Int. J. Mach. Learn. & Cyber. (2017) 8:547–569
DOI 10.1007/s13042-015-0349-2

ORIGINAL ARTICLE

Some intuitionistic trapezoidal fuzzy aggregation operators based on Einstein operations and their application in multiple attribute group decision making

Shuping Zhao[1,2] · Changyong Liang[1,2] · Junling Zhang[3]

Received: 12 May 2014 / Accepted: 10 March 2015 / Published online: 1 April 2015

Abstract Information aggregation is a key problem in decision making. The aim of this paper is to investigate information aggregation methods under intuitionistic trapezoidal fuzzy environment. Some Einstein operational laws on intuitionistic trapezoidal fuzzy numbers are defined based on Einstein sum and Einstein product. Then, some intuitionistic trapezoidal fuzzy aggregation operators based on Einstein operations are proposed, such as intuitionistic trapezoidal fuzzy Einstein weighted averaging operator, intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging operator, induced-intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging operator, intuitionistic trapezoidal fuzzy Einstein hybrid averaging operator, intuitionistic trapezoidal fuzzy Einstein weighted geometric operator, intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric operator, induced intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric operator and intuitionistic trapezoidal fuzzy Einstein hybrid geometric operator. Furthermore, we apply the proposed aggregation operators to deal with multiple attribute group decision making in which decision information takes the form of intuitionistic trapezoidal fuzzy numbers. Finally, an illustrative example is given to demonstrate its practicality and effectiveness.

Keywords Multiple attribute group decision making · Aggregation operation · Intuitionistic trapezoidal fuzzy numbers · Einstein operation

✉ Shuping Zhao
zhaoshuping1753@163.com

✉ Changyong Liang
cyliang@163.com

1 School of Management, Hefei University of Technology, Hefei 230009, China

2 Ministry of Education Engineering Research Center for Intelligent Decision-making and Information Systems Technologies, Hefei 230009, China

3 School of Economics and Management, Zhejiang Normal University, Jinhua 321004, China

1 Introduction

The concept of intuitionistic fuzzy sets (A-IFSs for short) is introduced by Atanassov [1], which is a generalization of the concept of fuzzy set which was proposed by Zadeh [2] to characterize fuzziness just by a membership degree. Since it is characterized by a membership degree and a non-membership degree, IFS comes to be more effective than Zadeh's Fuzzy sets to cope with uncertainty and vagueness originating from imprecise knowledge or information in real applications. A-IFSs has been investigated by many researchers and applied to many field since its appearance [3–22]. As for aggregation methods for A-IFSs information, Xu [5], Xu and Yager [3], Wei [23] and Liang [22] developed different A-IFSs aggregation operators, respectively, such as intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, intuitionistic fuzzy hybrid aggregation (IFHA) operator, intuitionistic fuzzy weighted geometric (IFWG) operator, intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, intuitionistic fuzzy hybrid geometric (IFHG) operator, and intuitionistic fuzzy weighted OWA (IFWOWA) operator. Later, Atanassov and Gargov [24], Atanassov [25] further extended the concept of intuitionistic fuzzy set to introduce interval-valued intuitionistic fuzzy sets (IVIFSs), which enhances greatly the representation ability of uncertainty than A-IFSs.

However, the domains of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets are discrete sets, which are used to indicate the extent to which certain criterion does or does not belong to some fuzzy concepts [26]. Later on, in order to address situations demanding domains of consecutive sets, the notion of A-IFSs has been further generalized to other forms, such as triangular intuitionistic fuzzy number (TIFN) introduced by Shu et al. [27] and intuitionistic trapezoidal fuzzy number (ITFN) introduced by Wang and Zhang [28]. Shu et al. [27] also developed operational laws of triangular intuitionistic fuzzy numbers (TIFNs) and proposed an algorithm for intuitionistic fuzzy fault-tree analysis. Then, Wang and Zhang [28] developed the definition of intuitionistic trapezoidal fuzzy numbers (ITFNs) and interval-valued intuitionistic trapezoidal fuzzy numbers (IV-ITFNs). Compared with intuitionistic fuzzy numbers, intuitionistic trapezoidal fuzzy numbers make their membership and non-membership degrees no longer relative to a fuzzy concept "Excellent" or "Good", but relative to the trapezoidal fuzzy number; thus can express decision information in different dimensions and avoid losing decision preference information [26]. Correspondingly on aggregation methods, Wang and Zhang [28] first proposed the definition and Hamming distance formula of intuitionistic trapezoidal fuzzy numbers, moreover, developed intuitionistic trapezoidal fuzzy weighted arithmetic averaging (ITFWAA) operator and multi-criteria decision-making method with incomplete certain information. Wei [29] developed the intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator and the intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator. Wan and Dong [30] defined the expectation and expectant score of intuitionistic trapezoidal fuzzy numbers from the geometric angle. Wu [31] further developed the intuitionistic trapezoidal fuzzy weighted geometric (ITFWG) operator, the intuitionistic trapezoidal fuzzy ordered weighted geometric (ITFOWG) operator, the induced intuitionistic trapezoidal fuzzy ordered weighted geometric (I-ITFOWG) operator and the intuitionistic trapezoidal fuzzy hybrid geometric (ITFHG) operator. Wan [32] developed the power average operator of intuitionistic trapezoidal fuzzy numbers, the weighted power average operator of intuitionistic trapezoidal fuzzy numbers, the power ordered weighted average operator of intuitionistic trapezoidal fuzzy numbers, and the power hybrid average operator of intuitionistic trapezoidal fuzzy numbers.

But it must be noticed that the above aggregation operators are all based on the most commonly used algebraic product and algebraic sum of ITFNs for carrying the combination process, which are not the only operations laws that can be chosen to model the intersection and union on ITFNs, And it is well known that Einstein t-norms and Einstein t-conorms are two prototypical examples of the class of strict Archimedean t-norms and t-conorms [33]. Moreover, to the best of our knowledge, in literatures there is still little research on aggregation operators using the Einstein operations for aggregating a collection of IFVs. Such as, Wang and Liu [34, 35] brought forward the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator, the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator, the intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator and the intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) operator successively. Zhao and Wei [36] developed the intuitionistic fuzzy Einstein hybrid averaging (IFEHA) operator and intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator. Zhang and Yu [37] proposed the Einstein based intuitionistic fuzzy Choquet geometric (EIFCG) operator and Einstein based interval-valued intuitionistic fuzzy Choquet geometric (EIIFCG) operator.

Therefore, in light of references [5, 34, 35], the aim of this paper is to enrich intuitionistic trapezoidal fuzzy theory by investigating information aggregation methods utilizing Einstein t-conorm and t-norm when the decision information takes the form of intuitionistic trapezoidal fuzzy numbers, and developing Einstein operations based on the operators. In order to do so, the rest of this paper is organized as follows. In Sect. 2, we briefly review some basic concepts of intuitionistic fuzzy trapezoidal numbers, and Einstein operations on intuitionistic fuzzy trapezoidal numbers. In Sect. 3, we propose some intuitionistic trapezoidal fuzzy aggregation operators based on Einstein operations, such as the intuitionistic trapezoidal fuzzy Einstein weighted averaging (ITFEWA) operator, the intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging (ITFEOWA) operator, the induced-intuitionistic trapezoidal fuzzy Einstein weighted averaging (I-ITFEOWA) operator and intuitionistic trapezoidal fuzzy Einstein hybrid averaging (ITFEHA) operator, intuitionistic trapezoidal fuzzy Einstein weighted geometric(ITFEWG) operator, intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (ITFEOWG) operator, induced intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (I-ITFEOWG) operator and intuitionistic trapezoidal fuzzy Einstein hybrid geometric (IFEHG) operator to aggregate the ITFNs, whose desirable properties are also studied in this section. In Sect. 4, we develop a multiple attribute group decision making method based on the proposed operators under intuitionistic trapezoidal fuzzy environment. In Sect. 5, numerical experiment on green supplier selection is provided to verify this decision making method. The last section concludes this paper.

2 Preliminaries

2.1 Intuitionistic trapezoidal fuzzy numbers

In the following, we shall introduce some basic concepts related to intuitionistic trapezoidal fuzzy numbers. First, we shall introduce the concept of intuitionistic fuzzy sets by Atanassov [1] as follows:

Definition 1 [1] Let X be a nonempty set. An Atanassov intuitionistic fuzzy set A of X is an object of the following form $A = \{\langle x, \mu_A(x), v_A(x)\rangle | x \in X\}$, where $\mu_A(x)$ means a membership function, and $v_A(x)$ means a non-membership, with the condition $0 \le \mu_A(x) + v_A(x) \le 1$, $\mu_A(x), v_A(x) \in [0,1]$ for all $x \in X$.

Given x, the pair $(\mu_A(x), v_A(x))$ in formulation (1) is called Atanassov intuitionistic fuzzy number [3] (A-IFN), which simply denoted as $\tilde{\alpha} = [\mu_{\tilde{\alpha}}, v_{\tilde{\alpha}}]$, where $\mu_{\tilde{\alpha}} \in [0,1]$, $v_{\tilde{\alpha}} \in [0,1]$, $\mu_{\tilde{\alpha}} + v_{\tilde{\alpha}} \le 1$.

Definition 2 [38] Let $\tilde{\alpha} = [\mu, v]$ be an A-IFN, a score function $s(\tilde{\alpha})$ of an A-IFN $\tilde{\alpha}$ can be represented as follows

$$s(\tilde{\alpha}) = \mu - v,\ s(\tilde{\alpha}) \in [-1,1] \tag{1}$$

to evaluate the degree of score of A-IFN $\tilde{\alpha} = [\mu, v]$, where $s(\tilde{\alpha}) \in [-1,1]$. The larger the value of $s(\tilde{\alpha})$, the more the degree of score of A-IFN $\tilde{\alpha}$.

Definition 3 [39] Let $\tilde{\alpha} = [\mu, v]$ be an A-IFN, an accuracy function $h(\tilde{\alpha})$ of an A-IFN $\tilde{\alpha}$ can be represented as follows

$$h(\tilde{\alpha}) = \mu + v,\ h(\tilde{\alpha}) \in [0,1] \tag{2}$$

to evaluate the degree of score of A-IFN $\tilde{\alpha} = [\mu, v]$, where $h(\tilde{\alpha}) \in [0,1]$. The larger the value of $h(\tilde{\alpha})$, the more the degree of accuracy of A-IFN $\tilde{\alpha}$.

Based on the score function and the accuracy function, Xu and Yager [3, 5] developed a method to compare any two A-IFNs as follows:

Definition 4 [3, 5] Let $\tilde{\alpha}_i = [\mu_{\tilde{\alpha}_i}, v_{\tilde{\alpha}_i}]$ $(i = 1, 2)$ be any two A-IFNs, and let $s(\tilde{\alpha}_i)$ and $h(\tilde{\alpha}_i)$ be the scores and accuracy degrees of A-IFN $\tilde{\alpha}_i$ $(i = 1, 2)$, respectively. Then, the following conditions hold:

(1) If $s(\tilde{\alpha}_1) > s(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
(2) If $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2)$, then

① If $h(\tilde{\alpha}_1) > h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
② If $h(\tilde{\alpha}_1) < h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.
③ If $h(\tilde{\alpha}_1) < h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

Definition 5 [26, 28] Let $\tilde{\alpha}$ is an intuitionistic trapezoidal fuzzy number, its membership function is defined as:

$$\mu_{\tilde{\alpha}}(x) = \begin{cases} (x-a)\mu_{\tilde{\alpha}}/(b-a) & (a \le x < b) \\ \mu_{\tilde{\alpha}} & (b \le x \le c) \\ (d-x)\mu_{\tilde{\alpha}}/(d-c) & (c < x \le d) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

its non-membership function is defined as:

$$v_{\tilde{\alpha}}(x) = \begin{cases} (b-x+v_{\tilde{\alpha}}(x-a'))/(b-a') & (a' \le x < b) \\ v_{\tilde{\alpha}} & (b \le x \le c) \\ (x-c+v_{\tilde{\alpha}}(d'-x))/(d'-c) & (c < x \le d') \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $0 \le \mu_{\tilde{\alpha}} \le 1; 0 \le v_{\tilde{\alpha}} \le 1; 0 \le u_{\tilde{\alpha}} + v_{\tilde{\alpha}} \le 1, a, b, c, d \in R$, R is the set of real numbers. Then $\tilde{\alpha} = \langle([a,b,c,d];\mu_{\tilde{\alpha}}), ([a',b,c,d'];v_{\tilde{\alpha}})\rangle$ is called an intuitionistic trapezoidal fuzzy number (ITFN for short). Generally, there exists $[a,b,c,d] = [a',b,c,d']$ in intuitionistic trapezoidal fuzzy number $\tilde{\alpha}$, here, denoted as $\tilde{\alpha} = ([a,b,c,d];\mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$. We let $\tilde{\alpha} = ([a,b,c,d];\mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ and only talk about this kind of fuzzy number in the remainder of this paper (As seen in Fig. 1).

The hesitation of intuitionistic trapezoidal fuzzy number $\tilde{\alpha}$ is denoted by $\pi_{\tilde{\alpha}}(x) = 1 - u_{\tilde{\alpha}}(x) - v_{\tilde{\alpha}}(x)$, the smaller the $\pi_{\tilde{\alpha}}$, the more certain is the intuitionistic trapezoidal fuzzy number. If $b = c$, then an intuitionistic trapezoidal fuzzy number become an intuitionistic triangular fuzzy number. When $u_{\tilde{\alpha}} = 1$ and $v_{\tilde{\alpha}} = 0$, $\tilde{\alpha}$ is called normal intuitionistic fuzzy number, namely, a traditional fuzzy number. If $a, b, c, d \in [0,1]$, then $\tilde{\alpha}$ is called a standardized intuitionistic fuzzy trapezoidal number.

Definition 6 [40] Let $\tilde{\alpha} = ([a,b,c,d];\mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ is an intuitionistic trapezoidal fuzzy number, the score function $s(\tilde{\alpha})$ of an intuitionistic trapezoidal fuzzy number can be represented as follows

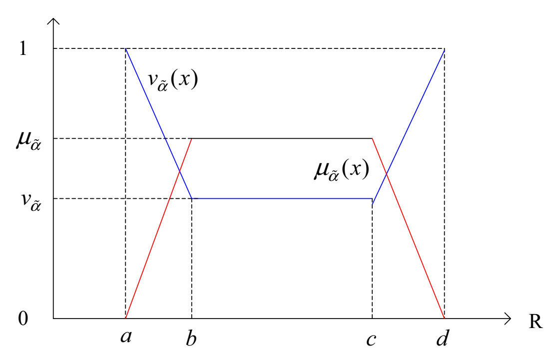

Fig. 1 An intuitionistic trapezoidal fuzzy number $\tilde{\alpha} = ([a,b,c,d];\ \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$

$$s(\tilde{\alpha}) = \frac{a+b+c+d}{4}(\mu_{\tilde{\alpha}} - v_{\tilde{\alpha}}),\ \ s(\tilde{\alpha}) \in [-1,1]. \tag{5}$$

Definition 7 [40] Let $\tilde{\alpha} = ([a,b,c,d];\ \ \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ is an intuitionistic trapezoidal fuzzy number, the accuracy function $h(\tilde{\alpha})$ of an intuitionistic trapezoidal fuzzy number can be represented as follows

$$h(\tilde{\alpha}) = \frac{a+b+c+d}{4}(\mu_{\tilde{\alpha}} + v_{\tilde{\alpha}}),\ \ h(\tilde{\alpha}) \in [0,1]. \tag{6}$$

Definition 8 [40] Let $\tilde{\alpha}_i$ $(i = 1,2)$ are two intuitionistic trapezoidal fuzzy numbers, and let $s(\tilde{\alpha}_i)$ and $h(\tilde{\alpha}_i)$ be the score and accuracy values of $\tilde{\alpha}_i$ $(i = 1,\ 2)$ respectively. Then, the following conditions hold:

(1) If $s(\tilde{\alpha}_1) > s(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
(2) If $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2)$, then

① If $h(\tilde{\alpha}_1) > h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
② If $h(\tilde{\alpha}_1) < h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.
③ If $h(\tilde{\alpha}_1) = h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

2.2 Einstein operations

The theory of aggregation operators has an important role since in the beginning of fuzzy set theory. Einstein operations is a kind of various t-norms and t-conorms families [41, 42] can be used to accomplish the corresponding intersections and unions of IFSs. Einstein operations includes the Einstein product and the Einstein sum, respectively. Einstein product $\otimes_\varepsilon$ and Einstein sum $\oplus_\varepsilon$ are defined as follows [33]:

$$a \otimes_\varepsilon b = \frac{a \cdot b}{1 + (1-a)\cdot(1-b)},\ \ a \oplus_\varepsilon b = \frac{a+b}{1+a\cdot b},\ \ \forall (a,b) \in [0,1]^2. \tag{7}$$

2.3 Einstein operations of intuitionistic trapezoidal fuzzy numbers

In this section, firstly, the operational laws of intuitionistic trapezoidal fuzzy numbers will be introduced based on algebraic t-conorm and t-norm, the Einstein product and Einstein sum on two A-IFSs are also introduced. Moreover, we will define the operational laws of intuitionistic trapezoidal fuzzy numbers based on Einstein t-norm and t-conorm on and will analyze some desirable properties of these operations respectively.

Definition 9 [26, 28] Let $\tilde{\alpha}_1 = ([a_1,b_1,c_1,d_1];\ \mu_{\tilde{\alpha}_1}, v_{\tilde{\alpha}_1})$ and $\tilde{\alpha}_2 = ([a_2,b_2,c_2,d_2];\ \mu_{\tilde{\alpha}_2}, v_{\tilde{\alpha}_2})$ be two intuitionistic trapezoidal fuzzy numbers, and $\lambda \geq 0$, then

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = ([a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2];\ \mu_{\tilde{\alpha}_1} + \mu_{\tilde{\alpha}_2} - \mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}, v_{\tilde{\alpha}_1}v_{\tilde{\alpha}_2})$;
(2) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = ([a_1a_2, b_1b_2, c_1c_2, d_1d_2];\ \mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}, v_{\tilde{\alpha}_1} + v_{\tilde{\alpha}_2} - v_{\tilde{\alpha}_1}v_{\tilde{\alpha}_2})$;
(3) $\lambda \cdot \tilde{\alpha} = ([\lambda a, \lambda b, \lambda c, \lambda d];\ 1-(1-\mu_{\tilde{\alpha}})^\lambda, (v_{\tilde{\alpha}})^\lambda)$;
(4) $\tilde{\alpha}^\lambda = ([a^\lambda, b^\lambda, c^\lambda, d^\lambda];\ (\mu_{\tilde{\alpha}})^\lambda, 1-(1-v_{\tilde{\alpha}})^\lambda)$.

