Int. J. Mach. Learn. & Cyber. (2017) 8:239–253
DOI 10.1007/s13042-014-0317-2

ORIGINAL ARTICLE

Ideals and fuzzy ideals on residuated lattices

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Received: 21 April 2014 / Accepted: 26 November 2014 / Published online: 11 December 2014

Abstract This paper mainly focus on building the ideals theory of non regular residuated lattices. Firstly, the notions of ideals and fuzzy ideals of a residuated lattice are introduced, their properties and equivalent characterizations are obtained; at the meantime, the relation between filter and ideal is discussed. Secondly, two types prime ideals of a residuated lattice are introduced, the relations between the two types ideals are studied, in some special residuated lattices (such as MTL-algebras, lattice implication algebras, BL-algebras), prime ideal and prime ideal of the second kind are coincide. At the meantime, the notions of fuzzy prime ideal and fuzzy prime ideal of the second kind on a residuated lattice are introduced, aiming at the relation between prime ideal and prime ideal of the second kind, we mainly investigate the fuzzy prime ideal of the second kind. Finally, we investigated the fuzzy congruence relations induced by fuzzy ideal, we construct a new residuated lattice induced by fuzzy congruences, the homomorphism theorem is given.

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Keywords Residuated lattice · (fuzzy) Ideals · (fuzzy) Prime ideals · (fuzzy) Prime ideals of the second kind · Fuzzy congruence · Homomorphism theorem

1 Introduction

As is known to all, one significant function of artificial intelligence is to make computer simulate human being in dealing with uncertain information. And logic establishes the foundations for it. However, certain information process is based on the classic logic. Non-classical logics consist of these logics handling a wide variety of uncertainties (such as fuzziness, randomness, and so on) and fuzzy reasoning. Therefore, non-classical logic has been proved to be a formal and useful technique for computer science to deal with fuzzy and uncertain information. Many-valued logic, as the extension and development of classical logic, has always been a crucial direction in non-classical logic. Lattice-valued logic, an important many-valued logic, has two prominent roles: One is to extend the chain-type truth-valued field of the current logics to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by the general lattice can more effectively reflect the uncertainty of human being's thinking, judging and decision. Hence, lattice-valued logic has been becoming a research field and strongly influencing the development of algebraic logic, computer science and artificial intelligent technology. At the same time, various logical algebras have been proposed as the structures of truth degrees associated with logic systems, for example, residuated lattices, MV-algebras, BL-algebras, G*ö*del algebras, lattice implication algebras, MTL-algebras, NM-algebras and R_0-algebras, etc. Among these logical algebras, residuated lattices are very basic and important algebraic structure because the other logical algebras are all particular cases of residuated lattices [3, 4].

Nonclassical logic is closely related to logic algebraic systems. A number of researches have motivated to develop nonclassical logics, and also to enrich the content of algebra [7, 18–20]. In modern fuzzy logic theory, residuated lattices and some related algebraic systems play an extremely important role because they provide an algebraic frameworks to fuzzy logic and fuzzy reasoning. By using the theory of residuated lattices, Pavelka has built up a more generalized logic systems, and he has successfully proved the semantical completeness of the Lukasiewicz's axiom system in 1979. From a logical point of view, various filters and ideals corresponding to various sets of provable formulae. The sets of provable formulas in the corresponding inference systems from the point of view of uncertain information can be described by fuzzy ideals of those algebraic semantics. In the meantime, ideal theory is a very effectively tool for investigating these various algebraic and logic systems. The notion of ideal has been introduced in many algebraic structure such as lattices, rings, MV-algebras, lattice implication algebras. In these algebraic structure, as filter, the ideal is in the center position. However, in BL-algebras and residuated lattices (especially non regular residuated lattice), the focus is shifted to deductive systems or filters [1, 2, 5, 6, 8–12, 14–16, 22, 24, 26, 27]. The study of residuated lattice have experienced a tremendous growth and the main focus has been on filters. For BL-algebras, Lele and Nganou [13] introduced the notion of ideal in BL-algebras as a natural generalization of that of ideal in MV-algebras. However, non regular residuated lattice as a more general important algebraic structure, the notion of ideal is missing.

But so far, mostly focus on filters and fuzzy filters while the study of ideals and fuzzy ideals in a residuated lattices have been completely ignored. We could not find and even a single paper on ideals and fuzzy ideals on non regular residuated lattices. Knowing the importance of ideals and congruences in classification problems, data organization, formal concept analysis, and so on; it is meaningful to make and intensive study of ideals in non regular residuated lattices. The fact that ideal is an dual of filter in some special logical algebras such that R_0-algebras, lattice implication algebras and so on. But, the dual of filter is not an ideal in MTL-algebras.

