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ORIGINAL ARTICLE

A comparison of three types of rough fuzzy sets based on two universal sets

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Abstract The extension of rough set model is an important research direction in rough set theory. This paper presents two new extensions of the rough set model over two different universes. By means of a binary relation between two universes of discourse, two pairs of rough fuzzy approximation operators are proposed. These models guarantee that the approximating sets and the approximated sets are on the same universes of discourse. Furthermore, some interesting properties are investigated, the connections between relations and rough fuzzy approximation operators are examined. Finally, the connections of these approximation operators are made, and conditions under which these approximation operators made equivalent are obtained.

Keywords Approximation operator · Inverse serial relation · Rough fuzzy set · Strong inverse serial relation

List of symbols

$\mathcal{P}(U)$	Power set of the universe set U
$\mathcal{F}(\mathrm{U})$	Fuzzy power set of the universe set U
$\mathrm{F}(\mathrm{x})$	Successor neighborhood of x
$G(y)$	Predecessor neighborhood of y
$\underline{R}_s$	Generalized rough fuzzy lower approximation operator with respect to the successor neighborhood
$\bar{R}_s$	Generalized rough fuzzy upper approximation operator with respect to the successor neighborhood
$\underline{R}_p$	Generalized rough fuzzy lower approximation operator with respect to the predecessor neighborhood
$\bar{R}_p$	Generalized rough fuzzy upper approximation operator with respect to the predecessor neighborhood
$\underline{R}^*$	Revised rough fuzzy lower approximation operator
$\bar{R}^*$	Revised rough fuzzy upper approximation operator
$\underline{R}'$	Weak rough fuzzy lower approximation operator
$\bar{R}'$	Weak rough fuzzy upper approximation operator
$\underline{R}''$	Strong rough fuzzy lower approximation operator
$\bar{R}''$	Strong rough fuzzy upper approximation operator

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1 Introduction

The theory of rough sets was originally proposed by Pawlak [25, 26] as a formal tool for modeling and processing incomplete information. The basic structure of the rough set theory is an approximation space consisting of a universe of discourse and an equivalence relation imposed on it. So equivalence relation is a key notion in Pawlak's rough set model. However, the requirement of an equivalence relation seems to be a very restrictive condition that may limit the applications of rough set theory. Therefore, some interesting and meaningful extensions of Pawlak's rough set model have been proposed in the literature. For example, the notions of approximation operators have been generalized by tolerance relations or similarity relations [2, 32, 34, 48], dominance relations [10, 11], general binary relations on the universe of discourse [15–17, 47, 48, 51], partitions and coverings of the universe of discourse [4, 27, 28], neighborhood systems and Boolean algebras [3, 19–21, 49], and general approximation spaces [32, 33, 35]. Based on the covering of the universe of discourse, Pomykala [27, 28], in particular, proposed two pairs of dual approximation operators. Yao [49, 50] discussed such extension by the notions of neighborhood and granulation. In addition, Yao and Lin [51] provided a systematic study on the constructive methods using binary relations.

On the other hand, the fuzzy generalization of rough sets is another topic receiving much attention in recent years [5, 7, 13, 22–24]. Based on an equivalence relation on the universe of discourse, Dubois and Prade [6] introduced the lower and upper approximations of fuzzy sets in the Pawlak approximation space to obtain an extended notion called rough fuzzy sets. Alternatively, a fuzzy similarity relation can be used to replace an equivalence relation, and the result is a deviation of rough set theory called fuzzy rough sets [6, 7]. In general, a rough fuzzy set is the approximation of a fuzzy set in a crisp approximation space [8, 12, 40]. Alternatively, a fuzzy rough set is the approximation of a crisp set or a fuzzy set in a fuzzy approximation space. Particular studies on rough fuzzy sets and fuzzy rough sets can be found in the literatures [9, 14, 29–31, 39, 41, 42, 45, 46, 48].

Rough fuzzy and fuzzy rough set models on two universes of discourse have been studied in the literature [18, 36, 39–44]. These models can be interpreted by the notions of interval structure [37, 38, 52] and generalized approximation space [32, 33, 35]. Though these models have different methods of computation, they start with almost the ''same'' framework (two universes of discourse with a compatibility relation). It should also be noted that the approximated sets and the approximating sets in these models always locate at two different universes of discourse. This is however un-natural and inconvenient for knowledge discovery by means of rough set theory. The main purpose of the present paper is to discuss this problem in the case of rough fuzzy sets on two universes. Many interesting properties of the new approximation operators, their associated relationships and the existing rough fuzzy approximation operators on two universes of discourse are examined. The new rough fuzzy approximation operators are then compared. The necessary and sufficient conditions for their equivalence are investigated.

2 Preliminaries

Let U be a finite and nonempty set called the universe. The class of all crisp subsets (respectively, fuzzy subsets) of U will be denoted by $\mathcal{P}(U)$ (respectively, by $\mathcal{F}(U)$).

Let U and V be two finite and nonempty universes and let R be a binary relation from U to V, the triple (U, V, R) is called (two-universe) approximation space. Then the relation R is called:

(i) Serial if for all $x \in U$, there exists $y \in V$ such that $(x, y) \in R$.
(ii) Inverse serial if for all $y \in V$, there exists $x \in U$ such that $(x, y) \in R$.
(iii) Compatibility relation, if R is both serial and inverse serial.

If $U = V$, R is referred to as a binary relation on U and is called:

(i) Reflexive if for all $x \in U$, $(x, x) \in R$.
(ii) Symmetric if for all $x, y \in U$, $(x, y) \in R$ implies that $(y, x) \in R$.
(iii) Transitive if for all $x, y, z \in U$, $(x, y) \in R$ and $(y, z) \in$ R imply that $(x, z) \in R$.
(iv) Euclidean if for all $x, y, z \in U$, $(x, y) \in$ R and $(x, z) \in R$ imply that $(y, z) \in R$.

Definition 2.1 [53] Let (U, V, R) be a (two-universe) approximation space. Then, a set-valued mappings F and G representing the successor and predecessor neighborhood operators, respectively, defined as follows:

$$F : U \to \mathcal{P}(V), F(x) = \{y \in V : (x, y) \in R\},$$

$$G : V \to \mathcal{P}(U), G(y) = \{x \in U : (x, y) \in R\}.$$

A natural mappings F and G can be introduced according to the following form

$$F : \mathcal{P}(U) \to \mathcal{P}(V), F(X) = \cup\{F(x) : x \in X\},$$

$$G : \mathcal{P}(V) \to \mathcal{P}(U), G(Y) = \cup\{G(y) : y \in Y\}.$$

Definition 2.2 Let (U, V, R) be a (two-universe) approximation space, an inverse serial relation $R \in \mathcal{P}(U \times V)$ is called strong inverse serial if for all $y_1, y_2 \in V$, $G(y_1) \cap G(y_2) \neq \emptyset$ implies that $G(y_1) = G(y_2)$.

Lemma 2.1 *Let (U, V, R) be a (two-universe) approximation space, if R is strong inverse serial, then for all $x_1, x_2 \in U$, $F(X_1) \cap F(X_2) \neq \emptyset$ implies that $F(x_1) = F(x_2)$.*

Proof Assume that $F(x_1) \neq F(x_2)$, then there exists $y_1 \in F(x_1)$, $y_1 \notin F(x_2)$. If $F(x_2) \neq \emptyset$, then there exists $y_2 \in F(x_2)$, i.e., $x_1 \in G(y_1)$ and $x_2 \in G(y_2)$ such that $G(y_1) \neq G(y_2)$. Since R is strong inverse serial, then $G(y_1) \cap G(y_2) = \emptyset$. Moreover, $G(y_1) = G(F(x_1))$ and

$G(y_2) = G(F(x_2))$. Hence $G(F(x_1)) \cap G(F(x_2)) \supseteq G(F(x_1) \cap F(x_2)) = \emptyset$. Because R is inverse serial, we get $F(x_1) \cap F(x_2) = \emptyset$.

3 Rough fuzzy generalizations

A rough fuzzy set is a generalization of rough set, derived from the approximation of fuzzy set in a crisp approximation space.