The above operations are based on the algebraic t-conorm and t-norm, in the following, we gave some the operations of intuitionistic trapezoidal fuzzy numbers based on Einstein t-norm and t-conorm.

Definition 10 Let $\tilde{\alpha}_1 = ([a_1,b_1,c_1,d_1];\ \mu_{\tilde{\alpha}_1}, v_{\tilde{\alpha}_1})$, $\tilde{\alpha}_2 = ([a_2,b_2,c_2,d_2];\ \mu_{\tilde{\alpha}_2}, v_{\tilde{\alpha}_2})$ and $\tilde{\alpha} = ([a,b,c,d];\ \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ be any three intuitionistic trapezoidal fuzzy numbers, and λ is positive real number, $\lambda \geq 0$, then

(1) $\tilde{\alpha}_1 \oplus_\varepsilon \tilde{\alpha}_2 = \left([a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2];\ \frac{\mu_{\tilde{\alpha}_1}+\mu_{\tilde{\alpha}_2}}{1+\mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}}, \frac{v_{\tilde{\alpha}_1}v_{\tilde{\alpha}_2}}{1+(1-v_{\tilde{\alpha}_1})(1-v_{\tilde{\alpha}_2})}\right)$;

(2) $\tilde{\alpha}_1 \otimes_\varepsilon \tilde{\alpha}_2 = \left([a_1a_2, b_1b_2, c_1c_2, d_1d_2];\ \frac{\mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}}{1+(1-\mu_{\tilde{\alpha}_1})(1-\mu_{\tilde{\alpha}_2})}, \frac{v_{\tilde{\alpha}_1}+v_{\tilde{\alpha}_2}}{1+v_{\tilde{\alpha}_1}v_{\tilde{\alpha}_2}}\right)$;

(3) $\lambda \cdot_\varepsilon \tilde{\alpha} = \left([\lambda a, \lambda b, \lambda c, \lambda d];\ \frac{(1+\mu_{\tilde{\alpha}})^\lambda - (1-\mu_{\tilde{\alpha}})^\lambda}{(1+\mu_{\tilde{\alpha}})^\lambda + (1-\mu_{\tilde{\alpha}})^\lambda}, \frac{2(v_{\tilde{\alpha}})^\lambda}{(2-v_{\tilde{\alpha}})^\lambda + (v_{\tilde{\alpha}})^\lambda}\right)$;

(4) $\tilde{\alpha}^{\wedge_\varepsilon \lambda} = \left([a^\lambda, b^\lambda, c^\lambda, d^\lambda];\ \frac{2(\mu_{\tilde{\alpha}})^\lambda}{(2-\mu_{\tilde{\alpha}})^\lambda + (\mu_{\tilde{\alpha}})^\lambda}, \frac{(1+v_{\tilde{\alpha}})^\lambda - (1-v_{\tilde{\alpha}})^\lambda}{(1+v_{\tilde{\alpha}})^\lambda + (1-v_{\tilde{\alpha}})^\lambda}\right)$.

Theorem 1 *Let $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and $\tilde{\alpha}$ be three any intuitionistic trapezoidal fuzzy number, n, m be any positive real number, then, $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2$, $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2$, $\lambda \cdot_\varepsilon \tilde{\alpha}$ and $\tilde{\alpha}^{\wedge_\varepsilon m}$ are also an intuitionistic trapezoidal fuzzy number.*

Proof (1)(2) This result is obvious.

(3) Let n be any positive integer and $\tilde{\alpha}$ is an intuitionistic trapezoidal fuzzy number; then

$$n \cdot_\varepsilon \tilde{\alpha} = \overbrace{\tilde{\alpha} \oplus_\varepsilon \tilde{\alpha} \oplus_\varepsilon \cdots \oplus_\varepsilon \tilde{\alpha}}^{n}. \tag{8}$$

Mathematical induction can be used to prove that the above Eq. (8) holds for all positive integers n.

First, we prove that Eq. (8) holds for $n = 2$. Since

$$\begin{aligned}\tilde{\alpha} + \tilde{\alpha} &= \left([a+a, b+b, c+c, d+d];\ \frac{\mu_{\tilde{\alpha}}+\mu_{\tilde{\alpha}}}{1+\mu_{\tilde{\alpha}}\mu_{\tilde{\alpha}}}, \frac{v_{\tilde{\alpha}}v_{\tilde{\alpha}}}{1+(1-v_{\tilde{\alpha}})(1-v_{\tilde{\alpha}})}\right)\\ &= \left([2a, 2b, 2c, 2d];\ \frac{(1+\mu_{\tilde{\alpha}})^2-(1-\mu_{\tilde{\alpha}})^2}{(1+\mu_{\tilde{\alpha}})^2+(1-\mu_{\tilde{\alpha}})^2}, \frac{2(v_{\tilde{\alpha}})^2}{(2-v_{\tilde{\alpha}})^2+(v_{\tilde{\alpha}})^2}\right)\\ &= 2 \cdot_\varepsilon \tilde{\alpha}\end{aligned}$$

Therefore, the Eq. (8) holds for $n = 2$.

Second, if Eq. (8) holds for $n = k$, that is

$$n \cdot_\varepsilon \tilde{\alpha} = \overbrace{\tilde{\alpha} \oplus_\varepsilon \tilde{\alpha} \oplus_\varepsilon \cdots \oplus_\varepsilon \tilde{\alpha}}^{n}.$$

then, when $n = k+1$, we have

$$\overbrace{\tilde{\alpha} \oplus_\varepsilon \tilde{\alpha} \oplus_\varepsilon \cdots \oplus_\varepsilon \tilde{\alpha}}^{n+1} = \overbrace{\tilde{\alpha} \oplus_\varepsilon \tilde{\alpha} \oplus_\varepsilon \cdots \oplus_\varepsilon \tilde{\alpha}}^{n} \oplus_\varepsilon \tilde{\alpha} = n \cdot_\varepsilon \tilde{\alpha} \oplus_\varepsilon \tilde{\alpha}$$
$$= \left([na, nb, nc, nd]; \frac{(1+\mu_{\tilde{\alpha}})^n - (1-\mu_{\tilde{\alpha}})^n}{(1+\mu_{\tilde{\alpha}})^n + (1-\mu_{\tilde{\alpha}})^n}, \frac{2(v_{\tilde{\alpha}})^n}{(2-v_{\tilde{\alpha}})^n + (v_{\tilde{\alpha}})^n}\right)$$
$$\oplus_\varepsilon ([a,b,c,d]; \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$$
$$= ([(n+1)a, (n+1)b, (n+1)c, (n+1)d];$$
$$\left.\frac{(1+\mu_{\tilde{\alpha}})^{n+1} - (1-\mu_{\tilde{\alpha}})^{n+1}}{(1+\mu_{\tilde{\alpha}})^{n+1} + (1-\mu_{\tilde{\alpha}})^{n+1}}, \frac{2(v_{\tilde{\alpha}})^{n+1}}{(2-v_{\tilde{\alpha}})^{n+1} + (v_{\tilde{\alpha}})^{n+1}}\right)$$
$$= (n+1) \cdot_\varepsilon \tilde{\alpha}$$

i.e. Equation (8) holds for $n = k+1$. Therefore, Eq. (8) holds for all n.

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that Eq. (7) holds for any positive integer n.

Since $0 \le \mu_{\tilde{\alpha}} \le 1$, $0 \le v_{\tilde{\alpha}} \le 1$, $0 \le \mu_{\tilde{\alpha}} + v_{\tilde{\alpha}} \le 1$, $1 - \mu_{\tilde{\alpha}} \ge v_{\tilde{\alpha}} \ge 0$, λ is any positive integer, then, we have

$$0 \le \frac{(1+\mu_{\tilde{\alpha}})^\lambda - (1-\mu_{\tilde{\alpha}})^\lambda}{(1+\mu_{\tilde{\alpha}})^\lambda + (1-\mu_{\tilde{\alpha}})^\lambda} \le \frac{(1+\mu_{\tilde{\alpha}})^\lambda - (v_{\tilde{\alpha}})^\lambda}{(1+\mu_{\tilde{\alpha}})^\lambda + (v_{\tilde{\alpha}})^\lambda},$$
$$0 \le \frac{2(v_{\tilde{\alpha}})^\lambda}{(2-v_{\tilde{\alpha}})^\lambda + (v_{\tilde{\alpha}})^\lambda} \le \frac{2(v_{\tilde{\alpha}})^\lambda}{(1+\mu_{\tilde{\alpha}})^\lambda + (v_{\tilde{\alpha}})^\lambda}.$$

Thus, $0 \le \frac{(1+\mu_{\tilde{\alpha}})^\lambda - (1-\mu_{\tilde{\alpha}})^\lambda}{(1+\mu_{\tilde{\alpha}})^\lambda + (1-\mu_{\tilde{\alpha}})^\lambda} + \frac{2(v_{\tilde{\alpha}})^\lambda}{(2-v_{\tilde{\alpha}})^\lambda + (v_{\tilde{\alpha}})^\lambda} \le 1$.

That is to say, the $\lambda \cdot_\varepsilon \tilde{\alpha}$ defined above is an intuitionistic trapezoidal fuzzy number for any positive real number λ.

(4) Let m be any positive integer and $\tilde{\alpha}$ is an intuitionistic trapezoidal fuzzy number; then

$$\tilde{\alpha}^{\wedge_\varepsilon m} = \overbrace{\tilde{\alpha} \otimes_\varepsilon \tilde{\alpha} \otimes_\varepsilon \cdots \otimes_\varepsilon \tilde{\alpha}}^{m}. \quad (9)$$

Mathematical induction can be used to prove that the above Equation holds for all positive integers m.

First, we prove that Eq. (9) holds for $m = 2$. Since

$$\tilde{\alpha} \otimes_\varepsilon \tilde{\alpha} = \left([a \cdot a, b \cdot b, c \cdot c, d \cdot d]; \frac{\mu_{\tilde{\alpha}}\mu_{\tilde{\alpha}}}{1 + (1-\mu_{\tilde{\alpha}})(1-\mu_{\tilde{\alpha}})}, \frac{v_{\tilde{\alpha}} + v_{\tilde{\alpha}}}{1 + v_{\tilde{\alpha}}v_{\tilde{\alpha}}}\right)$$
$$= \left([a^2, b^2, c^2, d^2]; \frac{2(\mu_{\tilde{\alpha}})^2}{(2-\mu_{\tilde{\alpha}})^2 + (\mu_{\tilde{\alpha}})^2}, \frac{(1+v_{\tilde{\alpha}})^2 - (1-v_{\tilde{\alpha}})^2}{(1+v_{\tilde{\alpha}})^2 + (1-v_{\tilde{\alpha}})^2}\right)$$
$$= \tilde{\alpha}^{\wedge_\varepsilon 2}$$

Therefore, the Eq. (9) holds for $m = 2$.

Second, if Eq. (9) holds for $m = k$, that is

$$\tilde{\alpha}_1^{\wedge_\varepsilon m} = \overbrace{\tilde{\alpha}_1 \otimes_\varepsilon \tilde{\alpha}_1 \otimes_\varepsilon \cdots \otimes_\varepsilon \tilde{\alpha}_1}^{m}.$$

then, when, we have

$$\overbrace{\tilde{\alpha} \otimes_\varepsilon \tilde{\alpha} \otimes_\varepsilon \cdots \otimes_\varepsilon \tilde{\alpha}}^{m+1} = \overbrace{\tilde{\alpha} \otimes_\varepsilon \tilde{\alpha} \otimes_\varepsilon \cdots \otimes_\varepsilon \tilde{\alpha}}^{m+1} \otimes_\varepsilon \tilde{\alpha} = \tilde{\alpha}^{\wedge_\varepsilon n} \otimes_\varepsilon \tilde{\alpha}$$
$$= \left([a^n, b^n, c^n, d^n]; \frac{2(\mu_{\tilde{\alpha}})^n}{(2-\mu_{\tilde{\alpha}})^n + (\mu_{\tilde{\alpha}})^n}, \frac{(1+v_{\tilde{\alpha}})^n - (1-v_{\tilde{\alpha}})^n}{(1+v_{\tilde{\alpha}})^n + (1-v_{\tilde{\alpha}})^n}\right)$$
$$\oplus_\varepsilon ([a,b,c,d]; \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$$
$$= \left([a^{n+1}, b^{n+1}, c^{n+1}, d^{n+1}]; \frac{2(\mu_{\tilde{\alpha}})^{n+1}}{(2-\mu_{\tilde{\alpha}})^{n+1} + (\mu_{\tilde{\alpha}})^{n+1}},\right.$$
$$\left.\frac{(1+v_{\tilde{\alpha}})^{n+1} - (1-v_{\tilde{\alpha}})^{n+1}}{(1+v_{\tilde{\alpha}})^{n} + (1-v_{\tilde{\alpha}})^{n+1}}\right)$$
$$= \tilde{\alpha}^{\wedge_\varepsilon n+1}$$

i.e. Eq. (9) holds for $m = k+1$. Therefore, Eq. (9) holds for all m.

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that Eq. (9) holds for any positive integer n.

Since $0 \le \mu_{\tilde{\alpha}} \le 1$, $0 \le v_{\tilde{\alpha}} \le 1$, $0 \le \mu_{\tilde{\alpha}} + v_{\tilde{\alpha}} \le 1$, $1 - \mu_{\tilde{\alpha}} \ge v_{\tilde{\alpha}} \ge 0$, λ is any positive integer, then, we have

$$0 \le \frac{2(\mu_{\tilde{\alpha}})^\lambda}{(2-\mu_{\tilde{\alpha}})^\lambda + (\mu_{\tilde{\alpha}})^\lambda} \le \frac{2(\mu_{\tilde{\alpha}})^\lambda}{(1+v_{\tilde{\alpha}})^\lambda + (\mu_{\tilde{\alpha}})^\lambda},$$
$$0 \le \frac{(1+v_{\tilde{\alpha}})^\lambda - (1-v_{\tilde{\alpha}})^\lambda}{(1+v_{\tilde{\alpha}})^\lambda + (1-v_{\tilde{\alpha}})^\lambda} \le \frac{(1+v_{\tilde{\alpha}})^\lambda - (\mu_{\tilde{\alpha}})^\lambda}{(1+v_{\tilde{\alpha}})^\lambda + (\mu_{\tilde{\alpha}})^\lambda}.$$

Thus, $0 \le \frac{2(\mu_{\tilde{\alpha}})^\lambda}{(2-\mu_{\tilde{\alpha}})^\lambda + (\mu_{\tilde{\alpha}})^\lambda} + \frac{(1+v_{\tilde{\alpha}})^\lambda - (1-v_{\tilde{\alpha}})v}{(1+v_{\tilde{\alpha}})^\lambda + (1-v_{\tilde{\alpha}})^\lambda} \le 1$.

That is to say, the $\tilde{\alpha}^{\wedge_\varepsilon \lambda}$ defined above is an intuitionistic trapezoidal fuzzy number for any positive real number λ.

By the Einstein operational laws of intuitionistic trapezoidal fuzzy numbers, we have.

Theorem 2 *Let* $\tilde{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\tilde{\alpha}_1}, v_{\tilde{\alpha}_1})$ *and* $\tilde{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\tilde{\alpha}_2}, v_{\tilde{\alpha}_2})$ *be two intuitionistic trapezoidal fuzzy numbers, then the operational laws between* $\tilde{\alpha}_1$ *and* $\tilde{\alpha}_2$ *are shown as follows*:

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \tilde{\alpha}_2 \oplus \tilde{\alpha}_1$;
(2) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \tilde{\alpha}_2 \otimes \tilde{\alpha}_1$;
(3) $\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2 \quad \lambda \ge 0$;
(4) $\lambda_1\tilde{\alpha}_1 \oplus \lambda_2\tilde{\alpha}_1 = (\lambda_1 + \lambda_2)\tilde{\alpha}_1 \quad \lambda_1, \lambda_2 \ge 0$;
(5) $(\tilde{\alpha}_1 \otimes \tilde{\alpha}_2)^\lambda = \tilde{\alpha}_1^\lambda \otimes \tilde{\alpha}_2^\lambda, \quad \lambda \ge 0$;
(6) $\tilde{\alpha}^{\lambda_1} \otimes \tilde{\alpha}^{\lambda_2} = \tilde{\alpha}^{\lambda_1 + \lambda_2} \quad \lambda_1, \lambda_2 \ge 0$.

Proof (1)(2) This result is obvious.