The main goal of this work is to fill this gap by introducing the notion of ideal and fuzzy ideals in a non regular residuated lattice. This notion must generalize the existing notion in MV-algebras, BL-algebras and lattice implication algebras. Firstly, the notions of ideals and fuzzy ideals of a residuated lattice are introduced in Sect. 3, their properties and equivalent characterizations are obtained; at the meantime, the relation between filter and ideal is discussed, unlike in lattice implication algebras and R_0-algebras, we observe that ideals and the dual of filters be quite differently in residuated lattices. Secondly, two types prime ideals of a residuated lattice are introduced in the Sect. 4, the relations between the two types ideals are studied, in some special residuated lattices (such as MTL-algebras, lattice implication algebras, BL-algebras), prime ideal and prime ideal of the second kind are coincide. At the meantime, the notions of fuzzy prime ideal and fuzzy prime ideal of the second kind on a residuated lattice are introduced, aiming at the relation between prime ideal and prime ideal of the second kind, we mainly investigate the fuzzy prime ideal of the second kind. Finally, we investigated the fuzzy congruence relations induced by fuzzy ideal in Sect. 5, we construct a new residuated lattice induced by fuzzy congruences, the homomorphism theorem is given.

2 Preliminaries

Definition 2.1 ([4]) A residuated lattice is an algebraic structure $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following axioms:

(C1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(C2) $(L, \otimes, 1)$ is a commutative semigroup (with the unit element 1).
(C3) $(\otimes, \rightarrow)$ is an adjoint pair.

Proposition 2.1 ([4]) A algebraic structure $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) is a residuated lattice if and only if it satisfies the following conditions, for any $x, y, z \in L$:

(R1) If $x \leq y$, then $x \otimes z \leq y \otimes z$.
(R2) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.
(R3) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$.
(R4) $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.
(R5) $x \otimes y = y \otimes x$.
(R6) $1 \otimes x = x$.

In what follows, let $\mathcal{L}$ denote a residuated lattice unless otherwise specified.

In a residuated lattice $\mathcal{L}$, denote $x' = x \rightarrow 0$. A residuated lattice is regular if $x'' = x$ for all $x \in L$.

A MTL-algebras is a residuated lattice with the prelinearity condition (i.e. $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for any $x, y \in L$)

Proposition 2.2 ([4, 17, 18]) In each residuated lattice $\mathcal{L}$, the following properties hold for all $x, y, z \in L$:

(P1) $(x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z)$.
(P2) $x \otimes (x \rightarrow y) \leq y$.

(P3) $(x \to y) \otimes x \leq x$.
(P4) $x \otimes y \leq x \wedge y$.
(P5) $(x \vee y) \otimes z = (x \otimes z) \vee (y \otimes z)$.
(P6) if $x \leq y$, then $y' \leq x'$.
(P7) $y \to z \leq (x \to y) \to (x \to z)$.
(P8) $(x \otimes y)' = x \to y'$.
(P9) $x^m \leq x^n$, $m, n \in N$, $m \geq n$.
(P10) $1 \to x = x, x \to x = 1$.
(P11) $x \to (y \to z) = y \to (x \to z)$.
(P12) $x \leq y \Leftrightarrow x \to y = 1$.
(P13) $0' = 1, 1' = 0, x' = x''', x \leq x''$.
(P14) $x \to y \leq (x \otimes y')'$.
(P15) $y \to x \leq (x \to z) \to (y \to z)$.
(P16) $(x \otimes y)'' = x'' \otimes y''$.
(P17) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
(P18) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

In a residuated lattice, the binary operation $\oplus$ defined by $x \oplus y = x' \to y$ for any $x, y \in L$.

Proposition 2.3 In each residuated lattice $\mathcal{L}$, the following properties hold for all $x, y, z \in L$:

(P19) if $x \leq y$, then $x \oplus z \leq y \oplus z$.
(P20) $x \oplus y \geq x$ and $x \oplus y \geq y$.
(P21) $x \oplus x' = 0$.
(P22) $x \oplus y \leq (x' \otimes y')'$.
(P23) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$.
(P24) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.
(P25) $(x \oplus y) \oplus z) = x \oplus (y \oplus z)$.

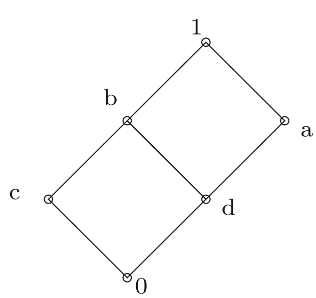

Fig. 1 Hasse diagram of L

Remark 2.1 $\oplus$ is associative and non commutative.

Definition 2.2 ([27]) A non-empty subset F of a residuated lattice is called a **filter** of $\mathcal{L}$ if it satisfies, for any x, y in L

(F1) $x, y \in F \Rightarrow x \otimes y \in F$.
(F2) $x \in F, x \leq y \Rightarrow y \in F$.

Proposition 2.4 ([27]) A non-empty subset F of a residuated lattice is a filter of $\mathcal{L}$ if and only if

(F3) $1 \in F$.
(F4) $x \in F, x \to y \in F \Rightarrow y \in F$.

3 Fuzzy ideals of residuated lattices

In this section, we will introduce the notions of ideal and fuzzy ideal in a residuated lattice which coincides with the notions of ideals in MTL-algebras, BL-algebras, Lattice implication algebras etc.

3.1 Ideals of residuated lattices

Definition 3.1 Let $\mathcal{L}$ be a residuated lattice and $\emptyset \neq I \subseteq L$. I is said to be an ideal of $\mathcal{L}$, if I satisfies:

(I1) for any $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$;
(I2) for any $x, y \in I$, $x \oplus y \in I$.

