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ORIGINAL ARTICLE

An approach to interval-valued intuitionistic uncertain linguistic multi-attribute group decision making

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Abstract The purpose of this paper is to investigate interval-valued intuitionistic uncertain linguistic multi-attribute group decision making with incomplete weight information and interactive conditions. In order to obtain the comprehensive attribute values of alternatives, the induced generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (IGS-IVIULHCA) operator is defined, which globally considers not only the importance of elements and the ordered positions but also their correlations. Based on gray relational analysis (GRA) method, models for the optimal fuzzy measures are constructed. Then, a new decision approach is developed, which considers the interactive characteristics between elements in a set. Finally, a numerical example is presented to illustrate the proposed approach and demonstrate its practicality and effectiveness.

Keywords Multi-attribute group decision making · Interval-valued intuitionistic uncertain linguistic set · Gray relational analysis (GRA) method · Choquet integral

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1 Introduction

Decision making is one of the most significant and omnipresent human activities in business, service, manufacturing, selection of products, etc. The key issues of decision making with incomplete weight information are to find the proper way to derive the weight vector and to aggregate the decision makers' preferences. For the former: there are several kinds of methods to derive the weight vector, such as TOPSIS method [11, 18, 56], correlation coefficient method [5–7], gray relational analysis method [21, 56] and deviation method [64]. For the latter: there are many aggregation operators. One of the most important aggregation operators is the ordered weighted averaging (OWA) operator [68]. Its fundamental aspect is a reordering step in which the input arguments are rearranged in descending order and the weight vector is merely associated with its ordered position. Since it was first introduced in 1988, many generalized forms have been developed [8, 12, 44, 59, 60, 66, 69, 73].

To address the uncertainties and fuzziness, Zadeh [74] introduced the concept of fuzzy sets, which have been successfully used in decision making. Later, some researchers found that fuzzy sets only consider the preference degrees of the decision maker, but cannot address the non-preference degrees of the decision maker. To cope with this issue, Atanassov [2] presented intuitionistic fuzzy sets (IFSs), which are characterized by a membership degree, a non-membership degree and a hesitancy degree. De et al. [15] defined some operations on IFSs. Xu and Yager [59] presented some operators in terms of geometrics, and Xu [61] defined some arithmetic aggregation operators. Xu and Wang [57] introduced several induced generalized intuitionistic fuzzy aggregation operators and studied their application to group decision making. Hung and Yang [19] researched the application of IFSs to pattern recognition; and Kharal [20] used IFSs to study homeopathic drug selection.

Considering the situations where there exist interactive characteristics, in references [45–47, 65] several intuitionistic fuzzy Choquet integral operators are defined. Because these intuitionistic fuzzy Choquet integral operators cannot address the importance of the ordered positions, Meng and Zhang [28] introduced the generalized intuitionistic fuzzy hybrid Choquet averaging operator and researched its application in decision making. Furthermore, Xia and Xu [67] discussed entropy and cross entropy of IFSs and studied their application to group decision making; Meng and Chen [29] studied entropy and similarity measure of IFSs by using the Shapley function and discussed their application to pattern recognition. Atanassov and Gargov [1] found that IFSs need the decision maker to give the exact values of the membership degree, the nonmembership degree and the hesitancy degree. This is still difficult in some situations. Thus, the authors further proposed the concept of interval-valued intuitionistic fuzzy sets (IVIFSs) that are characterized by an interval membership function, an interval non-membership function and an interval hesitancy function. Such a generalization further facilitates representing inherent imprecision and uncertainty of the decision makers. Atanassov [3] defined some operational laws on IVIFSs, and Xu and Chen [62, 63] proposed some aggregation operators in terms of geometrics and arithmetics, respectively. Researches about IVIFSs in multi-criteria decision making can be seen in the literature [9, 10, 54, 70–72]. To cope with the situations that the elements in a set are inter-dependent, the interval-valued intuitionistic fuzzy Choquet integral operators are studied in [27, 30, 31, 48, 65], and the interval-valued intuitionistic fuzzy Shapley operators are discussed in [32–34, 36].

However, IVIFSs can only address the quantitative preference of the decision maker, but cannot denote the decision maker's qualitative reference. To both denote the decision maker's quantitative and qualitative references, Liu [23] introduced the concept of interval-valued intuitionistic uncertain linguistic sets, which can be seen as a combination of IVIFSs and uncertain linguistic variables [58]. With respect to interval-valued intuitionistic uncertain linguistic sets, Liu [23] defined the interval-valued intuitionistic uncertain linguistic weighted geometric averaging (IVIULWGA) operator, the interval-valued intuitionistic uncertain linguistic ordered weighted geometric (IVIULOWG) operator and the interval-valued intuitionistic uncertain linguistic hybrid geometric (IVIULHG) operator. Furthermore, Meng et al. [35] defined several interval-valued intuitionistic uncertain linguistic Choquet operators, and Meng et al. [36] proposed several interval-valued intuitionistic uncertain linguistic hybrid Shapley operators. However, there are some disadvantages of the interval-valued intuitionistic uncertain linguistic aggregation operators in [35, 36]. For example, the aggregation operators in [35] only address the importance of elements and their interactions, but the importance of the ordered positions and their interactions do not consider. For the aggregation operators in [36], although they can address the importance of the elements and the ordered positions and their interactions, they are based on λ-fuzzy measures. As we know, λ-fuzzy measures can only address either complementary or redundant interactions, but they cannot reflect these two cases simultaneously.

With these issues, this paper continues to study interval-valued intuitionistic uncertain linguistic multi- attribute group decision-making problems with incomplete weight information, where the interactions between experts and between attributes are considered. The induced generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (IGS-IVIULHCA) operator is proposed, by which the comprehensive attribute values of the alternatives can be obtained. It is worth noticing that this operator addresses not only the importance of elements and the ordered positions but also their complementary and redundant interactions. In order to derive the weight vectors on the expert set, on the attribute set and on their ordered sets, we first define the gray relational coefficient for interval-valued intuitionistic uncertain linguistic numbers (IVIULNs), and then build models for the optimal fuzzy measures on the associated sets, respectively. Furthermore, an approach to interval-valued intuitionistic uncertain linguistic multi-attribute group decision making is developed, which considers the importance of the decision maker with respect to each attribute.

This paper is organized as follows: In Sect. 2, some basic notations and concepts are briefly reviewed, which will be used in the following. In Sect. 3, based on the Choquet integral [13], the induced generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (IGS-IVIULHCA) operator is defined, and some desirable properties are discussed. In Sect. 4, the gray relational coefficients for IVIULNs are presented. Then, the optimal models for the fuzzy measures on the expert set, on the attribute set and on their ordered sets are respectively constructed. Furthermore, a new algorithm for interval-valued intuitionistic uncertain linguistic multi-attribute group decision making with incomplete weight information and interactive conditions is developed. In Sect. 5, we illustrate our proposed algorithmic method with an example. The conclusion is made in the last section.

2 Some basic concepts

Let X be a no empty finite set. An IFS A in X is expressed as [1]:

$A = \{\langle x, u_A(x), v_A(x)\rangle | x \in X\}$,

where $u_A(x) \in [0, 1]$ and $v_A(x) \in [0, 1]$ respectively denote the degrees of membership and non-membership of the element $x \in X$ with the condition $u_A(x) + v_A(x) \leq 1$. The hesitancy degree is denoted by $\pi_A(x) = 1 - u_A(x) - v_A(x)$.

The linguistic approach is an approximate technique, which represents qualitative aspects as linguistic values [75] by means of linguistic variables. Let $S = \{s_i \mid i = 0, 2,\ldots,t\}$ be a linguistic term set with odd cardinality. Any label, s_i, represents a possible value for a linguistic variable, and it should satisfy the following characteristics [17]:

1. The set is ordered: $s_i > s_j$, if $i > j$;
2. Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$;
3. Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

For example, S can be defined as

$S = \{s_0$: extremely poor, s_1: very poor, s_2: poor, s_3: slightly poor, s_4: fair, s_5: slightly good, s_6: good, s_7: very good, s_8: extremely good$\}$.

It is worth noticing that linguistic values are a powerful tool to express the decision makers' qualitative preferences. Since it was introduced by Zadeh [75], many researchers have devoted themselves to decision making with linguistic information. For instance, Tapia García et al. [49] introduced a consensus model for group decision making with linguistic interval fuzzy preference relations. Alonso et al. [4] presented a linguistic consensus model for linguistic fuzzy preference relations and discussed its application to Web 2.0 communities. Massanet et al. [25] developed a new linguistic computational model by using discrete fuzzy numbers and researched its application to decision making. Morente-Molinera et al. [26] reviewed the studies about multi-granular fuzzy linguistic modelling in group decision making and pointed out the future trends. More researches about decision making with linguistic information can be seen in [37–40, 58].