(3) By the Einstein operational laws (1) in Definition 10, we have

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \frac{\mu_{\tilde{\alpha}_1} + \mu_{\tilde{\alpha}_2}}{1 + \mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}},\right.$$
$$\left.\frac{v_{\tilde{\alpha}_1}v_{\tilde{\alpha}_2}}{1 + (1 - v_{\tilde{\alpha}_1})(1 - v_{\tilde{\alpha}_2})}\right)$$

We transform the above equation into the following form:

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left(\begin{array}{l} [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \\ \frac{(1+\mu_{\tilde{\alpha}_1})(1+\mu_{\tilde{\alpha}_2}) - (1-\mu_{\tilde{\alpha}_1})(1-\mu_{\tilde{\alpha}_2})}{(1+\mu_{\tilde{\alpha}_1})(1+\mu_{\tilde{\alpha}_2}) + (1-\mu_{\tilde{\alpha}_1})(1-\mu_{\tilde{\alpha}_2})}, \frac{2 v_{\tilde{\alpha}_1} v_{\tilde{\alpha}_2}}{(2-v_{\tilde{\alpha}_1})(2-v_{\tilde{\alpha}_2}) + v_{\tilde{\alpha}_1} v_{\tilde{\alpha}_2}} \end{array} \right)$$

and let $x = (1+\mu_{\tilde{\alpha}_1})(1+\mu_{\tilde{\alpha}_2})$, $y = (1-\mu_{\tilde{\alpha}_1})(1-\mu_{\tilde{\alpha}_2})$, $z = v_{\tilde{\alpha}_1} v_{\tilde{\alpha}_2}$, $g = (2-v_{\tilde{\alpha}_1})(2-v_{\tilde{\alpha}_2})$; then

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \frac{x-y}{x+y}, \frac{2z}{g+z} \right)$$

By the Einstein operation (3) in Definition 10, we have

$$\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \left(\begin{array}{c} [\lambda(a_1+a_2), \lambda(b_1+b_2), \lambda(c_1+c_2), \lambda(d_1+d_2)]; \\ \frac{\left(1+\frac{x-y}{x+y}\right)^\lambda - \left(1-\frac{x-y}{x+y}\right)^\lambda}{\left(1+\frac{x-y}{x+y}\right)^\lambda + \left(1-\frac{x-y}{x+y}\right)^\lambda}, \frac{2\left(\frac{2z}{g+z}\right)^\lambda}{\left(2-\frac{2z}{g+z}\right)^\lambda + \left(\frac{2z}{g+z}\right)^\lambda} \end{array} \right)$$

$$= \left([\lambda(a_1+a_2), \lambda(b_1+b_2), \lambda(c_1+c_2), \lambda(d_1+d_2)]; \frac{(x)^\lambda - (y)^\lambda}{(x)^\lambda + (y)^\lambda}, \frac{2(z)^\lambda}{(g)^\lambda + (z)^\lambda} \right)$$

$$= \left(\begin{array}{r} [\lambda(a_1+a_2), \lambda(b_1+b_2), \lambda(c_1+c_2), \lambda(d_1+d_2)]; \\ \frac{(1+\mu_{\tilde{\alpha}_1})^\lambda (1+\mu_{\tilde{\alpha}_2})^\lambda - (1-\mu_{\tilde{\alpha}_1})^\lambda (1-\mu_{\tilde{\alpha}_2})^\lambda}{(1+\mu_{\tilde{\alpha}_1})^\lambda (1+\mu_{\tilde{\alpha}_2})^\lambda + (1-\mu_{\tilde{\alpha}_1})^\lambda (1-\mu_{\tilde{\alpha}_2})^\lambda}, \frac{2(v_{\tilde{\alpha}_1})^\lambda (v_{\tilde{\alpha}_2})^\lambda}{(2-v_{\tilde{\alpha}_1})^\lambda (2-v_{\tilde{\alpha}_2})^\lambda + (v_{\tilde{\alpha}_1})^\lambda (v_{\tilde{\alpha}_2})^\lambda} \end{array} \right)$$

In addition, since

$$\lambda\tilde{\alpha}_1 = \left([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; \frac{(1+\mu_{\tilde{\alpha}_1})^\lambda - (1-\mu_{\tilde{\alpha}_1})^\lambda}{(1+\mu_{\tilde{\alpha}_1})^\lambda + (1-\mu_{\tilde{\alpha}_1})^\lambda}, \frac{2(v_{\tilde{\alpha}_1})^\lambda}{(2-v_{\tilde{\alpha}_1})^\lambda + (v_{\tilde{\alpha}_1})^\lambda} \right)$$

$$\lambda\tilde{\alpha}_2 = \left([\lambda a_2, \lambda b_2, \lambda c_2, \lambda d_2]; \frac{(1+\mu_{\tilde{\alpha}_2})^\lambda - (1-\mu_{\tilde{\alpha}_2})^\lambda}{(1+\mu_{\tilde{\alpha}_2})^\lambda + (1-\mu_{\tilde{\alpha}_2})^\lambda}, \frac{2(v_{\tilde{\alpha}_2})^\lambda}{(2-v_{\tilde{\alpha}_2})^\lambda + (v_{\tilde{\alpha}_2})^\lambda} \right)$$

Let $x_1 = (1+\mu_{\tilde{\alpha}_1})^\lambda$, $y_1 = (1-\mu_{\tilde{\alpha}_1})^\lambda$, $z_1 = (v_{\tilde{\alpha}_1})^\lambda$, $g_1 = (2-v_{\tilde{\alpha}_1})^\lambda$, $x_2 = (1+\mu_{\tilde{\alpha}_2})^\lambda$, $y_2 = (1-\mu_{\tilde{\alpha}_2})^\lambda$, $z_2 = (v_{\tilde{\alpha}_2})^\lambda$, $g_2 = (2-v_{\tilde{\alpha}_2})^\lambda$; then

$$\lambda\tilde{\alpha}_1 = \left([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; \frac{x_1-y_1}{x_1+y_1}, \frac{2z_1}{g_1+z_1} \right),$$

$$\lambda\tilde{\alpha}_2 = \left([\lambda a_2, \lambda b_2, \lambda c_2, \lambda d_2]; \frac{x_2-y_2}{x_2+y_2}, \frac{2z_2}{g_2+z_2} \right)$$

By the Einstein operations (1) and (3) in Definition 10, we have

$$\lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2 = \left([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; \frac{x_1-y_1}{x_1+y_1}, \frac{2z_1}{g_1+z_1} \right) \oplus \left([\lambda a_2, \lambda b_2, \lambda c_2, \lambda d_2]; \frac{x_2-y_2}{x_2+y_2}, \frac{2z_2}{g_2+z_2} \right)$$

$$= \left([\lambda a_1 + \lambda a_2, \lambda b_1 + \lambda b_2, \lambda c_1 + \lambda c_2, \lambda d_1 + \lambda d_2]; \frac{\frac{x_1-y_1}{x_1+y_1} + \frac{x_2-y_2}{x_2+y_2}}{1 + \frac{x_1-y_1}{x_1+y_1} \cdot \frac{x_2-y_2}{x_2+y_2}}, \frac{\frac{2z_1}{g_1+z_1} \cdot \frac{2z_2}{g_2+z_2}}{1 + \left(1 - \frac{2z_1}{g_1+z_1}\right) \cdot \left(1 - \frac{2z_2}{g_2+z_2}\right)} \right)$$

$$= \left([\lambda a_1 + \lambda a_2, \lambda b_1 + \lambda b_2, \lambda c_1 + \lambda c_2, \lambda d_1 + \lambda d_2]; \frac{x_1 x_2 - y_1 y_2}{x_1 x_2 + y_1 y_2}, \frac{2 z_1 z_2}{g_1 g_2 + z_1 z_2} \right)$$

$$= \left(\begin{array}{l} [\lambda(a_1+a_2), \lambda(b_1+b_2), \lambda(c_1+c_2), \lambda(d_1+d_2)]; \\ \frac{(1+\mu_{\tilde{\alpha}_1})^\lambda (1+\mu_{\tilde{\alpha}_2})^\lambda - (1-\mu_{\tilde{\alpha}_1})^\lambda (1-\mu_{\tilde{\alpha}_2})^\lambda}{(1+\mu_{\tilde{\alpha}_1})^\lambda (1+\mu_{\tilde{\alpha}_2})^\lambda + (1-\mu_{\tilde{\alpha}_1})^\lambda (1-\mu_{\tilde{\alpha}_2})^\lambda}, \frac{2(v_{\tilde{\alpha}_1})^\lambda (v_{\tilde{\alpha}_2})^\lambda}{(2-v_{\tilde{\alpha}_1})^\lambda (2-v_{\tilde{\alpha}_2})^\lambda + (v_{\tilde{\alpha}_1})^\lambda (v_{\tilde{\alpha}_2})^\lambda} \end{array} \right)$$

Hence, $\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2$.

(4) Since $\lambda_1\tilde{\alpha}_1 = ([\lambda_1 a_1, \lambda_1 b_1, \lambda_1 c_1, \lambda_1 d_1]; \frac{(1+\mu_{\tilde{\alpha}_1})^{\lambda_1}-(1-\mu_{\tilde{\alpha}_1})^{\lambda_1}}{(1+\mu_{\tilde{\alpha}_1})^{\lambda_1}+(1-\mu_{\tilde{\alpha}_1})^{\lambda_1}}, \frac{2(v_{\tilde{\alpha}_1})^{\lambda_1}}{(2-v_{\tilde{\alpha}_1})^{\lambda_1}+(v_{\tilde{\alpha}_1})^{\lambda_1}})$

$$\lambda_2\tilde{\alpha}_1 = \left([\lambda_2 a_1, \lambda_2 b_1, \lambda_2 c_1, \lambda_2 d_1];\ \frac{(1+\mu_{\tilde{\alpha}_1})^{\lambda_2}-(1-\mu_{\tilde{\alpha}_1})^{\lambda_2}}{(1+\mu_{\tilde{\alpha}_1})^{\lambda_2}+(1-\mu_{\tilde{\alpha}_1})^{\lambda_2}}, \frac{2(v_{\tilde{\alpha}_1})^{\lambda_2}}{(2-v_{\tilde{\alpha}_1})^{\lambda_2}+(v_{\tilde{\alpha}_1})^{\lambda_2}}\right)$$

where $\lambda_1, \lambda_2 \geq 0$.

Let $x_1 = (1+\mu_{\tilde{\alpha}_1})^{\lambda_1}$, $y_1 = (1-\mu_{\tilde{\alpha}_1})^{\lambda_1}$, $z_1 = (v_{\tilde{\alpha}_1})^{\lambda_1}$, $g_1 = (2-v_{\tilde{\alpha}_1})^{\lambda_1}$, $x_2 = (1+\mu_{\tilde{\alpha}_1})^{\lambda_2}$, $y_2 = (1-\mu_{\tilde{\alpha}_1})^{\lambda_2}$, $z_2 = (v_{\tilde{\alpha}_1})^{\lambda_2}$, $g_2 = (2-v_{\tilde{\alpha}_1})^{\lambda_2}$; then

$$\lambda_1\tilde{\alpha}_1 = \left([\lambda_1 a_1, \lambda_1 b_1, \lambda_1 c_1, \lambda_1 d_1];\ \frac{x_1-y_1}{x_1+y_1}, \frac{2z_1}{g_1+z_1}\right),$$

$$\lambda_2\tilde{\alpha}_2 = \left([\lambda_2 a_1, \lambda_2 b_1, \lambda_2 c_1, \lambda_2 d_1];\ \frac{x_2-y_2}{x_2+y_2}, \frac{2z_2}{g_2+z_2}\right)$$

By the Einstein operations (1) and (3) in Definition 10, we obtain

$$\begin{aligned}
\lambda_1\tilde{\alpha}_1 \oplus \lambda_2\tilde{\alpha}_1 &= \left([\lambda_1 a_1, \lambda_1 b_1, \lambda_1 c_1, \lambda_1 d_1];\ \frac{x_1-y_1}{x_1+y_1}, \frac{2z_1}{g_1+z_1}\right) \oplus \left([\lambda_2 a_1, \lambda_2 b_1, \lambda_2 c_1, \lambda_2 d_1];\ \frac{x_2-y_2}{x_2+y_2}, \frac{2z_2}{g_2+z_2}\right) \\
&= \left([\lambda_1 a_1+\lambda_2 a_1, \lambda_1 b_1+\lambda_2 b_1, \lambda_1 c_1+\lambda_2 c_1, \lambda_1 d_1+\lambda_2 d_1];\ \frac{\frac{x_1-y_1}{x_1+y_1}+\frac{x_2-y_2}{x_2+y_2}}{1+\frac{x_1-y_1}{x_1+y_1}\cdot\frac{x_2-y_2}{x_2+y_2}}, \frac{\frac{2z_1}{g_1+z_1}\cdot\frac{2z_2}{g_2+z_2}}{1+\left(1-\frac{2z_1}{g_1+z_1}\right)\cdot\left(1-\frac{2z_2}{g_2+z_2}\right)}\right) \\
&= \left([\lambda_1 a_1+\lambda_2 a_1, \lambda_1 b_1+\lambda_2 b_1, \lambda_1 c_1+\lambda_2 c_1, \lambda_1 d_1+\lambda_2 d_1];\ \frac{x_1x_2-y_1y_2}{x_1x_2+y_1y_2}, \frac{2z_1z_2}{g_1g_2+z_1z_2}\right) \\
&= \left(\begin{array}{c}[a_1(\lambda_1+\lambda_2), b_1(\lambda_1+\lambda_2), c_1(\lambda_1+\lambda_2), d_1(\lambda_1+\lambda_2)]; \\ \frac{(1+\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}-(1-\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}{(1+\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}+(1-\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}, \frac{2(v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}{(2-v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}+(v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}\end{array}\right) = (\lambda_1+\lambda_2)\tilde{\alpha}_1
\end{aligned}.$$

Hence, $\lambda_1\tilde{\alpha}_1 \oplus \lambda_2\tilde{\alpha}_1 = (\lambda_1+\lambda_2)\tilde{\alpha}_1$.

(5) Since

$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left([a_1a_2, b_1b_2, c_1c_2, d_1d_2];\ \frac{2\mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}}{(2-\mu_{\tilde{\alpha}_1})(2-\mu_{\tilde{\alpha}_2})+\mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}}, \frac{(1+v_{\tilde{\alpha}_1})(1+v_{\tilde{\alpha}_2})-(1-v_{\tilde{\alpha}_1})(1-v_{\tilde{\alpha}_2})}{(1+v_{\tilde{\alpha}_1})(1+v_{\tilde{\alpha}_2})+(1-v_{\tilde{\alpha}_1})(1-v_{\tilde{\alpha}_2})}\right)$$

and let $x = \mu_{\tilde{\alpha}_1}\mu_{\tilde{\alpha}_2}$, $y = (2-\mu_{\tilde{\alpha}_1})(2-\mu_{\tilde{\alpha}_2})$, $z = (1+v_{\tilde{\alpha}_1})(1+v_{\tilde{\alpha}_2})$, $g = (1-v_{\tilde{\alpha}_1})(1-v_{\tilde{\alpha}_2})$; then

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left([a_1a_2, b_1b_2, c_1c_2, d_1d_2];\ \frac{2x}{y+x}, \frac{z-g}{z+g}\right)$$

$$(\tilde{\alpha}_1 \otimes \tilde{\alpha}_2)^{\lambda} = \left([(a_1a_2)^{\lambda}, (b_1b_2)^{\lambda}, (c_1c_2)^{\lambda}, (d_1d_2)^{\lambda}];\ \frac{2\left(\frac{2x}{y+x}\right)^{\lambda}}{\left(2-\frac{2x}{y+x}\right)^{\lambda}+\left(\frac{2x}{y+x}\right)^{\lambda}}, \frac{\left(1+\frac{z-g}{z+g}\right)^{\lambda}-\left(1-\frac{z-g}{z+g}\right)^{\lambda}}{\left(1+\frac{z-g}{z+g}\right)^{\lambda}+\left(1-\frac{z-g}{z+g}\right)^{\lambda}} \right)$$
$$= \left([(a_1a_2)^{\lambda}, (b_1b_2)^{\lambda}, (c_1c_2)^{\lambda}, (d_1d_2)^{\lambda}];\ \frac{2(x)^{\lambda}}{(x)^{\lambda}+(y)^{\lambda}}, \frac{(z)^{\lambda}-(g)^{\lambda}}{(z)^{\lambda}-(g)^{\lambda}} \right)$$
$$= \left(\begin{array}{l} [(a_1a_2)^{\lambda}, (b_1b_2)^{\lambda}, (c_1c_2)^{\lambda}, (d_1d_2)^{\lambda}]; \\ \frac{2(\mu_{\tilde{\alpha}_1})^{\lambda}(\mu_{\tilde{\alpha}_2})^{\lambda}}{(2-\mu_{\tilde{\alpha}_1})^{\lambda}(2-\mu_{\tilde{\alpha}_2})^{\lambda}+(\mu_{\tilde{\alpha}_1})^{\lambda}(\mu_{\tilde{\alpha}_2})^{\lambda}}, \frac{(1+v_{\tilde{\alpha}_1})^{\lambda}(1+v_{\tilde{\alpha}_2})^{\lambda}-(1-v_{\tilde{\alpha}_1})^{\lambda}(1-v_{\tilde{\alpha}_2})^{\lambda}}{(1+v_{\tilde{\alpha}_1})^{\lambda}(1+v_{\tilde{\alpha}_2})^{\lambda}+(1-v_{\tilde{\alpha}_1})^{\lambda}(1-v_{\tilde{\alpha}_2})^{\lambda}} \end{array} \right)$$

In addition, since

$$\tilde{\alpha}_1^{\lambda} = \left([a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}, d_1^{\lambda}];\ \frac{2(\mu_{\tilde{\alpha}_1})^{\lambda}}{(2-\mu_{\tilde{\alpha}_1})^{\lambda}+(\mu_{\tilde{\alpha}_1})^{\lambda}}, \frac{(1+v_{\tilde{\alpha}_1})^{\lambda}-(1-v_{\tilde{\alpha}_1})^{\lambda}}{(1+v_{\tilde{\alpha}_1})^{\lambda}+(1-v_{\tilde{\alpha}_1})^{\lambda}} \right)$$
$$\tilde{\alpha}_2^{\lambda} = \left([a_2^{\lambda}, b_2^{\lambda}, c_2^{\lambda}, d_2^{\lambda}];\ \frac{2(\mu_{\tilde{\alpha}_2})^{\lambda}}{(2-\mu_{\tilde{\alpha}_2})^{\lambda}+(\mu_{\tilde{\alpha}_2})^{\lambda}}, \frac{(1+v_{\tilde{\alpha}_2})^{\lambda}-(1-v_{\tilde{\alpha}_2})^{\lambda}}{(1+v_{\tilde{\alpha}_2})^{\lambda}+(1-v_{\tilde{\alpha}_2})^{\lambda}} \right)$$

and let $x_1 = (v_{\tilde{\alpha}_1})^{\lambda}$, $y_1 = (2-v_{\tilde{\alpha}_1})^{\lambda}$, $z_1 = (1+\mu_{\tilde{\alpha}_1})^{\lambda}$, $g_1 = (1-\mu_{\tilde{\alpha}_1})^{\lambda}$, $x_2 = (v_{\tilde{\alpha}_2})^{\lambda}$, $y_2 = (2-v_{\tilde{\alpha}_2})^{\lambda}$, $z_2 = (1+\mu_{\tilde{\alpha}_2})^{\lambda}$, $g_2 = (1-\mu_{\tilde{\alpha}_2})^{\lambda}$; then

$$\tilde{\alpha}_2^{\lambda} = \left([a_2^{\lambda}, b_2^{\lambda}, c_2^{\lambda}, d_2^{\lambda}];\ \frac{2(\mu_{\tilde{\alpha}_2})^{\lambda}}{(2-\mu_{\tilde{\alpha}_2})^{\lambda}+(\mu_{\tilde{\alpha}_2})^{\lambda}}, \frac{(1+v_{\tilde{\alpha}_2})^{\lambda}-(1-v_{\tilde{\alpha}_2})^{\lambda}}{(1+v_{\tilde{\alpha}_2})^{\lambda}+(1-v_{\tilde{\alpha}_2})^{\lambda}} \right)$$
$$\tilde{\alpha}_1^{\lambda} = \left([a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}, d_1^{\lambda}];\ \frac{2x_1}{x_1+y_1}, \frac{z_1-g_1}{z_1+g_1} \right),$$
$$\tilde{\alpha}_2^{\lambda} = \left([a_2^{\lambda}, b_2^{\lambda}, c_2^{\lambda}, d_2^{\lambda}];\ \frac{2x_2}{x_2+y_2} \frac{z_2-g_2}{z_2+g_2} \right),$$