From the Definition 3.1, for any residuated lattice $\mathcal{L}$, $\{0\}$ and $\mathcal{L}$ are ideals of $\mathcal{L}$. The ideal of lattice implication algebras is also called LI-ideal.

Example 3.1 Let $L = \{0, a, b, c, d, 1\}$, the Hasse diagram of L be defined as Fig. 1 and its implication operator $\to$ be defined as Table 1 and operator $\otimes$ be defined as Table 2. Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. $\mathcal{L}$ is also a regular residuated lattice.

It is routine to verify that $I_1 = \{0, c\}$ and $I_2 = \{0, d, a\}$ are ideals of $\mathcal{L}$.

Table 1 $\to$ of $\mathcal{L}$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 2 $\otimes$ of $\mathcal{L}$

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

Table 3 $\rightarrow$ of $\mathcal{L}$

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	c	1	1	1
f	0	b	c	c	c	f	1	1
1	0	a	b	c	c	e	f	1

Table 4 $\otimes$ of $\mathcal{L}$

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

Example 3.2 Let $L=\{0,a,b,c,d,e,f,1\}$ be such that $0<a<b<c<1$, $0<d<e<f<1$, $a<e$ and $b<f$. Its implication operator $\rightarrow$ and operator $\otimes$ as follows Table 3:

Then $\mathcal{L}=(L,\wedge,\vee,\rightarrow,\wedge,0,1)$ is a residuated lattice which is a non-regular residuated lattice Table 4. It is routine to verify that $I_3=\{0,d\}$ and $I_4=\{0,a,b,c\}$ are ideals of $\mathcal{L}$.

Theorem 3.1 Let $\mathcal{L}$ be a residuated lattice. I is an ideal of $\mathcal{L}$ if and only if I satisfies following conditions:

(I3) $0\in I$;

(I4) for any $x,y\in L$, if $x^{'}\otimes y\in I$ and $x\in I$, then $y\in I$.

Proof Suppose I is an ideal of $\mathcal{L}$. It follows from (I1) that $0\in I$, so (I3) holds. Let $x,y\in L$ such that $x^{'}\otimes y\in I$ and $x\in I$. Observe that $y\rightarrow(x\oplus(x^{'}\otimes y))=y\rightarrow(x^{'}\rightarrow(x^{'}\otimes y))=(y\otimes x^{'})\rightarrow(y\otimes x^{'})=1$, we have $y\leq x\oplus(x^{'}\otimes y)$. As $x^{'}\otimes y\in I$ and $x\in I$, it follows from (I2) that $x\oplus(x^{'}\otimes y)\in I$, by (I1), hence $y\in I$. Therefore, (I4) holds.

Conversely, Let $x,y\in L$ such that $x\leq y$ and $y\in I$, then $y^{'}\leq x^{'}$ and $y^{'}\otimes x\leq x^{'}\otimes x=0$, it follows that $y^{'}\otimes x=0\in I$, by $y\in I$, we have $x\in I$, that is, (I1) holds. Assume $x,y\in I$. Since $x^{'}\otimes(x\oplus y)=x^{'}\otimes(x^{'}\rightarrow y)\leq y$ and $y\in I$, by (I2), we have $x^{'}\otimes(x\oplus y)\in I$. It follows from (I4) that $x\oplus y\in I$. □

Theorem 3.2 Let $\mathcal{L}$ be a residuated lattice. I is an ideal of $\mathcal{L}$ if and only if I satisfies following conditions:

(I3) $0\in I$;

(I5) for any $x,y\in L$, if $(x^{'}\rightarrow y^{'})^{'}\in I$ and $x\in I$, then $y\in I$.

Proof Let I be an ideals of $\mathcal{L}$, so (I3) is obvious. Assume $(x^{'} \to y^{'})^{'} \in I$ and $x \in I$. Since $x^{'} \otimes y^{''} \leq (x^{'} \otimes y^{''})^{''} = ((x^{'} \otimes y^{''})^{'})^{'} = (x^{'} \to y^{'''})^{'} = (x^{'} \to y^{'})^{'} \in I$, by (I1), we have $x^{'} \otimes y^{''} \in I$, it follows from (I4) that $y^{''} \in I$. As $y^{''} \geq y$, we have $y \in I$.

Conversely, assume that (I5) holds, taking $y = x^{''}$ in (I5), we have $x^{''} \in I$. Let $x, y \in I$ such that $x^{'} \otimes y \in I$ and $x \in I$, we obtain $(x^{'} \otimes y)^{''} \in I$. Since $(x^{'} \otimes y)^{''} = ((x^{'} \otimes y)^{'})^{'} = (x^{'} \to y^{'})^{'}$, we have $(x^{'} \to y^{'})^{'} \in I$. By (I5), we have $y \in I$. Therefore I is an ideal of $\mathcal{L}$. □

Corollary 3.1 Let $\mathcal{L}$ be a residuated lattice and I is an ideal of $\mathcal{L}$. Then $x \in I$ if and only if $x^{''} \in I$.