In order to preserve all the given information, Xu [58] extended the discrete term set S to a continuous linguistic term set $\bar{S} = \{s_\alpha | s_0 \leq s_\alpha \leq s_t, \alpha \in [0, t]\}$, whose elements also meet all the characteristics above. If $s_\alpha \in S$, then it is called the original linguistic term, otherwise, it is called the virtual linguistic term [58].

Based on interval-valued intuitionistic fuzzy sets [1] and uncertain linguistic variables [58], Liu [23] defined the following interval-valued intuitionistic uncertain linguistic sets.

Definition 1 [23] An interval-valued intuitionistic uncertain linguistic set (IVIULS) A in $X = \{x_1, x_2, \ldots, x_n\}$ is defined by

$$A = \{\langle x_i, ([s_{\theta_A(x_i)}, s_{\tau_A(x_i)}], [u_{A_l}(x_i), u_{A_r}(x_i)], [v_{A_l}(x_i), v_{A_r}(x_i)])\rangle | x_i \in X\},$$

where $s_{\theta_A(x_i)}, s_{\tau_A(x_i)} \in \bar{S}$, the numbers $[u_{A_l}(x_i), u_{A_r}(x_i)]$ and $[v_{A_l}(x_i), v_{A_r}(x_i)]$ respectively represent the interval membership degree and the interval non-membership degree of the element $x_i \in X$ to the uncertain linguistic variable $[s_{\theta_A(x_i)}, s_{\tau_A(x_i)}]$ with $[u_{A_l}(x_i), u_{A_r}(x_i)] \subseteq [0, 1]$, $[v_{A_l}(x_i), v_{A_r}(x_i)] \subseteq [0, 1]$ and $u_{A_r}(x_i) + v_{A_r}(x_i) \leq 1$ for each $x_i \in X$.

When $u_{A_l}(x_i) = u_{A_u}(x_i)$ and $v_{A_l}(x_i) = v_{A_u}(x_i)$ for each $x_i \in X$, the IVIULS A degenerates to be the intuitionistic uncertain linguistic set $A = \{\langle x_i, ([s_{\theta_A(x_i)}, s_{\tau_A(x_i)}], u_A(x_i), v_A(x_i))\rangle | x_i \in X\}$ [22]. Furthermore, if $s_{\theta_A(x_i)} = s_{\tau_A(x_i)}$, then it reduces to the intuitionistic linguistic set (ILS) $A = \{\langle x_i, (s_{\theta_A(x_i)}, u_A(x_i), v_A(x_i))\rangle | x_i \in X\}$ [55].

Definition 2 [23] An interval-valued intuitionistic uncertain linguistic number (IVIULN)$\tilde{\alpha}$ is defined by $\tilde{\alpha} = ([s_{\theta(\alpha)}, s_{\tau(\alpha)}], [u_l(\alpha), u_r(\alpha)], [v_l(\alpha), v_r(\alpha)])$, where the interval numbers $[u_l(\alpha), u_r(\alpha)]$ and $[v_l(\alpha), v_r(\alpha)]$ respectively represent the interval membership and non-membership degrees to the uncertain linguistic variable $[s_{\theta(\alpha)}, s_{\tau(\alpha)}]$ with $[u_l(\alpha), u_r(\alpha)] \subseteq [0, 1]$,$[v_l(\alpha), v_r(\alpha)] \subseteq [0, 1]$ and $u_r(\alpha) + v_r(\alpha) \leq 1$.

From Definitions 1 and 2, we know that IVIULSs (or IVIULNs) consider both the quantitative and qualitative references. This means that IVIULSs (or IVIULNs) endow the decision maker more flexible to denote his/her judgment.

Let $\tilde{\alpha} = ([s_{\theta(\alpha)}, s_{\tau(\alpha)}], [u_l(\alpha), u_r(\alpha)], [v_l(\alpha), v_r(\alpha)])$ and $\tilde{\beta} = ([s_{\theta(\beta)}, s_{\tau(\beta)}], [u_l(\beta), u_r(\beta)], [v_l(\beta), v_r(\beta)])$ be any two IVIULNs, then some operations of $\tilde{\alpha}$ and $\tilde{\beta}$ are defined by [23]

(i) $\tilde{\alpha} \oplus \tilde{\beta} = ([s_{\theta(\alpha)+\theta(\beta)}, s_{\tau(\alpha)+\tau(\beta)}], [1 - (1 - u_l(\alpha))(1 - u_l(\beta)), 1 - (1 - u_r(\alpha))(1 - u_r(\beta))],.$
$[v_l(\alpha)v_l(\beta), v_r(\alpha)v_r(\beta)])$,

(ii) $\tilde{\alpha} \otimes \tilde{\beta} = ([s_{\theta(\alpha)\theta(\beta)}, s_{\tau(\alpha)\tau(\beta)}], [u_l(\alpha)u_l(\beta), u_r(\alpha)u_r(\beta)], [1 - (1 - v_l(\alpha))(1 - v_l(\beta)),$
$1 - (1 - v_r(\alpha))(1 - v_r(\beta))])$,

(iii) $\lambda\tilde{\alpha} = ([s_{\lambda\theta(\alpha)}, s_{\lambda\tau(\alpha)}], [1 - (1 - u_l(\alpha))^\lambda, 1 - (1 - u_r(\alpha))^\lambda], [v_l(\alpha)^\lambda, v_r(\alpha)^\lambda]) \quad \lambda \in [0, 1]$,

(iv) $\tilde{\alpha}^\lambda = ([s_{\theta(\alpha)^\lambda}, s_{\tau(\alpha)^\lambda}], [u_l(\alpha)^\lambda, u_r(\alpha)^\lambda], [1 - (1 - v_l(\alpha))^\lambda, 1 - (1 - v_r(\alpha))^\lambda]) \quad \lambda \in [0, 1]$.

Proposition 1 [21] *Let $\tilde{\alpha}$ and $\tilde{\beta}$ be any two IVIULNs, then*

(i) $\tilde{\alpha} \oplus \tilde{\beta} = \tilde{\beta} \oplus \tilde{\alpha}$,,

(ii) $\tilde{\alpha} \otimes \tilde{\beta} = \tilde{\beta} \otimes \tilde{\alpha}$,
(iii) $\lambda(\tilde{\alpha} \oplus \tilde{\beta}) = \lambda\tilde{\beta} \oplus \lambda\tilde{\alpha} \quad \lambda \in [0, 1]$,
(iv) $(\lambda_1 + \lambda_2)\tilde{\alpha} = \lambda_1\tilde{\alpha} \quad \lambda_1, \lambda_2 \in [0, 1]$,
(v) $(\tilde{\alpha} \otimes \tilde{\beta})^{\lambda} = \tilde{\beta}^{\lambda} \oplus \tilde{\alpha}^{\lambda} \quad \lambda \in [0, 1]$,
(vi) $\tilde{\alpha}^{\lambda_1+\lambda_2} = \tilde{\alpha}^{\lambda_1} \otimes \tilde{\alpha}^{\lambda_2} \quad \lambda_1, \lambda_2 \in [0, 1]$.

For any IVIULN $\tilde{\alpha} = \left([s_{\theta(\alpha)}, s_{\tau(\alpha)}], [u_l(\alpha), u_r(\alpha)], [v_l(\alpha), v_r(\alpha)]\right)$, Liu [23] defined the expected function $E(\tilde{\alpha})$ of $\tilde{\alpha}$ by
$E(\tilde{\alpha}) = s_{\frac{(\theta(\alpha)+\tau(\alpha))(u_l(\alpha)+u_r(\alpha)+2-v_l(\alpha)-v_r(\alpha))}{8}}$,

and presented the accuracy function

$H(\tilde{\alpha}) = s_{\frac{(\theta(\alpha)+\tau(\alpha))(u_l(\alpha)+u_r(\alpha)+v_l(\alpha)+v_r(\alpha))}{4}}$

to evaluate the accuracy degree of $\tilde{\alpha}$. Furthermore, Liu [23] gave the following order relationship between IVIULNs $\tilde{\alpha}$ and $\tilde{\beta}$.

If $E(\tilde{\alpha})\langle E(\tilde{\beta}))$, then $\tilde{\alpha} \prec \tilde{\beta}$.