By the Einstein operations (2) and (4) in Definition 10, we obtain

$$\tilde{\alpha}_1^{\lambda} \oplus \tilde{\alpha}_2^{\lambda} = \left([a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}, d_1^{\lambda}]; \frac{2x_1}{x_1+y_1}, \frac{z_1-g_1}{z_1+g_1} \right) \otimes \left([a_2^{\lambda}, b_2^{\lambda}, c_2^{\lambda}, d_2^{\lambda}]; \frac{2x_2}{x_2+y_2} \frac{z_2-g_2}{z_2+g_2} \right),$$
$$= \left([(a_1a_2)^{\lambda}, (b_1b_2)^{\lambda}, (c_1c_2)^{\lambda}, (d_1d_2)^{\lambda}]; \frac{\frac{2x_1}{x_1+y_1} \cdot \frac{2x_2}{x_2+y_2}}{1+\left(1-\frac{2x_1}{x_1+y_1}\right) \cdot \left(1-\frac{2x_2}{x_2+x_2}\right)}, \frac{\frac{z_1-g_1}{z_1+g_1}+\frac{z_2-g_2}{z_2+g_2}}{1+\frac{z_1-g_1}{z_1+g_1} \cdot \frac{z_2-g_2}{z_2+g_2}} \right)$$
$$= \left([(a_1a_2)^{\lambda}, (b_1b_2)^{\lambda}, (c_1c_2)^{\lambda}, (d_1d_2)^{\lambda}]; \frac{2x_1x_2}{x_1x_2+y_1y_2}, \frac{z_1z_2-g_1g_2}{z_1z_2+g_1g_2} \right)$$
$$= \left(\begin{array}{l} [(a_1a_2)^{\lambda}, (b_1b_2)^{\lambda}, (c_1c_2)^{\lambda}, (d_1d_2)^{\lambda}]; \\ \frac{2(\mu_{\tilde{\alpha}_1})^{\lambda}(\mu_{\tilde{\alpha}_2})^{\lambda}}{(2-\mu_{\tilde{\alpha}_1})^{\lambda}(2-\mu_{\tilde{\alpha}_2})^{\lambda}+(\mu_{\tilde{\alpha}_1})^{\lambda}(\mu_{\tilde{\alpha}_2})^{\lambda}}, \frac{(1+v_{\tilde{\alpha}_1})^{\lambda}(1+v_{\tilde{\alpha}_2})^{\lambda}-(1-v_{\tilde{\alpha}_1})^{\lambda}(1-v_{\tilde{\alpha}_2})^{\lambda}}{(1+v_{\tilde{\alpha}_1})^{\lambda}(1+v_{\tilde{\alpha}_2})^{\lambda}+(1-v_{\tilde{\alpha}_1})^{\lambda}(1-v_{\tilde{\alpha}_2})^{\lambda}} \end{array} \right)$$

Hence, $(\tilde{\alpha}_1 \otimes \tilde{\alpha}_2)^{\lambda} = \tilde{\alpha}_1^{\lambda} \otimes \tilde{\alpha}_2^{\lambda}$.

(6) Since $\tilde{\alpha}_1^{\lambda_1} = \left([a_1^{\lambda_1}, b_1^{\lambda_1}, c_1^{\lambda_1}, d_1^{\lambda_1}];\ \frac{2(\mu_{\tilde{\alpha}_1})^{\lambda_1}}{(2-\mu_{\tilde{\alpha}_1})^{\lambda_1}+(\mu_{\tilde{\alpha}_1})^{\lambda_1}}, \frac{(1+v_{\tilde{\alpha}_1})^{\lambda_1}-(1-v_{\tilde{\alpha}_1})^{\lambda_1}}{(1+v_{\tilde{\alpha}_1})^{\lambda_1}+(1-v_{\tilde{\alpha}_1})^{\lambda_1}}\right)$

$$\tilde{\alpha}_1^{\lambda_2} = \left([a_1^{\lambda_2}, b_1^{\lambda_2}, c_1^{\lambda_2}, d_1^{\lambda_2}];\ \frac{2(\mu_{\tilde{\alpha}_1})^{\lambda_2}}{(2-\mu_{\tilde{\alpha}_1})^{\lambda_2}+(\mu_{\tilde{\alpha}_1})^{\lambda_2}}, \frac{(1+v_{\tilde{\alpha}_1})^{\lambda_2}-(1-v_{\tilde{\alpha}_1})^{\lambda_2}}{(1+v_{\tilde{\alpha}_1})^{\lambda_2}+(1-v_{\tilde{\alpha}_1})^{\lambda_2}}\right)$$

where $\lambda_1, \lambda_2 \geq 0$, let $x_1 = (\mu_{\tilde{\alpha}_1})^{\lambda_1}$, $y_1 = (2-\mu_{\tilde{\alpha}_1})^{\lambda_1}$, $z_1 = (1+v_{\tilde{\alpha}_1})^{\lambda_1}$, $g_1 = (1-v_{\tilde{\alpha}_1})^{\lambda_1}$, $x_2 = (\mu_{\tilde{\alpha}_1})^{\lambda_2}$, $y_2 = (2-\mu_{\tilde{\alpha}_1})^{\lambda_2}$, $z_2 = (1+v_{\tilde{\alpha}_1})^{\lambda_2}$, $g_2 = (1-v_{\tilde{\alpha}_1})^{\lambda_2}$; then

By the Einstein operations (2) and (4) in Definition 10, we have

$$\begin{aligned}
\tilde{\alpha}_1^{\lambda_1} \otimes \tilde{\alpha}_1^{\lambda_2} &= \left([a_1^{\lambda_1}, b_1^{\lambda_1}, c_1^{\lambda_1}, d_1^{\lambda_1}];\ \frac{2x_1}{x_1+y_1}, \frac{z_1-g_1}{z_1+g_1}\right) \oplus \left([a_1^{\lambda_2}, b_1^{\lambda_2}, c_1^{\lambda_2}, d_1^{\lambda_2}];\ \frac{2x_2}{x_2+y_2}, \frac{z_2-g_2}{z_2+y_2}\right) \\
&= \left([a_1^{\lambda_1}a_1^{\lambda_2}, b_1^{\lambda_1}b_1^{\lambda_2}, c_1^{\lambda_1}c_1^{\lambda_2}, c_1^{\lambda_1}c_1^{\lambda_2}];\ \frac{\frac{2x_1}{x_1+y_1}\cdot\frac{2x_2}{g_2+y_2}}{1+\left(1-\frac{2x_1}{x_1+y_1}\right)\cdot\left(1-\frac{2x_2}{x_2+y_2}\right)}, \frac{\frac{z_1-g_1}{z_1+g_1}+\frac{z_2-g_2}{z_2+g_2}}{1+\frac{z_1-g_1}{z_1+g_1}\cdot\frac{z_2-g_2}{z_2+g_2}}\right) \\
&= \left([a_1^{\lambda_1}a_1^{\lambda_2}, b_1^{\lambda_1}b_1^{\lambda_2}, c_1^{\lambda_2}c_1^{\lambda_2}, c_1^{\lambda_1}c_1^{\lambda_2}];\ \frac{2x_1x_2}{x_1x_2+y_1y_2}, \frac{z_1z_2-g_1g_2}{z_1z_2+g_1g_2}\right) \\
&= \left(\begin{array}{l} [a_1^{\lambda_1+\lambda_2},\ b_1^{\lambda_1+\lambda_2}, c_1^{\lambda_1+\lambda_2}, d_1^{\lambda_1+\lambda_2}]; \\ \frac{2(\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}{(2-\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}+(\mu_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}, \frac{(1+v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}-(1-v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}}{(1+v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}+(1-v_{\tilde{\alpha}_1})^{\lambda_1+\lambda_2}} \end{array}\right) \\
&= \tilde{\alpha}_1^{\lambda_1+\lambda_2}
\end{aligned}$$

Hence, $\tilde{\alpha}_1^{\lambda_1} \otimes \tilde{\alpha}_1^{\lambda_2} = \tilde{\alpha}_1^{\lambda_1+\lambda_2}$.

3 Intuitionistic trapezoidal fuzzy Einstein aggregation operators

Based on the above Einstein operational laws of ITFNs, we will investigate the intuitionistic trapezoidal fuzzy information aggregation operators and give the definition of some aggregation operators with the intuitionistic trapezoidal fuzzy numbers based on Einstein operational laws as follows. Let Ω be the set of intuitionistic trapezoidal fuzzy numbers.

3.1 Intuitionistic trapezoidal fuzzy Einstein arithmetic aggregation operators

Definition 11 Let $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \cdots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy Einstein weighted averaging (ITFEWA) operator of dimension n is mapping ITFEWA: $\Omega^n \to \Omega$, and

$$\text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus_{\varepsilon}}_{j=1}^{n} (\omega_j \tilde{\alpha}_j). \tag{10}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$, with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

Based on the Definitions 10 and 11, we can derive the Theorem 3.

Theorem 3 *Let $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFEWA operator is also an intuitionistic trapezoidal fuzzy number, and*

$$\mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus_{\varepsilon}}_{j=1}^{n}(\omega_j \tilde{\alpha}_j) = \left(\begin{array}{l} \left[\sum_{j=1}^{n}\omega_j a_j, \sum_{j=1}^{n}\omega_j b_j, \sum_{j=1}^{n}\omega_j c_j, \sum_{j=1}^{n}\omega_j d_j\right]; \\ \dfrac{\prod_{j=1}^{n}(1+\mu_{\tilde{\alpha}_j})^{\omega_j} - \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\mu_{\tilde{\alpha}_j})^{\omega_j} + \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_j})^{\omega_j}}, \dfrac{2\prod_{j=1}^{n}(v_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{n}(2-v_{\tilde{\alpha}_j})^{\omega_j} + \prod_{j=1}^{n}(v_{\tilde{\alpha}_j})^{\omega_j}} \end{array} \right) \tag{11}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weight vector of* $\tilde{\alpha}_j$ *(*$j = 1, 2, \ldots, n$*), with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n}\omega_j = 1$.

Proof In the following, we prove the second result by using mathematical induction on n.

(1) We first prove that Eq. (11) holds for $n = 2$. Since

$$\omega_1\tilde{\alpha}_1 = \left([\omega_1 a_1, \omega_1 b_1, \omega_1 c_1, \omega_1 d_1]; \frac{(1+\mu_{\tilde{\alpha}_1})^{\omega_1} - (1-\mu_{\tilde{\alpha}_1})^{\omega_1}}{(1+\mu_{\tilde{\alpha}_1})^{\omega_1} + (1-\mu_{\tilde{\alpha}_1})^{\omega_1}}, \frac{2(v_{\tilde{\alpha}_1})^{\omega_1}}{(2-v_{\tilde{\alpha}_1})^{\omega_1} + (v_{\tilde{\alpha}_1})^{\omega_1}}\right)$$

$$\omega_2\tilde{\alpha}_2 = \left([\omega_2 a_2, \omega_2 b_2, \omega_2 c_2, \omega_2 d_2]; \frac{(1+\mu_{\tilde{\alpha}_2})^{\omega_2} - (1-\mu_{\tilde{\alpha}_2})^{\omega_2}}{(1+\mu_{\tilde{\alpha}_2})^{\omega_2} + (1-\mu_{\tilde{\alpha}_2})^{\omega_2}}, \frac{2(v_{\tilde{\alpha}_2})^{\omega_2}}{(2-v_{\tilde{\alpha}_2})^{\omega_2} + (v_{\tilde{\alpha}_2})^{\omega_2}}\right)$$

Let $x_1 = (1+\mu_{\tilde{\alpha}_1})^{\omega_1}$, $y_1 = (1-\mu_{\tilde{\alpha}_1})^{\omega_1}$, $z_1 = (v_{\tilde{\alpha}_1})^{\omega_1}$, $g_1 = (2-v_{\tilde{\alpha}_1})^{\omega_1}$, $x_2 = (1+\mu_{\tilde{\alpha}_2})^{\omega_2}$, $y_2 = (1-\mu_{\tilde{\alpha}_2})^{\omega_2}$, $z_1 = (v_{\tilde{\alpha}_2})^{\omega_2}$, $g_1 = (2-v_{\tilde{\alpha}_2})^{\omega_2}$; then,

$$\omega_1\tilde{\alpha}_1 = \left([\omega_1 a_1, \omega_1 b_1, \omega_1 c_1, \omega_1 d_1]; \frac{x_1 - y_1}{x_1 + y_1}, \frac{2z_1}{g_1 + z_1}\right)$$

$$\omega_2\tilde{\alpha}_2 = \left([\omega_2 a_2, \omega_2 b_2, \omega_2 c_2, \omega_2 d_2]; \frac{x_2 - y_2}{x_2 + y_2}, \frac{2z_2}{g_2 + z_2}\right)$$

Thus, by the Einstein operation (3) in Definition 10, we have

$$\begin{aligned} \mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2) &= \omega_1\tilde{\alpha}_1 \oplus_{\varepsilon} \omega_2\tilde{\alpha}_2 \\ &= \left([\omega_1 a_1 + \omega_2 a_2, \omega_1 b_1 + \omega_2 b_2, \omega_1 c_1 + \omega_2 c_2, \omega_1 d_1 + \omega_2 d_2]; \ \frac{x_1 x_2 - y_1 y_2}{x_1 x_2 + y_1 y_2}, \ \frac{2 z_1 z_2}{g_1 g_2 + z_1 z_2}\right) \\ &= \left(\left[\sum_{j=1}^{2}\omega_j a_j, \sum_{j=1}^{2}\omega_j b_j, \sum_{j=1}^{2}\omega_j c_j, \sum_{j=1}^{2}\omega_j d_j\right]; \frac{\prod_{j=1}^{2}(1+\mu_{\tilde{\alpha}_j})^{\omega_j} - \prod_{j=1}^{2}(1-\mu_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{2}(1+\mu_{\tilde{\alpha}_j})^{\omega_j} + \prod_{j=1}^{2}(1-\mu_{\tilde{\alpha}_j})^{w_j}}, \ \frac{2\prod_{j=1}^{2}(v_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{2}(2-v_{\tilde{\alpha}_j})^{\omega_j} + \prod_{j=1}^{2}(v_{\tilde{\alpha}_j})^{\omega_j}}\right) \end{aligned}$$

(2) If Eq. (11) holds for $n = k$, that is

$$\mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) = \mathop{\oplus_{\varepsilon}}_{j=1}^{k}(\omega_j \tilde{\alpha}_j)$$

$$= \left(\left[\sum_{j=1}^{k} \omega_j a_j, \sum_{j=1}^{k} \omega_j b_j, \sum_{j=1}^{k} \omega_j c_j, \sum_{j=1}^{k} \omega_j d_j \right]; \frac{\prod_{j=1}^{k}\left(1+\mu_{\tilde{\alpha}_j}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\mu_{\tilde{\alpha}_j}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\mu_{\tilde{\alpha}_j}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\mu_{\tilde{\alpha}_j}\right)^{\omega_j}}, \frac{2\prod_{j=1}^{k}\left(v_{\tilde{\alpha}_j}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(2-v_{\tilde{\alpha}_j}\right)^{\omega_j} + \prod_{j=1}^{k}\left(v_{\tilde{\alpha}_j}\right)^{\omega_j}} \right).$$

then, when $n = k + 1$, we have

$$\mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{k+1}) = \mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots \tilde{\alpha}_k) \oplus_{\varepsilon} (\omega_{k+1}\tilde{\alpha}_{k+1})$$

$$= \mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) \oplus_{\varepsilon}$$

$$\left([\omega_{k+1}a_{k+1}, \omega_{k+1}b_{k+1}, \omega_{k+1}c_{k+1}, \omega_{k+1}d_{k+1}]; \frac{(1+\mu_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}} - (1-\mu_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}}}{(1+\mu_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}} + (1-\mu_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}}}, \frac{2(v_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}}}{(2-v_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}} + (v_{\tilde{\alpha}_{k+1}})^{\omega_{k+1}}} \right)$$