Definition 3.1 [42] Let U and V be two finite non-empty universes of discourse and $R \in \mathcal{P}(U \times V)$ a binary relation from U to V. The ordered triple (U, V, R) is called a (two-universe) approximation space. For any fuzzy set $Y \in \mathcal{F}(V)$, the lower and upper approximations of Y, $\underline{R}_s(Y)$ and $\bar{R}_s(Y)$, with respect to the approximation space are fuzzy sets of U whose membership functions, for each $x \in U$, are defined, respectively, by

$$\underline{R}_s(Y)(x) = min\{Y(y) \mid y \in F(x)\},$$

$$\bar{R}_s(Y)(x) = max\{Y(y) \mid y \in F(x)\}.$$

where $F(x)$ is the successor neighborhood of x defined in Definition 2.1.

The ordered set-pair $(\underline{R}_s(Y), \bar{R}_s(Y))$ is referred to as a generalized rough fuzzy set with respect to successor neighborhood, and $\underline{R}_s$ and $\bar{R}_s : \mathcal{F}(V) \to \mathcal{F}(U)$ are referred to as lower and upper generalized rough fuzzy approximation operators, respectively.

Proposition 3.1 [42] *In a (two-universe) approximation space* (U, V, R) *with compatibility relation R, the approximation operators satisfy the following properties for all* $Y, Y_1, Y_2 \in \mathcal{F}(V)$:

(L$_1$) $\underline{R}_s(Y) = (\bar{R}_s(Y^c))^c$, where Y^c denotes the complement of the fuzzy subset Y in V
(L$_2$) $\underline{R}_s(V) = U$
(L$_3$) $\underline{R}_s(Y_1 \cap Y_2) = \underline{R}_s(Y_1) \cap \underline{R}_s(Y_2)$
(L$_4$) $\underline{R}_s(Y_1 \cup Y_2) \supseteq \underline{R}_s(Y_1) \cup \underline{R}_s(Y_2)$
(L$_5$) $Y_1 \subseteq Y_2 \Rightarrow \underline{R}_s(Y_1) \subseteq \underline{R}_s(Y_2)$
(L$_6$) $\underline{R}_s(\emptyset) = \emptyset$
(U$_1$) $\bar{R}_s(Y) = (\underline{R}_s(Y^c))^c$
(U$_2$) $\bar{R}_s(\emptyset) = \emptyset$
(U$_3$) $\bar{R}_s(Y_1 \cup Y_2) = \bar{R}_s(Y_1) \cup \bar{R}_s(Y_2)$
(U$_4$) $\bar{R}_s(Y_1 \cap Y_2) \subseteq \bar{R}_s(Y_1) \cap \bar{R}_s(Y_2)$
(U$_5$) $Y_1 \subseteq Y_2 \Rightarrow \bar{R}_s(Y_1) \subseteq \bar{R}_s(Y_2)$
(U$_6$) $\bar{R}_s(V) = U$
(LU) $\underline{R}_s(Y) \subseteq \bar{R}_s(Y)$

Properties (L$_1$) and (U$_1$) *show that the approximation operators* $\underline{R}_s$ *and* $\bar{R}_s$ *are dual to each other. Properties with the same number may be considered as dual properties. With respect to certain special types, say, reflexive, symmetric, transitive, and Euclidean binary relations on the universe U, the approximation operators have additional properties.*

Proposition 3.2 [42] *Let* $R \in \mathcal{P}(U \times U)$ *be an arbitrary binary relation on U. Then* $\forall X \in \mathcal{F}(U)$,

$$R \text{ is reflexive} \Leftrightarrow (L_7)\underline{R}_s(X) \subseteq X, \quad \Leftrightarrow (U_7)X \subseteq \bar{R}_s(X).$$

$$R \text{ is symmetric} \Leftrightarrow (L_8)\bar{R}_s(\underline{R}_s(X)) \subseteq X, \quad \Leftrightarrow (U_8)X \subseteq \underline{R}_s(\bar{R}_s(X).$$

$$R \text{is transitive} \Leftrightarrow (L_9)\underline{R}_s(X) \subseteq \underline{R}_s(\underline{R}_s(X)), \quad \Leftrightarrow (U_9)\bar{R}_s(\bar{R}_s(X)) \subseteq \bar{R}_s(X).$$

$$R \text{ is Euclidean} \Leftrightarrow (L_{10})\bar{R}_s(\underline{R}_s(X)) \subseteq \underline{R}_s(X), \quad \Leftrightarrow (U_{10})\bar{R}_s(X) \subseteq \underline{R}_s(\bar{R}_s(X)).$$

Definition 3.2 Let (U, V, R) be a (two-universe) approximation space. Then the lower and upper approximations of $X \in \mathcal{F}(U)$ are defined respectively as follows:

$$\underline{R}_p(X)(y) = min\{X(x) \mid x \in G(y)\},$$

$$\bar{R}_p(X)(y) = max\{X(x) \mid x \in G(y)\}.$$

where $G(y)$ is the predecessor neighborhood of y in Definition 2.1.

The pair $(\underline{R}_p(X), \bar{R}_p(X))$ is referred to as a generalized rough fuzzy set with respect to predecessor neighborhood, and $\underline{R}_p$ and $\bar{R}_p : \mathcal{F}(U) \to \mathcal{F}(V)$ are referred to as lower and upper rough fuzzy approximation operators, respectively.

Proposition 3.3 *In a (two-universe) approximation space* (U, V, R) *with a binary relation R, the approximation operators* $\underline{R}_p$ *and* $\bar{R}_p$ *satisfy the following properties for all* $X, X_1, X_2 \in \mathcal{F}(U)$:

(L$_1$) $\underline{R}_p(X) = (\bar{R}_p(X^c))^c$
(L$_2$) $\underline{R}_p(U) = V$
(L$_3$) $\underline{R}_p(X_1 \cap X_2) = \underline{R}_p(X_1) \cap \underline{R}_p(X_2)$
(L$_4$) $\underline{R}_p(X_1 \cup X_2) \supseteq \underline{R}_p(X_1) \cup \underline{R}_p(X_2)$
(L$_5$) $X_1 \subseteq X_2 \Rightarrow \underline{R}_p(X_1) \subseteq \underline{R}_p(X_2)$
(U$_1$) $\bar{R}_p(X) = (\underline{R}_p(X^c))^c$
(U$_2$) $\bar{R}_p(\emptyset) = \emptyset$
(U$_3$) $\bar{R}_p(X_1 \cup X_2) = \bar{R}_p(X_1) \cup \bar{R}_p(X_2)$
(U$_4$) $\bar{R}_p(X_1 \cap X_2) \subseteq \bar{R}_p(X_1) \cap \bar{R}_p(X_2)$
(U$_5$) $X_1 \subseteq X_2 \Rightarrow \bar{R}_p(X_1) \subseteq \bar{R}_p(X_2)$

Proof By the duality of approximation operators, we only need to prove the properties $L_1 - L_{15}$.

(L$_1$) For all $y \in V$, according to Definition 3.2, we can obtain

$$
\begin{aligned}
(\bar{R}_p(X^c))^c(y) &= 1 - \{max\{X^c(x) : x \in G(y)\}\} \\
&= 1 - \{max\{1 - X(x) : x \in G(y)\}\} \\
&= 1 - \{1 - min\{X(x) : x \in G(y)\}\} \\
&= 1 - \{max\{X(x) : x \in G(y)\} \\
&= min\{X(x) : x \in G(y)\} \\
&= \underline{R}_p(X)(y).
\end{aligned}
$$

Therefore $\underline{R}_p(X) = (\bar{R}_p(X^c))^c$.

(L_2) Since $\forall x \in U$, $U(x) = 1$ and $G(y) \subseteq U$, then $min\{U(x) : x \in G(y)\} = 1$. Thus, $\underline{R}_p(U)(y) = min\{U(x) : x \in G(y)\} = 1$ for all $y \in V$. Therefore $\underline{R}_p(U) = V$.

(L_3) Since $\forall y \in V$

$$
\begin{aligned}
\underline{R}_p(X_1 \cap X_2)(y) &= min\{(X_1 \cap X_2)(x) : x \in G(y)\} \\
&= min\{min\{X_1(x), X_2(x)\} : x \in G(y)\} \\
&= min\{X_1(x) : x \in G(y)\} \\
&\quad \wedge min\{X_2(x) : x \in G(y)\} \\
&= \min\{\underline{R}_p(X_1)(y) \wedge \underline{R}_p(X_2)(y)\} \\
&= \underline{R}_p(X_1) \cap \underline{R}_p(X_2)(y).
\end{aligned}
$$

Therefore $\underline{R}_p(X_1 \cap X_2) = \underline{R}_p(X_1) \cap \underline{R}_p(X_2)$.

