Supermarket 2 (the dates of the two supermarkets providing ingredients are uncertain, but there is at least one supermarket providing ingredients everyday), $D = \{h_1, h_2, h_3\}$ means three kinds of food: Sandwich, Hamburg, Pizza, and $C_D = \{c_3, c_4\}$ means two restaurants: Restaurant 1 and Restaurant 2.

Firstly, we design Algorithm 1 to compute the triadic concepts of the triadic context (U, A, C, I) whose main idea is as follows: for every condition set $H \subseteq C$, check whether the concepts of the projected context $(U, A \times H, I^{(3)})$ are triadic concepts of (U, A, C, I).

Algorithm 1. Computing triadic concepts of a triadic context.

Input: A triadic context (U, A, C, I).
Output: All triadic concepts of (U, A, C, I).

1. Initialize $\Omega = \emptyset$.
2. For every condition set $H \subseteq C$, store the projected context $(U, A \times H, I^{(3)})$.
3. Use the existing concept lattice construction algorithm [3] to build the concept lattice $\underline{\mathfrak{B}}(U, A \times H, I^{(3)})$ of each projected context $(U, A \times H, I^{(3)})$.
4. For each element $(X, B \times H)$ of every concept lattice $\underline{\mathfrak{B}}(U, A \times H, I^{(3)})$, if $B \not\subseteq X^{(C\setminus H)^{(3)}}$, then $\Omega \leftarrow \Omega \cup \{(X, B, H)\}$.
5. Output Ω and end the algorithm.

It is easy to observe that the time complexity of Algorithm 1 is exponential.

Table 2 All triadic concepts of (U, A, C_A, I)

	Extent	Intent	Modus
1	$\{x_1, x_2, x_3, x_4\}$	$\{a_1, a_2, a_3, a_4, a_5\}$	$\emptyset$
2	$\{x_1\}$	$\{a_1, a_2, a_4, a_5\}$	$\{c_1\}$
3	$\{x_2, x_4\}$	$\{a_1, a_2, a_3\}$	$\{c_1\}$
4	$\{x_1, x_2, x_4\}$	$\{a_1, a_2\}$	$\{c_1\}$
5	$\{x_1\}$	$\{a_2, a_3, a_4\}$	$\{c_2\}$
6	$\{x_2\}$	$\{a_1, a_5\}$	$\{c_2\}$
7	$\{x_3\}$	$\{a_1, a_3, a_4\}$	$\{c_2\}$
8	$\{x_1, x_3\}$	$\{a_3, a_4\}$	$\{c_2\}$
9	$\{x_1, x_4\}$	$\{a_2, a_4\}$	$\{c_2\}$
10	$\{x_2, x_3\}$	$\{a_1\}$	$\{c_2\}$
11	$\{x_1, x_3, x_4\}$	$\{a_4\}$	$\{c_2\}$
12	$\{x_1\}$	$\{a_2, a_4\}$	$\{c_1, c_2\}$
13	$\{x_2\}$	$\{a_1\}$	$\{c_1, c_2\}$
14	$\{x_1, x_3\}$	$\{a_4\}$	$\{c_1, c_2\}$
15	$\{x_1, x_4\}$	$\{a_2\}$	$\{c_1, c_2\}$
16	$\emptyset$	$\{a_1, a_2, a_3, a_4, a_5\}$	$\{c_1, c_2\}$
17	$\{x_1, x_2, x_3, x_4\}$	$\emptyset$	$\{c_1, c_2\}$

Now, we employ Algorithm 1 to compute all triadic concepts of (U, A, C_A, I) and (U, D, C_D, J) which can be found in Tables 2 and 3, respectively.

Moreover, we depict the diagrams of $\underline{\mathfrak{B}}(U, A, C, I)$ and $\underline{\mathfrak{B}}(U, D, C_D, J)$ in Figs. 1 and 2, respectively. In the first figure, the geometric structure of the triadic concepts is represented by the triangular pattern in the center of the diagram. The circles represent the triadic concepts and the lines the equivalence classes. The perforated lines indicate the connection to the extent diagram on the right, to the intent diagram on the lower left, and to the modus diagram above. A circle of the line diagram on the right represents the extent consisting of those objects whose signs are attached to this circle below. The intents and modi can analogously be read from the diagram where the intents get larger from the upper left to the lower right and the modi get larger from the upper right to the lower left. For instance, the second circle from bottom to top connects horizontally with the extent $\{x_3\}$, to the lower left with the

Table 3 All triadic concepts of (U, D, C_D, J)

	Extent	Intent	Modus
1	$\{x_1, x_2, x_3, x_4\}$	$\{h_1, h_2, h_3\}$	$\emptyset$
2	$\{x_4\}$	$\{h_1, h_2\}$	$\{c_3\}$
3	$\{x_2, x_3\}$	$\{h_1, h_3\}$	$\{c_3\}$
4	$\{x_1, x_4\}$	$\{h_2\}$	$\{c_3\}$
5	$\{x_2, x_3, x_4\}$	$\{h_1\}$	$\{c_3\}$
6	$\{x_1\}$	$\{h_2, h_3\}$	$\{c_4\}$
7	$\{x_1, x_2, x_4\}$	$\{h_3\}$	$\{c_4\}$
8	$\{x_1\}$	$\{h_2\}$	$\{c_3, c_4\}$
9	$\{x_2\}$	$\{h_1, h_3\}$	$\{c_3, c_4\}$
10	$\{x_2, x_3\}$	$\{h_1\}$	$\{c_3, c_4\}$
11	$\emptyset$	$\{h_1, h_2, h_3\}$	$\{c_3, c_4\}$
12	$\{x_1, x_2, x_3, x_4\}$	$\emptyset$	$\{c_3, c_4\}$

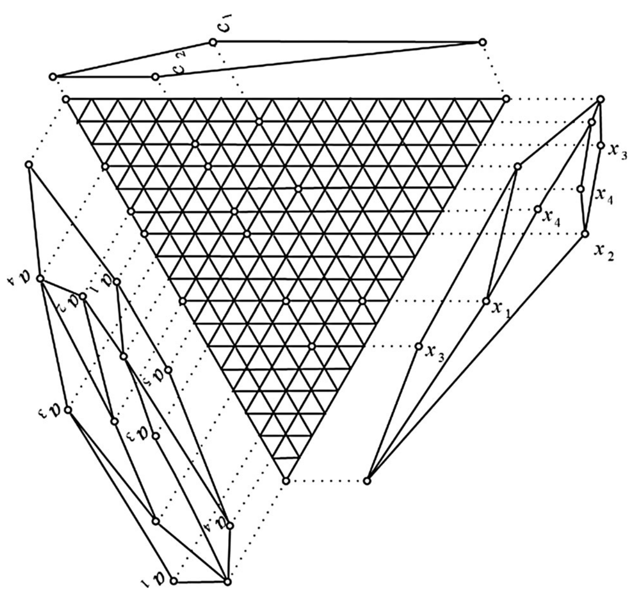

Fig. 1 The diagram of $\underline{\mathfrak{B}}(U, A, C, I)$

intent $\{a_1, a_3, a_4\}$, and to the upper left with the modus $\{c_2\}$; hence it represents the triadic concept $(\{x_3\}, \{a_1, a_3, a_4\}, \{c_2\})$.