If $E(\tilde{\alpha}) = E(\tilde{\beta}))$, then $\begin{cases} H(\tilde{\alpha}) = H(\tilde{\beta}) \Rightarrow \tilde{\alpha} = \tilde{\beta} \\ H(\tilde{\alpha})\langle H(\tilde{\beta}) \Rightarrow \tilde{\alpha} \prec \tilde{\beta} \end{cases}$.

3 The IGS-IVIULHCA operator

In practical decision-making problems, the independence between elements is usually violated. Thus, it is unsuitable to use the additive measures to measure the importance of them. Because the importance of an element is not only determined by itself but also influenced by the other elements. Fuzzy measures [43] seem to be a good choice to cope with this issue.

Definition 3 [43] A fuzzy measure μ on finite set $N = \{1, 2,\ldots,n\}$ is a set function $\mu : P(N) \to [0, 1]$ satisfying

(i) $\mu(\emptyset) = 0, \mu(N) = 1$,
(ii) If A, $B \in P(N)$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$,

where $P(N)$ is the power set of N.

In multi-attribute group decision making, $\mu(A)$ can be viewed as the importance of the attribute (or expert) set A. The Choquet integral [13] is one of the most important fuzzy integrals, which has been used in many fields. Grabisch [16] defined the following Choquet integral on discrete sets.

Definition 4 [16] Let f be a positive real-valued function on $X = \{x_1, x_2, \ldots, x_n\}$, and μ be a fuzzy measure on X. The discrete Choquet integral of f with respect to μ is defined by

$$C_\mu(f(x_{(1)}), f(x_{(2)}), \ldots, f(x_{(n)})) = \sum_{i=1}^{n} f(x_{(i)})(\mu(A_{(i)}) - \mu(A_{(i+1)})), \quad (1)$$

where $(\cdot)$ indicates a permutation on X such that $f(x_{(1)}) \leq f(x_{(2)}) \leq \cdots \leq f(x_{(n)})$, and $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$ with $A_{(n+1)} = \emptyset$.

When there is no interaction between elements, namely, $\mu(A) = \sum_{x_i \in A} \mu(x_i)$ for any $A \subseteq X$, then the Choquet integral given in Eq. (1) reduces to the OWA operator [68] with respect to f on X.

In order to measure the overall influence of each coalition rather than each player in a cooperative game, Marichal [24] defined the following generalized Shapley function

$$\Phi_S(\mu, N) = \sum_{T \subseteq N \setminus S} \frac{(n-s-t)!t!}{(n-s+1)!}(\mu(S \cup T) - \mu(T)) \quad \forall S \subseteq N, \quad (2)$$

where s, t and n respectively denote the cardinalities of S, T and N.

When $s = 1$, then Eq. (2) degenerates to the Shapley function [42]

$$\varphi_i(\mu, N) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!}(\mu(S \cup i) - \mu(S)) \quad \forall i \in N, \quad (3)$$

From Eq. (2), we know that the generalized Shapley function is an expected value of all marginal contributions between the coalition S and every coalition in $N\backslash S$, and the Shapley function is an expected value of all marginal contributions between the element i and every coalition in $N\backslash i$. When there is no interaction between the coalition S and every coalition in $N\backslash S$, from Eq. (2) we obtain $\Phi_S(\mu, N) = \mu(S)$.

Definition 5 The induced generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (IGS-IVIULHCA) operator of dimension n is a mapping IGS-IVIULHCA: IVIULNs^n → IVIULNs defined on the set of second arguments of two tuples $\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle$ with a set of order-inducing variables ρ_i (i = 1, 2,…,n), denoted by

$$\text{IGS-IVIULHCA}_{\Phi,\varphi}(\langle \rho_1, \tilde{\alpha}_1\rangle, \langle \rho_2, \tilde{\alpha}_2\rangle, \ldots, \langle \rho_n, \tilde{\alpha}_n\rangle) = \frac{\overset{n}{\underset{j=1}{\oplus}} \left(\Phi_{A_{(j)}}(\mu, N) - \Phi_{A_{(j+1)}}(\mu, N)\right) \varphi_{\tilde{a}_{(j)}}(\eta, \vartheta)\tilde{\alpha}_{(j)}}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu, N) - \Phi_{A_{(j+1)}}(\mu, N)\right) \varphi_{\tilde{a}_{(j)}}(\eta, \vartheta)},$$

where $(\cdot)$ is a permutation on ρ_i (i = 1, 2,…,n) such that $\rho_{(j)}$ being the jth largest value of ρ_i (i = 1, 2,…,n), $\Phi_{A_{(j)}}(\mu, N)$ is the generalized Shapley value with respect to the fuzzy measure μ on ordered set N = {1, 2,…,n} for $A_{(j)}$ = {j,…,n} with $A_{(n+1)} = \emptyset$, and $\varphi_{\tilde{a}_i}(\eta, \vartheta)$ is the Shapley value with respect to the fuzzy measure η on $\vartheta = \{\tilde{\alpha}_i\}_{i \in N}$ for $\tilde{\alpha}_i$ (i = 1, 2,…,n).

Remark 1 When μ and η are both an additive measure, then the IGS-IVIULHCA operator reduces to the induced interval-valued intuitionistic uncertain linguistic hybrid weighed averaging (I-IVIULHWA) operator

$$\text{I-IVIULHWA}_{w,\omega}(\langle \rho_1, \tilde{\alpha}_1 \rangle, \langle \rho_2, \tilde{\alpha}_2 \rangle, \ldots, \langle \rho_n, \tilde{\alpha}_n \rangle) = \frac{\overset{n}{\underset{j=1}{\oplus}} w_j \omega_{(j)} \tilde{\alpha}_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}},$$

where $\mu(i) = w_i$ and $\eta(\tilde{\alpha}_i) = \omega_i$ (i = 1, 2,…,n).

Remark 2 When $\rho_i = \varphi_{\tilde{a}_i}(\eta, \vartheta)\tilde{\alpha}_i$ (i = 1, 2,…,n), then the IGS-IVIULHCA operator reduces to the generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (GS-IVIULHCA) operator

$$\text{GS-IVIULHCA}_{\Phi,\varphi}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \frac{\overset{n}{\underset{j=1}{\oplus}} \left(\Phi_{A_{(j)}}(\mu, N) - \Phi_{A_{(j+1)}}(\mu, N) \right) \varphi_{\tilde{a}_{(j)}}(\eta, \vartheta) \tilde{\alpha}_{(j)}}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu, N) - \Phi_{A_{(j+1)}}(\mu, N) \right) \varphi_{\tilde{a}_{(j)}}(\eta, \vartheta)},$$

Remark 3 When $\varphi_{\tilde{a}_i}(\eta, \vartheta) = 1/n$ (i = 1, 2,…,n), then the IGS-IVIULHCA operator reduces to the induced generalized Shapley interval-valued intuitionistic uncertain linguistic Choquet averaging (IGS-IVIULCA) operator

$$\text{IGS-IVIULCA}_{\Phi}(\langle \rho_1, \tilde{\alpha}_1 \rangle, \langle \rho_2, \tilde{\alpha}_2 \rangle, \ldots, \langle \rho_n, \tilde{\alpha}_n \rangle) = \overset{n}{\underset{j=1}{\oplus}} \left(\Phi_{A_{(j)}}(\mu, N) - \Phi_{A_{(j+1)}}(\mu, N) \right) \tilde{\alpha}_{(j)}.$$

Remark 4 When $\Phi_{A_{(i)}}(\mu, N) - \Phi_{A_{(i+1)}}(\mu, N) = 1/n$ (i = 1, 2,…,n), then the IGS-IVIULHCA operator reduces to the induced interval-valued intuitionistic uncertain linguistic Shapley averaging (I-IVIULSA) operator

$$\text{I-IVIULSA}_{\varphi}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \overset{n}{\underset{j=1}{\oplus}} \varphi_{\tilde{a}_{(j)}}(v, \vartheta) \tilde{\alpha}_{(j)}.$$