(L_4) For all $y \in V$, we can have

$$
\begin{aligned}
\underline{R}_p(X_1 \cup X_2)(y) &= min\{(X_1 \cup X_2)(x) : x \in G(y)\} \\
&= min\{max\{X_1(x), X_2(x)\} : x \in G(y)\} \\
&\geq max\{min\{X_1(x) : x \in G(y)\}, \\
&\quad min\{X_2(x) : x \in G(y)\} \\
&= max\{\underline{R}_p(X_1)(y), \underline{R}_p(X_2)(y)\} \\
&= (\underline{R}_p(X_1) \cup \underline{R}_p(X_2))(y).
\end{aligned}
$$

Hence $\underline{R}_p(X_1 \cup X_2) \supseteq \underline{R}_p(X_1) \cup \underline{R}_p(X_2)$.

(L_5) Since $X_1 \subseteq X_2$, then $\forall x \in U, X_1(x) \leq X_2(x)$. Thus $\underline{R}_p(X_1)(y) = min\{X_1(x) : x \in G(y)\} \leq min\{X_2(x) : x \in G(y)\} = \underline{R}_p(X_2)(y)$

Therefore $\underline{R}_p(X_1) \subseteq \underline{R}_p(X_2)$.

Proposition 3.4 *Let $R \in \mathcal{P}(U \times V)$ be an arbitrary binary relation on. Then $\forall X \in \mathcal{F}(U)$*

$$
\begin{aligned}
R \text{ is inverse serial } &\Leftrightarrow (L_6)\underline{R}_p(\emptyset) = \emptyset \\
&\Leftrightarrow (U_6)\bar{R}_p(U) = V, \\
&\Leftrightarrow (LU)\underline{R}_p(X) \subseteq \bar{R}_p(X).
\end{aligned}
$$

Proof Suppose that R is an inverse serial relation, for any $y \in V$, we have $G(y) \neq \emptyset$. Then $min\{\emptyset(x) : x \in G(y)\} = 0 \, \forall x \in U$. Therefore, $\underline{R}_p(\emptyset) = \emptyset$.

Conversely, assume that $\underline{R}_p(\emptyset) = \emptyset$, i.e., $min\{\emptyset(x) : x \in G(y)\} = 0 \, \forall x \in U$. If there exists $y_\circ \in V$ such that $G(y_\circ) = \emptyset$, then $min\{\emptyset(x) : x \in G(y_\circ)\} = 1$ which contradicts the assumption. Thus $G(y) \neq \emptyset \, \forall y \in V$. i.e., R is inverse serial. We can prove that R is inverse serial$\Leftrightarrow (U_6)\bar{R}_p(U) = V$ by the duality of approximation operators.

For the third part, R *is inverse serial* $\Leftrightarrow (LU)\underline{R}_p(X) \subseteq \bar{R}_p(X)$, let R is inverse serial, then $G(y) \neq \emptyset$ for every $y \in V$. So $min\{X(x) : x \in G(y)\} \leq max\{X(x) : x \in G(y)\}$. Therefore $\underline{R}_p(X) \subseteq \bar{R}_p(X)$.

On the other hand, if we assume that $(LU)\underline{R}_U(X) \subseteq \bar{R}_U(X)$ holds, let $X = U$, then by Proposition 2.3 we have $\bar{R}_p(U) \supseteq \underline{R}_p(U) = V$, which follows that R is fuzzy inverse serial.

Proposition 3.5 *Let (U, V, R) be a (two-universe) approximation space, then the following hold for all $X \in \mathcal{F}(U)$ and $Y \in \mathcal{F}(V)$:*

(i) $\bar{R}_s(\underline{R}_p(X)) \subseteq X, X \subseteq \underline{R}_s(\bar{R}_p(X))$.
(ii) $\bar{R}_p(\underline{R}_s(Y)) \subseteq Y, Y \subseteq \underline{R}_p(\bar{R}_s(Y))$.
(iii) $\underline{R}_s(Y) = \underline{R}_s(\bar{R}_p(\underline{R}_s(Y)))$.
(iv) $\bar{R}_s(Y) = \bar{R}_s(\underline{R}_p(\bar{R}_s(Y)))$.
(v) $\underline{R}_p(X) = \underline{R}_p(\bar{R}_s(\underline{R}_p(X)))$.
(vi) $\bar{R}_p(X) = \bar{R}_p(\underline{R}_s(\bar{R}_p(X)))$.

Proof (i) Since for every $x \in U$, we have either $F(x) = \emptyset$ or $F(x) \neq \emptyset$.

If $F(x) = \emptyset$, then $\bar{R}_s(\underline{R}_p(X))(x) = max\{min\{X(z) : z \in G(y)\} : y \in F(x)\} = 0$ and hence $\bar{R}_s(\underline{R}_p(X)) \subseteq X$. If $F(x) \neq \emptyset$, then we have $x \in G(y) \ \forall y \in F(x)$. Thus $max\{min\{X(z) : z \in G(y)\} : y \in F(x)\} \leq X(x)$, and hence $\bar{R}_s(\underline{R}_p(X)) \subseteq X$.

We can easily prove the other part by the duality of approximation operators.

(ii) Similarly as (i).

(iii)–(vi) can be proved by properties (i) and (ii).

Proposition 3.6 *Let (U, V, R) be a (two-universe) approximation space with strong inverse serial relation, then the following hold for all $X \in \mathcal{F}(U)$ and $Y \in \mathcal{F}(V)$:*

(i) $\bar{R}_p(\underline{R}_s(Y)) = \underline{R}_p(\underline{R}_s(Y))$,

(ii) $\underline{R}_p(\bar{R}_s(Y)) = \bar{R}_p(\bar{R}_s(Y))$.

Proof The proof comes directly from Definition 2.2 and Lemma 2.1.

Because reflexivity, symmetry and transitivity are meaningless for binary relations from U to V, the properties (L_7) – (L_{10}) and (U_7) – (U_{10}) which are true in various generalized rough fuzzy set models do not hold in two-universe models. However, In the above model for generalized rough fuzzy sets, fuzzy subsets of the universe V (resp. *U*) are approximated by fuzzy subsets of the other universe U (resp. *V*). This seems to be very unreasonable. Thus a more natural form for rough fuzzy sets on two universes is proposed so that the approximations of subsets of the universe are subsets of the same universe. Therefore we will modify these models in the next sections.

4 Revised rough fuzzy sets

Definition 4.1 [2] Let (*U*, *V*, *R*) be a (two-universe) approximation space. Then we can define a set valued mapping G^* from *V* to $\mathcal{P}(V)$ induced by *R* as follows:

$$G^* : V \to \mathcal{P}(V),$$

$$G^*(y) = \begin{cases} \cap F(x) & if \exists x \in U : \{y\} \subseteq F(x), \\ \emptyset & otherwise. \end{cases}$$

Definition 4.2 [1] Let (U, V, R) be a (two-universe) approximation space. Then the lower and upper approximations of $Y \in \mathcal{F}(V)$ are defined respectively as follows:

$$\underline{R}^*(Y)(y) = min\{Y(z) \mid z \in G^*(y)\}$$
$$\bar{R}^*(Y)(y) = max\{Y(z) \mid z \in G^*(y)\}.$$

The pair $(\underline{R}^*(Y), \bar{R}^*(Y))$ is referred to as a revised rough fuzzy set, and $\underline{R}^*$ and $\bar{R}^* : \mathcal{F}(V) \to \mathcal{F}(V)$ are referred to as revised lower and upper rough fuzzy approximation operators, respectively.