3 A rule acquisition based knowledge reduction framework for triadic decision contexts

In this section, we discuss how to derive implication rules from a triadic decision context and put forward a corresponding knowledge reduction framework.

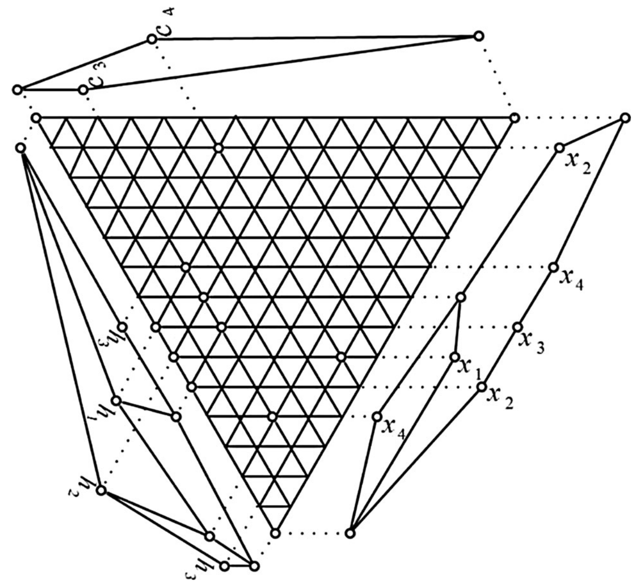

Fig. 2 The diagram of $\underline{\mathfrak{B}}(U, D, C_D, J)$

Definition 8 Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, E be a nonempty subset of A, H be a nonempty subset of C_A, and K be a nonempty subset of C_D. For any $(X, B \times H) \in \underline{\mathfrak{B}}(U, E \times H, I_E^{(3)})$ and $(Y, G \times K) \in \underline{\mathfrak{B}}(U, D \times K, J^{(3)})$, if $X \subseteq Y$ and X, Y, B, G are all nonempty, we say that $(Y, G \times K)$ can be implied by $(X, B \times H)$, which is denoted by $(X, B \times H) \to (Y, G \times K)$.

The following conclusion can be drawn from $(X, B \times H) \to (Y, G \times K)$: if $x \in U$ is shared by all the attributes in B under every condition of H, then x possesses all the attributes in G under each condition of K. That is, we can obtain a constrained decision rule "If B under H, then G under K" which is denoted by $B \times H \to G \times K$. Note that constrained decision rules in triadic decision contexts are special implication rules [9] in triadic contexts.

Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context. For any nonempty subset $E \subseteq A$, $H \subseteq C_A$ and $K \subseteq C_D$, we denote by $\mathfrak{R}(E, H, D, K) = \{B \times H \to G \times K \mid (X, B \times H) \in \mathfrak{B}(U, E \times H, I_E^{(3)}), (Y, G \times K) \in \mathfrak{B}(U, D \times K, J^{(3)})\}$ the set of all the constrained decision rules between the concepts of $\mathfrak{B}(U, E \times H, I_E^{(3)})$ and those of $\mathfrak{B}(U, D \times K, J^{(3)})$.

Definition 9 Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, $E \subseteq A$, $H \subseteq C_A$ and $K \subseteq C_D$. For $B' \times H \to G' \times K \in \mathfrak{R}(E, H, D, K)$ and $B'' \times H \to G'' \times K \in \mathfrak{R}(A, H, D, K)$, if $B' \subseteq B''$ and $G'' \subseteq G'$, we say that $B'' \times H \to G'' \times K$ can be implied by $B' \times H \to G' \times K$ and denote this implication relationship by $B' \times H \to G' \times K \Rightarrow B'' \times H \to G'' \times K$.

Definition 10 Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, $E \subseteq A$, $H \subseteq C_A$, $K \subseteq C_D$, $\Sigma \subseteq \mathfrak{R}(E, H, D, K)$ and $\Omega \subseteq \mathfrak{R}(A, H, D, K)$. If for any $B \times H \to G \times K \in \Omega$, there exists $B_0 \times H \to G_0 \times K \in \Sigma$ such that $B_0 \times H \to G_0 \times K \Rightarrow B \times H \to G \times K$, we say that Ω can be implied by Σ, which is denoted by $\Sigma \Rightarrow \Omega$.

Definition 11 For a triadic decision context $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$, $E \subseteq A$, $H \subseteq C_A$ and $K \subseteq C_D$, if $\mathfrak{R}(E, H, D, K) \Rightarrow \mathfrak{R}(A, H, D, K)$, then E is called an *HK*-consistent set of $\mathbb{F}$; otherwise, E is called an *HK*-inconsistent set of $\mathbb{F}$. Furthermore, if E is an *HK*-consistent set and any $S \subset E$ is not an *HK*-consistent set of $\mathbb{F}$, then E is called an *HK*-reduct of $\mathbb{F}$. The intersection of all the *HK*-reducts is called the *HK*-core of $\mathbb{F}$.

Thus, a *HK*-reduct E of a given triadic decision context $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ is such a minimal subset of A that all the constrained decision rules of $\mathbb{F}$ can be implied by the constrained decision rules of the reduced triadic decision context (U, E, H, I_E, D, K, J).

Definition 12 Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, $H \subseteq C_A$ and $K \subseteq C_D$. $a \in A$ is called an *HK*-dispensable attribute of $\mathbb{F}$ if $A \backslash \{a\}$ is an *HK*-consistent set of $\mathbb{F}$; otherwise, $a \in A$ is called an *HK*-indispensable attribute of $\mathbb{F}$.

Definition 13 Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, $E \subseteq A$, $H \subseteq C_A$ and $K \subseteq C_D$. $B \times H \to G \times K \in \mathfrak{R}(E, H, D, K)$ is said to be redundant in $\mathfrak{R}(E, H, D, K)$ if there exists another $B_0 \times H \to G_0 \times K \in \mathfrak{R}(E, H, D, K)$ such that $B_0 \times H \to G_0 \times K \Rightarrow B \times H \to G \times K$. Otherwise, $B \times H \to G \times K$ is said to be non-redundant.

Generally speaking, it is more appealing to derive non-redundant constrained decision rules from a triadic decision context because the redundant ones can be implied by others.

Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, $E \subseteq A$, $H \subseteq C_A$ and $K \subseteq C_D$. We denote

$$\mathfrak{U}^*(U, E \times H, I_E^{(3)}) = \{X \in \mathfrak{U}(U, E \times H, I_E^{(3)}) \mid X \neq \emptyset, X^{H_E^{(3)}} \neq \emptyset\},$$
$$\mathfrak{U}^*(U, D \times K, J^{(3)}) = \{Y \in \mathfrak{U}(U, D \times K, J^{(3)}) \mid Y \neq \emptyset, Y^{K^{(3)}} \neq \emptyset\}.$$

Then three mappings α_E, β_E, γ_E: $\mathfrak{U}^*(U, E \times H, I_E^{(3)}) \times \mathfrak{U}^*(U, D \times K, J^{(3)}) \to \{0, 1\}$ are defined as follows:

$$\alpha_E^{HK}(X, Y) = \begin{cases} 1, & \text{for any } X' \in \mathfrak{U}^*(U, E \times H, I_E^{(3)}), \text{ if} \\ & X \subseteq Y \text{ and } X \subset X', \text{ then } X' \nsubseteq Y, \\ 0, & \text{otherwise,} \end{cases}$$
$$\beta_E^{HK}(X, Y) = \begin{cases} 1, & \text{for any } Y' \in \mathfrak{U}^*(U, D \times K, J^{(3)}), \text{ if} \\ & X \subseteq Y \text{ and } Y' \subset Y, \text{ then } X \nsubseteq Y', \\ 0, & \text{otherwise,} \end{cases}$$
$$\gamma_E^{HK}(X, Y) = \begin{cases} 1, & \text{if } \alpha_E^{HK}(X, Y) = 1 \text{ and } \beta_E^{HK}(X, Y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1 *For a triadic decision context* $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$, $E \subseteq A$, $H \subseteq C_A$ *and* $K \subseteq C_D$, *let* $(X, B \times H) \in \mathfrak{B}(U, E \times H, I_E^{(3)})$, $(Y, G \times K) \in \mathfrak{B}(U, D \times K, J^{(3)})$ *and* $B \times H \to G \times K$. *Then* $B \times H \to G \times K$ *is redundant in* $\mathfrak{R}(E, H, D, K)$ *if and only if* $\gamma_E^{HK}(X, Y) = 0$, *or equivalently,* $B \times H \to G \times K$ *is non-redundant in* $\mathfrak{R}(E, H, D, K)$ *if and only if* $\gamma_E^{HK}(X, Y) = 1$.

Proof Since $(X, B \times H) \in \mathfrak{B}(U, E \times H, I_E^{(3)})$, $(Y, G \times K) \in \mathfrak{B}(U, D \times K, J^{(3)})$ and $B \times H \to G \times K$, it follows from Definition 8 that $X \neq \emptyset$, $X^{H_E^{(3)}} \neq \emptyset$, $Y \neq \emptyset$, $Y^{K^{(3)}} \neq \emptyset$. Thus, $X \in \mathfrak{U}^*(U, E \times H, I_E^{(3)})$ and $Y \in \mathfrak{U}^*(U, D \times K, J^{(3)})$.

"$\Leftarrow$" If $\gamma_E^{HK}(X, Y) = 0$, we have $\alpha_E^{HK}(X, Y) = 0$ or $\beta_E^{HK}(X, Y) = 0$. If $\alpha_E^{HK}(X, Y) = 0$, then there exists $(X_0, B_0 \times H) \in \mathfrak{B}(U, E \times H, I_E^{(3)})$ such that $X \subset X_0$ and $X_0 \subseteq Y$. Thus, $B_0 \times H \to G \times K \in \mathfrak{R}(E, H, D, K)$ and $B_0 \subset B$. It can be known from Definition 13 that $B \times H \to G \times K$ is redundant in $\mathfrak{R}(E, H, D, K)$ since $B_0 \times H \to G \times K \Rightarrow B \times H \to G \times K$. If $\beta_E^{HK}(X, Y) = 0$, we can prove in a similar manner that $B \times H \to G \times K$ is redundant in $\mathfrak{R}(E, H, D, K)$.

"$\Rightarrow$" If the constrained decision rule $B \times H \to G \times K$ is redundant in $\mathfrak{R}(E, H, D, K)$, there exists $B_0 \times H \to G_0 \times K \in \mathfrak{R}(E, H, D, K) \setminus \{B \times H \to G \times K\}$ such that $B_0 \times H \to G_0 \times K \Rightarrow B \times H \to G \times K$. Hence, $B_0 \subset B$ and $G \subseteq G_0$ or $B_0 \subseteq B$ and $G \subset G_0$. That is, $X \subset X_0$ and $Y_0 \subseteq Y$, or $X \subseteq X_0$ and $Y_0 \subset Y$, where $(X_0, B_0 \times H) \in \mathfrak{B}(U, E \times H, I_E^{(3)})$ and $(Y_0, G_0 \times K) \in \mathfrak{B}(U, D \times K, J^{(3)})$. Noting that $X \subseteq Y$ and $X_0 \subseteq Y_0$, we have $X \subseteq Y$, $X \subset X_0$ and $X_0 \subseteq Y_0 \subseteq Y$, or $X \subseteq Y$, $Y_0 \subset Y$ and $X \subseteq X_0 \subseteq Y_0$. Therefore, $\alpha_E^{HK}(X, Y) = 0$ or $\beta_E^{HK}(X, Y) = 0$, which implies $\gamma_E^{HK}(X, Y) = 0$. □

Let $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ be a triadic decision context, $E\subseteq A$, $H\subseteq C_A$ and $K\subseteq C_D$. Then we denote

$$\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}=\{X\in\mathfrak{U}^*(U,E\times H,I_E^{(3)})\mid \text{there is } Y\in\mathfrak{U}^*(U,D\times K,J^{(3)}) \text{ such that } \gamma_E^{HK}(X,Y)=1\},$$

$$\overline{\mathfrak{R}(E,H,D,K)}=\{B\times H\to G\times K\in\mathfrak{R}(E,H,D,K)\mid \gamma(B^{H_E^{(3)}},G^{K^{(3)}})=1\}.$$

That is, $\overline{\mathfrak{R}(E,H,D,K)}$ is just the set of all non-redundant constrained decision rules of $\mathfrak{R}(E,H,D,K)$.

Example 2 Continued with Example 1. Take $E=A$, $H=C_A$ and $K=C_D$. It can be seen from Tables 2 and 3 that

$$\mathfrak{U}^*(U,E\times H,I_E^{(3)})=\{\{x_1\},\{x_2\},\{x_1,x_3\},\{x_1,x_4\}\}$$

and

$$\mathfrak{U}^*(U,D\times K,J^{(3)})=\{\{x_1\},\{x_2\},\{x_2,x_3\}\}.$$

Then, we obtain

$$\gamma_E^{HK}(\{x_1\},\{x_1\})=1,$$
$$\gamma_E^{HK}(\{x_2\},\{x_2\})=1,$$

and

$$\gamma_E^{HK}(X,Y)=0 \text{ for other } X \text{ and } Y,$$

which leads to $\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}=\{\{x_1\},\{x_2\}\}$. Moreover, it is easy to verify that

$$\overline{\mathfrak{R}(E,H,D,K)}=\{\{a_2,a_4\}\times\{c_1,c_2\}\to\{h_2\}\times\{c_3,c_4\},\ \{a_1\}\times\{c_1,c_2\}\to\{h_1,h_3\}\times\{c_3,c_4\}\}$$

In what follows, we put forward an equivalent condition for judging whether a given attribute set is an *HK*-consistent set of a triadic decision context.

Proposition 4 *Let $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ be a triadic decision context, $E\subseteq A$, $H\subseteq C_A$ and $K\subseteq C_D$. Then both $\mathfrak{R}(E,H,D,K)\Rightarrow\overline{\mathfrak{R}(E,H,D,K)}$ and $\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\mathfrak{R}(E,H,D,K)$ are true.*

Proof Since $\overline{\mathfrak{R}(E,H,D,K)}$ is a subset of $\mathfrak{R}(E,H,D,K)$, it follows $\mathfrak{R}(E,H,D,K)\Rightarrow\overline{\mathfrak{R}(E,H,D,K)}$ according to Definition 10. The remainder is to prove $\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\mathfrak{R}(E,H,D,K)$.