Remark 3.1 If the residuated lattice is a *MTL*-algebras, the notion of ideal as well as the concept of ideals in lattice implication algebras, R_0-algebras are coincidence.

Theorem 3.3 Let $\mathcal{L}$ be a residuated lattice. I is an ideal of $\mathcal{L}$ if and only if I satisfies following conditions:

(I2) for any $x, y \in I$, $x \oplus y \in I$;
(I6) for any $x, y \in L$, if $x \vee y \in I$, then $x \in I$ and $y \in I$.

Proof This proof is straightforward from the Definition 3.1. □

Theorem 3.4 Let $\mathcal{L}$ be a residuated lattice. I is an ideal of $\mathcal{L}$ if and only if I satisfies following conditions:

(I2) for any $x, y \in I$, $x \oplus y \in I$;
(I7) for any $x, y \in L$, if $x \in I$, then $x \wedge y \in I$.

Proof If I is an ideal of $\mathcal{L}$, then it is clear that I satisfies (I7). Let I satisfy (I2) and (I7). Let $x \in I$, $y \in L$ and $y \leq x$. Then $0 = x \wedge 0 \in I$ and $y = x \wedge y \in I$. Thus I is an ideal of $\mathcal{L}$. □

Definition 3.2 Let $\mathcal{L}$ be a residuated lattice. I is an lattice ideal of $\mathcal{L}$ if and only if

(I1) for any $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$;
(I6) for any $x, y \in I$, $x \vee y \in I$.

Theorem 3.5 Let $\mathcal{L}$ be a residuated lattice and I an ideal of $\mathcal{L}$. Then I is a lattice ideal of $\mathcal{L}$.

Proof Let I be an ideal of $\mathcal{L}$, so (I1) is obvious. For any $x, y \in I$, then $x \oplus y \in I$. Since $x \oplus y = x^{'} \to y \geq y$ and $x \oplus y = x^{'} \to y \geq x^{''} \geq x$, we have $x \oplus y \geq x \vee y$, by (I1), we have $x \vee y \in I$. Therefore I is a lattice ideal of $\mathcal{L}$.

In general, the converse of Theorem may not be true. In fact, In Example 3.2, $\{0, a\}$ is a lattice ideals of $\mathcal{L}$, but it is not an ideal of $\mathcal{L}$.

Lattice implication algebra, MV-algebras, MTL-algebras and *BL*-algebra are residuated lattice. $x = x^{''}$ is true in lattice implication algebras and *MV*-algebras. But it may not be true in *BL*-algebras and *MTL*-algebras. In lattice implication algebras $\mathcal{L}$, $F \subseteq L$ is a filter of $\mathcal{L}$ if and only if $F^{'} = \{x^{'} | x \in F\}$ is an LI-ideal. But the result may not be true in non-regular residuated lattices, the main reason is the involution law does not hold in general in non-regular residuated lattice such as *MTL*-algebras and *BL*-algebras and so on. □

Example 3.3 In Example 3.1, I_1, I_2 are ideals of $\mathcal{L}$, meanwhile, $I_1^{'} = \{1, a\}, I_2^{'} = \{1, b, c\}$ are all filters. But in Example 3.2, $\mathcal{L}$ is non-regular residuated lattice, the set $I_4^{'} = \{1, d\}$ is not a filter of $\mathcal{L}$. At the meantime, $F = \{1, d, e, f\}$ is a filter of $\mathcal{L}$, but $F^{'} = \{0, c\}$ not an ideal of $\mathcal{L}$.

Example 3.4 Let $L = \{0, a, b, c, d, 1\}$, the Hasse diagram of L be defined as Fig. 2 and its operator $\to$ be defined as Table 5 and implication operator $\otimes$ be defined as Table 6:

Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice, but not a regular residuated lattice, because $(a \to 0) \to 0 \neq a$. Obviously, $L = \{0, a, b, c, d, 1\}$ is an ideal, but $L^{'} = \{0, 1, d\}$ is not an ideal of $\mathcal{L}$.

The following Theorems 3.6, 3.7 will reveal the relations between ideal and filter in a non regular residuated lattice.

Theorem 3.6 Let F be a filter of a residuated lattice $\mathcal{L}$. Then F_* is an ideal of $\mathcal{L}$, where $F_* = \{x \in L | \text{thereexists } y \in F \text{ suchthat } x^{''} \leq y^{'}\}$

Proof Let $x, y \in L$ such that $x \leq y$ and $y \in F_*$, then there exist $y_0 \in F$ such that $y^{''} \leq y_0^{'}$. As $x \leq y \leq y^{''} \leq y_0^{'}$, so $x \in F_*$.