Proposition 4.1 [1] *In a (two-universe) approximation space (U, V, R), the approximation operators have the following properties for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:*

(L_1) $\underline{R}^*(Y) = (\bar{R}^*(Y^c))^c$
(L_2) $\underline{R}^*(V) = V$
(L_3) $\underline{R}^*(Y_1 \cap Y_2) = \underline{R}^*(Y_1) \cap \underline{R}^*(Y_2)$
(L_4) $\underline{R}^*(Y_1 \cup Y_2) \supseteq \underline{R}^*(Y_1) \cup \underline{R}^*(Y_2)$
(L_5) $Y_1 \subseteq Y_2 \Rightarrow \underline{R}^*(Y_1) \subseteq \underline{R}^*(Y_2)$
(L_9) $\underline{R}^*(Y) \subseteq \underline{R}^*(\underline{R}^*(Y))$
(U_1) $\bar{R}^*(Y) = (\underline{R}^*(Y^c))^c$
(U_2) $\bar{R}^*(\emptyset) = \emptyset$
(U_3) $\bar{R}^*(Y_1 \cup Y_2) = \bar{R}^*(Y_1) \cup \bar{R}^*(Y_2)$
(U_4) $\bar{R}^*(Y_1 \cap Y_2) \subseteq \bar{R}^*(Y_1) \cap \bar{R}^*(Y_2)$
(U_5) $Y_1 \subseteq Y_2 \Rightarrow \bar{R}^*(Y_1) \subseteq \bar{R}^*(Y_2)$
(U_9) $\bar{R}^*(\bar{R}^*(Y)) \subseteq \bar{R}^*(Y)$

Proposition 4.2 [1] *In a (two-universe) approximation space (U, V, R) with inverse serial relation R, the approximation operators have the following properties for all $Y \in \mathcal{F}(V)$:*

(L_6) $\underline{R}^*(\emptyset) = \emptyset$
(L_7) $\underline{R}^*(Y) \subseteq Y$
(U_6) $\bar{R}^*(V) = V$
(U_7) $Y \subseteq \bar{R}^*(Y)$
(LU) $\underline{R}^*(Y) \subseteq \bar{R}^*(Y)$

Proposition 4.3 [1] *In a (two-universe) approximation space (U, V, R) with strong inverse serial relation R, the approximation operators have the following properties for all $Y \in \mathcal{F}(V)$:*

(L_8) $Y \subseteq \underline{R}^*(\bar{R}^*(Y))$
(L_{10}) $\bar{R}^*(Y) \subseteq \underline{R}^*(\bar{R}^*(Y))$
(U_8) $\bar{R}^*(\underline{R}^*(Y)) \subseteq Y$
(U_{10}) $\bar{R}^*(\underline{R}^*(Y)) \subseteq \underline{R}^*(Y)$

5 Another two new generalizations of rough fuzzy sets

Definition 5.1 Let (U, V, R) be a (two-universe) approximation space. Then the lower and upper approximations of $Y \in \mathcal{F}(V)$ are defined respectively as follows:

$$\underline{R}'(Y)(y) = max\{min\{Y(z) : z \in F(x)\} : x \in G(y)\}$$

$$\bar{R}'(Y)(y) = min\{max\{Y(z) : z \in F(x)\} : y \in G(y)\}.$$

The pair $(\underline{R}'(Y), \bar{R}'(Y))$ is referred to as a weak rough fuzzy set, and $\underline{R}'$ and $\bar{R}' : \mathcal{F}(V) \to \mathcal{F}(V)$ are referred to as weak lower and upper rough fuzzy approximation operators, respectively.

Proposition 5.1 *Let (U, V, R) be a (two-universe) approximation space. Then*

$$\underline{R}'(Y) = \bar{R}_p(\underline{R}_s(Y)),$$

$$\bar{R}'(Y) = \underline{R}_p(\bar{R}_s(Y)).$$

Proof Straightforward.

Proposition 5.2 *In a (two-universe) approximation space (U, V, R), the approximation operators have the following properties for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:*

(L_1) $\underline{R}'(Y) = (\bar{R}'(Y^c))^c$
(L_4) $\underline{R}'(Y_1 \cup Y_2) \supseteq \underline{R}'(Y_1) \cup \underline{R}'(Y_2)$
(L_5) $Y_1 \subseteq Y_2 \Rightarrow \underline{R}'(Y_1) \subseteq \underline{R}'(Y_2)$
(L_6) $\underline{R}'(\emptyset) = \emptyset$
(L_7) $\underline{R}'(Y) \subseteq Y$
(L_9) $\underline{R}'(Y) \subseteq \underline{R}'(\underline{R}'(Y))$
(U_1) $\bar{R}'(Y) = (\underline{R}'(Y^c))^c$
(U_4) $\bar{R}'(Y_1 \cap Y_2) \subseteq \bar{R}'(Y_1) \cap \bar{R}'(Y_2)$
(U_5) $Y_1 \subseteq Y_2 \Rightarrow \bar{R}'(Y_1) \subseteq \bar{R}'(Y_2)$
(U_6) $\bar{R}'(V) = V$
(U_7) $Y \subseteq \bar{R}'(Y)$
(U_9) $\bar{R}'(\bar{R}'(Y)) \subseteq \bar{R}'(Y)$
(LU) $\underline{R}'(Y) \subseteq \bar{R}'(Y)$

Proof By the duality of approximation operators, we only need to prove the properties $(L_1), (L_4) - (L_7)$ and (L_9).

(L_1) Since $\forall y \in V$

$$\begin{aligned}(\bar{R}'(Y^c))^c(y) &= 1 - \{min\{max\{Y^c(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= 1 - \{min\{max\{1 - Y(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= 1 - \{min\{1 - min\{Y(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= 1 - \{1 - max\{min\{Y(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= max\{min\{Y(z) : z \in F(x)\} : x \in G(y)\} \\ &= \underline{R}'(Y)(y).\end{aligned}$$

Therefore $\underline{R}'(Y) = (\bar{R}'(Y^c))^c$.

(L_4) $\forall y \in V$, we can have

$$\begin{aligned}\underline{R}'(Y_1 \cup Y_2)(y) &= max\{min\{(Y_1 \cup Y_2)(z) : z \in F(x)\} : x \in G(y)\} \\ &= max\{min\{max\{Y_1(z), Y_2(z)\} : z \in F(x)\} : x \in G(y)\} \\ &\geq max\{max\{min\{Y_1(z) : z \in F(x)\} : x \in G(y)\}, \\ &\quad max\{min\{Y_2(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= max\{\underline{R}'(Y_1)(y), \underline{R}'(Y_2)(y)\} \\ &= (\underline{R}'(Y_1) \cup \underline{R}'(Y_2))(y).\end{aligned}$$

Hence $\underline{R}'(Y_1 \cup Y_2) \supseteq \underline{R}'(Y_1) \cup \underline{R}'(Y_2)$.

(L_5) Since $Y_1 \subseteq Y_2$, then $\forall y \in V, Y_1(y) \leq Y_2(y)$.

Thus

$$\begin{aligned}\underline{R}'(Y_1)(y) &= max\{min\{Y_1(z) : z \in F(x)\} : x \in G(y)\} \\ &\leq max\{min\{Y_2(z) : z \in F(x)\} : x \in G(y)\} \\ &= \underline{R}'(Y_2)(y)\end{aligned}$$

Therefore $\underline{R}'(Y_1) \subseteq \underline{R}'(Y_2)$.

(L_7) and (L_9) Obvious from Propositions 3.5 and 5.1

(L_6) Comes from (L_7) and the fact that empty set is a subset from any set.

Remark 5.1 If $R \in \mathcal{P}(U \times V)$ is a binary relation in a (two-universe) approximation space (U, V, R), then the following properties do not hold for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:

(L_2) $\underline{R}'(V) = V$
(L_3) $\underline{R}'(Y_1 \cap Y_2) = \underline{R}'(Y_1) \cap \underline{R}'(Y_2)$
(L_8) $Y \subseteq \underline{R}'(\bar{R}'(Y))$
(L_{10}) $\bar{R}'(Y) \subseteq \underline{R}'(\bar{R}'(Y))$
(U_2) $\bar{R}'(\emptyset) = \emptyset$
(U_3) $\bar{R}'(Y_1 \cup Y_2) = \bar{R}'(Y_1) \cup \bar{R}'(Y_2)$
(U_8) $\bar{R}'(\underline{R}'(Y)) \subseteq Y$
(U_{10}) $\bar{R}'(\underline{R}'(Y)) \subseteq \underline{R}'(Y)$

The following example shows Remark 5.1.