Theorem 1 *Let* $\tilde{\alpha}_i = \left([s_{\theta(\alpha_i)}, s_{\tau(\alpha_i)}], [u_l(\alpha_i), u_u(\alpha_i)], [v_l(\alpha_i), v_u(\alpha_i)] \right)$ *be a collection of IVIULNs, and let* μ *and* η *be a fuzzy measure on* $N = \{1, 2, \ldots, n\}$ *and* $\vartheta = \{\tilde{\alpha}_i\}_{i \in N}$, *respectively. Then their collective value by using the IGS-IVIULHCA operator is also an IVIULN, denoted by*

$$\begin{aligned} &\text{IGS-IVIULHCA}_{\Phi,\varphi}(\langle \rho_1, \tilde{\alpha}_1 \rangle, \langle \rho_2, \tilde{\alpha}_2 \rangle, \ldots, \langle \rho_n, \tilde{\alpha}_n \rangle) \\ &= \left(\left[s_{\frac{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta) \theta(\alpha_{(j)})}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}}, s_{\frac{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta) \tau(\alpha_{(j)})}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}} \right], \right. \\ &\left[1 - \prod_{j=1}^{n} (1 - u_l(\alpha_{(j)}))^{\frac{\left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}}, 1 - \prod_{j=1}^{n} (1 - u_r(\alpha_{(j)}))^{\frac{\left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}} \right], \\ &\left. \left[\prod_{j=1}^{n} v_l(\alpha_{(j)})^{\frac{\left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}}, \prod_{j=1}^{n} v_r(\alpha_{(j)})^{\frac{\left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}{\sum_{j=1}^{n} \left(\Phi_{A_{(j)}}(\mu,N) - \Phi_{A_{(j+1)}}(\mu,N) \right) \varphi_{\tilde{a}_{(j)}}(\eta,\vartheta)}} \right] \right) \end{aligned} \tag{4}$$

where $\varphi_{\tilde{a}_{(j)}}(\eta, \vartheta)\tilde{\alpha}_{(j)}$ is the jth largest value of $\varphi_{\tilde{a}_i}(\eta, \vartheta)\tilde{\alpha}_i$($i$ = 1, 2,…,n).

Proof From the operations on IVIULNs and Proposition 1, it is not difficult to get the conclusion. □

From Definition 5, it is easy to know that the IGS-IVIULHCA operator satisfies *commutative, monotonic, bounded* and *idempotent.*

4 A decision making method based on grey relational analysis (GRA) method

Consider a multi-attribute group decision-making problem, in which the experts and attributes are correlative. Let $A = \{a_1, a_2,\ldots,a_m\}$ be the set of alternatives, $C = \{c_1, c_2,\ldots,c_n\}$ be the set of attributes, and $E = \{e_1, e_2,\ldots,e_q\}$ be the set of the decision makers. Assume that $A^k = \left(\tilde{a}_{ij}^k\right)_{m\times n}$ is the IVIULN matrix, where $\tilde{a}_{ij}^k = \left([s_{\theta(a_{ij}^k)}, s_{\tau(a_{ij}^k)}], [u_l(a_{ij}^k), u_r(a_{ij}^k)], .[v_l(a_{ij}^k), v_r(a_{ij}^k)]\right)$ is the IVIULN given by the decision maker e_k ($k = 1, 2,\ldots,q$) for the alternative $a_i \in A$ ($i = 1, 2,\ldots,m$) with respect to the attribute $c_j \in C$ ($j = 1, 2,\ldots,n$). When the weight information of the decision makers, the attributes and their ordered positions is exactly known, then we can use an aggregation operator to obtain the comprehensive attribute values of the alternatives. However, because of various reasons, such as time pressure and the expert's limited expertise about the problem domain, the information about the weights may be incompletely known [27–41, 50].

4.1 Models for the optimal fuzzy measures

Grey relational analysis (GRA) method [14] is an important multi-attribute decision making method that has been studied by researchers. In order to define the grey relational coefficient for IVIULNs, we first introduce the distance between any two IVIULNs.

Definition 6 Let $\tilde{\alpha} = \left([s_{\theta(\alpha)}, s_{\tau(\alpha)}], [u_l(\alpha), u_r(\alpha)],[v_l(\alpha), v_r(\alpha)]\right)$ and $\tilde{\beta} = \left([s_{\theta(\beta)}, s_{\tau(\beta)}],[u_l(\beta), u_r(\beta)], [v_l(\beta), v_r(\beta)]\right)$ be any two IVIULNs, the distance between $\tilde{\alpha}$ and $\tilde{\beta}$ is defined by

$$d(\tilde{\alpha},\tilde{\beta}) = \frac{(|\theta(\alpha)-\theta(\beta)| + |\tau(\alpha)-\tau(\beta)|)/t + |u_l(\alpha)-u_l(\beta)| + |u_r(\alpha)-u_r(\beta)| + |v_l(\alpha)-v_l(\beta)| + |v_r(\alpha)-v_r(\beta)|}{6}.$$

Let $\bar{A} = \left(\tilde{b}_{ij}\right)_{m\times n}$ be the averaging IVIULN matrix, where $\tilde{b}_{ij} = \left(\left[s_{\sum_{k=1}^q \theta(a_{ij}^k)/q}, s_{\sum_{k=1}^q \tau(a_{ij}^k)/q}\right], \left[\sum_{k=1}^q u_l(a_{ij}^k)/q, \sum_{k=1}^q u_r(a_{ij}^k)/q\right], \left[\sum_{k=1}^q v_l(a_{ij}^k)/q, \sum_{k=1}^q v_r(a_{ij}^k)/q\right]\right)$. Similar to the grey relational coefficient of intuitionistic fuzzy sets [56], we present the following grey relational coefficient between individual preferences and the averaging preference with respect to the attribute c_j ($j = 1, 2,\ldots,n$).

$$\xi_{e_k}^{ij} = \frac{\min\limits_{1\le k\le q}\min\limits_{1\le i\le m} d(\tilde{a}_{ij}^k,\tilde{b}_{ij}) + \rho \max\limits_{1\le k\le q}\max\limits_{1\le i\le m} d(\tilde{a}_{ij}^k,\tilde{b}_{ij})}{d(\tilde{a}_{ij}^k,\tilde{b}_{ij}) + \rho \max\limits_{1\le k\le q}\max\limits_{1\le i\le m} d(\tilde{a}_{ij}^k,\tilde{b}_{ij})} \tag{5}$$

for each pair (i, j) ($i = 1, 2,\ldots,m$; $j = 1, 2,\ldots,n$), where ρ is usually equal to 0.5.

Based on the grey relational coefficient given in Eq. (5), we build the following model for the optimal fuzzy measure on the decision maker set E with respect to the attribute c_j ($j = 1, 2,\ldots,n$).

$$\min \sum_{i=1}^m \sum_{k=1}^q \varphi_{e_k}(\eta^j, E)(1-\xi_{e_k}^{ij})$$
$$s.t.\begin{cases} \eta^j(e_k) \in H_{e_k}^j, & k = 1,2,\ldots,q \\ \eta^j(\emptyset) = 0, & \mu^j(E) = 1 \\ \eta^j(S) \le \eta^j(T) & \forall S,T \subseteq E, S\subseteq T, \end{cases} \tag{6}$$

where $\varphi_{e_k}(\eta^j, E)$ is the Shapley value of the decision maker e_k ($k = 1, 2,\ldots,q$) with respect to the attribute c_j, and $H_{e_k}^j$ is the known weight information.

Furthermore, by Eq. (5) we build the following model for the optimal fuzzy measure on ordered set $K = \{1, 2,\ldots,q\}$ with respect to the attribute c_j ($j = 1, 2,\ldots,n$).

$$\min \sum_{i=1}^m \sum_{k=1}^q \varphi_{(k)}(\mu^j, K)(1-\xi_{e_{(k)}}^{ij})$$
$$s.t.\begin{cases} \mu^j((k)) \in H_{(k)}^j, & k = 1,2,\ldots,q \\ \mu^j(\emptyset) = 0, & \mu^j(K) = 1 \\ \mu^j(S) \le \mu^j(T) & \forall S,T \subseteq K, S\subseteq T, \end{cases} \tag{7}$$

where $(1-\xi_{e_{(k)}}^{ij})$ is the kth largest value of $(1-\xi_{e_l}^{ij})$ ($l = 1, 2,\ldots,q$) for each pairs (i, j) ($i = 1, 2,\ldots,m$; $j = 1, 2,\ldots,n$),

$\varphi_{(k)}(\mu^j, K)$ is the Shapley value of the kth position ($k = 1, 2,\ldots,q$) with respect to the attribute c_j, and $H^j_{(k)}$ is the known weight information.