Let $x_1 = \prod_{j=1}^{k}(1+\mu_{\tilde{\alpha}_j})^{\omega_j}$, $y_1 = \prod_{j=1}^{k}(1-\mu_{\tilde{\alpha}_j})^{\omega_j}$, $z_1 = \prod_{j=1}^{k}(v_{\tilde{\alpha}_j})^{\omega_j}$, $g_1 = \prod_{j=1}^{k}(2-v_{\tilde{\alpha}_j})^{\omega_j}$, $x_2 = (1+\mu_{\tilde{\alpha}_{k+1}})^{\omega_j}$, $y_2 = (1-\mu_{\tilde{\alpha}_{k+1}})^{\omega_j}$, $z_1 = (v_{\tilde{\alpha}_{k+1}})^{\omega_j}$, $g_1 = (2-v_{\tilde{\alpha}_{k+1}})^{\omega_j}$; then,

thus, by the Einstein operation (3) in Definition 10, we have

$$\mathrm{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_k) = \left(\left[\sum_{j=1}^{k} \omega_j a_j, \sum_{j=1}^{k} \omega_j b_j, \sum_{j=1}^{k} \omega_j c_j, \sum_{j=1}^{k} \omega_j d_j \right]; \frac{x_1 - y_1}{x_1 + y_1}, \frac{2z_1}{g_1 + z_1} \right),$$

$$w_{k+1}\tilde{\alpha}_{k+1} = \left([\omega_{k+1}a_{k+1}, \omega_{k+1}b_{k+1}, \omega_{k+1}c_{k+1}, \omega_{k+1}d_{k+1}]; \frac{x_2 - y_2}{x_2 + y_2}, \frac{2z_2}{g_2 + z_2} \right).$$

$$\begin{aligned}
&\mathrm{ITFEWA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_{k+1}) = \mathrm{ITFEWA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_k)\oplus_\varepsilon(\omega_{k+1}\tilde{\alpha}_{k+1})\\
&= \left(\left[\sum_{j=1}^{k}\omega_j a_j,\sum_{j=1}^{k}\omega_j b_j,\sum_{j=1}^{k}\omega_j c_j,\sum_{j=1}^{k}\omega_j d_j\right];\frac{x_1-y_1}{x_1+y_1},\frac{2z_1}{g_1+z_1}\right)\\
&\quad\oplus_\varepsilon\left([\omega_{k+1}a_{k+1},\omega_{k+1}b_{k+1},\omega_{k+1}c_{k+1},\omega_{k+1}d_{k+1}];\frac{x_2-y_2}{x_2+y_2},\frac{2z_2}{g_2+z_2}\right)\\
&= \left(\left[\sum_{j=1}^{k+1}\omega_j a_j,\sum_{j=1}^{k+1}\omega_j b_j,\sum_{j=1}^{k+1}\omega_j c_j,\sum_{j=1}^{k+1}\omega_j d_j\right];\frac{x_1x_2-y_1y_2}{x_1x_2+y_1y_2},\frac{2z_1z_2}{g_1g_2+z_1z_2}\right)\\
&= \left(\left[\sum_{j=1}^{k+1}\omega_j a_j,\sum_{j=1}^{k+1}\omega_j b_j,\sum_{j=1}^{k+1}\omega_j c_j,\sum_{j=1}^{k+1}\omega_j d_j\right];\frac{\prod_{j=1}^{k+1}(1+\mu_{\tilde{\alpha}_j})^{\omega_j}-\prod_{j=1}^{k+1}(1-\mu_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{k+1}(1+\mu_{\tilde{\alpha}_j})^{\omega_j}+\prod_{j=1}^{k+1}(1-\mu_{\tilde{\alpha}_j})^{\omega_j}},\frac{2\prod_{j=1}^{k+1}(v_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{k+1}(2-v_{\tilde{\alpha}_j})^{\omega_j}+\prod_{j=1}^{k+1}(v_{\tilde{\alpha}_j})^{\omega_j}}\right)
\end{aligned}$$

i.e. Eq. (11) holds for $n=k+1$.

Therefore, Eq. (11) holds for all n, which completes the proof of Theorem 3.

Example 1 Let $\tilde{\alpha}_1=([0.3,0.4,0.5,0.6];0.1,0.7)$, $\tilde{\alpha}_2=([0.2,0.3,0.4,0.5];0.4,0.3)$, $\tilde{\alpha}_3=([0.2,0.3,0.5,0.6];0.6,0.1)$, $\tilde{\alpha}_4=([0.6,0.7,0.8,0.9];0.2,0.5)$ be four intuitionistic trapezoidal fuzzy numbers, and let $\omega=(0.2,0.3,0.1,0.4)^T$ be the weighted vector of $\tilde{\alpha}_j(j=1,2,3,4)$; then, by Eq. (11), it follows that $ITFEWA(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3,\ \tilde{\alpha}_4)=([0.38,0.48,0.59,0.69];0.289,0.403)$.

Based on the Theorem 3, the ITFEWA operator satisfies the following properties:

Theorem 4 *Let $\tilde{\alpha}_j$ $(j=1,2,\ldots,n)$ be a collection of ITFNs, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ $(j=1,2,\ldots,n)$, with $\omega_j\in[0,1]$ and $\sum_{j=1}^n\omega_j=1$, then the ITFEWA operator satisfies the following properties:*

(1) (Idempotency) If all $\tilde{\alpha}_j(j=1,2,\ldots,n)$ are equal, i.e. $\tilde{\alpha}_j=\tilde{\alpha}$ for all j, then

$$\mathrm{ITFEWA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\tilde{\alpha} \tag{12}$$

Proof By Definition 11, we have

$$\begin{aligned}
&\mathrm{ITFEWA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=\omega_1\tilde{\alpha}_1\oplus_\varepsilon\omega_2\tilde{\alpha}_2\oplus_\varepsilon\cdots\oplus_\varepsilon\omega_n\tilde{\alpha}_n\\
&=\omega_1\tilde{\alpha}\oplus_\varepsilon\omega_2\tilde{\alpha}\oplus_\varepsilon\cdots\oplus_\varepsilon\omega_n\tilde{\alpha}=\sum_{j=1}^n\omega_j\tilde{\alpha}=\tilde{\alpha}\,.
\end{aligned}$$

(2) (Boundary) Let $\tilde{\alpha}_{\min}=\min\{\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\}$ and $\tilde{\alpha}_{\max}=\max\{\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\}$, then

$$\tilde{\alpha}_{\min}\le\mathrm{ITFEWA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)\le\tilde{\alpha}_{\max}. \tag{13}$$

Proof Let $f(x)=\frac{1-x}{1+x}$, $x\in[0,1]$; then, $f'(x)=\frac{-2}{(1+x)^2}<0$; thus $f(x)$ is a decreasing function.

Since $\mu_{\tilde{\alpha}_{\min}}\le\mu_{\tilde{\alpha}_j}\le\mu_{\tilde{\alpha}_{\max}}$, for all j, then $f(\mu_{\tilde{\alpha}_{\max}})\le f(\mu_{\tilde{\alpha}_j})\le f(\mu_{\tilde{\alpha}_{\min}})$, thus,

$$\begin{aligned}
&\frac{1-\mu_{\tilde{\alpha}_{\max}}}{1+\mu_{\tilde{\alpha}_{\max}}}\le\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\le\frac{1-\mu_{\tilde{\alpha}_{\min}}}{1+\mu_{\tilde{\alpha}_{\min}}},\left(\frac{1-\mu_{\tilde{\alpha}_{\max}}}{1+\mu_{\tilde{\alpha}_{\max}}}\right)^{\omega_j}\le\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}\\
&\le\left(\frac{1-\mu_{\tilde{\alpha}_{\min}}}{1+\mu_{\tilde{\alpha}_{\min}}}\right)^{\omega_j},\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_{\max}}}{1+\mu_{\tilde{\alpha}_{\max}}}\right)^{\omega_j}\le\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}\\
&\le\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_{\min}}}{1+\mu_{\tilde{\alpha}_{\min}}}\right)^{\omega_j},\left(\frac{1-\mu_{\tilde{\alpha}_{\max}}}{1+\mu_{\tilde{\alpha}_{\max}}}\right)^{\sum_{j=1}^n\omega_j}\le\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}\\
&\le\left(\frac{1-\mu_{\tilde{\alpha}_{\min}}}{1+\mu_{\tilde{\alpha}_{\min}}}\right)^{\sum_{j=1}^n\omega_j},\left(\frac{1-\mu_{\tilde{\alpha}_{\max}}}{1+\mu_{\tilde{\alpha}_{\max}}}\right)\le\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}\\
&\le\left(\frac{1-\mu_{\tilde{\alpha}_{\min}}}{1+\mu_{\tilde{\alpha}_{\min}}}\right),\left(\frac{2}{1+\mu_{\tilde{\alpha}_{\max}}}\right)\le1+\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}\\
&\le\left(\frac{2}{1+\mu_{\tilde{\alpha}_{\min}}}\right),\left(\frac{1+\mu_{\tilde{\alpha}_{\max}}}{2}\right)\le\frac{1}{1+\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}}\\
&\le\left(\frac{1+\mu_{\tilde{\alpha}_{\min}}}{2}\right),(1+\mu_{\tilde{\alpha}_{\max}})\le\frac{2}{1+\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}}\le(1+\mu_{\tilde{\alpha}_{\min}}),\\
&\mu_{\tilde{\alpha}_{\max}}\le\frac{2}{1+\prod_{j=1}^n\left(\frac{1-\mu_{\tilde{\alpha}_j}}{1+\mu_{\tilde{\alpha}_j}}\right)^{\omega_j}}-1\le\mu_{\tilde{\alpha}_{\min}}
\end{aligned}$$

Thus, $\mu_{\tilde{\alpha}_{\max}}\le\dfrac{\prod_{j=1}^n\left(1+\mu_{\tilde{\alpha}_j}\right)^{\omega_j}-\prod_{j=1}^n\left(1-\mu_{\tilde{\alpha}_j}\right)^{\omega_j}}{\prod_{j=1}^n\left(1+\mu_{\tilde{\alpha}_j}\right)^{\omega_j}+\prod_{j=1}^n\left(1-\mu_{\tilde{\alpha}_j}\right)^{\omega_j}}\le\mu_{\tilde{\alpha}_{\min}}$.

Let $g(y)=\frac{2-y}{y}$, $y\in(0,1]$; then, $g'(y)=\frac{-2}{y^2}<0$; thus $g(y)$ is a decreasing function.

Since $v_{\tilde{\alpha}_{\max}} \le v_{\tilde{\alpha}_j} \le v_{\tilde{\alpha}_{\min}}$, for all j, where $v_{\tilde{\alpha}_{\max}} > 0$, then $g(v_{\tilde{\alpha}_{\min}}) \le g(v_{\tilde{\alpha}_j}) \le g(v_{\tilde{\alpha}_{\max}})$, thus,

$$\frac{2-v_{\tilde{\alpha}_{\min}}}{v_{\tilde{\alpha}_{\min}}} \le \frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}} \le \frac{2-v_{\tilde{\alpha}_{\max}}}{v_{\tilde{\alpha}_{\max}}}, \left(\frac{2-v_{\tilde{\alpha}_{\min}}}{v_{\tilde{\alpha}_{\min}}}\right)^{\omega_j} \le \left(\frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}}\right)^{\omega_j}$$

$$\le \left(\frac{2-v_{\tilde{\alpha}_{\max}}}{v_{\tilde{\alpha}_{\max}}}\right)^{\omega_j}, \left(\frac{2-v_{\tilde{\alpha}_{\min}}}{v_{\tilde{\alpha}_{\min}}}\right)^{\sum_{j=1}^{n}\omega_j} \le \prod_{j=1}^{n}\left(\frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}}\right)^{\omega_j}$$

$$\le \left(\frac{2-v_{\tilde{\alpha}_{\max}}}{v_{\tilde{\alpha}_{\max}}}\right)^{\sum_{j=1}^{n}\omega_j}, \left(\frac{2-v_{\tilde{\alpha}_{\min}}}{v_{\tilde{\alpha}_{\min}}}\right) \le \prod_{j=1}^{n}\left(\frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}}\right)^{\omega_j}$$

$$\le \left(\frac{2-v_{\tilde{\alpha}_{\max}}}{v_{\tilde{\alpha}_{\max}}}\right), \left(\frac{2}{v_{\tilde{\alpha}_{\min}}}\right) \le \prod_{j=1}^{n}\left(\frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}}\right)^{\omega_j} + 1 \le \left(\frac{2}{v_{\tilde{\alpha}_{\max}}}\right),$$

$$\left(\frac{v_{\tilde{\alpha}_{\min}}}{2}\right) \le \frac{1}{\prod_{j=1}^{n}\left(\frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}}\right)^{\omega_j} + 1} \le \left(\frac{v_{\tilde{\alpha}_{\max}}}{2}\right),$$

$$v_{\tilde{\alpha}_{\min}} \le \frac{2}{\prod_{j=1}^{n}\left(\frac{2-v_{\tilde{\alpha}_j}}{v_{\tilde{\alpha}_j}}\right)^{\omega_j} + 1} \le v_{\tilde{\alpha}_{\max}}.$$

Thus, $v_{\tilde{\alpha}_{\min}} \le \frac{2\prod_{j=1}^{n}\left(v_{\tilde{\alpha}_j}\right)^{\omega_j}}{\prod_{j=1}^{n}\left(2-v_{\tilde{\alpha}_j}\right)^{\omega_j}+\prod_{j=1}^{n}\left(v_{\tilde{\alpha}_j}\right)^{\omega_j}} \le v_{\tilde{\alpha}_{\max}}$.

Since $a_{\min} \le a_j \le a_{\max}$, $b_{\min} \le b_j \le b_{\max}$, $c_{\min} \le c_j \le c_{\max}$, $d_{\min} \le d_j \le d_{\max}$, for all j, then, $\omega_j a_{\min} \le \omega_j a_j \le \omega_j\ a_{\max}$, $\omega_j b_{\min} \le \omega_j b_j \le \omega_j b_{\max}$, $\omega_j c_{\min} \le \omega_j c_j \le \omega_j c_{\max}$, $\omega_j d_{\min} \le \omega_j d_j \le \omega_j d_{\max}$, $\sum_{j=1}^{n}\omega_j a_{\min} \le \sum_{j=1}^{n}\omega_j a_j \le \sum_{j=1}^{n}\omega_j a_{\max}$, $\sum_{j=1}^{n}\omega_j b_{\min} \le \sum_{j=1}^{n}\omega_j b_j \le \sum_{j=1}^{n}\omega_j b_{\max}$, $\sum_{j=1}^{n}\omega_j c_{\min} \le \sum_{j=1}^{n}\omega_j c_j \le \sum_{j=1}^{n}\omega_j c_{\max}$, $\sum_{j=1}^{n}\omega_j d_{\min} \le \sum_{j=1}^{n}\omega_j d_j \le \sum_{j=1}^{n}\omega_j d_{\max}$.Then, $a_{\min} \le \sum_{j=1}^{n}\omega_j a_j \le a_{\max}$,$b_{\min} \le \sum_{j=1}^{n}\omega_j b_j \le b_{\max}$, $b_{\min} \le \sum_{j=1}^{n}\omega_j b_j \le b_{\max}$, $d_{\min} \le \sum_{j=1}^{n}\omega_j d_j \le d_{\max}$.

Let $\text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$, then we have $a_{\min} \le a \le a_{\max}$, $b_{\min} \le b \le b_{\max}$, $c_{\min} \le c \le c_{\max}$, $d_{\min} \le d \le d_{\max}$,$\mu_{\tilde{\alpha}_{\min}} \le \mu_{\tilde{\alpha}} \le \mu_{\tilde{\alpha}_{ax}}$, $v_{\tilde{\alpha}_{\max}} \le v_{\tilde{\alpha}} \le v_{\tilde{\alpha}_{\min}}$.

From the above analysis, we can get easily.

$$a_{\min} + b_{\min} + c_{\min} + d_{\min} \le a + b + c + d \le a_{\max} + b_{\max} + c_{\max} + d_{\max}.$$

$$\mu_{\tilde{\alpha}_{\min}} - v_{\tilde{\alpha}_{\min}} \le \mu_{\tilde{\alpha}} - v_{\tilde{\alpha}} \le \mu_{\tilde{\alpha}_{ax}} - v_{\tilde{\alpha}_{\max}}.$$

Therefore, $\tilde{\alpha}_{\min} \le \text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \le \tilde{\alpha}_{\max}$.

(3) (Monotonicity) Let $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. If $\tilde{\alpha}_j \le \tilde{\alpha}_j^*$, for all j, then

$$\text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \le \text{ITFEWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*). \quad (14)$$

Proof Let $\text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \sum_{j=1}^{n}\omega_j\tilde{\alpha}_j$, $\text{ITFEWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*) = \sum_{j=1}^{n}\omega_j\tilde{\alpha}_j^*$.

Since $\tilde{\alpha}_j \le \tilde{\alpha}_j^*$ for all j, then we have $\omega_j\tilde{\alpha}_j \le \omega_j\tilde{\alpha}_j^*$. Therefore, we have $\text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \le \text{ITFEWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*)$.

Definition 12 Let $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging (ITFEOWA) operator of dimension n is mapping ITFEOWA: $\Omega^n \to \Omega$, that has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus_{\varepsilon}}_{j=1}^{n}(w_j\tilde{\alpha}_{\sigma(j)}). \quad (15)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{\alpha}_{\sigma(j)} \le \tilde{\alpha}_{\sigma(j-1)}$ for all $j = 1, 2, \ldots, n$.

The fundamental aspect of the ITFEOWA operator is its re-ordering step. More specifically, the ITFEOWA operator first ranks all the given ITFNs in descending order, and then additively aggregates these ITFNs together with the weights of their ordered positions, where the corresponding operations are Einstein operations.

Based on the Definitions 10 and 12, we can derive the Theorem 5.

Theorem 5 *Let $\tilde{\alpha}_i (i = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFEOWA operator is also an intuitionistic trapezoidal fuzzy number, and*

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus_{\varepsilon}}_{j=1}^{n}(w_j\tilde{\alpha}_{\sigma(j)})$$
$$= \left(\begin{array}{l}\left[\sum_{j=1}^{n} w_j a_{\sigma(j)}, \sum_{j=1}^{n} w_j b_{\sigma(j)}, \sum_{j=1}^{n} w_j c_{\sigma(j)}, \sum_{j=1}^{n} w_j d_{\sigma(j)}\right]; \\ \frac{\prod_{j=1}^{n}(1+\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^{n}(1+\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}, \frac{2\prod_{j=1}^{n}(v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^{n}(2-v_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n}(v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}\end{array}\right) \quad (16)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{\alpha}_{\sigma(j)} \le \tilde{\alpha}_{\sigma(j-1)}$ for all $j = 1, 2, \ldots, n$. $w = (w_1, w_2, \ldots, w_n)$ is the weight vector of the ITFEOWA operator, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof Similarly as proof of Theorem 3, it is omitted here.