Example 5.1 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ and $R \in \mathcal{P}(U \times V)$ be a binary relation defined as:

R	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0	1	1	0	0	0
x_2	1	0	1	0	1	0
x_3	0	0	0	0	0	0
x_4	0	1	1	0	0	1
x_5	1	0	0	0	1	0
x_6	0	1	0	0	1	1
x_7	1	0	0	0	0	1

If Y and Z are two fuzzy subsets of V defined as:

$Y(y_1) = 0.5, Y(y_2) = 0.3, Y(y_3) = 0.7, Y(y_4) = 0.1,$
$Y(y_5) = 0.8, Y(y_6) = 0.4,$
$Z(y_1) = 0.3, Z(y_2) = 0.4, Z(y_3) = 0.9, Z(y_4) = 0.2,$
$Z(y_5) = 0.1, Z(y_6) = 0.6,$ then we have

	y_1	y_2	y_3	y_4	y_5	y_6
$\underline{R}'(Y)(y)$	0.5	0.3	0.5	0	0.5	0.4
$\bar{R}'(\underline{R}'(Y))(y)$	0.5	0.5	0.5	1	0.5	0.5
$\bar{R}'(Y)(y)$	0.5	0.7	0.7	1	0.8	0.5
$\underline{R}'(\bar{R}'(Y))(y)$	0.5	0.7	0.7	0	0.5	0.5
$\bar{R}'(\emptyset)(y)$	0	0	0	1	0	0
$\underline{R}'(V)(y)$	1	1	1	0	1	1
$\underline{R}'(Z)(y)$	0.3	0.4	0.4	0	0.1	0.4
$\underline{R}'(Y \cap Z)(y)$	0.3	0.3	0.3	0	0.1	0.3
$\bar{R}'(Z)(y)$	0.3	0.6	0.9	1	0.3	0.6
$\bar{R}'(Y \cup Z)(y)$	0.6	0.8	0.9	1	0.8	0.6

Hence we have $\underline{R}'(V) \neq V$, $\bar{R}'(\emptyset) \neq \emptyset$, $\underline{R}'(Y \cap Z) \neq \underline{R}'(Y) \cap \underline{R}'(Z)$, $\bar{R}'(Y \cup Z) \neq \bar{R}'(Y) \cup \bar{R}'(Z)$, $Y \not\subset \underline{R}'(\bar{R}'(Y))$, $\bar{R}'(\underline{R}'(Y)) \not\subset Y$, $\bar{R}'(Y) \not\subset \underline{R}'(\bar{R}'(Y))$ and $\bar{R}'(\underline{R}'(Y)) \not\subset \underline{R}'(Y)$, i.e., L_2, U_2, L_3, U_3, L_8, U_8, L_{10} and U_{10} do not hold.

Proposition 5.3 *In a (two-universe) approximation space (U, V, R) with inverse serial relation R, the approximation operators have the following properties for all $Y \in \mathcal{F}(V)$:*

(L_2) $\underline{R}'(V) = V$
(U_2) $\bar{R}'(\emptyset) = \emptyset$

Proof Obvious.

Remark 5.2 If $R \in \mathcal{P}(U \times V)$ is an inverse serial relation in a (two-universe) approximation space (U, V, R), then the following properties do not hold for all $Y \in \mathcal{F}(V)$:

(L_3) $\underline{R}'(Y_1 \cap Y_2) = \underline{R}'(Y_1) \cap \underline{R}'(Y_2)$
(L_8) $Y \subseteq \underline{R}'(\bar{R}'(Y))$
(L_{10}) $\bar{R}'(Y) \subseteq \underline{R}'(\bar{R}'(Y))$
(U_3) $\bar{R}'(Y_1 \cup Y_2) = \bar{R}'(Y_1) \cup \bar{R}'(Y_2)$
(U_8) $\bar{R}'(\underline{R}'(Y)) \subseteq Y$

(U_{10}) $\bar{R}'(\underline{R}'(Y)) \subseteq \underline{R}'(Y)$

The following example shows Remark 5.2.

Example 5.2 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ and $R \in \mathcal{P}(U \times V)$ be an inverse serial relation defined as:

R	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0	1	1	0	0	0
x_2	1	0	0	1	1	0
x_3	0	0	0	0	0	0
x_4	0	1	1	1	0	0
x_5	1	0	0	0	1	0
x_6	0	1	0	0	1	1
x_7	0	0	0	1	0	1

If Y and Z are two fuzzy subsets of V defined as:
$Y(y_1) = 0.2, Y(y_2) = 0.7, Y(y_3) = 0.3, Y(y_4) = 0.9,$
$Y(y_5) = 0.5, Y(y_6) = 0.8,$
$Z(y_1) = 0.9, Z(y_2) = 0.5, Z(y_3) = 0.6, Z(y_4) = 0.8,$
$Z(y_5) = 0.1, Z(y_6) = 0.3$, then we have

	y_1	y_2	y_3	y_4	y_5	y_6
$\underline{R}'(Y)(y)$	0.2	0.5	0.3	0.8	0.5	0.8
$\bar{R}'(\underline{R}'(Y))(y)$	0.5	0.5	0.5	0.8	0.5	0.8
$\bar{R}'(Y)(y)$	0.5	0.7	0.7	0.9	0.5	0.8
$\underline{R}'(\bar{R}'(Y))(y)$	0.5	0.7	0.7	0.8	0.5	0.8
$\underline{R}'(Z)(y)$	0.1	0.5	0.5	0.5	0.1	0.3
$\underline{R}'(Y \cap Z)(y)$	0.1	0.3	0.3	0.3	0.1	0.3
$\bar{R}'(Z)(y)$	0.9	0.5	0.6	0.8	0.5	0.5
$\bar{R}'(Y \cup Z)(y)$	0.9	0.7	0.7	0.9	0.8	0.8

Hence we have $\underline{R}'(Y \cap Z) \neq \underline{R}'(Y) \cap \underline{R}'(Z)$, $\bar{R}'(Y \cup Z) \neq \bar{R}'(Y) \cup \bar{R}'(Z)$, $Y \not\subseteq \underline{R}'(\bar{R}'(Y))$, $\bar{R}'(\underline{R}'(Y)) \not\subseteq Y$, $\bar{R}'(Y) \not\subseteq \underline{R}'(\bar{R}'(Y))$ and $\bar{R}'(\underline{R}'(Y)) \not\subseteq \underline{R}'(Y)$, i.e., L_3, U_3, L_8, U_8, L_{10} and U_{10} do not hold.

Proposition 5.4 *In a (two-universe) approximation space (U, V, R) with strong inverse serial relation R, the approximation operators have the following properties for all $Y \in \mathcal{F}(V)$:*

(L_3) $\underline{R}'(Y_1 \cap Y_2) = \underline{R}'(Y_1) \cap \underline{R}'(Y_2)$
(L_8) $Y \subseteq \underline{R}'(\bar{R}'(Y))$
(L_{10}) $\bar{R}'(Y) \subseteq \underline{R}'(\bar{R}'(Y))$
(U_3) $\bar{R}'(Y_1 \cup Y_2) = \bar{R}'(Y_1) \cup \bar{R}'(Y_2)$
(U_8) $\bar{R}'(\underline{R}'(Y)) \subseteq Y$
(U_{10}) $\bar{R}'(\underline{R}'(Y)) \subseteq \underline{R}'(Y)$

Proof (L_3) Assume that R is strong inverse serial. Then by Proposition 3.6, we have $\forall x \in G(y)$, $max\{min\{Y(z) : z \in F(x)\} : x \in G(y)\} = min\{min\{Y(z) : z \in F(x)\} : x \in G(y)\}$. Thus,

$$\begin{aligned} \underline{R}'(Y_1 \cap Y_2)(y) &= max\{min\{(Y_1 \cap Y_2)(z) : z \in F(x)\} : x \in G(y)\} \\ &= min\{min\{min\{Y_1(z), Y_2(z)\} : z \in F(x)\} : x \in G(y)\} \\ &= min\{min\{min\{Y_1(z) : z \in F(x)\} : x \in G(y)\}, \\ &\quad min\{min\{Y_2(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= min\{max\{min\{Y_1(z) : z \in F(x)\} : x \in G(y)\}, \\ &\quad max\{min\{Y_2(z) : z \in F(x)\} : x \in G(y)\}\} \\ &= min\{\underline{R}'(Y_1)(y), \underline{R}'(Y_2)(y)\} \\ &= (\underline{R}'(Y_1) \cap \underline{R}'(Y_2))(y). \end{aligned}$$

(L_{10}) Since R is a strong inverse serial. Then by Proposition 3.6, $\forall y \in V$, we can have

$$\begin{aligned} \underline{R}'(\bar{R}'(Y))(y) &= max\{min\{min\{max\{Y(u) : u \in F(w)\} : \\ &\quad w \in G(z)\} : z \in F(x)\} : x \in G(y)\} \\ &= max\{min\{max\{max\{Y(u) : u \in F(w)\} : \\ &\quad w \in G(z)\} : z \in F(x)\} : x \in G(y)\}. \end{aligned}$$

In terms of Proposition 3.5 and Proposition 3.6 we have

$$\begin{aligned} \underline{R}'(\bar{R}'(Y)) &= \bar{R}_p(\underline{R}_s(\bar{R}_p(\bar{R}_s(Y)))) = \bar{R}_p(\bar{R}_s(Y)) = \underline{R}_p(\bar{R}_s(Y)) \\ &= \bar{R}'(Y). \end{aligned}$$

(L_8) The proof follows from (U_9) of Proposition 5.2 and (L_{10}) of Proposition 5.4. In the same manner we can also prove (U_3), (U_8) and (U_{10}).