Let $A = (\tilde{a}_{ij})_{m\times n}$ be the comprehensive IVIULN matrix, where $\tilde{a}_{ij} = \left([s_{\theta(a_{ij})}, s_{\tau(a_{ij})}], [u_l(a_{ij}), u_r(a_{ij})], [v_l(a_{ij}), v_r(a_{ij})]\right)$. Let

$$\tilde{a}_j^+ = \left([s_{\max_{i=1}^m \theta(a_{ij})}, s_{\max_{i=1}^m \tau(a_{ij})}], [\max_{i=1}^m u_l(a_{ij}), \max_{i=1}^m u_r(a_{ij})], [\min_{i=1}^m v_l(a_{ij}), \min_{i=1}^m v_r(a_{ij})]\right)$$

and $\tilde{a}_j^- = \left([s_{\min_{i=1}^m \theta(a_{ij})}, s_{\min_{i=1}^m \tau(a_{ij})}], [\min_{i=1}^m u_l(a_{ij}), \min_{i=1}^m u_r(a_{ij})], .[\max_{i=1}^m v_l(a_{ij}), \max_{i=1}^m v_r(a_{ij})]\right)$.

We define the grey relational coefficients from the positive-ideal solution (PIS) and the negative-ideal solution (NIS) as follows:

$$\xi_{ij}^+ = \frac{\min\limits_{1\le i\le m}\min\limits_{1\le j\le n} d(\tilde{a}_{ij}, \tilde{a}_j^+) + \rho \max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{a}_{ij}, \tilde{a}_j^+)}{d(\tilde{a}_{ij}, \tilde{a}_j^+) + \rho \max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{a}_{ij}, \tilde{a}_j^+)}, \tag{8}$$

$$\xi_{ij}^- = \frac{\min\limits_{1\le i\le m}\min\limits_{1\le j\le n} d(\tilde{a}_{ij}, \tilde{a}_j^-) + \rho \max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{a}_{ij}, \tilde{a}_j^-)}{d(\tilde{a}_{ij}, \tilde{a}_j^-) + \rho \max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{a}_{ij}, \tilde{a}_j^-)} \tag{9}$$

for each pair (i, j) ($i = 1, 2,\ldots,m; j = 1, 2,\ldots,n$), where ρ is usually equal to 0.5.

Because all alternatives are non inferior, and the weight vector makes the comprehensive attribute values for the alternatives the bigger the better, we build the following model for the optimal fuzzy measure on attribute set C.

$$\min \sum_{i=1}^m \sum_{j=1}^n \frac{\xi_{ij}^-}{\xi_{ij}^- + \xi_{ij}^+} \varphi_{c_j}(\eta, C)$$
$$s.t.\begin{cases} \eta(C) = 1 \\ \eta(S) \le \eta(T) \qquad \forall S, T \subseteq C, S \subseteq T \\ \eta(c_j) \in W_{c_j}, \quad \eta(c_j) \ge 0 \quad \forall c_j \in C, \end{cases} \tag{10}$$

where $\varphi_{c_j}(\eta, C)$ is the Shapley value of the attribute c_j ($j = 1, 2,\ldots,n$), and W_{c_j} is the known weight information.

Similarly to model (9), we build the following model for the optimal fuzzy measure on ordered set $N = \{1, 2,\ldots,n\}$.

$$\min \sum_{i=1}^m \sum_{j=1}^n \frac{\xi_{i(j)}^-}{\xi_{i(j)}^- + \xi_{i(j)}^+} \varphi_{(j)}(\mu, N)$$
$$s.t.\begin{cases} \mu((j)) \in W_{(j)}, \qquad j = 1, 2, \ldots, n \\ \mu(\emptyset) = 0, \qquad \mu(N) = 1 \\ \mu(S) \le \mu(T) \qquad \forall S, T \subseteq N, S \subseteq T, \end{cases} \tag{11}$$

where $\xi_{i(j)}^- \big/ (\xi_{i(j)}^- + \xi_{i(j)}^+)$ is the jth largest value of $\xi_{ip}^- \big/ (\xi_{ip}^- + \xi_{ip}^+)(p = 1, 2,\ldots,n)$ for each i ($i = 1, 2,\ldots,m$), $\varphi_{(j)}(\mu, N)$ is the Shapley value of the jth position ($j = 1, 2,\ldots,n$), and $W_{(j)}$ is the known weight information.

When there is no interaction between elements, then we get the corresponding models for the optimal additive weight vectors.

4.2 A new algorithm

Based on above discussions, this subsection introduces an algorithm to group decision making with interval- valued intuitionistic uncertain linguistic information.

Step 1 Assume that the evaluation of the alternative a_i with respect to the attribute c_j given by the decision maker e_k ($k = 1, 2,\ldots,q$) is an IVIULN $\tilde{a}_{ij}^k = \left([s_{\theta(a_{ij}^k)}, s_{\tau(a_{ij}^k)}], [u_l(a_{ij}^k), u_r(a_{ij}^k)], .[v_l(a_{ij}^k), v_r(a_{ij}^k)]\right)$ ($i = 1, 2,\ldots,m$; $j = 1, 2,\ldots,n$). Then, we obtain the following IVIULN matrix

$$A^k = \begin{pmatrix} \tilde{a}_{11}^k & \tilde{a}_{12}^k & \ldots & \tilde{a}_{1n}^k \\ \tilde{a}_{21}^k & \tilde{a}_{22}^k & \ldots & \tilde{a}_{2n}^k \\ \ldots & \ldots & \ldots & \\ \tilde{a}_{m1}^k & \tilde{a}_{m2}^k & \ldots & \tilde{a}_{mn}^k \end{pmatrix}.$$

Step 2 Confirm the optimal fuzzy measure η^j on the decision maker set E with respect to the attribute c_j ($j = 1, 2,\ldots,n$) by applying model (6). Use Eq. (3) to calculate the decision makers' Shapley values with respect to the attribute c_j.

Step 3 Confirm the optimal fuzzy measure μ^j on ordered set $K = \{1, 2,\ldots,q\}$ with respect to the attribute c_j ($j = 1, 2,\ldots,n$) by applying model (7).

Step 4 Let $u_l = (1 - \xi_{e_l}^{ij})$ ($l = 1, 2,\ldots,q$) for each pair (i, j) ($i = 1, 2,\ldots,m$; $j = 1, 2,\ldots,n$), use the IGS- IVIULHCA operator to calculate the comprehensive IVIULN $\tilde{a}_{ij} = \left([s_{\theta(a_{ij})}, s_{\tau(a_{ij})}], [u_l(a_{ij}), u_r(a_{ij})], [v_l(a_{ij}), v_r(a_{ij})]\right)$, and the comprehensive matrix

$$A = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \ldots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \ldots & \tilde{a}_{2n} \\ \ldots & \ldots & \ldots & \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \ldots & \tilde{a}_{mn} \end{pmatrix},$$

where $\vartheta_i = \{\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in}\}$ for each pair (i, j) ($i = 1, 2,\ldots,m$; $j = 1, 2,\ldots,n$).

Step 5 Confirm the optimal fuzzy measure η on attribute set C by applying model (10). Use Eq. (3) to calculate the attribute Shapley values.

Step 6 Confirm the optimal fuzzy measure μ on the ordered set $N = \{1, 2,\ldots,n\}$ by applying model (11).

Step 7 Let $u_j = \xi_{ij}^- \big/ (\xi_{ij}^- + \xi_{ij}^+)$ (j = 1, 2,…,n) for each i (i = 1, 2,…,m), use the IGS-IVIULHCA operator to calculate the comprehensive IVIULN $\tilde{a}_i = \left([s_{\theta(a_i)}, s_{\tau(a_i)}], [u_l(a_i), u_r(a_i)], .[v_l(a_i), v_r(a_i)]\right)$ of the alternative a_i (i = 1, 2,…,m).

Step 8 For the comprehensive IVIULNs $\tilde{a}_i = \left([s_{\theta(a_i)}, s_{\tau(a_i)}], [u_l(a_i), u_r(a_i)], .[v_l(a_i), v_r(a_i)]\right)$ (i = 1, 2,…,m), let

$$\tilde{a}^+ = \left([s_{\max_{i=1}^m \theta(a_i)}, s_{\max_{i=1}^m \tau(a_i)}], [\max_{i=1}^m u_l(a_i), \max_{i=1}^m u_r(a_i)], [\min_{i=1}^m v_l(a_i), \min_{i=1}^m v_r(a_i)]\right)$$

and $\tilde{a}^- = \left([s_{\min_{i=1}^m \theta(a_i)}, s_{\min_{i=1}^m \tau(a_i)}], [\min_{i=1}^m u_l(a_i), \min_{i=1}^m u_r(a_i)], .[\max_{i=1}^m v_l(a_i), \max_{i=1}^m v_r(a_i)]\right)$.