For any $B\times H\to G\times K\in\mathfrak{R}(E,H,D,K)$, if it is non-redundant in $\mathfrak{R}(E,H,D,K)$, then $B\times H\to G\times K\in\overline{\mathfrak{R}(E,H,D,K)}$ and it can imply itself; if $B\times H\to G\times K$ is redundant in $\mathfrak{R}(E,H,D,K)$, then there exists $B_1\times H\to G_1\times K\in\mathfrak{R}(E,H,D,K)\setminus\{B\times H\to G\times K\}$ such that $B_1\times H\to G_1\times K\Rightarrow B\times H\to G\times K$. If $B_1\times H\to G_1\times K$ is non-redundant in $\mathfrak{R}(E,H,D,K)$, then $B_1\times H\to G_1\times K\in\overline{\mathfrak{R}(E,H,D,K)}$ and $B_1\times H\to G_1\times K\Rightarrow B\times H\to G\times K$; otherwise, there exists $B_2\times H\to G_2\times K\in\mathfrak{R}(E,H,D,K)\setminus\{B\times H\to G\times K,B_1\times H\to G_1\times K\}$ such that $B_2\times H\to G_2\times K\Rightarrow B\times H\to G\times K$. Since $\mathfrak{R}(E,H,D,K)$ is a finite set of constrained decision rules, it is true even in the worst case that there exists $B_n\times H\to G_n\times K\in\mathfrak{R}(E,H,D,K)\setminus\{B\times H\to G\times K,B_1\times H\to G_1\times K,\ldots,B_{n-1}\times H\to G_{n-1}\times K\}$ such that $B_n\times H\to G_n\times K$ is non-redundant in $\mathfrak{R}(E,H,D,K)$ and $B_n\times H\to G_n\times K\Rightarrow B\times H\to G\times K$. Therefore, $\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\mathfrak{R}(E,H,D,K)$. □

Theorem 2 *For a triadic decision context $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$, $E\subseteq A$ is an HK -consistent set of $\mathbb{F}$ if and only if $\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}=\overline{\mathfrak{U}(U,A\times H,I^{(3)})}$.*

Proof "⇒" If $E\subseteq A$ is an *HK*-consistent set of $\mathbb{F}$, it follows from Definition 11 that $\mathfrak{R}(E,H,D,K)\Rightarrow\mathfrak{R}(A,H,D,K)$. According to Proposition 4, we obtain $\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\mathfrak{R}(E,H,D,K)\Rightarrow\mathfrak{R}(A,H,D,K)\Rightarrow\overline{\mathfrak{R}(A,H,D,K)}$. Now, we are to prove $\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}=\overline{\mathfrak{U}(U,A\times H,I^{(3)})}$.

For any $X\in\overline{\mathfrak{U}(U,A\times H,I^{(3)})}$, there exist $B\subseteq A$ and $(Y,G\times K)\in\mathfrak{B}(U,D\times K,J^{(3)})$ such that $\gamma_A^{HK}(X,Y)=1$ and $B\times H\to G\times K\in\overline{\mathfrak{R}(A,H,D,K)}$. Since $\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\overline{\mathfrak{R}(A,H,D,K)}$, there exists $B_0\times H\to G_0\times K\in\overline{\mathfrak{R}(E,H,D,K)}$ such that $B_0\subseteq B$ and $G\subseteq G_0$. That is, $X_0\in\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$, $X\subseteq X_0$ and $Y_0\subseteq Y$, where $(X_0,B_0\times H)\in\mathfrak{B}(U,E\times H,I_E^{(3)})$ and $(Y_0,G_0\times K)\in\mathfrak{B}(U,D\times K,J^{(3)})$. If $X_0\neq X$, we have $X\subset X_0\subseteq Y_0\subseteq Y$. Noting that $X_0\in\mathfrak{U}(U,E\times H,I_E^{(3)})\subseteq\mathfrak{U}(U,A\times H,I^{(3)})$, we obtain $\alpha_A^{HK}(X,Y)=0$, which is in contradiction with $\gamma_A^{HK}(X,Y)=1$. So, $X=X_0\in\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$. Therefore, we conclude $\overline{\mathfrak{U}(U,A\times H,I^{(3)})}\subseteq\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$.

On the other hand, for any $X\in\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$, there exist $B\subseteq E$ and $(Y,G\times K)\in\mathfrak{B}(U,D\times K,J^{(3)})$ such that $\gamma_E^{HK}(X,Y)=1$. If $X\notin\overline{\mathfrak{U}(U,A\times H,I^{(3)})}$, we obtain $\gamma_A^{HK}(X,Y)=0$. Noting that $\beta_A^{HK}(X,Y)=\beta_E^{HK}(X,Y)=1$, we have $\alpha_A^{HK}(X,Y)=0$. Then, there exists $X_0\in\mathfrak{U}(U,A\times H,I^{(3)})$ such that $\alpha_A^{HK}(X_0,Y)=1$ and $X\subset X_0$. Furthermore, it is easy to conclude that $\beta_A^{HK}(X_0,Y)=1$ based on $\alpha_A^{HK}(X_0,Y)=1$, $\beta_A^{HK}(X,Y)=1$ and $X\subset X_0$. Hence, $\gamma_A^{HK}(X_0,Y)=1$ and $X\subset X_0$. Since $\overline{\mathfrak{U}(U,A\times H,I^{(3)})}\subseteq\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$ has been proved above, it follows $X_0\in\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$ and $X\subset X_0\subseteq Y$. Therefore,

$\alpha_E^{HK}(X,Y)=0$, which is in contradiction with $\alpha_E^{HK}(X,Y)=1$.

"$\Leftarrow$" If $\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}=\overline{\mathfrak{U}(U,A\times H,I^{(3)})}$, then for any $B\times H\to G\times K\in\overline{\mathfrak{R}(A,H,D,K)}$, there exists $B_0\times H\to G\times K\in\overline{\mathfrak{R}(E,H,D,K)}$ such that $X=X_0$, where $(X,B\times H)\in\mathfrak{B}(U,A\times H,I^{(3)})$, $(Y,G\times K)\in\mathfrak{B}(U,D\times K,J^{(3)})$ and $(X_0,B_0\times H)\in\mathfrak{B}(U,E\times H,I_E^{(3)})$. So, $B_0=X_0^{H_E^{(3)}}=X_0^{H^{(3)}}\cap E=X^{H^{(3)}}\cap E=B\cap E\subseteq B$, which means $B_0\times H\to G\times K\Rightarrow B\times H\to G\times K$. Therefore, we get $\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\overline{\mathfrak{R}(A,H,D,K)}$ and hence $\mathfrak{R}(E,H,D,K)\Rightarrow\overline{\mathfrak{R}(E,H,D,K)}\Rightarrow\overline{\mathfrak{R}(A,H,D,K)}$. Consequently, E is an HK-consistent set of $\mathbb{F}$. □

Finally, we design Algorithm 2 to derive the non-redundant constrained decision rules from a triadic decision context.

Algorithm 2. A rule acquisition algorithm for a triadic decision context.