Definition 5.2 Let (U, V, R) be a (two-universe) approximation space. Then the lower and upper approximations of $Y \in \mathcal{F}(V)$ are defined respectively as follows:

$$\underline{R}''(Y)(y) = min\{min\{Y(z) : z \in F(x)\} : x \in G(y)\}$$

$$\bar{R}''(Y)(y) = max\{max\{Y(z) : z \in F(x)\} : y \in G(y)\}.$$

The pair $(\underline{R}''(Y), \bar{R}''(Y))$ is referred to as a strong rough fuzzy set, and $\underline{R}''$ *and* $\bar{R}'' : \mathcal{F}(V) \to \mathcal{F}(V)$ are referred to as strong lower and upper rough fuzzy approximation operators, respectively.

Proposition 5.5 *Let (U, V, R) be a (two-universe) approximation space. Then*

$$\underline{R}''(Y) = \underline{R}_p(\underline{R}_s(Y)),$$

$$\bar{R}''(Y) = \bar{R}_p(\bar{R}_s(Y)).$$

Proposition 5.6 *In a (two-universe) approximation space (U, V, R), the approximation operators have the following properties for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:*

(L_1) $\underline{R}''(Y) = (\bar{R}''(Y^c))^c$

(L_2) $\underline{R}''(V) = V$
(L_3) $\underline{R}''(Y_1 \cap Y_2) = \underline{R}''(Y_1) \cap \underline{R}''(Y_2)$
(L_4) $\underline{R}''(Y_1 \cup Y_2) \supseteq \underline{R}''(Y_1) \cup \underline{R}''(Y_2)$
(L_5) $Y_1 \subseteq Y_2 \Rightarrow \underline{R}''(Y_1) \subseteq \underline{R}''(Y_2)$
(L_8) $Y \subseteq \underline{R}''(\bar{R}''(Y))$
(U_1) $\bar{R}''(Y) = (\underline{R}''(Y^c))^c$
(U_2) $\bar{R}''(\emptyset) = \emptyset$
(U_3) $\bar{R}''(Y_1 \cup Y_2) = \bar{R}''(Y_1) \cup \bar{R}''(Y_2)$
(U_4) $\bar{R}''(Y_1 \cap Y_2) \subseteq \bar{R}''(Y_1) \cap \bar{R}''(Y_2)$
(U_5) $Y_1 \subseteq Y_2 \Rightarrow \bar{R}''(Y_1) \subseteq \bar{R}''(Y_2)$
(U_8) $\bar{R}''(\underline{R}''(Y)) \subseteq Y$

Proof We can obtain them according to Propositions 3.1, 3.3, 3.5 and 5.5.

Remark 5.3 If $R \in \mathcal{P}(U \times V)$ is a binary relation in a (two-universe) approximation space (U, V, R), then the following properties do not hold for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:

(L_6) $\underline{R}''(\emptyset) = \emptyset$
(L_7) $\underline{R}''(Y) \subseteq Y$
(L_9) $\underline{R}''(Y) \subseteq \underline{R}''(\underline{R}''(Y))$
(L_{10}) $\bar{R}''(Y) \subseteq \underline{R}''(\bar{R}''(Y))$
(U_6) $\bar{R}''(V) = V$
(U_7) $Y \subseteq \bar{R}''(Y)$
(U_9) $\bar{R}''(\bar{R}''(Y)) \subseteq \bar{R}''(Y)$
(U_{10}) $\bar{R}''(\underline{R}''(Y)) \subseteq \underline{R}''(Y)$
(LU) $\underline{R}''(Y) \subseteq \bar{R}''(Y)$

The following example shows Remark 5.3.

Example 5.3 In Example 5.1, if Y and Z are two fuzzy subsets of V defined as:

$Y(y_1) = 0.5, Y(y_2) = 0.3, Y(y_3) = 0.7, Y(y_4) = 0.1,$
$Y(y_5) = 0.8, Y(y_6) = 0.4,$
$Z(y_1) = 0.8, Z(y_2) = 0.9, Z(y_3) = 0.2,$
$Z(y_4) = 0.3, Z(y_5) = 0.6, Z(y_6) = 0.5,$ then we have

	y_1	y_2	y_3	y_4	y_5	y_6
$\underline{R}''(Y)(y)$	0.4	0.3	0.3	1	0.3	0.3
$\bar{R}''(\underline{R}''(Y))(y)$	0.4	0.3	0.4	0	0.4	0.4
$\underline{R}''(\underline{R}''(Y))(y)$	0.3	0.3	0.3	1	0.3	0.3
$\bar{R}''(Y)(y)$	0.8	0.8	0.8	0	0.8	0.8
$\underline{R}''(\emptyset)(y)$	0	0	0	1	0	0
$\bar{R}''(V)(y)$	1	1	1	0	1	1
$\bar{R}''(Z)(y)$	0.8	0.9	0.9	0	0.9	0.9
$\underline{R}''(\bar{R}''(Z))(y)$	0.8	0.9	0.8	1	0.8	0.8
$\bar{R}''(\bar{R}''(Z))(y)$	0.9	0.9	0.9	0	0.9	0.9

Hence we have $\underline{R}''(\emptyset) \neq \emptyset$, $\bar{R}''(V) \neq V$, $\underline{R}''(Y) \not\subseteq Y$, $Z \not\subseteq \bar{R}''(Z)$, $\underline{R}''(Y) \not\subseteq \underline{R}''(\underline{R}''(Y))$, $\bar{R}''(\bar{R}''(Z)) \not\subseteq \bar{R}''(Z)$, $\bar{R}''(Z) \not\subseteq \underline{R}''(\bar{R}''(Z))$, $\bar{R}''(\underline{R}''(Y)) \not\subseteq \underline{R}''(Y)$ *and* $\underline{R}''(Y) \subseteq \bar{R}''(Y)$ i.e., L_6, U_6, L_7, U_7, L_9, U_9, L_{10}, U_{10} and LU do not hold.

Proposition 5.7 *In a (two-universe) approximation space (U, V, R) with inverse serial relation R, the approximation operators have the following properties for all $Y \in \mathcal{F}(V)$:*

(L_6) $\underline{R}''(\emptyset) = \emptyset$
(L_7) $\underline{R}''(Y) \subseteq Y$
(U_6) $\bar{R}''(V) = V$
(U_7) $Y \subseteq \bar{R}''(Y)$
(LU) $\underline{R}''(Y) \subseteq \bar{R}''(Y)$

Proof (L_7) Since R is an inverse serial. Then $\forall y \in V$, we can have

$$\begin{aligned}\underline{R}''(Y)(y) &= min\{min\{Y(z) : z \in F(x)\} : x \in G(y)\}.\\ &\leq max\{min\{Y(z) : z \in F(x)\} : x \in G(y)\}.\\ &= \bar{R}_p(\underline{R}_s(Y))(y).\end{aligned}$$

Then from Proposition 3.4, $\underline{R}''(Y) \subseteq Y$.

(L_6) follows directly from (L_7).

(U_6) and (U_7) can be proved by the duality of approximation operators.

(LU) comes from (L_7) and (U_7).

Remark 5.4 If $R \in \mathcal{P}(U \times V)$ is an inverse serial relation in a (two-universe) approximation space (U, V, R), then the following properties do not hold for all $Y \in \mathcal{F}(V)$:

(L_9) $\underline{R}''(Y) \subseteq \underline{R}''(\underline{R}''(Y))$
(L_{10}) $\bar{R}''(Y) \subseteq \underline{R}''(\bar{R}''(Y))$
(U_9) $\bar{R}''(\bar{R}''(Y)) \subseteq \bar{R}''(Y)$
(U_{10}) $\bar{R}''(\underline{R}''(Y)) \subseteq \underline{R}''(Y)$

The following example shows Remark 5.4.