Calculate

$$d_i = \frac{\xi_i^+}{\xi_i^- + \xi_i^+}$$

for each i = 1, 2,…,m, where

$$\xi_i^+ = \frac{\min_{1 \le i \le m} d(\tilde{a}_i, \tilde{a}^+) + \rho \max_{1 \le i \le m} d(\tilde{a}_i, \tilde{a}^+)}{d(\tilde{a}_i, \tilde{a}^+) + \rho \max_{1 \le i \le m} d(\tilde{a}_i, \tilde{a}^+)}$$

and

$$\xi_i^- = \frac{\min_{1 \le i \le m} d(\tilde{a}_i, \tilde{a}^-) + \rho \max_{1 \le i \le m} d(\tilde{a}_i, \tilde{a}^-)}{d(\tilde{a}_i, \tilde{a}^-) + \rho \max_{1 \le i \le m} d(\tilde{a}_i, \tilde{a}^-)}$$

with $\rho = 0.5$.

Step 9 Rank the alternatives according to d_i (i = 1, 2,…,m), and then select the biggest one(s).

Step 10 End.

5 An illustrative example

There is a panel with four possible alternatives to invest the money: a_1 (car company), a_2 (food company), a_3 (computer company) and a_4 (arms company). The investment company must take a decision according to the following three attributes: c_1 (the risk analysis), c_2 (the growth analysis) and c_3 (the environmental impact analysis). The four possible alternatives a_i (i = 1, 2, 3, 4) are to be evaluated by three decision makers $\{e_1, e_2, e_3\}$ using the linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ under the above four attributes. The IVIULN matrices are listed as follows:

$$A^1 = \begin{pmatrix} ([s_2,s_4],[0.3,0.5],[0.2,0.4]) & ([s_3,s_4],[0.4,0.5],[0.2,0.3]) & ([s_3,s_5],[0.4,0.6],[0.1,0.3]) \\ ([s_2,s_3],[0.3,0.4],[0.4,0.6]) & ([s_1,s_2],[0.2,0.3],[0.4,0.6]) & ([s_1,s_2],[0.2,0.3],[0.5,0.6]) \\ ([s_3,s_5],[0.4,0.6],[0.2,0.3]) & ([s_4,s_5],[0.4,0.6],[0.1,0.3]) & ([s_2,s_3],[0.2,0.4],[0.4,0.5]) \\ ([s_2,s_3],[0.3,0.5],[0.3,0.4]) & ([s_4,s_5],[0.4,0.6],[0.2,0.4]) & ([s_4,s_5],[0.5,0.7],[0.1,0.2]) \end{pmatrix},$$

$$A^2 = \begin{pmatrix} ([s_1,s_2],[0.2,0.3],[0.5,0.6]) & ([s_4,s_5],[0.5,0.7],[0.1,0.2]) & ([s_4,s_5],[0.5,0.7],[0.1,0.2]) \\ ([s_2,s_3],[0.3,0.4],[0.3,0.5]) & ([s_2,s_4],[0.3,0.5],[0.2,0.4]) & ([s_2,s_3],[0.3,0.5],[0.3,0.4]) \\ ([s_3,s_5],[0.4,0.6],[0.3,0.4]) & ([s_2,s_3],[0.3,0.5],[0.2,0.4]) & ([s_1,s_2],[0.2,0.3],[0.5,0.7]) \\ ([s_4,s_5],[0.5,0.6],[0.2,0.3]) & ([s_4,s_5],[0.5,0.7],[0.1,0.2]) & ([s_4,s_6],[0.6,0.8],[0.1,0.2]) \end{pmatrix},$$

$$A^3 = \begin{pmatrix} ([s_2,s_4],[0.3,0.5],[0.3,0.5]) & ([s_3,s_5],[0.4,0.7],[0.2,0.3]) & ([s_5,s_5],[0.6,0.7],[0.1,0.2]) \\ ([s_4,s_5],[0.5,0.7],[0.1,0.2]) & ([s_4,s_5],[0.5,0.6],[0.2,0.4]) & ([s_3,s_5],[0.4,0.6],[0.2,0.3]) \\ ([s_2,s_4],[0.3,0.5],[0.2,0.4]) & ([s_3,s_5],[0.4,0.6],[0.2,0.3]) & ([s_0,s_1],[0.1,0.2],[0.6,0.8]) \\ ([s_2,s_3],[0.2,0.4],[0.3,0.5]) & ([s_3,s_5],[0.4,0.6],[0.2,0.3]) & ([s_4,s_5],[0.5,0.6],[0.2,0.3]) \end{pmatrix}.$$

Assume that the uncertain weight vectors of the decision makers with respect to each attribute are given as follows:

$$H^1 = ([0.3,0.4],[0.2,0.3],[0.2,0.3]),$$

$$H^2 = ([0.3,0.4],[0.4,0.5],[0.5,0.6]),$$

$$H^3 = ([0.2,0.3],[0.4,0.5],[0.3,0.4]),$$

and the uncertain weight vector of the ordered set $K = \{1, 2, 3\}$ with respect to each attribute is given by $H = ([0.2,0.3],[0.3,0.4],[0.4,0.5])$. Furthermore, the uncertain weight vector of attributes is defined by $W = ([0.3,0.5],[0.4,0.6],[0.4,0.5])$, and the uncertain weight vector of the ordered set $N = \{1, 2, 3\}$ is defined by $G = ([0.2,0.3],[0.3,0.4],[0.4,0.5])$. In the following, we can utilize the proposed procedure to derive the most desirable alternative(s).

Step 1 From A^k (k = 1, 2, 3), we obtain the averaging matrix $\bar{A}$ as follows:

$$\bar{A} = \begin{pmatrix} ([s_{1.66}, s_{3.33}], [0.27, 0.43], [0.33, 0.5]) & ([s_{3.33}, s_{4.67}], [0.43, 0.63], [0.17, 0.27]) & ([s_4, s_5], [0.5, 0.67], [0.1, 0.23]) \\ ([s_{2.67}, s_{3.67}], [0.37, 0.5], [0.27, 0.43]) & ([s_{2.33}, s_{3.67}], [0.33, 0.4], [0.27, 0.47]) & ([s_2, s_{3.3}], [0.3, 0.47], [0.33, 0.43]) \\ ([s_{2.67}, s_{4.67}], [0.37, 0.57], [0.23, 0.37]) & ([s_3, s_{4.33}], [0.37, 0.57], [0.17, 0.33]) & ([s_1, s_2], [0.17, 0.3], [0.5, 0.67]) \\ ([s_{2.67}, s_{3.67}], [0.33, 0.5], [0.27, 0.4]) & ([s_{3.67}, s_5], [0.43, 0.63], [0.17, 0.3]) & ([s_4, s_{5.33}], [0.53, 0.7], [0.13, 0.23]) \end{pmatrix}$$

By Eq. (5), we obtain the following grey relational coefficient matrices

$$\xi^1 = \begin{pmatrix} 0.79 & 0.8 & 0.62 \\ 0.67 & 0.46 & 0.4 \\ 1 & 0.76 & 0.46 \\ 0.98 & 0.94 & 0.87 \end{pmatrix},$$

$$\xi^2 = \begin{pmatrix} 0.62 & 0.75 & 1 \\ 0.79 & 0.87 & 0.79 \\ 1 & 0.63 & 1 \\ 0.58 & 0.82 & 0.67 \end{pmatrix},$$

$$\xi^3 = \begin{pmatrix} 0.97 & 0.91 & 0.68 \\ 0.49 & 0.5 & 0.37 \\ 0.85 & 0.96 & 0.46 \\ 0.73 & 1 & 0.57 \end{pmatrix}.$$

Let $\rho = 0.5$. From model (6), the following linear programming for the optimal fuzzy measure on expert set E with respect to the attribute c_1 is constructed.

$$\min -0.142(\eta^1(e_1) - \eta^1(e_2, e_3)) + 0.083(\eta^1(e_2) - \eta^1(e_1, e_3)) + 0.058(\eta^1(e_3) - \eta^1(e_1, e_2)) + 0.843$$

$$s.t. \begin{cases} \eta^1(S) \le \eta^1(T) \quad \forall S, T \subseteq \{e_1, e_2, e_3\}, S \subseteq T, \\ \eta^1(e_1) \in [0.3, 0.4], \eta^1(e_2) \in [0.2, 0.3], \\ \eta^1(e_3) \in [0.2, 0.3], \eta^1(e_1, e_2, e_3) = 1. \end{cases}$$