Example 2 Let $\tilde{\alpha}_1 = ([0.6, 0.7, 0.8, 0.9]; 0.6, 0.3)$, $\tilde{\alpha}_2 = ([0.2, 0.3, 0.4, 0.5]; 0.1, 0.6)$, $\tilde{\alpha}_3 = ([0.3, 0.4, 0.5, 0.6]; 0.4, 0.5)$, $\tilde{\alpha}_4 = ([0.2, 0.3, 0.5, 0.6]; 0.3, 0.6)$ and $\tilde{\alpha}_5 = ([0.2, 0.3, 0.5, 0.6]; 0.3, 0.6)$ be five intuitionistic trapezoidal fuzzy numbers, and let $\omega = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$ be the weighted vector of $\tilde{\alpha}_j (j = 1, 2, 3, 4, 5)$. Since $s(\tilde{\alpha}_2) < s(\tilde{\alpha}_4) < s(\tilde{\alpha}_3) < s(\tilde{\alpha}_1) < s(\tilde{\alpha}_5)$, then $\tilde{\alpha}_{\sigma(1)} = \tilde{\alpha}_5$, $\tilde{\alpha}_{\sigma(2)} = \tilde{\alpha}_1$, $\tilde{\alpha}_{\sigma(3)} = \tilde{\alpha}_3$, $\tilde{\alpha}_{\sigma(4)} = \tilde{\alpha}_4$, $\tilde{\alpha}_{\sigma(5)} = \tilde{\alpha}_2$, then, by Eq. (5), it follows that $\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \tilde{\alpha}_5) = ([0.35, 0.45, 0.57, 0.67]; 0.4413, 0.4060)$.

Based on the Theorem 5, the ITFEOWA operator satisfies the following properties:

Theorem 6 *Let $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \ldots, n)$ be a collection of ITFNs, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the ITFEOWA operator, with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, then the ITFEOWA operator satisfies the following properties:*

(1) (*Idempotency*) *If all $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ are equal, i.e. $\tilde{\alpha}_j = \tilde{\alpha}$ for all j, then*

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha} \tag{17}$$

(2) (*Boundary*) *Let $\tilde{\alpha}_{\min} = \min\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$ and $\tilde{\alpha}_{\max} = \max\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$, then*

$$\tilde{\alpha}_{\min} \le \text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \le \tilde{\alpha}_{\max}. \tag{18}$$

(3) (*Monotonicity*) *Let $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. If $\tilde{\alpha}_j \le \tilde{\alpha}_j^*$, for all j, then*

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \le \text{ITFEOWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*). \tag{19}$$

(4) (*Commutativity*) *Let $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers, then*

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \text{ITFEOWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*) \tag{20}$$

where $(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*)$ is any permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$.

Proof Let

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus_\varepsilon}_{j=1}^{n} (w_j \tilde{\alpha}_{\sigma(j)}),$$

$$\text{ITFEOWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*) = \mathop{\oplus_\varepsilon}_{j=1}^{n} (w_j \tilde{\alpha}_{\sigma(j)}^*).$$

Since $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$ is any a permutation of $(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*)$, we have $\tilde{\alpha}_{\sigma(j)} = \tilde{\alpha}_{\sigma(j)}^*$, then

$$\text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \text{ITFEOWA}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*).$$

From the Theorems 4 and 6, we know that the ITFEOWA operator and ITFEWA operator have has the properties such as idempotency, boundary, and monotonicity. The ITFEOWA operator has a kind of commutativity, but the ITFWA operator does not have this property.

In the following, we shall develop the induced intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging (I-ITFEOWA) operator, which is an extension of the ITFEOWA operators.

Definition 13 Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An induced intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging (I-ITFEOWA) operator of dimension n is a mapping I-ITFEOWA: $\Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and

$$\begin{aligned} &\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) \\ &= \mathop{\oplus_\varepsilon}_{j=1}^{n} (w_j \tilde{\alpha}_{\sigma(j)}) \end{aligned} \tag{21}$$

where $\tilde{\alpha}_{\sigma(j)}$ is the $\tilde{\alpha}_j$ value of ITFOWA pair $< u_j, \tilde{\alpha}_j >$ having the jth largest u_j $\left(u_j \in [0, 1]\right)$, u_j in $< u_j, \tilde{\alpha}_j >$ is referred to as the order inducing variable and $\tilde{\alpha}_j$ as the intuitionistic trapezoidal fuzzy numbers.

Based on the Definitions 10 and 13, we can derive the Theorem 7.

Theorem 7 *Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a be a collection of the intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the I-ITFEOWA operator is also an intuitionistic trapezoidal fuzzy number and*

$$\begin{aligned} &\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \mathop{\oplus_\varepsilon}_{j=1}^{n} (w_j \tilde{\alpha}_{\sigma(j)}) \\ &= \left(\begin{array}{l} \left[\sum_{j=1}^{n} w_j a_{\sigma(j)}, \sum_{j=1}^{n} w_j b_{\sigma(j)}, \sum_{j=1}^{n} w_j c_{\sigma(j)}, \sum_{j=1}^{n} w_j d_{\sigma(j)} \right]; \\ \dfrac{\prod_{j=1}^{n} (1 + \mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n} (1 - \mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^{n} (1 + \mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n} (1 - \mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}, \dfrac{2 \prod_{j=1}^{n} (v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^{n} (2 - v_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n} (v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}} \end{array} \right) \end{aligned} \tag{22}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the I-ITFEOWA operator, such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Proof Similarly as proof of Theorem 3, it is omitted here.

Based on the Theorem 7, the I-ITFEOWA operator satisfies the following properties:

Theorem 8 *Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of the intuitionistic trapezoidal fuzzy numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the I-ITFEOWA operator, with $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, then the I-ITFEOWA operator satisfies the following properties:*

(1) (*Commutativity*)

$$\begin{aligned} &\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) \\ &= \text{I} - \text{ITFEOWA}\left(\left\langle u_1, \tilde{\alpha}_1^* \right\rangle, \left\langle u_2, \tilde{\alpha}_2^* \right\rangle, \ldots, \left\langle u_n, \tilde{\alpha}_n^* \right\rangle\right) \end{aligned} \tag{23}$$

where $(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle)$ *is any permutation of* $(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle)$.

Proof Let

$$\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\alpha}_{\sigma(j)}),$$

$$\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\alpha}_{\sigma(j)}^*).$$

Since $(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle)$ is any permutation of $(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle)$, we have $\tilde{\alpha}_{\sigma(j)} = \tilde{\alpha}_{\sigma(j)}^*$, and then

$$\begin{aligned}&\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle)\\ &\quad = \text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle)\end{aligned}$$

(2) (Idempotency) If all $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ are equal, i.e.$\tilde{\alpha}_j = \tilde{\alpha}$ for all j, then

$$\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \tilde{\alpha} \tag{24}$$

Proof Since $\tilde{\alpha}_j = \tilde{\alpha}$, for all j, we have

$$\begin{aligned}&\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle)\\ &\quad = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\alpha}_{\sigma(j)}) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\alpha}) = \tilde{\alpha}.\end{aligned}$$

(3) (Monotonicity) Let $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. If $\tilde{\alpha}_j \le \tilde{\alpha}_j^*$, for all j, then

$$\begin{aligned}&\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) \le \text{I}\\ &\quad - \text{ITFEOWA}(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle)\end{aligned} \tag{25}$$

Proof Let

$$\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\alpha}_{\sigma(j)}),$$

$$\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\alpha}_{\sigma(j)}^*).$$

Since $\tilde{\alpha}_j \le \tilde{\alpha}_j^*$, for all j, it follows that $\tilde{\alpha}_{\sigma(j)} \le \tilde{\alpha}_{\sigma(j)}^*$, then

$$\begin{aligned}&\text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle)\\ &\quad \le \text{I - ITFEOWA}(\langle u_1, \tilde{\alpha}_1^*\rangle, \langle u_2, \tilde{\alpha}_2^*\rangle, \ldots, \langle u_n, \tilde{\alpha}_n^*\rangle)\end{aligned}$$

Note that if $u_j = -j$ is the ordered position of $\tilde{\alpha}_j$, that is to say, $u_1 > u_2 > \ldots u_n$,then the I-ITFEOWA operator is reduced to the ITFEWA operator; if $u_j = \tilde{\alpha}_j$ for all j, then the I-ITFEOWA operator is reduced to the ITFEOWA operator.

From Definitions 11 and 12, we know that the ITFEWA operator weights only the ITFNs, while the ITFEOWA operator weights only the ordered positions of the ITFNs instead of weighting the ITFNs themselves. To solve this drawback both the operators consider only one of them, we shall propose an intuitionistic trapezoidal fuzzy Einstein hybrid averaging (ITFEHA) operator as follows.

Definition 14 Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy Einstein hybrid averaging (ITFEHA) operator of dimension n is mapping ITFEHA: $\Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, $j = 1, 2, \ldots, n$, if

$$\text{ITFEHA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\beta}_{\sigma(j)}). \tag{26}$$

where $\tilde{\beta}_{\sigma(j)}$ is the jth largest of the weighted intuitionistic trapezoidal fuzzy numbers $\tilde{\beta}_j$ $(\tilde{\beta}_j = n\omega_j\tilde{\alpha}_j, j = 1, 2, \ldots, n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{\alpha}_j (j = 1, 2, \cdots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, n is the balancing coefficient.

Based on the Definitions 10 and 14, we can derive the Theorem 9.

Theorem 9 *Let* $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ *e a collection of intuitionistic trapezoidal fuzzy numbers; then their aggregated value by using the ITFEHA operator is also an intuitionistic trapezoidal fuzzy number, and*

$$\text{ITFEHA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \overset{n}{\underset{j=1}{\oplus_\varepsilon}}(w_j\tilde{\beta}_{\sigma(j)})$$
$$= \left(\begin{array}{l}\left[\sum_{j=1}^n w_j\dot{a}_{\sigma(j)}, \sum_{j=1}^n w_j\dot{b}_{\sigma(j)}, \sum_{j=1}^n w_j\dot{c}_{\sigma(j)}, \sum_{j=1}^n w_j\dot{d}_{\sigma(j)}\right];\\ \dfrac{\prod_{j=1}^n (1+\dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{w_j} - \prod_{j=1}^n (1-\dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n (1+\dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (1-\dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{w_j}}, \dfrac{2\prod_{j=1}^n (\dot{v}_{\tilde{\beta}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n (2-\dot{v}_{\tilde{\beta}_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (\dot{v}_{\tilde{\beta}_{\sigma(j)}})^{w_j}}\end{array}\right). \tag{27}$$

Proof Similarly as proof of Theorem 3, it is omitted here.

Theorem 10 *The ITFEWA operator is a special case of the ITFEHA operator.*

Proof Let $w = (1/n, 1/n, \ldots, 1/n)^T$, then

$$\begin{aligned}\text{ITFEHA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= w_1\tilde{\beta}_{\sigma(1)} \oplus w_2\tilde{\beta}_{\sigma(2)} \ldots \oplus w_n\tilde{\beta}_{\sigma(n)}\\ &= \frac{1}{n}\tilde{\beta}_{\sigma(1)} \oplus \frac{1}{n}\tilde{\beta}_{\sigma(2)} \ldots \oplus \frac{1}{n}\tilde{\beta}_{\sigma(n)}\\ &= \omega_1\tilde{\alpha}_1 \oplus \omega_2\tilde{\alpha}_2 \ldots \oplus \omega_n\tilde{\alpha}_n\\ &= \text{ITFEWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n).\end{aligned}$$

Theorem 11 *The ITFEOWA operator is a special case of the ITFEHG operator.*

Proof Let $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then

$$\begin{aligned}\text{ITFEHA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= w_1\tilde{\beta}_{\sigma(1)} \oplus w_2\tilde{\beta}_{\sigma(2)} \ldots \oplus w_n\tilde{\beta}_{\sigma(n)}\\ &= w_1\tilde{\alpha}_{\sigma(1)} \oplus w_2\tilde{\alpha}_{\sigma(1)} \ldots \oplus w_n\tilde{\alpha}_{\sigma(1)}\\ &= \text{ITFEOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n).\end{aligned}$$

3.2 Intuitionistic trapezoidal fuzzy Einstein geometric aggregation operators

Definition 15 Let $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy Einstein weighted geometric (ITFEWG) operator of dimension n is mapping ITFEWG: $\Omega^n \to \Omega$, and

$$\mathrm{ITFEWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\otimes_{\varepsilon}}_{j=1}^{n} (\tilde{\alpha}_j)^{\omega_j}. \tag{28}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Based on the Definitions 10 and 15, we can derive the Theorem 12.

Theorem 12 *Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFEWG operator is also an intuitionistic trapezoidal fuzzy number, and*

$$\mathrm{ITFEWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\otimes_{\varepsilon}}_{j=1}^{n} (\tilde{\alpha}_j)^{\omega_j} = \left(\begin{array}{l} \left[\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} b_j^{\omega_j}, \prod_{j=1}^{n} c_j^{\omega_j}, \prod_{j=1}^{n} d_j^{\omega_j} \right]; \\ \dfrac{2\prod_{j=1}^{n} (\mu_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{n} (2-\mu_{\tilde{\alpha}_j})^{\omega_j} + \prod_{j=1}^{n} (\mu_{\tilde{\alpha}_j})^{\omega_j}}, \dfrac{\prod_{j=1}^{n} (1+v_{\tilde{\alpha}_j})^{\omega_j} - \prod_{j=1}^{n} (1-v_{\tilde{\alpha}_j})^{\omega_j}}{\prod_{j=1}^{n} (1+v_{\tilde{\alpha}_j})^{\omega_j} + \prod_{j=1}^{n} (1-v_{\tilde{\alpha}_j})^{\omega_j}} \end{array} \right) \tag{29}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$, and $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof Similarly as proof of Theorem 3, it is omitted here.

Based on the Theorem 12, the ITFEWG operator satisfies the following properties:

Theorem 13 *Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of ITFNs, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$, and $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, then the ITFEWG operator satisfies the following properties*:

(1) (*Idempotency*) *If all $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ are equal, i.e. $\tilde{\alpha}_j = \tilde{\alpha}$ for all j, then*

$$\mathrm{ITFEWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha}. \tag{30}$$

(2) (*Boundary*) *Let $\tilde{\alpha}_{\min} = \min\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$ and $\tilde{\alpha}_{\max} = \max\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$, then*

$$\tilde{\alpha}_{\min} \leq \mathrm{ITFEWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leq \tilde{\alpha}_{\max}. \tag{31}$$

(3) (*Monotonicity*) *Let $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ be a collection of ITFNs. If $\tilde{\alpha}_j \leq \tilde{\alpha}_j^*$, for all j, then*

$$\mathrm{ITFEWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leq \mathrm{ITFEWG}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*). \tag{32}$$

Proof Similarly as proof of Theorem 4, it is omitted here.

Definition 16 Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (ITFEOWG) operator of dimension n is a mapping ITFEOWG: $\Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, if

$$\mathrm{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\otimes_{\varepsilon}}_{j=1}^{n} (\tilde{\alpha}_{\sigma(j)})^{w_j} \tag{33}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{\alpha}_{\sigma(j)} \leq \tilde{\alpha}_{\sigma(j-1)}$ for all $j = 1, 2, \ldots, n$.

Based on the Definitions 10 and 16, we can derive the Theorem 14.

Theorem 14 *Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFEOWG operator is also an intuitionistic trapezoidal fuzzy number, and*

$$\mathrm{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\otimes_{\varepsilon}}_{j=1}^{n} (\tilde{\alpha}_{\sigma(j)})^{w_j} = \left(\begin{array}{l} \left[\prod_{j=1}^{n} a_{\sigma(j)}^{w_j}, \prod_{j=1}^{n} b_{\sigma(j)}^{w_j}, \prod_{j=1}^{n} c_{\sigma(j)}^{w_j}, \prod_{j=1}^{n} d_{\sigma(j)}^{w_j} \right]; \\ \dfrac{2\prod_{j=1}^{n} (\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^{n} (2-\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n} (\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}, \dfrac{\prod_{j=1}^{n} (1+v_{\tilde{\alpha}_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n} (1-v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^{n} (1+v_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n} (1-v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}} \end{array} \right). \tag{34}$$

Proof Similarly as proof of Theorem 3, it is omitted here.

Based on the Theorem 14, the ITFEOWG operator satisfies the following properties:

Theorem 15 *Let $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \ldots, n)$ be a collection of ITFNs, where $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the ITFEOWG operator, and $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, then the ITFEOWG operator satisfies the following properties*:

(1) (*Idempotency*) *If all $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ are equal, i.e. $\tilde{\alpha}_j = \tilde{\alpha}$ for all j, then*

$$\mathrm{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha} \tag{35}$$

(2) (*Boundary*) *Let* $\tilde{\alpha}_{\min} = \min\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$ *and* $\tilde{\alpha}_{\max} = \max\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$, *then*

$$\tilde{\alpha}_{\min} \leq \text{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leq \tilde{\alpha}_{\max} \quad (36)$$

(3) (*Monotonicity*) *Let* $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ *be a collection of intuitionistic trapezoidal fuzzy numbers. If* $\tilde{\alpha}_j \leq \tilde{\alpha}_j^*$, *for all j, then*

$$\text{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leq \text{ITFEOWG}(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*) \quad (37)$$

(4) (*Commutativity*) *Let* $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ *be a collection of intuitionistic trapezoidal fuzzy numbers, then*

$$\text{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \text{ITFEOWG}_w(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*) \quad (38)$$

where $(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \ldots, \tilde{\alpha}_n^*)$ *is any permutation of* $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$.

Proof Similarly as proof of Theorem 7, it is omitted here.

In the following, we shall develop the induced intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (I-ITFEOWG) operator, which is an extension of the ITFEOWG operators.

Definition 17 Let $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An induced intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (I-ITFEOWG) operator and let I-ITFEOWG: $\Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Furthermore

$$\text{I - ITFEOWG}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \underset{j=1}{\overset{n}{\otimes}}_{\varepsilon} (\tilde{\alpha}_{\sigma(j)})^{w_j} \quad (39)$$

where $\tilde{\alpha}_{\sigma(j)}$ is the $\tilde{\alpha}_j$ value of ITFOWG pair $< u_j, \tilde{\alpha}_j >$ having the *j*th largest u_j $(u_j \in [0, 1])$, u_j in $< u_j, \tilde{\alpha}_j >$ is referred to as the order inducing variable and $\tilde{\alpha}_j$ as the intuitionistic trapezoidal fuzzy numbers.