Example 5.4 In Example 5.2, we have

	y_1	y_2	y_3	y_4	y_5	y_6
$\underline{R}''(Z)(y)$	0.1	0.1	0.5	0.1	0.1	0.1
$\bar{R}''(\underline{R}''(Z))(y)$	0.1	0.5	0.5	0.5	0.1	0.1
$\underline{R}''(\underline{R}''(Z))(y)$	0.1	0.1	0.1	0.1	0.1	0.1
$\bar{R}''(Z)(y)$	0.9	0.8	0.8	0.9	0.9	0.8
$\underline{R}''(\bar{R}''(Z))(y)$	0.9	0.8	0.8	0.8	0.8	0.8
$\bar{R}''(\bar{R}''(Z))(y)$	0.9	0.9	0.9	0.9	0.9	0.9

Hence we have $\underline{R}''(Z) \not\subset \underline{R}''(\underline{R}''(Z))$, $\bar{R}''(\bar{R}''(Z)) \not\subset \bar{R}''(Z)$, $\bar{R}''(Z) \not\subset \underline{R}''(\bar{R}''(Z))$ and $\bar{R}''(\underline{R}''(Z)) \not\subset \underline{R}''(Z)$, i.e., L_9, U_9, L_{10} and U_{10} do not hold.

Proposition 5.8 *In a (two-universe) approximation space (U, V, R) with strong inverse serial relation R, the approximation operators have the following properties for all $Y \in \mathcal{F}(V)$:*

(L_9) $\underline{R}''(Y) \subseteq \underline{R}''(\underline{R}''(Y))$
(L_{10}) $\bar{R}''(Y) \subseteq \underline{R}''(\bar{R}''(Y))$
(U_9) $\bar{R}''(\bar{R}''(Y)) \subseteq \bar{R}''(Y)$
(U_{10}) $\bar{R}''(\underline{R}''(Y)) \subseteq \underline{R}''(Y)$

Proof The proof is similar to Proposition 5.4.

In Table 1 we compare the properties that are satisfied by the different definitions of rough set.

6 Connections of the rough fuzzy approximation operators

Proposition 6.1 *Let $R \in \mathcal{P}(U \times V)$ be a binary relation from U to V. Then $\forall Y \in \mathcal{F}(V)$,*

(1) $\underline{R}'(Y) \subseteq \underline{R}^*(Y), \bar{R}^*(Y) \subseteq \bar{R}'(Y)$.

(2) $\underline{R}''(Y) \subseteq \underline{R}.^*(Y), \bar{R}^*(Y) \subseteq \bar{R}''(Y)$.

Proof By duality of approximation operators we only need to prove the first part of each property.

(1) Since for every y ∈ Y, we have

$$\begin{aligned}\underline{R}^*(Y)(y) &= min\{Y(z) : z \in G^*(y)\} = min\{Y(z) : z \in \cap F(x), x \in G(y)\} \\ &\geq max\{min\{Y(z) : z \in F(x)\} : x \in G(y)\} \\ &= \underline{R}'(Y)(y)\end{aligned}$$

Hence $\underline{R}'(Y) \subseteq \underline{R}^*(Y)$.

(2) the proof is similar as (1)

Remark 6.1 Let $R \in \mathcal{P}(U \times V)$ be a binary relation from U to V. Then Definitions 5.1 and 5.2 are independent.

The following example shows Remark 6.1. Moreover, the inclusion in Proposition 6.1 can not be replaced by equality,

Example 6.1 From Example 5.1, we get:

	y_1	y_2	y_3	y_4	y_5	y_6
$\underline{R}'(Y)(y)$	0.5	0.3	0.5	0	0.5	0.4
$\underline{R}''(Y)(y)$	0.4	0.3	0.3	1	0.3	0.3
$\underline{R}^*(Y)(y)$	0.5	0.3	0.7	1	0.8	0.4
$\bar{R}'(Y)(y)$	0.5	0.7	0.7	1	0.8	0.5
$\bar{R}''(Y)(y)$	0.8	0.8	0.8	0	0.8	0.8
$\bar{R}^*(Y)(y)$	0.5	0.3	0.7	0	0.8	0.4

Proposition 6.2 *Let $R \in \mathcal{P}(U \times V)$ be an inverse serial relation from U to V. Then $\forall Y \in \mathcal{F}(V)$,*

$\underline{R}''(Y) \subseteq \underline{R}'(Y) \subseteq \underline{R}^*(Y) \subseteq Y \subseteq \bar{R}^*(Y) \subseteq \bar{R}'(Y) \subseteq \bar{R}''(Y)$.

Table 1 comparison between the properties of rough fuzzy sets depending on Definitions 3.2, 4.1, 5.1 and 5.2 by using binary, inverse serial and strong inverse serial relations

	Binary relations				Inverse serial relations				Strong inverse serial relations			
	Def. 3.2	Def. 4.1	Def. 5.1	Def. 5.2	Def. 3.2	Def. 4.1	Def. 5.1	Def. 5.2	Def. 3.2	Def. 4.1	Def. 5.1	Def. 5.2
L_1, U_1	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_2, U_2	$\mathcal{Y}$	$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_3, U_3	$\mathcal{Y}$	$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_4, U_4	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_5, U_5	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_6, U_6			$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_7, U_7			$\mathcal{Y}$			$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_8, U_8				$\mathcal{Y}$				$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_9, U_9		$\mathcal{Y}$	$\mathcal{Y}$			$\mathcal{Y}$	$\mathcal{Y}$			$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
L_{10}, U_{10}										$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
LU			$\mathcal{Y}$		$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$

($\mathcal{Y}$) indicates that the property is satisfied

Proof Obvious.

We can introduce an example to show that the converse of Proposition 6.2 is not true in general.

Example 6.2 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ and $R \in \mathcal{P}(U \times V)$ be an inverse serial relation defined as:

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	0	1	0	1	1	0	0
x_2	1	1	0	0	1	1	0
x_3	0	0	1	1	0	0	0
x_4	1	0	0	1	0	0	1
x_5	0	0	1	1	1	0	0
x_6	0	0	0	0	0	0	0

If Y is a fuzzy subset of V defined as:

$Y(y_1)=0.2, Y(y_2)=0.5, Y(y_3)=0.1, Y(y_4)=0.7, Y(y_5)=0.3, Y(y_6)=0.8, Y(y_7)=0.4$, then we have

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
$\underline{R}'(Y)(y)$	0.2	0.3	0.1	0.3	0.3	0.2	0.2
$\underline{R}''(Y)(y)$	0.2	0.2	0.1	0.1	0.1	0.2	0.2
$\underline{R}^*(Y)(y)$	0.2	0.3	0.1	0.7	0.3	0.2	0.2
$\bar{R}'(Y)(y)$	0.7	0.7	0.7	0.7	0.7	0.8	0.7
$\bar{R}''(Y)(y)$	0.8	0.8	0.7	0.7	0.8	0.8	0.7
$\bar{R}^*(Y)(y)$	0.2	0.5	0.7	0.7	0.3	0.8	0.7

Proposition 6.3 Three pairs of lower and upper approximation operators in Definition 4.1, Definition 5.1 and Definition 5.2 are equivalent if R is a strong inverse serial relation.

Proof For a strong inverse serial relation, by Propositions 3.6, 5.1 and 5.5 we have $\underline{R}''(Y) = \underline{R}'(Y)$ and $\bar{R}''(Y) = \bar{R}'(Y)$. We only need to show $\underline{R}^*(Y) \subseteq \underline{R}'(Y)$. The other relation $\bar{R}'(Y) \subseteq \bar{R}^*(Y)$ can be obtained by duality. Since for every $y \in Y$, we have

$$\begin{aligned} \underline{R}^*(Y)(y) &= min\{Y(z) : z \in G^*(y)\} = min\{Y(z) : z \in \cap F(x), x \in G(y)\} \\ &= min\{Y(z) : z \in F(x), x \in G(y)\} \\ &\leq max\{min\{Y(z) : z \in F(x)\}, x \in G(y)\} \\ &= \underline{R}'(Y)(y) \end{aligned}$$

7 Conclusion

In this paper we presented two new definitions of the lower approximation and upper approximation operators on two universes through the combination of successor and predecessor neighborhood operators. It should be pointed out that the approximating sets and the approximated sets in these rough set models are on the same universe of discourse V, and each type of the approximation operator captures different aspects of approximating a subset of the universe of discourse. All the properties of rough sets have been simulated by employing these notions, and the relationships between some of them and the existing rough fuzzy approximation operators on two universes of discourse have also been examined. By comparing these approximation operators, some conditions on the relation R under which all of the rough fuzzy approximation operators made equivalent are identified.