By solving the above linear programming model, it derives the following fuzzy measure

$$\eta^1(e_1) = 0.4, \eta^1(e_2) = \eta^1(e_3) = \eta^1(e_2, e_3) = 0.2, \eta^1(e_1, e_2) = \eta^1(e_1, e_3) = \eta^1(e_1, e_2, e_3) = 1.$$

Similar to the calculation of the optimal fuzzy measure η^1, the following optimal fuzzy measures are obtained.

$$\eta^2(e_1) = \eta^2(e_3) = \eta^2(e_1, e_2) = 0.3, \eta^2(e_2) = 0.2, \eta^2(e_1, e_3) = \eta^2(e_2, e_3) = \eta^2(e_1, e_2, e_3) = 1;$$

$$\eta^3(e_1) = \eta^3(e_2) = \eta^3(e_1, e_3) = 0.3, \eta^3(e_3) = 0.2, \eta^3(e_1, e_2) = \eta^3(e_2, e_3) = \eta^3(e_1, e_2, e_3) = 1.$$

From Eq. (3), the decision makers' Shapley values with respect to each attribute c_j ($j = 1, 2, 3$) are derived as follows:

$$\varphi_{e_1}(\eta^1, E) = 0.67, \varphi_{e_2}(\eta^1, E) = 0.17, \varphi_{e_3}(\eta^1, E) = 0.17;$$

$$\varphi_{e_1}(\eta^2, E) = 0.23, \varphi_{e_2}(\eta^2, E) = 0.18, \varphi_{e_3}(\eta^2, E) = 0.58;$$

$$\varphi_{e_1}(\eta^3, E) = 0.23, \varphi_{e_2}(\eta^3, E) = 0.58, \varphi_{e_3}(\eta^3, E) = 0.18.$$

Step 2 Let $\rho = 0.5$. From model (7), the following linear programming for the optimal fuzzy measure on ordered set $K = \{1, 2, 3\}$ is constructed.

$$\min 0.308(\mu^1(1) - \mu^1(2, 3)) - 0.017(\mu^1(2) - \mu^1(1, 3)) - 0.292(\mu^1(3) - \mu^1(1, 2)) + 0.843$$

$$s.t. \begin{cases} \mu^1(S) \le \mu^1(T) \quad \forall S, T \subseteq \{1, 2, 3\}, S \subseteq T, \\ \mu^1(1) \in [0.2, 0.3], \mu^1(2) \in [0.3, 0.4], \\ \mu^1(3) \in [0.4, 0.5], \mu^1(1, 2, 3) = 1. \end{cases}$$

By solving the above linear programming model, it derives the following optimal fuzzy measure

$$\mu^1(1) = 0.2, \mu^1(2) = \mu^1(1, 2) = 0.3, \mu^1(3) = \mu^1(1, 3) = 0.5, \mu^1(2, 3) = \mu^1(1, 2, 3) = 1.$$

Similar to the calculation of the optimal fuzzy measure μ^1, the following optimal fuzzy measures with respect to the attributes c_j ($j = 2, 3$) are obtained.

$$\mu^j(1) = 0.2, \mu^j(2) = \mu^j(1, 2) = 0.3, \mu^j(3) = 0.5, \mu^j(1, 3) = \mu^j(2, 3) = \mu^j(1, 2, 3) = 1.$$

From Eq. (2), the following generalized Shapley values with respect to each attribute are obtained.

$$\varphi_1(\mu^1, K) = 0.07, \varphi_{\{1,2\}}(\mu^1, K) = 0.4, \varphi_K(\mu^1, K) = 1;$$

$$\varphi_1(\mu^j, K) = 0.15, \varphi_{\{1,2\}}(\mu^j, K) = 0.4, \varphi_K(\mu^j, K) = 1,$$

where $j = 2, 3$.

Step 3 Let $u_l = (1 - \xi_{e_l}^{ij})$ ($l = 1, 2, 3$) for each pair (i, j) ($i = 1, 2, 3, 4$; $j = 1, 2, 3$), use the IGS-IVIULHCA operator to calculate the comprehensive value $\tilde{a}_{ij}$, *i.g.*,

$$
\begin{aligned}
\tilde{a}_{11} &= \text{IGS-IVIULHCA}_{\Phi,\varphi}\left(\tilde{\alpha}_{11}^{1}, \tilde{\alpha}_{11}^{2}, \tilde{\alpha}_{11}^{3}\right) \\
&= \Bigg(\left[s_{\frac{0.07\times0.17\times1+0.33\times0.67\times2+0.6\times0.17\times2}{0.07\times0.17+0.33\times0.67+0.6\times0.17}}, s_{\frac{0.07\times0.17\times2+0.33\times0.67\times4+0.6\times0.17\times4}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \right], \\
&\quad \Big[1-(1-0.2)^{\frac{0.07\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \\
&\quad \times (1-0.3)^{\frac{0.33\times0.67}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \times (1-0.3)^{\frac{0.6\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}}, \\
&\quad 1-(1-0.3)^{\frac{0.07\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \\
&\quad \times (1-0.5)^{\frac{0.33\times0.67}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \times (1-0.5)^{\frac{0.6\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \Big], \\
&\quad [0.5^{\frac{0.07\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \times 0.2^{\frac{0.33\times0.67}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \\
&\quad \times 0.3^{\frac{0.6\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}}, 0.6^{\frac{0.07\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \\
&\quad \times 0.4^{\frac{0.33\times0.67}{0.07\times0.17+0.33\times0.67+0.6\times0.17}} \times 0.5^{\frac{0.6\times0.17}{0.07\times0.17+0.33\times0.67+0.6\times0.17}}] \Bigg) \\
&= ([s_{1.96}, s_{3.93}], [0.3, 0.49], [0.23, 0.43]).
\end{aligned}
$$

Similar to the calculation of $\tilde{a}_{11}$, the following comprehensive IVIULN matrix is obtained.

$$
A = \begin{pmatrix}
([s_{1.96}, s_{3.93}], [0.30, 0.49], [0.23, 0.43]) & ([s_{3.06}, s_{4.87}], [0.41, 0.68], [0.19, 0.29]) & ([s_{4.02}, s_{5.00}], [0.50, 0.69], [0.10, 0.21]) \\
([s_{2.07}, s_{3.07}], [0.31, 0.41], [0.35, 0.55]) & ([s_{2.89}, s_{4.26}], [0.40, 0.53], [0.22, 0.42]) & ([s_{1.93}, s_{3.00}], [0.29, 0.48], [0.31, 0.41]) \\
([s_{2.96}, s_{4.96}], [0.40, 0.60], [0.23, 0.33]) & ([s_{3.07}, s_{4.88}], [0.39, 0.59], [0.18, 0.31]) & ([s_{1.07}, s_{2.07}], [0.19, 0.31], [0.49, 0.67]) \\
([s_{2.05}, s_{3.05}], [0.29, 0.49], [0.30, 0.41]) & ([s_{3.20}, s_{5.00}], [0.41, 0.61], [0.19, 0.30]) & ([s_{3.91}, s_{5.47}], [0.55, 0.75], [0.11, 0.21])
\end{pmatrix}
$$

Step 4 Let $\rho = 0.5$. From model (10), the following linear programming for the optimal fuzzy measure η on attribute set C is constructed.

$$
\begin{aligned}
&\min 0.063(\eta(c_1) - \eta(c_2, c_3)) - 0.012(\eta(c_2) - \eta(c_1, c_3)) \\
&\quad - 0.052(\eta(c_3) - \eta(c_1, c_2)) + 1.95
\end{aligned}
$$

$$
s.t.\begin{cases} \eta(S) \leq \eta(T) \quad \forall S, T \subseteq \{c_1, c_2, c_3\}, S \subseteq T, \\ \eta(c_1) \in [0.3, 0.5], \eta(c_2) \in [0.4, 0.6], \\ \eta(c_3) \in [0.4, 0.5], \eta(c_1, c_2, c_3) = 1. \end{cases}
$$

By solving the above linear programming model, it derives the following optimal fuzzy measure

$$
\begin{aligned}
\eta(c_1) &= 0.3, \eta(c_2) = \eta(c_1, c_2) = 0.4, \eta(c_3) = \eta(c_1, c_3) \\
&= 0.5, \eta(c_2, c_3) = \eta(c_1, c_2, c_3) = 1.
\end{aligned}
$$

From Eq. (3), the attribute Shapley values are derived by $\varphi_{c_1}(\eta, C) = 0.1, \varphi_{c_2}(\eta, C) = 0.4, \varphi_{c_3}(\eta, C) = 0.5$.