Input: A triadic decision context $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ with $H\subseteq C_A$ and $K\subseteq C_D$.
Output: All the non-redundant constrained decision rules of $\mathbb{F}$.

1. Initialize $\Omega=\emptyset$.
2. Construct the concept lattices $\underline{\mathfrak{B}}(U,A\times H,I^{(3)})$ and $\underline{\mathfrak{B}}(U,D\times K,J^{(3)})$.
3. For every $((X,B\times H),(Y,G\times K))\in\underline{\mathfrak{B}}(U,A\times H,I^{(3)})\times\underline{\mathfrak{B}}(U,D\times K,J^{(3)})$ with $X\neq\emptyset$, $B\neq\emptyset$, $Y\neq\emptyset$ and $G\neq\emptyset$, if $\alpha_A^{HK}(X,Y)=1$ and $\beta_A^{HK}(X,Y)=1$, then $\Omega\cup\{B\times H\to G\times K\}\to\Omega$.
4. Output Ω and end the algorithm.

It is easy to observe that the time complexity of Algorithm 2 is exponential.

Example 3 Continued with Example 1. Take $H=C_A=\{c_1,c_2\}$ and $K=C_D=\{c_3,c_4\}$. Then, according to Algorithm 2, all the non-redundant constrained decision rules hidden in $\mathbb{F}$ are as follows:

r_1^{HK}: If Supermarket 1 and Supermarket 2 provide Tomatoes and Milk, then Restaurant 1 and Restaurant 2 sell Hamburg.

r_2^{HK}: If Supermarket 1 and Supermarket 2 provide Potatoes, then Restaurant 1 and Restaurant 2 sell Sandwich and Pizza.

Moreover, it can be verified with Theorem 2 that $E=\{a_1,a_2,a_4\}$ is an HK-consistent set of $\mathbb{F}$.

4 Knowledge reduction in triadic decision contexts

In the previous section, we have put forward a novel reduction framework for triadic decision contexts. With this framework, we can obtain all the non-redundant constrained decision rules from a triadic decision context. Note that, generally speaking, the implementation of knowledge reduction makes the discovery of constrained decision rules easier. Motivated by this, we discuss below how to implement knowledge reduction under the proposed framework.

Definition 14 Let $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ be a triadic decision context, $H\subseteq C_A$ and $K\subseteq C_D$. For $(X_i,B_i\times H),(X_j,B_j\times H)\in\mathfrak{B}(U,A\times H,I^{(3)})$, we denote

$$P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=\begin{cases}B_i\Diamond B_j, & \text{if } X_i\nsubseteq X_j\in\overline{\mathfrak{U}(U,A\times H,I^{(3)})} \text{ or}\\ & X_j\nsubseteq X_i\in\overline{\mathfrak{U}(U,A\times H,I^{(3)})},\\ \emptyset, & \text{otherwise,}\end{cases}$$

where $B_i\Diamond B_j=(B_i\cup B_j)\setminus(B_i\cap B_j)$. We call $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))$ the HK-discernibility attribute set of $(X_i,B_i\times H)$ and $(X_j,B_j\times H)$, and $\mathfrak{P}^{\triangle}=\{P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\mid(X_i,B_i\times H),(X_j,B_j\times H)\in\mathfrak{B}(U,A\times H,I^{(3)})\}$ the HK-discernibility matrix of $\mathbb{F}$.

Proposition 5 *Let $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ be a triadic decision context, $H\subseteq C_A$, $K\subseteq C_D$ and $(X_i,B_i\times H),(X_j,B_j\times H),(X_k,B_k\times H)\in\mathfrak{B}(U,A\times H,I^{(3)})$. Then*

(i) $P^{\triangle}((X_i,B_i\times H),(X_i,B_i\times H))=\emptyset$;
(ii) $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=P^{\triangle}((X_j,B_j\times H),(X_i,B_i\times H))$;
(iii) $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\subseteq P^{\triangle}((X_i,B_i\times H),(X_k,B_k\times H))\cup P^{\triangle}((X_k,B_k\times H),(X_j,B_j\times H))$ *if* $X_i,X_j,X_k\in\overline{\mathfrak{U}(U,A\times H,I^{(3)})}$.

Proof According to Definition 14, the proofs of (i) and (ii) are trivial. The remainder is to prove (iii).

For any $a\in P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))$, we have that $a\in B_i$ but $a\notin B_j$, or $a\in B_j$ but $a\notin B_i$.

If $a\in B_i$, $a\notin B_j$ and $a\notin B_k$, then $a\in P^{\triangle}((X_i,B_i\times H),(X_k,B_k\times H))$; if $a\in B_i$, $a\notin B_j$ and $a\in B_k$, then $a\in P^{\triangle}((X_k,B_k\times H),(X_j,B_j\times H))$. Thus, $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\subseteq P^{\triangle}((X_i,B_i\times H),(X_k,B_k\times H))\cup P^{\triangle}((X_k,B_k\times H),(X_j,B_j\times H))$ when $a\in B_i$ but $a\notin B_j$.

Similarly, we can prove $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\subseteq P^{\triangle}((X_i,B_i\times H),(X_k,B_k\times H))\cup P^{\triangle}((X_k,B_k\times H),(X_j,B_j\times H))$ when $a\in B_j$ but $a\notin B_i$. □

In what follows, from the viewpoint of the HK-discernibility attribute set, we put forward an equivalent condition for judging whether a given attribute set is an *HK*-consistent set of a triadic decision context. Before embarking on this issue, we explain the relationship between the *HK*-discernibility attribute set and an *HK*-consistent set.

For any nonempty set $P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H)) \in \mathfrak{P}^{\triangle}$, it follows from Definition 14 that the concepts $(X_i, B_i \times H)$ and $(X_j, B_j \times H)$ can be distinguished from one to another by the attribute set $P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H))$. Now, consider such a case that we remove the attributes in the complement set of E in A in order to avoid redundancy. Under such a circumstance, if $E \cap P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H)) = \emptyset$, then both $(X_i, B_i \times H)$ and $(X_j, B_j \times H)$ will be degenerated into the same concept. Without loss of generality, we assume that this degenerated concept is $(X_k, B_k \times H)$. The following discussion is divided into two cases.

Case 1 $X_i \nsubseteq X_j \in \overline{\mathfrak{U}(U, A \times H, I^{(3)})}$

Under this case, there are two circumstances for the relationship among $(X_i, B_i \times H)$, $(X_j, B_j \times H)$ and $(X_k, B_k \times H)$. See Figs. 3 and 4 for details. No matter which circumstance it occurs, it always follows $X_j \notin \overline{\mathfrak{U}(U, E \times H, I_E^{(3)})}$. As a result, we can conclude that $\overline{\mathfrak{U}(U, E \times H, I_E^{(3)})} \neq \overline{\mathfrak{U}(U, A \times H, I^{(3)})}$. Then, according to Theorem 2, E is not an *HK*-consistent set of $\mathbb{F}$.

Case 2 $X_j \nsubseteq X_i \in \overline{\mathfrak{U}(U, A \times H, I^{(3)})}$

In a similar manner, $X_i \notin \overline{\mathfrak{U}(U, E \times H, I_E^{(3)})}$ but $X_i \in \overline{\mathfrak{U}(U, A \times H, I^{(3)})}$ can be proved, which leads to $\overline{\mathfrak{U}(U, E \times H, I_E^{(3)})} \neq \overline{\mathfrak{U}(U, A \times H, I^{(3)})}$. Consequently, E is not an *HK*-consistent set of $\mathbb{F}$.