Based on the Definitions 10 and 17, we can derive the Theorem 16.

Theorem 16 *Let* $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ *be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the I-ITFEOWG operator is also an intuitionistic trapezoidal fuzzy number and*

$$\text{I - ITFEOWG}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \underset{j=1}{\overset{n}{\otimes}}_{\varepsilon} (\tilde{\alpha}_{\sigma(j)})^{w_j}$$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^n a_{\sigma(j)}^{w_j}, \prod_{j=1}^n b_{\sigma(j)}^{w_j}, \prod_{j=1}^n c_{\sigma(j)}^{w_j}, \prod_{j=1}^n d_{\sigma(j)}^{w_j} \right]; \\ \dfrac{2 \prod_{j=1}^n (\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n (2 - \mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (\mu_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}, \dfrac{\prod_{j=1}^n (1 + v_{\tilde{\alpha}_{\sigma(j)}})^{w_j} - \prod_{j=1}^n (1 - v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}}{\prod_{j=1}^n (1 + v_{\tilde{\alpha}_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_{\sigma(j)}})^{w_j}} \end{array} \right) \quad (40)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^n w_j = 1$.

Proof Similarly as proof of Theorem 3, it is omitted here.

Based on the Theorem 16, the I-ITFEOWG operator satisfies the following properties:

Theorem 17 *Let* $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \ldots, n)$ *be a collection of intuitionistic trapezoidal fuzzy numbers, where* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of the ITFEOWG operator, with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^n w_j = 1$, *then I-ITFEOWG operator satisfies the following properties*:

(1) (*Idempotency*) *If all* $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$ *are equal, i.e.* $\tilde{\alpha}_j = \tilde{\alpha}$ *for all j, then*

$$\text{I - ITFEOWG}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \tilde{\alpha}. \quad (41)$$

(2) (*Monotonicity*) *Let* $\tilde{\alpha}_j^* = ([a_j^*, b_j^*, c_j^*, d_j^*], \mu_{\tilde{\alpha}_j}^*, v_{\tilde{\alpha}_j}^*)$ $(j = 1, 2, \ldots, n)$ *be a collection of intuitionistic trapezoidal fuzzy numbers. If* $\tilde{\alpha}_j \leq \tilde{\alpha}_j^*$, *for all j, then*

$$\begin{aligned} &\text{I - ITFEOWG}_w(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) \\ &\quad \leq \text{I - ITFEOWG}_w(\langle u_1, \tilde{\alpha}_1^* \rangle, \langle u_2, \tilde{\alpha}_2^* \rangle, \ldots, \langle u_n, \tilde{\alpha}_n^* \rangle) \end{aligned} \quad (42)$$

(3) (*Commutativity*)

$$\begin{aligned} &\text{I - ITFEOWG}_w(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) \\ &\quad = \text{I - ITFEOWG}_w(\langle u_1, \tilde{\alpha}_1^* \rangle, \langle u_2, \tilde{\alpha}_2^* \rangle, \ldots, \langle u_n, \tilde{\alpha}_n^* \rangle) \end{aligned} \quad (43)$$

where $(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \cdots, \langle u_n, \tilde{\alpha}_n \rangle)$ *is any permutation of* $(\langle u_1, \tilde{\alpha}_1^* \rangle, \langle u_2, \tilde{\alpha}_2^* \rangle, \cdots, \langle u_n, \tilde{\alpha}_n^* \rangle)$.

Proof Similarly as proof of Theorem 8, it is omitted here.

Note that if $u_j = -j$ is the ordered position of $\tilde{\alpha}_j$, that is to say, $u_1 > u_2 > \ldots u_n$, then the I-ITFEOWG operator is reduced to the ITFEWG operator; if $u_j = \tilde{\alpha}_j$ for all *j*, then the I-ITFEOWG operator is reduced to the ITFEOWG operator.

From Definitions 15 and 16, we know that the ITFEWG operator weights only the intuitionistic trapezoidal fuzzy numbers, while the ITFEOWG operator weights only the ordered positions of the trapezoidal fuzzy numbers. Therefore, weights represent different aspects in both the ITFEWG and ITFEOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose an intuitionistic trapezoidal fuzzy Einstein hybrid geometric (ITFEHG) operator.

Definition 18 Let $\tilde{\alpha}_j = ([a_j, b_j, c_j, d_j], \mu_{\tilde{\alpha}_j}, v_{\tilde{\alpha}_j})$ $(j = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy hybrid geometric (ITFEHG) operator of dimension n is a mapping ITFEHG: $\Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, $j = 1, 2, \ldots, n$, if

$$\mathrm{ITFEHG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \underset{j=1}{\overset{n}{\otimes_\varepsilon}} (\tilde{\beta}_{\sigma(j)})^{w_j}. \tag{44}$$

where $\tilde{\beta}_{\sigma(j)}$ is the jth largest of the weighted intuitionistic trapezoidal fuzzy values $\tilde{\beta}_j$ $(\tilde{\beta}_j = \tilde{\alpha}_j^{n\omega_j}, j = 1, 2, \ldots, n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{\alpha}_j (j = 1, 2, \ldots, n)$, and $\omega_j > 0$, $\sum_{i=1}^n \omega_i = 1$.

Based on the Definitions 10 and 18, we can derive the Theorem 18.

Theorem 18 *Let $\tilde{\alpha}_i (i = 1, 2, \ldots, n)$ be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFEHG operator is also an intuitionistic trapezoidal fuzzy number, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and*

$$\mathrm{ITFEHG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \underset{j=1}{\overset{n}{\oplus_\varepsilon}} (\omega_j \tilde{\beta}_{\sigma(j)})$$
$$= \left(\begin{array}{l} \left[\prod_{j=1}^n \dot{a}_{\sigma(j)}^{w_j}, \prod_{j=1}^n \dot{b}_{\sigma(j)}^{w_j}, \prod_{j=1}^n \dot{c}_{\sigma(j)}^{w_j}, \prod_{j=1}^n \dot{d}_{\sigma(j)}^{w_j} \right]; \\ \dfrac{2\prod_{j=1}^n (\dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^n (2 - \dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^n (\dot{\mu}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j}}, \dfrac{\prod_{j=1}^n (1 + \dot{v}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^n (1 - \dot{v}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^n (1 + \dot{v}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^n (1 - \dot{v}_{\tilde{\beta}_{\sigma(j)}})^{\omega_j}} \end{array} \right). \tag{45}$$

Proof Similarly as proof of Theorem 3, it is omitted here.

Theorem 19 *The ITFEWG operator is a special case of the ITFEHG operator.*

Proof Let $w = (1/n, 1/n, \ldots, 1/n)^T$, then

$$\begin{aligned} \mathrm{ITFEHG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \tilde{\beta}_{\sigma(1)}^{w_1} \otimes \tilde{\beta}_{\sigma(2)}^{w_2} \cdots \otimes \tilde{\beta}_{\sigma(n)}^{w_n} \\ &= \tilde{\beta}_{\sigma(1)}^{1/n} \otimes \tilde{\beta}_{\sigma(2)}^{1/n} \cdots \otimes \tilde{\beta}_{\sigma(n)}^{1/n} \\ &= \tilde{\alpha}_1^{\omega_1} \otimes \tilde{\alpha}_2^{\omega_2} \otimes \cdots \otimes \tilde{\alpha}_n^{\omega_n} \\ &= \mathrm{ITFEWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n). \end{aligned}$$

Theorem 20 *The ITFEOWG operator is a special case of the ITFEHG operator.*

Proof Let $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then

$$\begin{aligned} \mathrm{ITFEHG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \tilde{\beta}_{\sigma(1)}^{w_1} \otimes \tilde{\beta}_{\sigma(2)}^{w_2} \otimes \cdots \otimes \tilde{\beta}_{\sigma(n)}^{w_n} \\ &= \tilde{\alpha}_{\sigma(1)}^{w_1} \otimes \tilde{\alpha}_{\sigma(2)}^{w_2} \otimes \cdots \otimes \tilde{\alpha}_{\sigma(n)}^{w_n} \\ &= \mathrm{ITFEOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n). \end{aligned}$$

4 An approach to multiple attribute group decision making problem with intuitionistic trapezoidal fuzzy environment

In this section, we apply the proposed aggregation operators to develop an approach for dealing with multiple attribute group decision making problems under intuitionistic trapezoidal fuzzy environment.

For a group decision-making problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite set of alternatives, $C = \{c_1, c_2, \ldots, c_n\}$ be a finite set of attributes, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the attributes, such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Let $E = \{e_1, e_2, \ldots, e_p\}$ be the finite set of decision makers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ be the weighting vector of decision makers, $\lambda_k \in [0, 1]$ and $\sum_{k=1}^p \lambda_k = 1$. Let $H^{(k)} = (\tilde{h}_{ij}^{(k)})_{m \times n}$ $(k = 1, 2, \ldots, p)$ be an intuitionistic trapezoidal fuzzy decision matrix, where $\tilde{h}_{ij}^{(k)} = \left(\left[h_{1ij}^{(k)}, h_{2ij}^{(k)}, h_{3ij}^{(k)}, h_{4ij}^{(k)} \right]; \mu_{ij}^{(k)}, v_{ij}^{(k)} \right)$ is intuitionistic trapezoidal fuzzy number provided by decision maker $e_k \in E$, such that $0 \le \mu_{ij}^{(k)} \le 1$, $0 \le v_{ij}^{(k)} \le 1$, $0 \le \mu_{ij}^{(k)} + v_{ij}^{(k)} \le 1$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

In order to eliminate the effect from different physical dimensions to decision results, so the matrix $H^{(k)} = \left(\tilde{h}_{ij}^{(k)} \right)_{m \times n}$ needs to be normalized into $R^{(k)} = \left(\tilde{r}_{ij}^{(k)} \right)_{m \times n}$, where $\tilde{r}_{ij}^{(k)} = \left(\left[r_{1ij}^{(k)}, r_{2ij}^{(k)}, r_{3ij}^{(k)}, r_{4ij}^{(k)} \right]; \mu_{ij}^{(k)}, v_{ij}^{(k)} \right)$. Consider that there are benefit attributes and cost attributes in multiple attribute group decision making problems. In this paper, the normalization method is chosen as follows:

For cost criteria:

$$r_{qij}^{(k)} = \frac{\max_j \left(h_{4ij}^{(k)} \right) - h_{(5-q)ij}^{(k)}}{\max_j \left(h_{4ij}^{(k)} \right) - \min_j \left(h_{1ij}^{(k)} \right)} \quad q = 1, 2, 3, 4 \tag{46}$$

For benefit criteria:

$$r_{qij}^{(k)} = \frac{h_{qij}^{(k)} - \max_j\left(h_{4ij}^{(k)}\right)}{\max_j\left(h_{4ij}^{(k)}\right) - \min_j\left(h_{1ij}^{(k)}\right)} \quad q = 1,2,3,4 \tag{47}$$

In the following, we apply the ITFEWA and ITFEHA operator to develop an approach to deal with multiple attribute group decision making problems when decision information is intuitionistic trapezoidal fuzzy number. The method mainly involves the following steps:

Step 1 Transform the intuitionistic trapezoidal fuzzy decision matrix $H^{(k)} = \left(\tilde{h}_{ij}^{(k)}\right)_{m\times n}$ into the normalized intuitionistic trapezoidal fuzzy decision matrix $R^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}$ using Eqs. (46) and (47).

Step 2 Utilize the decision information given in the normalized intuitionistic trapezoidal fuzzy decision matrix $R^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}$, and the ITFEWA operator to derive the individual overall preference values $\tilde{r}_i^{(k)}$ of the alternative x_i, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighing vector of the attributes.

$$\tilde{r}_i^{(k)} = \text{ITFEWA}_\omega(\tilde{r}_{i1}^{(k)}, \tilde{r}_{i2}^{(k)}, \ldots, \tilde{r}_{in}^{(k)}), i = 1,2,\ldots,m, k = 1,2,\ldots,p \tag{48}$$

Step 3 Utilize the ITFEHA operator to aggregate all the individual overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i^{(k)}$ $(k = 1,2,\ldots,p)$ into the collective overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i$, where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated vector of the ITFEHA operator, such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ is the weight vector of decision makers with $\lambda_k \in [0,1]$, $\lambda_k > 0$, $\sum_{k=1}^p \lambda_k = 1$.

$$\tilde{r}_i = \text{ITFEHA}_{\lambda,w}(\tilde{r}_i^{(1)}, \tilde{r}_i^{(2)}, \cdots, \tilde{r}_i^{(p)}). i = 1,2,\cdots,m, k = 1,2,\cdots,p \tag{49}$$

Step 4 Calculate the score $s(\tilde{r}_i)$ of the collective overall values $\tilde{r}_i(i = 1,2\ldots,m)$ to rank all the alternatives x_i.

Step 5 Rank all the alternatives x_i $(i = 1,2\ldots,m)$ and select the best one in accordance with $s(\tilde{r}_i)$ and $h(\tilde{r}_i)$ $(i = 1,2\ldots,m)$.

5 Numerical example

5.1 The green supplier selection problem and its analysis process

In order to illustrate the application of the developed method, we give an example about the green supplier of a car company, which wants to select the most appropriate green supplier for one of the key elements in its manufacturing process. The production department manager e_1, the quality inspection department manager e_2 and the purchasing department manager e_3 sets up the panel of decision makers (whose weighting vector $\lambda = (0.35, 0.4, 0.25)^T$) which will take the whole responsibility for this supplier selection. For five supplier candidates $x_i(i = 1,2,3,4,5)$, they made strict evaluation from four aspects: product quality c_1, technology capability c_2, pollution control c_3, environment management c_4. The supplier candidates $x_i(i = 1,2,3,4,5)$ are to be evaluated by the decision makers under the above four attributes (whose weighting vector $\omega = (0.2, 0.1, 0.3, 0.4)^T$), and construct the following three normalized intuitionistic trapezoidal fuzzy decision matrix $R^{(k)} = (\tilde{r}_{ij}^{(k)})_{5\times 4}(k = 1,2,3)$.

$$R^{(1)} = \begin{bmatrix} \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.5,0.4\end{pmatrix} & \begin{pmatrix}[0.1,0.2,0.3,0.4]; \\ 0.6,0.3\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.8,0.9]; \\ 0.3,0.6\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.2,0.7\end{pmatrix} \\ \begin{pmatrix}[0.6,0.7,0.8,0.9]; \\ 0.7,0.3\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.7,0.2\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.7,0.8]; \\ 0.7,0.2\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.7,0.9]; \\ 0.4,0.5\end{pmatrix} \\ \begin{pmatrix}[0.1,0.2,0.4,0.5]; \\ 0.6,0.4\end{pmatrix} & \begin{pmatrix}[0.2,0.3,0.5,0.6]; \\ 0.5,0.4\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.5,0.3\end{pmatrix} & \begin{pmatrix}[0.3,0.5,0.7,0.9]; \\ 0.2,0.3\end{pmatrix} \\ \begin{pmatrix}[0.3,0.4,0.5,0.6]; \\ 0.8,0.1\end{pmatrix} & \begin{pmatrix}[0.1,0.3,0.4,0.5]; \\ 0.6,0.3\end{pmatrix} & \begin{pmatrix}[0.1,0.3,0.5,0.7]; \\ 0.3,0.4\end{pmatrix} & \begin{pmatrix}[0.6,0.7,0.8,0.9]; \\ 0.2,0.6\end{pmatrix} \\ \begin{pmatrix}[0.2,0.3,0.4,0.5]; \\ 0.6,0.2\end{pmatrix} & \begin{pmatrix}[0.3,0.4,0.5,0.6]; \\ 0.4,0.3\end{pmatrix} & \begin{pmatrix}[0.2,0.3,0.4,0.5]; \\ 0.7,0.1\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.1,0.3\end{pmatrix} \end{bmatrix}$$

$$R^{(2)} = \begin{bmatrix} \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.4,0.3\end{pmatrix} & \begin{pmatrix}[0.1,0.2,0.3,0.4]; \\ 0.5,0.2\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.7,0.8]; \\ 0.2,0.5\end{pmatrix} & \begin{pmatrix}[0.3,0.4,0.5,0.6]; \\ 0.1,0.6\end{pmatrix} \\ \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.6,0.2\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.6,0.1\end{pmatrix} & \begin{pmatrix}[0.3,0.4,0.6,0.7]; \\ 0.6,0.1\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.6,0.8]; \\ 0.3,0.4\end{pmatrix} \\ \begin{pmatrix}[0.1,0.2,0.3,0.4]; \\ 0.5,0.3\end{pmatrix} & \begin{pmatrix}[0.1,0.2,0.4,0.5]; \\ 0.4,0.3\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.4,0.2\end{pmatrix} & \begin{pmatrix}[0.2,0.4,0.6,0.8]; \\ 0.5,0.2\end{pmatrix} \\ \begin{pmatrix}[0.2,0.3,0.4,0.5]; \\ 0.7,0.1\end{pmatrix} & \begin{pmatrix}[0.1,0.2,0.3,0.5]; \\ 0.5,0.2\end{pmatrix} & \begin{pmatrix}[0.1,0.2,0.4,0.6]; \\ 0.2,0.3\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.1,0.5\end{pmatrix} \\ \begin{pmatrix}[0.1,0.2,0.3,0.4]; \\ 0.5,0.1\end{pmatrix} & \begin{pmatrix}[0.2,0.3,0.4,0.5]; \\ 0.3,0.2\end{pmatrix} & \begin{pmatrix}[0.1,0.2,0.3,0.4]; \\ 0.6,0.2\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.4,0.2\end{pmatrix} \end{bmatrix}$$

$$R^{(3)} = \begin{bmatrix} \begin{pmatrix}[0.6,0.7,0.8,0.9]; \\ 0.4,0.5\end{pmatrix} & \begin{pmatrix}[0.2,0.3,0.4,0.5]; \\ 0.5,0.4\end{pmatrix} & \begin{pmatrix}[0.6,0.7,0.9,1.0]; \\ 0.2,0.7\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.7,0.8]; \\ 0.1,0.8\end{pmatrix} \\ \begin{pmatrix}[0.7,0.8,0.9,1.0]; \\ 0.6,0.4\end{pmatrix} & \begin{pmatrix}[0.6,0.7,0.8,0.9]; \\ 0.6,0.3\end{pmatrix} & \begin{pmatrix}[0.5,0.6,0.8,0.9]; \\ 0.6,0.3\end{pmatrix} & \begin{pmatrix}[0.6,0.7,0.8,1.0]; \\ 0.3,0.6\end{pmatrix} \\ \begin{pmatrix}[0.2,0.3,0.5,0.6]; \\ 0.5,0.5\end{pmatrix} & \begin{pmatrix}[0.3,0.4,0.6,0.7]; \\ 0.4,0.5\end{pmatrix} & \begin{pmatrix}[0.6,0.7,0.8,0.9]; \\ 0.4,0.4\end{pmatrix} & \begin{pmatrix}[0.4,0.6,0.8,1.0]; \\ 0.5,0.4\end{pmatrix} \\ \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.7,0.2\end{pmatrix} & \begin{pmatrix}[0.2,0.4,0.5,0.6]; \\ 0.5,0.4\end{pmatrix} & \begin{pmatrix}[0.2,0.4,0.6,0.8]; \\ 0.2,0.5\end{pmatrix} & \begin{pmatrix}[0.7,0.8,0.9,1.0]; \\ 0.6,0.3\end{pmatrix} \\ \begin{pmatrix}[0.3,0.4,0.5,0.6]; \\ 0.5,0.3\end{pmatrix} & \begin{pmatrix}[0.4,0.5,0.6,0.7]; \\ 0.3,0.4\end{pmatrix} & \begin{pmatrix}[0.3,0.4,0.5,0.6]; \\ 0.6,0.2\end{pmatrix} & \begin{pmatrix}[0.6,0.7,0.8,0.9]; \\ 0.4,0.4\end{pmatrix} \end{bmatrix}$$

In the following, we apply the ITFEWA and ITFEHA operator to multiple attribute group decision making based on intuitionistic trapezoidal fuzzy information. The method involves the following steps.