Step 5 Let $\rho = 0.5$. From model (11), the following linear programming for the optimal fuzzy measure μ on ordered set $N = \{1, 2, 3\}$ is built.

$$
\begin{aligned}
&\min 0.263(\mu(1) - \mu(2, 3)) - 0.012(\mu(2) - \mu(1, 3)) \\
&\quad - 0.252(\mu(3) - \mu(1, 2)) + 1.95
\end{aligned}
$$

$$
s.t.\begin{cases} \mu(S) \leq \mu(T) \quad \forall S, T \subseteq \{1, 2, 3\}, S \subseteq T, \\ \mu(1) \in [0.2, 0.3], \mu(2) \in [0.3, 0.4], \\ \mu(3) \in [0.4, 0.5], \mu(1, 2, 3) = 1. \end{cases}
$$

By solving the above linear programming model, the following optimal fuzzy measure is obtained

$$
\begin{aligned}
\mu(1) &= 0.2, \mu(2) = \mu(1, 2) = 0.3, \mu(3) = \mu(1, 3) \\
&= 0.5, \mu(2, 3) = \mu(1, 2, 3) = 1.
\end{aligned}
$$

From Eq. (2), the following generalized Shapley values are derived

$$
\varphi_1(\mu, N) = 0.07, \varphi_{\{1,2\}}(\mu, N) = 0.4, \varphi_N(\mu, N) = 1.
$$

Step 6 Let $u_p = \xi_{ip}^{-} \big/ (\xi_{ip}^{-} + \xi_{ip}^{+})$ $(p = 1, 2, 3)$ for each i $(i = 1, 2, 3, 4)$, use the IGS-IVIULHCA operator, it derives the comprehensive value $\tilde{a}_i$ $(i = 1, 2, 3, 4)$. For example,

$$
\begin{aligned}
\tilde{a}_1 &= \text{IGS-IVIULHCA}_{\Phi,\varphi}(\langle u_1, \tilde{\alpha}_{11} \rangle, \langle u_2, \tilde{\alpha}_{12} \rangle, \langle u_n, \tilde{\alpha}_{13} \rangle) \\
&= \Bigg(\left[s_{\frac{0.07\times0.1\times1.96+0.33\times0.4\times3.06+0.6\times0.5\times4.02}{0.07\times0.1+0.33\times0.4+0.6\times0.5}}, s_{\frac{0.07\times0.1\times3.93+0.33\times0.4\times4.87+0.6\times0.5\times5}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \right], \\
&\quad [1-(1-0.3)^{\frac{0.07\times0.1}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \\
&\quad \times (1-0.41)^{\frac{0.33\times0.4}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \times (1-0.5)^{\frac{0.6\times0.5}{0.07\times0.1+0.33\times0.4+0.6\times0.5}}, \\
&\quad 1-(1-0.49)^{\frac{0.07\times0.1}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \\
&\quad \times (1-0.68)^{\frac{0.33\times0.4}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \times (1-0.69)^{\frac{0.6\times0.5}{0.07\times0.1+0.33\times0.4+0.6\times0.5}}], \\
&\quad [0.23^{\frac{0.07\times0.1}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \times 0.19^{\frac{0.33\times0.4}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \\
&\quad \times 0.1^{\frac{0.6\times0.5}{0.07\times0.1+0.33\times0.4+0.6\times0.5}}, 0.43^{\frac{0.07\times0.1}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \\
&\quad \times 0.29^{\frac{0.33\times0.4}{0.07\times0.1+0.33\times0.4+0.6\times0.5}} \times 0.21^{\frac{0.6\times0.5}{0.07\times0.1+0.33\times0.4+0.6\times0.5}}] \Bigg) \\
&= ([s_{3.7}, s_{4.94}], [0.47, 0.68], [0.12, 0.23]).
\end{aligned}
$$

Similar to the calculation of $\tilde{a}_1$, the comprehensive IVIULNs of the alternatives are obtained as follows:

$$\tilde{a}_2 = ([s_{2.22}, s_{3.98}], [0.33, 0.49], [0.28, 0.41]),$$

$$\tilde{a}_3 = ([s_{2.73}, s_{4.47}], [0.37, 0.56], [0.22, 0.35]),$$

$$\tilde{a}_4 = ([s_{3.67}, s_{5.29}], [0.51, 0.71], [0.13, 0.24]).$$

Step 7 According to the comprehensive IVIULNs $\tilde{a}_i (i = 1,2,3)$, it derives

Table 1 Ranking orders with respect to different aggregation operators

The aggregation operator	d_1	d_2	d_3	d_4	The ranking order
The IVIULWAA operator [23]	0.69	0.36	0.25	0.75	$d_4 > d_1 > d_2 > d_3$
The IVIULWGA operator [23]	0.69	0.47	0.25	0.75	$d_4 > d_1 > d_2 > d_3$
The IVIULHG operator [23]	0.67	0.49	0.25	0.75	$d_4 > d_1 > d_2 > d_3$
The GS-IVIUCA operator [35]	0.72	0.26	0.42	0.75	$d_4 > d_1 > d_3 > d_2$
The GS-IVIULCGM operator [35]	0.71	0.37	0.25	0.74	$d_4 > d_1 > d_2 > d_3$
The I-IVIULHSA operator [36]	0.61	0.25	0.44	0.75	$d_4 > d_1 > d_3 > d_2$
The I-IVIULHSG operator [36]	0.67	0.42	0.26	0.72	$d_4 > d_1 > d_2 > d_3$

$$\tilde{a}^{+} = ([s_{3.7}, s_{5.29}], [0.51, 0.71],[0.12, 0.23]),$$

$$\tilde{a}^{-} = ([s_{2.22}, s_{3.98}], [0.33, 0.49], [0.28, 0.41])$$

and

$$d_1 = 0.68, d_2 = 0.26, d_3 = 0.43, d_4 = 0.75.$$

Step 8 From Step 7, it obtains $d_4 > d_1 > d_3 > d_2$. Thus, a_4 (arms company) is the best choice.

In Step 8, when we adopt the Liu's ranking method [15], then the expected values of the alternatives are

$$E(\tilde{\alpha}_1) = 2.87, E(\tilde{a}_2) = 1.65, E(\tilde{a}_3) = 2.12, E(\tilde{a}_4) = 3.19.$$

From $E(\tilde{\alpha}_i)$ (i = 1, 2, 3, 4), the same ranking results are obtained, and a_4 (arms company) is the best choice.

In this example, when the aggregation operators in [23, 35, 36] are applied to calculate the comprehensive attribute values of alternatives, ranking orders are obtained as shown in Table 1.

From Table 1, one can see that the same ranking order is derived by using the IVIULWAA operator [23], the IVIULWGA operator [23], the IVIULHG operator [23], the GS-IVIULCGM operator [35] and the I-IVIULHSG operator [36]. Furthermore, it obtains the same ranking order by applying the IGS- IVIULHCA operator, the GS-IVIUCA operator [35], and the I-IVIULHSA operator [36]. However, all ranking order shows that the alternative a_4 is the best choice. Because all these aggregation operators are based on different points of view, we need to determine the weight vector with respect to each aggregation operator. Although all ranking results show that the alternative a_4 is the best choice, the values of the ranking indices are different.

6 Conclusion

We have researched interval-valued intuitionistic uncertain linguistic multi-attribute group decision making with incomplete weight information and interactive conditions. In order to derive the comprehensive attribute values for the alternatives and reflect the interactions between elements in a set, the induced generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (IGS-IVIULHCA) operator is defined, which globally considers not only the importance of elements but also their correlations. By using the defined grey relational coefficient for interval-valued intuitionistic uncertain linguistic sets, models for the optimal fuzzy measures on the expert set, on the attribute set and on their ordered sets are respectively established, which use their Shapley values as their weights. It is worth noticing that when there is no interaction between elements, then we get their corresponding models for the optimal weight vectors. Furthermore, an approach to interval-valued intuitionistic uncertain linguistic multi-attribute group decision making problems is developed, which considers the decision maker's weight for each attribute. It is worth noticing that we can also have other methods to decision making with interval-valued intuitionistic uncertain linguistic information such as maximum fuzziness [51], maximum ambiguity [52] a maximum entropy [53].

However, this paper only develops an approach to group decision making with interval-valued intuitionistic uncertain linguistic information, and we will further consider the application of IVIULSs in the other fields such as pattern recognition, expert system, social sciences, and economics.

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