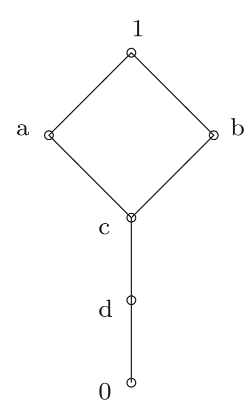

Fig. 2 Hasse diagram of L

Table 5 $\rightarrow$ of $\mathcal{L}$

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	b	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

Table 6 $\otimes$ of $\mathcal{L}$

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

Let $x, y \in L$ such that $x, y \in F_*$, then there exist $x_0, y_0 \in F$ such that $x'' \leq x_0', y'' \leq y_0'$. Since F is a filter, we have $x_0 \otimes y_0 \in F$. We observe that $(x \oplus y)'' \leq (x' \otimes y')''' = (x' \otimes y')' = x' \rightarrow y'' \leq x' \rightarrow y_0' \leq x_0 \rightarrow y_0' = (x_0 \otimes y_0)'$, thus $x \oplus y \in F_*$. Therefore F_* is an ideal of $\mathcal{L}$. □

Theorem 3.7 Let I be an ideal of a residuated lattice $\mathcal{L}$. Then I^* is a filter of $\mathcal{L}$, where $I^* = \{x \in L | \text{thereexists } y \in I \text{ suchthat } x'' \geq y'\}$.

Proof Let $x, y \in L$ such that $x \leq y$ and $x \in I^*$, then there exist $a \in I$ such that $x'' \geq a'$. We observe that $x'' \leq y''$, we have $y'' \geq a'$. Therefore $y \in I^*$.

Let $x, y \in L$ such that $x, y \in I^*$, then there exist $a, b \in I$ such that $x'' \geq a'$ and $y'' \geq b'$. We observe that $x \rightarrow y' \leq y'' \rightarrow x' \leq b' \rightarrow x' \leq x'' \rightarrow b'' \leq a' \rightarrow b'' = a \oplus b''$, we have $(x \rightarrow y')' \geq (a \oplus b'')'$, that is, $(x \otimes y)'' \geq (a \oplus b'')'$. Since I is an ideal of $\mathcal{L}$ and $a, b \in I$, we have $a \oplus b \in I$. Since $a \oplus b'' \geq a \oplus b$, by I is an ideal, we have $a \oplus b'' \in I$, and so $x \otimes y \in I^*$. Therefore I^* is a filter of $\mathcal{L}$. □

Theorem 3.8 Let $I_i (i \in \Gamma)$ be ideals of $\mathcal{L}$. Then $\bigcap_{i \in \Gamma} I_i$ is an ideals of $\mathcal{L}$, where Γ is an index set.

Proof This proof is straightforward. □

Definition 3.3 Let A be nonempty set of a residuated lattice $\mathcal{L}$. The least ideal containing A is called the ideal generalized by A, written $\langle A \rangle$.

Theorem 3.9 Let A be nonempty set of a residuated lattice $\mathcal{L}$. Then

$$\langle A \rangle = \{a \in L | a \leq \big(\cdots((x_1 \oplus x_2) \oplus x_3) \cdots\big) \oplus x_n; \\ x_i \in A, i = 1, 2, \ldots, n\}.$$

Proof Let $U = \{a \in L | a \leq \big(\cdots((x_1 \oplus x_2) \oplus x_3) \cdots\big) \oplus x_n; x_i \in A, i = 1, 2, \ldots, n\}$. Obviously, $0 \in U$. Let $x' \otimes y \in U$ and $x \in U$, then there exists $a_i, b_j \in A (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ such that

$$x' \otimes y \leq \big(\cdots((a_1 \oplus a_2) \oplus a_3) \cdots\big) \oplus a_n,$$

$$x \leq \big(\cdots((b_1 \oplus b_2) \oplus b_3) \cdots\big) \oplus b_m.$$

we have

$$x' \geq \big((\cdots((b_1 \oplus b_2) \oplus b_3) \cdots) \oplus b_m\big)'.$$

and

$$\begin{aligned} y &\leq x' \rightarrow \big((\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_n\big) \\ &\leq \big((\cdots((b_1 \oplus b_2) \oplus b_3) \cdots) \oplus b_m\big)' \\ &\rightarrow \big((\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_n\big) \\ &= \big((\cdots((b_1 \oplus b_2) \oplus b_3) \cdots) \oplus b_m\big) \\ &\oplus \big((\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_n\big). \end{aligned}$$

Therefore $y \in U$ and so U is an ideal of $\mathcal{L}$ and $A \subseteq U$.

Let V be any ideal of $\mathcal{L}$ and $A \subseteq V$. For any $x \in U$, then there exists $a_i \in A (i = 1, 2, \ldots, n)$ such that

$$x \leq (\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_n,$$

then $x \otimes \big((\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_{n-1}\big)' \leq a_n$. Since A is an ideal of $\mathcal{L}$ and $a_n \in A$, therefore $((\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_{n-1})' \in A \subseteq V$, as $((a_1 \oplus$

$a_2) \oplus a_3) \cdots) \oplus a_{n-1}) \in A \subseteq V$ and V is an ideal, so $x \in V$. □

Corollary 3.2 For any element a of a residuated lattice $\mathcal{L}$, we have

$$\langle a \rangle = \{x \in L | x \leq (\cdots((\underbrace{a \oplus a) \oplus a) \cdots) \oplus a}_{n\text{terms}},$$

n is a natural number.$\}$.