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References

1. Abd El-Monsef ME, Kozae AM, Salama AS, Aqeel RM (2012) A comprehensive study of rough sets and rough fuzzy sets on two universes. J Comput 4(3):94–101
2. Allam AA, Bakeir MY, Abo-Tabl EA (2006) New approach for closure spaces by relations. Acta Mathematica Academiae Paedagogicae Nyregyhziensis 22:285–304
3. Boixader D, Jacas J, Recasens J (2000) Upper and lower approximations of fuzzy sets. Int J Gen Syst 29:555–568
4. Bonikowski Z (1994) Algebraic structure of rough sets. In: Ziarko WP (ed) Rough sets, fuzzy sets and knowledge discovery. Springer, London, pp 242–247
5. Chakrabarty K, Biawas R, Nanda S (2000) Fuzziness in rough sets. Fuzzy Sets Syst 10:247–251
6. Dubois D, Prade H (1990) Rough fuzzy sets and fuzzy rough sets. Int J Gen Syst 17:191–208
7. Dubois D, Prade H (1992) Putting fuzzy sets and rough sets together. In: Slowinski R (ed) Intelligence Decision Support. Kluwer Academic, Dordrecht, pp 203–232
8. Feng T, Mi J, Wu W (2006) Covering-based generalized rough fuzzy sets. In: Wang G et al (eds) RSKT 2006, LNAI 4062, pp 208–215
9. Gong Z, Zhang X (2014) Variable precision intuitionistic fuzzy rough sets model and its application. Int J Mach Learn Cybern 5(2):263–280
10. Greco S, Matarazzo B, Slowinski R (1999) Theory and methodology: rough approximation of a preference relation by dominance relations. Eur J Oper Res 117:63–83
11. Greco S, Matarazzo B, Slowinski R (2002) Rough approximation by dominance relations. Int J Intell Syst 17:153–171
12. Huynh VN, Nakamori Y (2005) A roughness measure for fuzzy sets. Inf Sci 173:255–275
13. Inuiguchi M, Tanino T (2002) New fuzzy rough set based on certainty qualification. In: Pal SK, Polkowski L, Skowron A (eds) Rough-neural computing: techniques for computing with words. Springer, Berlin, pp 277–296
14. Ju H, Yang X, Song X, Qi Y (2014) Dynamic updating multigranulation fuzzy rough set: approximations and reducts. Int J Mach Learn Cybern 5(6):981–990

15. Kondo M (2005) Algebraic approach to generalized rough sets. Lect Notes Artif Intell 3641:132–140
16. Kondo M (2006) On the structure of generalized rough sets. Inf Sci 176:589–600
17. Kryszkiewicz M (1998) Rough set approach to incomplete information systems. Inf Sci 112:39–49
18. Li TJ, Zhang WX (2008) Rough fuzzy approximations on two universes of discourse. Inf Sci 178:892–906
19. Lin TY (1989) Neighborhood systems and approximation in database and knowledge bases. In: Proceedings of the fourth international symposium on methodologies of intelligent systems, pp 75–86
20. Lin TY, Huang KJ, Liu Q, Chen W (1990) Rough sets, neighborhood systems and approximation. In: Proceedings of the fifth international symposium on methodologies of intelligent systems, selected papers, Knoxville, Tennessee, pp 130–141
21. Liu GL (2005) Rough sets over the boolean algebras. Lecture Notes in Artificial Intelligence 3641:24–131
22. Mi J-S, Leung Y, Wu W-Z (2005) An uncertainty measure in partition-based fuzzy rough sets. Int J Gen Syst 34:77–90
23. Nakamura A (1998) Fuzzy rough set. Note Mult-Valued Log Jpn 9(8):1–8
24. Nanda S, Majumda S (1992) Fuzzy rough set. Fuzzy Sets Syst 45:157–160
25. Pawlak Z (1982) Rough sets. Int J Comput Inform Sci 11:341–356
26. Pawlak Z (1991) Rough sets-theoretical aspects of reasoning about data. Kluwer Academic Publishers, Boston
27. Pomykala JA (1987) Approximation operations in approximation space. Bull Pol Acad Sci Math 35:653–662
28. Pomykala JA (1988) On definability in the nondeterministic information system. Bull Pol Acad Sci Math 36:193–210
29. Qian Y, Liang J, Wei W (2013) Consistency-preserving attribute reduction in fuzzy rough set framework. Int J Mach Learn Cybern 4(4):287–299
30. Radzikowska AM, Kerre EE (2002) A comparative study of fuzzy rough sets. Fuzzy Sets Syst 126:137–155
31. Radzikowska AM, Kerre EE (2004) L-fuzzy rough sets. In: Peter JF et al (eds) Artificial intelligence and soft computing—ICAISC2004, LNCS 3135, vol 3070, pp. 526–531
32. Skowron A, Stepaniuk J (1996) Tolerance approximation spaces. Fundam Inform 27:245–253
33. Skowron A, Swiniarski R, Synak P (2005) Approximation spaces and information granulation, In: Peters JF, Skowron A (eds) Transactions on rough sets III, LNCS 3400, pp 175–189
34. Slowinski R, Vanderpooten D (2000) A generalized definition of rough approximations based on similarity. IEEE Trans Knowl Data Eng 12(2):331–336
35. Stepaniuk J (1998) Approximation spaces in extensions of rough set theory. In: Polkowski L, Skowron A (eds) RSCTC'98, LNAI 1424, pp 290–297
36. Sun BZ, Ma WM (2011) Fuzzy rough set model on two different universes and its application. Appl Math Model 35:1798–1809
37. Wong SK, Wang LS, Yao YY (1992) Interval structure: a framework for representing uncertain information. In: Proceedings of conference on uncertain artificial intelligence, pp 336–343
38. Wong SK, Wang LS, Yao YY (1995) On modeling uncertainty with interval structure. Comput Intell 11:406–426
39. Wu W-Z, Leung Y, Mi J-S (2005) On characterizations of (i, t)-fuzzy rough approximation operators. Fuzzy Sets Syst 154:76–102
40. Wu WZ, Leung Y, Zhang WX (2006) On generalized rough fuzzy approximation operators. In: Peters JF, Skowron A (eds) Transactions on rough sets V, LNCS 4100, pp 263–284
41. Wu W-Z, Zhang W-X (2003) Generalized fuzzy rough sets. Inf Sci 15:263–282
42. Wu W-Z, Zhang W-X (2004) Constructive and axiomatic approaches of fuzzy approximation operators. Inf Sci 159:233–254
43. Xu W, Sun W, Liu Y, Zhang W (2013) Fuzzy rough set models over two universes. Int J Mach Learn Cybern 4(6):631–645
44. Yang H-L (2012) S-G. Li, S. Wang, J. Wang, Bipolar fuzzy rough set model on two different universes and its application. Knowl-Based Syst 35:94–101
45. Yang X, Yang Y (2013) Independence of axiom sets on intuitionistic fuzzy rough approximation operators. Int J Mach Learn Cybern 4(5):505–513
46. Yao YY (1997) Combination of rough and fuzzy sets based on α-level sets. In: Lin TY, Cercone N (eds) Rough sets and data mining: analysis for imprecise data. Kluwer Academic Publishers, Boston, pp 301–321
47. Yao YY (1998) Generalized rough set models. In: Polkowski L et al (eds) Rough sets in knowledge discovery, methodology and applications, 1. Physical-Verlag, Heidelberg, pp 286–318
48. Yao YY (1998) Constructive and algebraic methods of the theory of rough sets. Inf Sci 109:21–47
49. Yao YY (1998) Relational interpretations of neighborhood operators and rough set approximation operators. Inf Sci 111:239–259
50. Yao YY (1999) Rough sets, neighborhood systems, and granular computing. In: Meng M (ed) Proceedings of the 1999 IEEE Canadian conference on electrical and computer engineering, Edmonton, Canada. IEEE press, pp 1553–1558
51. Yao YY, Lin TY (1996) Generalization of rough sets using modal logic. Intell Autom Soft Comput Int J 2:103–120
52. Yao YY, Wong SKM, Wang LS (1995) A non-numeric approach to uncertain reasoning. Int J Gen Syst 23:343–359
53. Yao YY, Wong SKM, Lin TY (1997) A review of rough set models. In: Lin TY, Cercone N (eds) Rough sets and data mining: analysis for imprecise data. Kluwer Academic Publishers, Boston, pp 47–75