Step 1 Utilize the decision information and the ITFEWA operator to derive the individual overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i^{(k)}$ $(k = 1, 2, 3)$ of the alternative x_i.

$\tilde{r}_1^{(1)} = ([0.42, 0.52, 0.65, 0.75]; 0.3391, 0.5568)$,
$\tilde{r}_2^{(1)} = ([0.49, 0.59, 0.72, 0.86]; 0.5979, 0.3181)$,
$\tilde{r}_3^{(1)} = ([0.31, 0.45, 0.62, 0.76]; 0.4132, 0.3276)$,
$\tilde{r}_4^{(1)} = ([0.34, 0.48, 0.61, 0.63]; 0.4473, 0.3598)$,
$\tilde{r}_5^{(1)} = ([0.33, 0.43, 0.53, 0.63]; 0.4473, 0.2013)$
$\tilde{r}_1^{(2)} = ([0.33, 0.44, 0.57, 0.67]; 0.2361, 0.4506)$,
$\tilde{r}_2^{(2)} = ([0.39, 0.49, 0.62, 0.76]; 0.4928, 0.2046)$,
$\tilde{r}_3^{(2)} = ([0.23, 0.37, 0.52, 0.66]; 0.4614, 0.2264)$,
$\tilde{r}_4^{(2)} = ([0.28, 0.38, 0.51, 0.65]; 0.3179, 0.2915)$,
$\tilde{r}_5^{(2)} = ([0.23, 0.33, 0.43, 0.53]; 0.4763, 0.1747)$
$\tilde{r}_1^{(3)} = ([0.52, 0.62, 0.75, 0.85]; 0.2361, 0.6609)$,
$\tilde{r}_2^{(3)} = ([0.59, 0.69, 0.82, 0.96]; 0.4928, 0.4250)$,
$\tilde{r}_3^{(3)} = ([0.41, 0.55, 0.72, 0.86]; 0.4614, 0.4283)$,
$\tilde{r}_4^{(3)} = ([0.44, 0.58, 0.71, 0.84]; 0.3179, 0.4791)$,
$\tilde{r}_5^{(3)} = ([0.43, 0.53, 0.63, 0.73]; 0.4763, 0.3091)$.

Step 2 Utilize the ITFEHA operator which has associated weighting vector $w = (0.2, 0.5, 0.3)^T$, we obtain the collective overall preference values $\tilde{r}_i$ of the alternatives $x_i (i = 1, 2, 3, 4, 5)$.

$\tilde{r}_1 = ([0.4167, 0.5181, 0.6484, 0.7458]; 0.2888, 0.5557)$,
$\tilde{r}_2 = ([0.4836, 0.5826, 0.7113, 0.8499]; 0.5463, 0.3115)$
$\tilde{r}_3 = ([0.3102, 0.4488, 0.6123, 0.7509]; 0.4323, 0.3243)$,
$\tilde{r}_4 = ([0.3447, 0.4737, 0.6024, 0.7335]; 0.3767, 0.3707)$
$\tilde{r}_5 = ([0.3252, 0.4242, 0.5232, 0.6222]; 0.4571, 0.2191)$.

Step 3 Calculate the score $s(\tilde{r}_i)$ of the overall intuitionistic trapezoidal fuzzy preference values $\tilde{r}_i$ $(i = 1, 2, 3, 4, 5)$, then we have the result as follows:

$$s(\tilde{r}_1) = -0.155,\ s(\tilde{r}_2) = 0.154,\ s(\tilde{r}_3) = 0.057,\ s(\tilde{r}_4) = 0.003,\ s(\tilde{r}_5) = 0.113.$$

Step 4 Rank all the alternatives x_i $(i = 1, 2, 3, 4, 5)$, and select the best one according to the scores $s(\tilde{r}_i)(i = 1, 2, 3, 4, 5)$ of the overall preference values $\tilde{r}_i(i = 1, 2, 3, 4, 5)$: $x_2 \succ x_5 \succ x_3 \succ x_4 \succ x_1$, and thus the most appropriate green supplier is x_2.

5.2 Further discussion

Additionally, in order to further illustrate the validity and superiority of the decision method proposed in this paper, we apply the aggregation operators developed in this paper, the methods in [28, 31], and Wei [29] respectively to solve the above green supplier selection problem, we can get the ranking result $x_2 \succ x_3 \succ x_5 \succ x_4 \succ x_1$, the most appropriate green supplier is x_2.

In the above illustrated example, if we use A-IFSs to express the decision makers' evaluations, then decision matrix $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ can be written as decision matrix $R'^{(1)}$, $R'^{(2)}$ and $R'^{(3)}$through deleting the corresponding trapezoidal fuzzy numbers in intuitionistic trapezoidal fuzzy numbers.

Wang and Liu [35] proposed the IFEWA operator, and Zhao [36] utilized the IFEHA operator to deal with multiple attribute decision making with intuitionistic fuzzy information, respectively. In order to further explain the importance of ITFNs which represent the decision information, we utilize the method by Wang and Liu [35] and Zhao and Wei [36] to deal with the green supplier selection problem. After computation, the overall preference values of green supplier candidates $\tilde{h}_i (i = 1, 2, 3, 4, 5)$ are obtained as follows:

$$\tilde{h}_1 = (0.2888, 0.5557), \tilde{h}_2 = (0.5463, 0.3115), \tilde{h}_3 = (0.4323, 0.3243), \tilde{h}_4 = (0.3767, 0.3707), \tilde{h}_5 = (0.4571, 0.2191).$$

The scores of $\tilde{h}_i (i = 1, 2, 3, 4, 5)$ are as follows:

$$s(\tilde{h}_1) = -0.2669, s(\tilde{h}_2) = 0.2348, s(\tilde{h}_3) = 0.1081, s(\tilde{h}_4) = 0.0060, s(\tilde{h}_5) = 0.2380.$$

Since $s(\tilde{h}_5) > s(\tilde{h}_2) > s(\tilde{h}_3) > s(\tilde{h}_4) > s(\tilde{h}_1)$, the ranking order is $x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$ by the method by Wang and Liu [35] and Zhao and Wei [36], the most desirable candidate is x_5.

It is noted that the ranking orders obtained by this paper and by Wang and Liu [35] and Zhao and Wei [36] are very different. This is because that all trapezoidal fuzzy numbers are lost from the ITFNs, which weakens the ability of information representation for intuitionistic fuzzy sets. Therefore, intuitionistic trapezoidal fuzzy numbers may better reflect the decision information than A-IFSs by adding trapezoidal fuzzy numbers under real decision making environment. Therefore, the multiple attribute group decision making method in this paper is reasonable than methods by Wang and Liu [35] and Zhao [36].

$$R'^{(1)} = \begin{bmatrix} (0.5, 0.4) & (0.6, 0.3) & (0.3, 0.6) & (0.2, 0.7) \\ (0.7, 0.3) & (0.7, 0.2) & (0.7, 0.2) & (0.4, 0.5) \\ (0.6, 0.4) & (0.5, 0.4) & (0.5, 0.3) & (0.2, 0.3) \\ (0.8, 0.1) & (0.6, 0.3) & (0.3, 0.4) & (0.2, 0.6) \\ (0.6, 0.2) & (0.4, 0.3) & (0.7, 0.1) & (0.1, 0.3) \end{bmatrix},$$

$$R'^{(2)} = \begin{bmatrix} (0.4,0.3) & (0.5,0.2) & (0.2,0.5) & (0.1,0.6) \\ (0.6,0.2) & (0.6,0.1) & (0.6,0.1) & (0.3,0.4) \\ (0.5,0.3) & (0.4,0.3) & (0.4,0.2) & (0.5,0.2) \\ (0.7,0.1) & (0.5,0.2) & (0.2,0.3) & (0.1,0.5) \\ (0.5,0.1) & (0.3,0.2) & (0.6,0.2) & (0.4,0.2) \end{bmatrix}$$

$$R'^{(3)} = \begin{bmatrix} (0.4,0.5) & (0.5,0.4) & (0.2,0.7) & (0.1,0.8) \\ (0.6,0.4) & (0.6,0.3) & (0.6,0.3) & (0.3,0.6) \\ (0.5,0.5) & (0.4,0.5) & (0.4,0.4) & (0.5,0.4) \\ (0.7,0.2) & (0.5,0.4) & (0.2,0.5) & (0.6,0.3) \\ (0.5,0.3) & (0.3,0.4) & (0.6,0.2) & (0.4,0.4) \end{bmatrix}.$$

6 Conclusions

In this paper, we have investigated the multiple attribute group decision making problems in which attribute values are in the form of ITFNs. We first define some operations of ITFNs based on Einstein operations, and some corresponding operational laws. Further, we have proposed some new Einstein aggregation operators for ITFNs, including intuitionistic trapezoidal fuzzy Einstein weighted averaging (ITFEWA) operator, intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging (ITFEOWA) operator, induced intuitionistic trapezoidal fuzzy Einstein ordered weighted averaging (I-ITFEOWA) operator, intuitionistic trapezoidal fuzzy Einstein hybrid averaging (ITFEHA) operator, intuitionistic trapezoidal fuzzy Einstein weighted geometric(ITFEWG) operator, intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (ITFEOWG) operator, induced intuitionistic trapezoidal fuzzy Einstein ordered weighted geometric (I-ITFEOWG) operator and intuitionistic trapezoidal fuzzy Einstein hybrid geometric (ITFEHG) operator. And desirable properties of the operators have also been analyzed. Then, an approach based on these operators has been constructed to deal with multiple attribute group decision making problems under intuitionistic trapezoidal fuzzy environment. Numerical experiments on green supplier selection and comparative studies have shown the practicality and advantages of our presented approach. In future studies, we will further intuitionistic trapezoidal fuzzy aggregation operators which take into account the various interactions or priority among the decision criteria based on the Einstein t-conorm and t-norm operation laws. At the same time, the application of these aggregation operators in many actual fields, such as decision making, patter recognition and clustering analysis, are open questions for future research.

Acknowledgments This paper is supported by the National Natural Science Foundation of China (No. 71331002, No. 71271072 and No. 71201145), the Doctoral Foundation of Ministry of Education of China (No. 20110111110006), the Social Science Foundation of Ministry of Education of China (No. 11YJC630283). The authors also would like to express appreciation to the editors and the anonymous reviewers for their insightful and constructive comments and suggestions, which have been very helpful in improving the paper.

Conflict of interest The authors declared that they have no conflicts of interest to this work.

References

1. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96
2. Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353
3. Xu Z, Yager RR (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst 35(4):417–433
4. Liu H-W, Wang G-J (2007) Multi-criteria decision-making methods based on intuitionistic fuzzy sets. Eur J Oper Res 179(1):220–233
5. Xu Z (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15(6):1179–1187
6. Chen T-Y, Li C-H (2010) Determining objective weights with intuitionistic fuzzy entropy measures: a comparative analysis. Inf Sci 180(21):4207–4222
7. Tan C, Chen X (2010) Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. Expert Syst Appl 37(1):149–157
8. Wei G (2010) Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Appl Soft Comput 10(2):423–431
9. Xu Z (2010) Choquet integrals of weighted intuitionistic fuzzy information. Inf Sci 180(5):726–736
10. Li D-F (2011) Closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information. Appl Soft Comput 11(4):3402–3418
11. Tan C (2011) A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS. Expert Syst Appl 38(4):3023–3033
12. Hwang C-M, Yang M-S, Hung W-L, Lee M-G (2012) A similarity measure of intuitionistic fuzzy sets based on the Sugeno integral with its application to pattern recognition. Inf Sci 189:93–109
13. Li J, Deng G, Li H, Zeng W (2012) The relationship between similarity measure and entropy of intuitionistic fuzzy sets. Inf Sci 188:314–321
14. Xia M, Xu Z (2012) Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment. Inf Fusion 13(1):31–47
15. Xu Z (2012) Intuitionistic fuzzy multiattribute decision making: an interactive method. IEEE Trans Fuzzy Syst 20(3):514–525
16. Farhadinia B (2013) A theoretical development on the entropy of interval-valued fuzzy sets based on the intuitionistic distance and its relationship with similarity measure. Knowl Based Syst 39:79–84
17. Meng F, Zhang Q, Cheng H (2013) Approaches to multiple-criteria group decision making based on interval-valued intuitionistic fuzzy Choquet integral with respect to the generalized λ-Shapley index. Knowl Based Syst 37:237–249
18. Wang J-q, Nie R-r, Zhang H-y, Chen X-h (2013) Intuitionistic fuzzy multi-criteria decision-making method based on evidential reasoning. Appl Soft Comput 13(4):1823–1831
19. Wang J-Q, Zhang H-Y (2013) Multicriteria decision-making approach based on Atanassov's intuitionistic fuzzy sets with incomplete certain information on weights. IEEE Trans Fuzzy Syst 21(3):510–515

20. Wu J, Chen F, Nie C, Zhang Q (2013) Intuitionistic fuzzy-valued Choquet integral and its application in multicriteria decision making. Inf Sci 222:509–527
21. Liang X, Wei C (2014) An Atanassov's intuitionistic fuzzy multi-attribute group decision making method based on entropy and similarity measure. Int J Mach Learn Cybern 5(3):435–444
22. Liang X, Wei C, Chen Z (2013) An intuitionistic fuzzy weighted OWA operator and its application. Int J Mach Learn Cybern 4(6):713–719
23. Wei G (2009) Some geometric aggregation functions and their application to dynamic multiple attribute decision making in the intuitionistic fuzzy setting. Int J Uncertain Fuzziness Knowl Based Syst 17(02):179–196
24. Atanassov K, Gargov G (1989) Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31(3):343–349
25. Atanassov KT (1994) Operators over interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 64(2):159–174
26. Jianqiang W, Zhong Z (2009) Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems. J Syst Eng Electron 20(2):321–326
27. Shu M-H, Cheng C-H, Chang J-R (2006) Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly. Microelectron Reliab 46(12):2139–2148
28. Wang J, Zhang Z (2009) Multi-criteria decision-making method with incomplete certain information based on intuitionistic fuzzy number. Control Decis 24(2):226–230
29. Wei G (2010) Some arithmetic aggregation operators with intuitionistic trapezoidal fuzzy numbers and their application to group decision making. J Comput 5(3):345–351
30. Wan S-P, Dong J-Y (2010) Method of intuitionistic trapezoidal fuzzy number for multi-attribute group decision. Control Decis 25(5):773–776
31. Wu J, Cao Q-w (2013) Same families of geometric aggregation operators with intuitionistic trapezoidal fuzzy numbers. Appl Math Model 37(1–2):318–327
32. Wan S-p (2013) Power average operators of trapezoidal intuitionistic fuzzy numbers and application to multi-attribute group decision making. Appl Math Model 37(6):4112–4126
33. Klement EP, Mesiar R, Pap E (2004) Triangular norms. Position paper I: basic analytical and algebraic properties. Fuzzy Sets Syst 143(1):5–26
34. Wang W, Liu X (2011) Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. Int J Intell Syst 26(11):1049–1075
35. Wang W, Liu X (2012) Intuitionistic fuzzy information aggregation using Einstein operations. IEEE Trans Fuzzy Syst 20(5):923–938
36. Zhao X, Wei G (2013) Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making. Knowl Based Syst 37:472–479
37. Zhang S, Yu D (2014) Some geometric Choquet aggregation operators using Einstein operations under intuitionistic fuzzy environment. J Intell Fuzzy Syst 26(1):491–500
38. Chen S-M, Tan J-M (1994) Handling multicriteria fuzzy decision-making problems based on vague set theory. Fuzzy Sets Syst 67(2):163–172
39. Hong DH, Choi C-H (2000) Multicriteria fuzzy decision-making problems based on vague set theory. Fuzzy Sets Syst 114(1):103–113
40. Wang J, NIE R (2012) Multi-criteria group decision-making method based on intuitionistic trapezoidal fuzzy information. Syst Eng Theory Pract 8(32):1747–1753
41. Deschrijver G, Kerre EE (2002) A generalization of operators on intuitionistic fuzzy sets using triangular norms and conorms. Notes Intuit Fuzzy Sets 8(1):19–27
42. Lowen R (1996) Fuzzy set theory. Springer, New York