Let I be an ideal of a residuated lattice $\mathcal{L}$. We define a binary relation $\sim$ on $\mathcal{L}$ as follows:

$x \sim_I y$ if and only if $(x \rightarrow y)' \in I$ and $(y \rightarrow x)' \in I$.

From the definition of the binary $'' \sim_I''$, we have the fact that $x \sim_I x''$ for any $x \in L$.

Lemma 3.1 $'' \sim_I''$ is an equivalence relation on $\mathcal{L}$.

Proof It is obvious that $\sim_I$ is reflexive and symmetric. Now, to prove the transitivity. Assume $x \sim_I y$ and $y \sim_I z$, then $(x \rightarrow y)' \in I, (y \rightarrow x)' \in I$ and $(y \rightarrow z)' \in I, (z \rightarrow y)' \in I$. Since $z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x) \leq (z \rightarrow x)' \rightarrow (y \rightarrow x)'$, we have $(z \rightarrow y)' \geq ((z \rightarrow x)' \rightarrow (y \rightarrow x)')' = (((y \rightarrow x)')' \rightarrow ((z \rightarrow x)')')'$. Since $(z \rightarrow y)' \in I$ and I is an ideal, we have $(((y \rightarrow x)')' \rightarrow ((z \rightarrow x)')')' \in I$. It follows from Theorem 3.2 that we have $(z \rightarrow x)' \in I$. Similarly, we have $(x \rightarrow z)' \in I$. Therefore $x \sim_I z$. This completes the proof. □

Theorem 3.10 $'' \sim_I''$ is an congruence relation on $\mathcal{L}$.

Proof Assume $x \sim_I y$, then $(x \rightarrow y)' \in I$ and $(y \rightarrow x)' \in I$. For any $z \in L$, since $(x \vee z) \rightarrow (y \vee z) = (x \rightarrow (y \vee z)) \wedge (z \rightarrow (y \vee z)) = (x \rightarrow (y \vee z)) \geq x \rightarrow y$, we have $((x \vee z) \rightarrow (y \vee z))' \leq (x \rightarrow y)'$. As I is an ideal of $\mathcal{L}$ and $(x \rightarrow y)' \in I$, so $((x \vee z) \rightarrow (y \vee z))' \in I$. Similarly, we have $((y \vee z) \rightarrow (x \vee z))' \in I$, therefore $x \vee z \sim y \vee z$.

Suppose $x \sim_I y$, then $(x \rightarrow y)' \in I$ and $(y \rightarrow x)' \in I$. For any $z \in L$, since $(x \wedge z) \rightarrow (y \wedge z) \geq x \rightarrow y$, we have $((x \wedge z) \rightarrow (y \wedge z))' \leq (x \rightarrow y)'$. As I is an ideal of $\mathcal{L}$ and $(x \rightarrow y)' \in I$, so $((x \wedge z) \rightarrow (y \wedge z))' \in I$. Similarly, we have $((y \wedge z) \rightarrow (x \wedge z))' \in I$, therefore $x \wedge z \sim y \wedge z$.

Assume $x \sim_I y$, then $(x \rightarrow y)' \in I$ and $(y \rightarrow x)' \in I$. For any $z \in L$, since $x \rightarrow y \leq x \otimes z \rightarrow y \otimes z$, it follows that $(x \rightarrow y)' \geq ((x \otimes z) \rightarrow (y \otimes z))'$, and so $((y \otimes z) \rightarrow (x \otimes z))' \in I$. Similarly, we have $((y \otimes z) \rightarrow (x \otimes z))' \in I$, hence $x \otimes z \sim_I y \otimes z$.

Assume $x \sim_I y$, then $(x \rightarrow y)' \in I$ and $(y \rightarrow x)' \in I$. For any $z \in L$, since $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, we have $(x \rightarrow y)' \geq ((y \rightarrow z) \rightarrow (x \rightarrow z))'$. We observe that I is an ideal and $(x \rightarrow y)' \in I$, hence $((y \rightarrow z) \rightarrow (x \rightarrow z))' \in I$, similarly, $((x \rightarrow z) \rightarrow (y \rightarrow z))' \in I$. Therefore $x \rightarrow z \sim_I y \rightarrow z$.

Therefore, $\sim_I$ is a congruence relation on a residuated lattice $\mathcal{L}$. □

Theorem 3.11 Let I be an ideal of a residuated lattice $\mathcal{L}$. Then $I = \{x \in L | x \sim_I 0\}$.

Proof Let $B = \{x \in L | x \sim_I 0\}$. Now we will prove B is an ideal of $\mathcal{L}$. Obviously, $0 \in B$. Let $x, y \in L$ such that $x \in B$ and $x' \otimes y \in B$, it follows that $x \sim_I 0$ and $x' \otimes y \sim_I 0$, then $x' = x \rightarrow 0 \sim_I 0 \rightarrow 0 = 1$ and $x' \otimes y \sim_I 1 \otimes y = y$. By the transitivity of $\sim_I$, we have $y \sim_I 0$, hence $y \in B$. Therefore B is an ideal of $\mathcal{L}$.