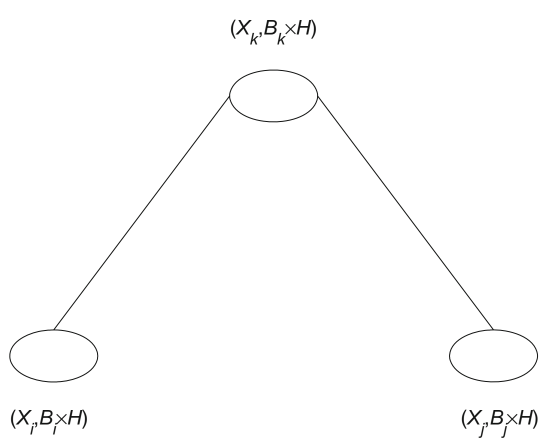

Fig. 3 The relationship among $(X_i, B_i \times H)$, $(X_j, B_j \times H)$ and $(X_k, B_k \times H)$

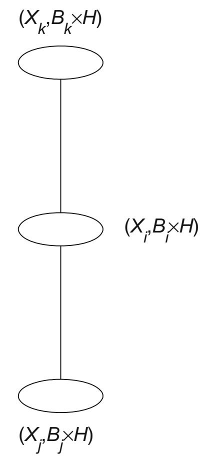

Fig. 4 The relationship among $(X_i, B_i \times H)$, $(X_j, B_j \times H)$ and $(X_k, B_k \times H)$

To sum up, in order to obtain that E is an *HK*-consistent set of $\mathbb{F}$, it is necessary to guarantee $E \cap P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H)) \neq \emptyset$ for every nonempty set $P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H)) \in \mathfrak{P}^{\triangle}$.

Theorem 3 *Let $\mathbb{F} = (U, A, C_A, I, D, C_D, J)$ be a triadic decision context, $H \subseteq C_A$ and $K \subseteq C_D$. Then $E \subseteq A$ is an HK -consistent set of $\mathbb{F}$ if and only if $E \cap P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H)) \neq \emptyset$ for every nonempty set $P^{\triangle}((X_i, B_i \times H), (X_j, B_j \times H)) \in \mathfrak{P}^{\triangle}$.*

Proof "⇒" It is trivial according to the aforementioned analysis.

"⇐" For any $X \in \overline{\mathfrak{U}(U, A \times H, I^{(3)})}$, there exists $B \subseteq A$ such that $(X, B \times H) \in \mathfrak{B}(U, A \times H, I^{(3)})$. Moreover, it can be known from Proposition 2 that $((B \cap E)^{H^{(3)}}, (B \cap E)^{H^{(3)}H^{(3)}} \times H) \in \mathfrak{B}(U, A \times H, I^{(3)})$. If $(B \cap E)^{H^{(3)}} = X$, then $(X, (B \cap E) \times H) \in \mathfrak{B}(U, E \times H, I_E^{(3)})$ and hence $X \in \overline{\mathfrak{U}(U, E \times H, I_E^{(3)})}$ is at hand. Otherwise, $(B \cap E)^{H^{(3)}} \neq X$. Under such a case, it in fact satisfies $(B \cap E)^{H^{(3)}} \nsubseteq X$. Based on Definition 14, $P^{\triangle}((X, B \times H), ((B \cap E)^{H^{(3)}}, (B \cap E)^{H^{(3)}H^{(3)}} \times H)) \neq \emptyset$. According to the assumption, we obtain $E \cap P^{\triangle}((X, B \times H), ((B \cap E)^{H^{(3)}}, (B \cap E)^{H^{(3)}H^{(3)}} \times H)) \neq \emptyset$, which yields $E \cap B \neq E \cap (B \cap E)^{H^{(3)}H^{(3)}}$. However,

$$\begin{aligned} E \cap B \subseteq B \quad &\Rightarrow (E \cap B)^{H^{(3)}} \supseteq B^{H^{(3)}} = X \\ &\Rightarrow (E \cap B)^{H^{(3)}H^{(3)}} \subseteq B \\ &\Rightarrow E \cap (E \cap B)^{H^{(3)}H^{(3)}} \subseteq E \cap B. \end{aligned}$$

On the other hand,

$$E \cap B \subseteq (E \cap B)^{H^{(3)}H^{(3)}} \Rightarrow E \cap B \subseteq E \cap (E \cap B)^{H^{(3)}H^{(3)}}.$$

Table 4 The HK-discernibility matrix of $\mathbb{F}$

	$(U,\emptyset)$	(x_1x_3,a_4)	(x_1x_4,a_2)	(x_1,a_2a_4)	(x_2,a_1)	$(\emptyset,A)$
$(U,\emptyset)$	$\emptyset$	$\emptyset$	$\emptyset$	a_2a_4	a_1	$\emptyset$
(x_1x_3,a_4)	$\emptyset$	$\emptyset$	$\emptyset$	a_2	a_1a_4	$\emptyset$
(x_1x_4,a_2)	$\emptyset$	$\emptyset$	$\emptyset$	a_4	a_1a_2	$\emptyset$
(x_1,a_2a_4)	a_2a_4	a_2	a_4	$\emptyset$	$a_1a_2a_4$	$\emptyset$
(x_2,a_1)	a_1	a_1a_4	a_1a_2	$a_1a_2a_4$	$\emptyset$	$\emptyset$
$(\emptyset,A)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 5 All triadic concepts of (U,E,C_A,I_E)

	Extent	Intent	Modus
1	$\{x_1,x_2,x_3,x_4\}$	$\{a_1,a_2,a_4\}$	$\emptyset$
3	$\{x_1\}$	$\{a_1,a_2,a_4\}$	$\{c_1\}$
2	$\{x_1,x_2,x_4\}$	$\{a_1,a_2\}$	$\{c_1\}$
4	$\{x_3\}$	$\{a_1,a_4\}$	$\{c_2\}$
5	$\{x_1,x_4\}$	$\{a_2,a_4\}$	$\{c_2\}$
6	$\{x_2,x_3\}$	$\{a_1\}$	$\{c_2\}$
7	$\{x_1,x_3,x_4\}$	$\{a_4\}$	$\{c_2\}$
8	$\{x_1\}$	$\{a_2,a_4\}$	$\{c_1,c_2\}$
9	$\{x_2\}$	$\{a_1\}$	$\{c_1,c_2\}$
10	$\{x_1,x_3\}$	$\{a_4\}$	$\{c_1,c_2\}$
11	$\{x_1,x_4\}$	$\{a_2\}$	$\{c_1,c_2\}$
12	$\emptyset$	$\{a_1,a_2,a_4\}$	$\{c_1,c_2\}$
13	$\{x_1,x_2,x_3,x_4\}$	$\emptyset$	$\{c_1,c_2\}$

Therefore, $E\cap B = E\cap (B\cap E)^{H^{(3)}H^{(3)}}$, a contradiction. As a result, $\overline{\mathfrak{U}(U,A\times H,I^{(3)})} \subseteq \overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$.