For any $x \in I$, we have $(x \rightarrow 0)' = x'' \in I$ and $(0 \rightarrow x)' = 0 \in I$, therefore $x \sim_I 0$, hence $x \in B$. Conversely, For any $x \in B$, we have $x \sim_I 0$, that is, $x \leq (x \rightarrow 0)' = x'' \in I$. Since I is an ideal of $\mathcal{L}$, we have $x \in I$. Consequently, $I = B$. □

Remark 3.2 In Theorem 3.11, the ideal $\{x \in L | x \sim_I 0\}$ denoted by $I_\sim$. This expression $(x, y) \in \sim_I$ means $x \sim_I y$.

Theorem 3.12 Let I be an ideal of $\mathcal{L}$ and $\sim$ be a congruence relation on $\mathcal{L}$. Then $\sim_{I_\sim} = * \sim$ and $I_{\sim_I} = I$.

Proof

(1) For any $(x, y) \in \sim_{I_\sim}$ if and only if $(x \rightarrow y)' \in I_\sim$ and $(y \rightarrow x)' \in I_\sim$ if and only if $((y \rightarrow x)', 0) \in \sim_I$ and $((x \rightarrow y)', 0) \in \sim_I$ if and only if $(y \rightarrow x)' \in I$ and $(x \rightarrow y)' \in I$ if and only if $(x, y) \in \sim_I$.

(2) $x \in I_{\sim_I}$ if and only if $(x, 0) \in \sim_I$ if and only if $x \leq (x \rightarrow 0)' = x'' \in I$ and $(0 \rightarrow x)' \in I$ if and only if $x \in I$. Hence $I_{\sim_I} = I$.

□

Remark 3.3 Theorem 3.12 shows that there is a bijection between the set of ideals and the set of congruence relations in a residuated lattice.

3.2 Fuzzy ideals on a residuated lattice

Let $[0, 1]$ be the closed unit interval of reals and $L \neq \emptyset$ be a set. Recall that a fuzzy set ([21]) in L is any function $\mu : L \rightarrow [0, 1]$.

If μ and v are fuzzy sets in L, define $\mu \le v$ iff $\mu(x) \le v(x)$ for all $x \in L$. Level set μ_t defined by $\mu_t = \{x \in L | \mu(x) \ge t\}$, where $t \in [0,1]$, the μ_t is also denoted by $U(\mu; t)$.

If $\Gamma \subseteq [0,1]$, put $\bigwedge \Gamma = inf\Gamma$ and $\bigvee \Gamma = sup\Gamma$; In particular, if $a, b \in [0,1]$, then $a \wedge b = min\{a, b\}$ and $a \vee b = max\{a, b\}$. Recall that $[0,1]$ is a complete Heyting algebra.

Definition 3.4 Let μ be a fuzzy subset of a residuated lattice $\mathcal{L}$. μ is called a fuzzy ideal of $\mathcal{L}$, if μ satisfies the following condition:

(FI1) for any $x, y \in L$, if $x \le y$, then $\mu(x) \ge \mu(y)$;
(FI2) for any $x, y \in L$, $\mu(x \oplus y) \ge min\{\mu(x), \mu(y)\}$.

Example 3.5 In Example 3.2, we define a fuzzy set μ on $\mathcal{L}$ as follows :

$$\mu(0) = 0.9, \mu(a) = \mu(b) = \mu(c) = 0.6, \mu(d) = \mu(e) = \mu(f) = \mu(1) = 0.2.$$

It is routine to verify μ is a fuzzy ideal of $\mathcal{L}$.

Corollary 3.3 Let μ be a fuzzy ideal of $\mathcal{L}$. The the following hold for any $x, y \in L$:

(1) $\mu(x \vee y) = \min\{\mu(\mathrm{x}), \mu(\mathrm{y})\}$,
(2) $\mu(x \wedge y) \ge \min\{\mu(\mathrm{x}), \mu(\mathrm{y})\}$,
(3) $\mu(x \otimes y) \ge \min\{\mu(\mathrm{x}), \mu(\mathrm{y})\}$,
(4) $\mu(x \oplus y) = \min\{\mu(\mathrm{x}), \mu(\mathrm{y})\}$.

Proof We observe that $x \otimes y \le x \wedge y \le x \vee y \le x \oplus y$ for any $x, y \in L$. We have $\mu(x \otimes y) \ge \mu(x \wedge y) \ge \mu(x \vee y) \ge \mu(x \oplus y) \ge min\{\mu(x), \mu(y)\}$. Since $x \oplus y \ge x \vee y \ge x, y$, it follows that $\mu(x \oplus y) \le \mu(x \vee y) \le \mu(x), \mu(y)$, and so $\mu(x \oplus y) \le \mu(x \vee y) \le min\{\mu(x), \mu(y)\}$. This completes the proof. □

Theorem 3.13 Let μ be a fuzzy subset of a residuated lattice $\mathcal{L}$. Then μ is a fuzzy ideal of $\mathcal{L}$ if and only if the level set $\mu_t (\ne \emptyset)$ is an ideal of $\mathcal{L}$.