Moreover, similar to the case in the proof of Theorem 2, we can draw the conclusion $\overline{\mathfrak{U}(U,E\times H,I_E^{(3)})} \subseteq \overline{\mathfrak{U}(U,A\times H,I^{(3)})}$ with the help of $\overline{\mathfrak{U}(U,A\times H,I^{(3)})} \subseteq \overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$.

To sum up, $\overline{\mathfrak{U}(U,A\times H,I^{(3)})} = \overline{\mathfrak{U}(U,E\times H,I_E^{(3)})}$ is established. According to Theorem 2, E is an HK-consistent set of $\mathbb{F}$. □

Proposition 6 *Let $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ be a triadic decision context, $H\subseteq C_A$ and $K\subseteq C_D$. $a\in A$ is an HK -indispensable attribute of $\mathbb{F}$ if and only if there exists $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\in\mathfrak{P}^{\triangle}$ such that $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=\{a\}$.*

Proof "⇒" If $a\in A$ is an HK-indispensable attribute of $\mathbb{F}$, it follows from Definition 12 that $A\setminus\{a\}$ is an HK-inconsistent set of $\mathbb{F}$. Based on Theorem 3, there exists $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\in\mathfrak{P}^{\triangle}$ such that $(A\setminus\{a\})\cap P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=\emptyset$, which yields $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=\{a\}$.

"⇐" If there exists $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))\in\mathfrak{P}^{\triangle}$ such that $P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=\{a\}$, then $(A\setminus\{a\})\cap P^{\triangle}((X_i,B_i\times H),(X_j,B_j\times H))=\emptyset$. Hence, it follows from Theorem 3 that $A\setminus\{a\}$ is an HK-inconsistent set of $\mathbb{F}$. Consequently, a is an HK-indispensable attribute of $\mathbb{F}$. □

Definition 15 For a triadic decision context $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ with $H\subseteq C_A$ and $K\subseteq C_D$, let $\mathfrak{P}^{\triangle}$ be the HK-discernibility matrix of $\mathbb{F}$. We call $F_{\triangle}=\bigwedge_{P^{\triangle}\in\mathfrak{P}^{\triangle},P^{\triangle}\neq\emptyset}\bigvee P^{\triangle}$ the HK-discernibility function of $\mathbb{F}$.

Theorem 4 *For a triadic decision context $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$ with $H\subseteq C_A$ and $K\subseteq C_D$, let $F_{\triangle}=\bigvee_{t=1}^{k}(\bigwedge_{s=1}^{r_t}a_s)$ be the minimal disjunctive normal form of the HK -discernibility function $F_{\triangle}$, where $\bigwedge_{s=1}^{r_t}(a_s)$ $(t\leqslant k)$ are all the prime implicants of the HK -discernibility function $F_{\triangle}$. Then $E_t=\{a_s \mid s\leqslant r_t\}$ $(t\leqslant k)$ are all the HK -reducts of $\mathbb{F}$.*

Proof According to the definition of the minimal disjunctive normal form of a Boolean function, the proof is immediate from Theorem 3 and Proposition 6. □

Example 4 Continued with Example 1. Take $H=C_A$ and $K=C_D$. Then, Table 4 shows the HK-discernibility matrix of $\mathbb{F}=(U,A,C_A,I,D,C_D,J)$. According to Theorem 4, we can compute all the HK-reducts of $\mathbb{F}$ as follows:

$$\begin{aligned} F_{\triangle} &= \bigwedge_{P^{\triangle}\in\mathfrak{P}^{\triangle},P^{\triangle}\neq\emptyset}\bigvee P^{\triangle} \\ &= (a_1\vee a_4)\wedge(a_2\vee a_4)\wedge(a_1\vee a_2) \\ &\quad \wedge(a_1\vee a_2\vee a_4)\wedge a_1\wedge a_2\wedge a_4 \\ &= a_1\wedge a_2\wedge a_4. \end{aligned}$$

Thus, $\mathbb{F}$ has one HK-reduct $E=\{a_1,a_2,a_4\}$. By this HK-reduct, we can obtain the reduced triadic decision context (U,E,C_A,I_E,D,C_D,J). Table 5 shows all triadic concepts of (U,E,C_A,I_E), and Fig. 5 depicts the diagram of $\underline{\mathfrak{B}}(U,E,C_A,I_E)$. Based on Tables 3 and 5, all the non-redundant constrained decision rules hidden in (U,E,C_A,I_E,D,C_D,J) are as follows:

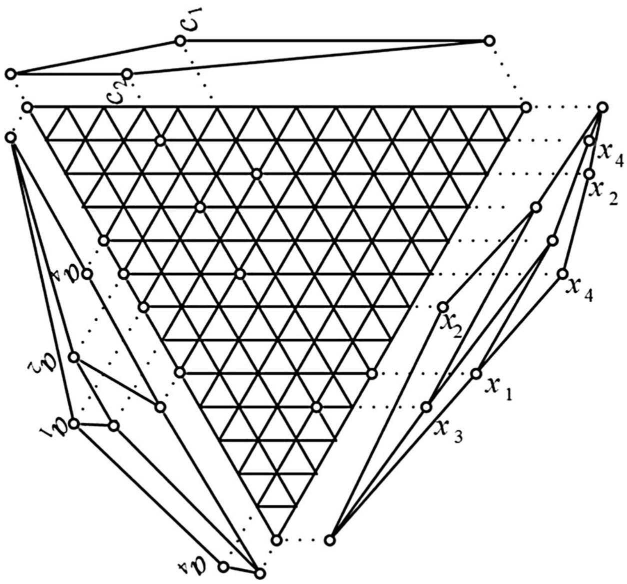

Fig. 5 The diagram of $\underline{\mathfrak{B}}(U, E, C_A, I_E)$

r_3^{HK}: If Supermarket 1 and Supermarket 2 provide Tomatoes and Milk, then Restaurant 1 and Restaurant 2 sell Hamburg.

r_4^{HK}: If Supermarket 1 and Supermarket 2 provide Potatoes, then Restaurant 1 and Restaurant 2 sell Sandwich and Pizza.

Comparing r_3^{HK} and r_4^{HK} with r_1^{HK} and r_2^{HK} in Example 3, we find that the non-redundant constrained decision rules derived from the reduced triadic decision context are the same as those derived from the original dataset $\mathbb{F}$.

5 Conclusion

This study has put forward a novel rule acquisition method for triadic decision contexts and discussed the issue of attribute reduction to make the original constrained decision rules more compact for making better decision analysis of the data. It deserves to be mentioned that the proposed method can be regarded as an information fusion technology since a triadic decision context can be viewed as multi-source data if one takes the data under each condition to be a single-source data.

The study of the triadic concept analysis is still in its infancy. At present, this theory has been mainly applied to mining triadic hierarchy structure [34], association rules [35], searching for optimal patterns [36] and analyzing triadic security context [37] from ternary relation. So, the proposed method will also be applied to these potential applications in the near future.

In fact, the discussion of triadic decision contexts is a challenging problem since the dimension of the data is three. As is well known, building a concept lattice of a two-dimension data is computationally expensive, let alone a three-dimension data. So, it is still necessary to improve the efficiency of mining knowledge from triadic decision contexts. Maybe concept learning [38] can provide a feasible way of solving this kind of problems.

Last but not least, the results in this paper on constrained decision rules of a triadic decision context should be generalized into the fuzzy environment because the data under consideration may often be fuzzy in the real world. Note that there have been many excellent studies (e.g. [39–41]) for our reference on this aspect. In our opinion, the problems to be investigated include fuzzy decision tree induction, generalization of fuzzy decision rules, and so on. We will try to find possible solutions to these problems in our forthcoming work.

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References

1. Wille R (1982) Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I (ed) Ordered sets. Reidel, Dordrecht, pp 445–470
2. Zhang WX, Wei L, Qi JJ (2005) Attribute reduction theory and approach to concept lattice. Science in China F 48(6):713–726
3. Ganter B, Wille R (1999) Formal concept analysis. Mathematical foundations. Springer, Berlin
4. Wang LD, Liu XD (2008) Concept analysis via rough set and AFS algebra. Inf Sci 178:4125–4137
5. Aswani Kumar Ch, Srinivas S (2010) Concept lattice reduction using fuzzy K-means clustering. Expert Syst Appl 37(3):2696–2704
6. Ma JM, Zhang WX (2013) Axiomatic characterizations of dual concept lattices. Int J Approx Reason 54(5):690–697
7. Zou LG, Zhang ZP, Long J (2015) A fast incremental algorithm for constructing concept lattices. Expert Syst Appl 42(9):4474–4481
8. Li TJ, Wu WZ (2011) Attribute reduction in formal contexts: a covering rough set approach. Fundam Inform 111(1):15–32
9. Lehmann F, Wille R (1995) A triadic approach to formal concept analysis. Lecture Notes in Computer Science 954:32–43
10. Wille R (1995) The basic theorem of triadic concept analysis. Order 12(2):149–158
11. Dau F, Wille R (2000) On the modal understanding of triadic contexts. In: Decker R, Gaul W (eds) Studies in classification, data analysis, and knowledge organization. Springer, Berlin, pp 83–94
12. Konecny J, Osicka P (2010) General approach to triadic concept analysis. In: Proceedings of CLA, pp 116–126

13. Belohlavek R, Glodeanu C, Vychodil V (2013) Optimal factorization of three-way binary data using triadic concepts. Order 30(2):437–454
14. Wei L, Wan Q, Qian T, Qi JJ (2014) An overview of triadic concept analysis. J Northwest Univ 44(5):689–699 (in Chinese)
15. Tang YQ, Fan M, Li JH (2014) Cognitive system model and approach to transformation of information granules under triadic formal concept analysis. J Shandong Univ 49(8):102–106 (in Chinese)
16. Zhang WX, Qiu GF (2005) Uncertain decision making based on rough sets. Tsinghua University Press, Beijing
17. Shao MW (2007) Knowledge acquisition in decision formal contexts. In: Proceedings of the sixth international conference on machine learning and cybernetics, Hong Kong, pp 4050–4054
18. Qu KS, Zhai YH, Liang JY, Chen M (2007) Study of decision implications based on formal concept analysis. Int J Gen Syst 36(2):147–156
19. Wei L, Qi JJ, Zhang WX (2008) Attribute reduction theory of concept lattice based on decision formal contexts. Sci China F 51(7):910–923
20. Wang H, Zhang W (2008) Approaches to knowledge reduction in generalized consistent decision formal context. Math Comput Model 48(11–12):1677–1684
21. Wu WZ, Leung Y, Mi JS (2009) Granular computing and knowledge reduction in formal contexts. IEEE Trans Knowl Data Eng 21(10):1461–1474
22. Shao MW, Leung Y, Wu WZ (2013) Rule acquisition and complexity reduction in formal decision contexts. Int J Approx Reason 55(1):259–274
23. Li J, Mei C, Lv Y (2012a) Knowledge reduction in formal decision contexts based on an order-preserving mapping. Int J Gen Syst 41(2):143–161
24. Li J, Mei C, Lv Y (2011a) A heuristic knowledge-reduction method for decision formal contexts. Comput Math Appl 61(4):1096–1106
25. Li J, Mei C, Lv Y (2011b) Knowledge reduction in decision formal contexts. Knowl Based Syst 24:709–715
26. Li J, Mei C, Wang J, Zhang X (2014) Rule-preserved object compression in formal decision contexts using concept lattices. Knowl Based Syst 71:435–445
27. Hong WX, Yu JP, Cai F, Song JL (2012) A new method of attribute reduction for decision formal context. ICIC Express Lett B Appl 3(5):1061–1068
28. Li J, Mei C, Lv Y (2013) Incomplete decision contexts: approximate concept construction, rule acquisition and knowledge reduction. Int J Approx Reason 54(1):149–165
29. Pei D, Li MZ, Mi JS (2011) Attribute reduction in fuzzy decision formal contexts. In: International conference on machine learning and cybernetics. IEEE Press, New York, pp 204–208
30. Kang XP, Li DY, Wang SG, Qu KS (2012) Formal concept analysis based on fuzzy granularity base for different granulations. Fuzzy Sets Syst 203:33–48
31. Li J, Mei C, Lv Y (2012) Knowledge reduction in real decision formal contexts. Inf Sci 189:191–207
32. Li J, Mei C, Lv Y, Zhang X (2012) A heuristic knowledge reduction algorithm for real decision formal contexts. In: Yao JT et al (eds) Proceedings of RSCTC, Lecture Notes in Artificial Intelligence, vol 7413. Springer, Berlin, pp 303–312
33. Yang HZ, Leung Y, Shao MW (2011) Rule acquisition and attribute reduction in real decision formal contexts. Soft Comput 15(6):1115–1128
34. Jschke R, Hotho A, Schmitz C, et al (2006) TRIAS—an algorithm for mining iceberg tri-lattices. In: Proceeding of the sixth international conference on data mining. Hong Kong, pp 907–911
35. Missaoui R, Kwuida L (2011) Mining triadic association rules from ternary relations. In: Proceeding of ICFCA, Lecture Notes in Computer Science, vol 6628. Springer, Berlin, pp 204–218
36. Ignatov DI, Gnatyshak DV, Kuznetsov SO et al (2015) Triadic formal concept analysis and triclustering: searching for optimal patterns. Mach Learn. doi:10.1007/s10994-015-5487-y
37. Aswani Kumar C (2013) Designing role-based access control using formal concept analysis. Secur Commun Netw 6(3): 373–383
38. Li J, Mei C, Xu W, Qian Y (2015) Concept learning via granular computing: a cognitive viewpoint. Inf Sci 298:447–467
39. Wang XZ, Xing HJ, Li Y et al (2014) A study on relationship between generalization abilities and fuzziness of base classifiers in ensemble learning. IEEE Trans Fuzzy Syst. doi:10.1109/TFUZZ.2014.2371479
40. Wang XZ, Dong LC, Yan JH (2012) Maximum ambiguity based sample selection in fuzzy decision tree induction. IEEE Trans Knowl Data Eng 24(8):1491–1505
41. Wang XZ, Dong CR (2009) Improving generalization of fuzzy if–then rules by maximizing fuzzy entropy. IEEE Trans Fuzzy Syst 17(3):556–567