Proof Let μ be a fuzzy ideal of $\mathcal{L}$ and $\mu_t \ne \emptyset$. Assume $x, y \in L$ such that $x \le y$ and $y \in \mu_t$, then $\mu(y) \ge t$. Since μ is a fuzzy ideal and $x \le y$, it follows that $\mu(x) \ge \mu(y) \ge t$, we have $x \in \mu_t$, and so (I1) holds. Let $x, y \in \mu_t$, we have $\mu(x) \ge t$ and $\mu(y) \ge t$, then $\mu(x \oplus y) \ge min\{\mu(x), \mu(y)\} \ge t$. And so $x \oplus y \in \mu_t$. Therefore μ_t is a ideal of $\mathcal{L}$.

Conversely, assume that μ_t is an ideal of $\mathcal{L}$. Let $x, y \in L$, taking $t = min\{\mu(x), \mu(y)\}$, we can obtain $x \in \mu_t$ and $y \in \mu_t$. By μ_t is an ideal, we have $x \oplus y \in \mu_t$, and so $\mu(x \oplus y) \ge min\{\mu(x), \mu(y)\} = t$. Let $x, y \in L$ such that $x \le y$. Taking $t = \mu(y)$, we have $y \in \mu_t$, It follows (I1) that $x \in \mu_t$, and so $\mu(x) \ge \mu(y)$. Therefore μ is a fuzzy ideal of $\mathcal{L}$. □

Theorem 3.14 Let μ be a fuzzy subset of a residuated lattice $\mathcal{L}$. μ is a fuzzy ideal of $\mathcal{L}$, if μ satisfies the following condition:

(FI3) for any $x \in L$, $\mu(0) \ge \mu(x)$;
(FI4) for any $x, y \in L$, $\mu(y) \ge min\{\mu(x), \mu(x' \otimes y)\}$.

Proof Let μ be a fuzzy ideal of $\mathcal{L}$. Since $0 \le x$ for any $x \in L$, it follows that $\mu(0) \ge \mu(x)$. So (FI3) holds. Since $x \oplus (x' \otimes y) = x' \to (x' \to y) \ge y$ and μ is a fuzzy ideal, we have $\mu(y) \ge \mu(x \oplus (x' \otimes y)) \ge min\{\mu(x), \mu(x' \otimes y)\}$. And so (FI4) holds

Conversely, assume that (FI3) and (FI4) hold. Let $x, y \in L$ such that $x \le y$, then $y' \le x'$ and $x \otimes y' \le x \otimes x' = 0$, and so $\mu(0) = \mu(y' \otimes x)$. By (FI4), we have $\mu(x) \ge min\{\mu(y), \mu(y' \otimes x)\} = min\{\mu(y), \mu(0)\} \ge \mu(y)$. And hence (FI1) holds. Let $x, y \in L$, since $x' \otimes (x \oplus y) = x' \otimes (x' \to y) \le y$, we have $\mu(x' \otimes (x \oplus y)) \ge \mu(y)$. And $\mu(x \oplus y) \ge min\{\mu(x), \mu(x' \otimes (x \oplus y))\} \ge min\{\mu(x), \mu(y)\}$. Therefore, μ is a fuzzy ideal of $\mathcal{L}$. □

Theorem 3.15 Let μ be a fuzzy subset of a residuated lattice $\mathcal{L}$. μ is a fuzzy ideal of $\mathcal{L}$, if μ satisfies the following condition:

(FI3) for any $x \in L$, $\mu(0) \ge \mu(x)$;
(FI5) for any $x, y \in L$, $\mu(y) \ge min\{\mu(x), \mu((x' \to y')')\}$.

Proof Let μ be a fuzzy ideal of $\mathcal{L}$. Since $0 \le x$ for any $x \in L$, it follows that $\mu(0) \ge \mu(x)$. So (FI3) holds. Since $x' \otimes y'' \le (x' \otimes y'')'' = (x' \to y')'$, and so $\mu((x' \to y')') \le \mu(x' \otimes y'')$. It follows that $\mu(y'') \ge \mu(x), \mu(x' \otimes y'') \ge min\{\mu(x), \mu(x' \otimes y)\} \ge min\{\mu(x), \mu((x' \to y')')\}$. Since $y'' \ge y$, we have $\mu(y) \ge \mu(y'')$. And so $\mu(y) \ge min\{\mu(x), \mu((x' \to y')')\}$, that is, (FI5) holds.

Conversely, assume that (FI3) and (FI5) hold. In (FI5), taking $y = x''$, we have $\mu(x'') \ge \mu(x)$. Let $x, y \in L$, since $x' \otimes y'' \le (x' \otimes y'')'' = (x' \to y')'$, we have $\mu(y) \ge min\{\mu(x), \mu((x' \to y')')\} \ge min\{\mu(x), \mu(x' \otimes y)\}$. Therefore, μ is a fuzzy ideal of $\mathcal{L}$. □

Corollary 3.4 Let μ be a fuzzy ideal of a residuated lattice $\mathcal{L}$. Then $\mu(x'') = \mu(x)$ for any $x \in L$.

Theorem 3.16 Let μ be a fuzzy subset of a residuated lattice $\mathcal{L}$. μ is a fuzzy ideal of $\mathcal{L}$, if μ satisfies the following condition:

(FI2) for any $x, y \in L$, $\mu(x \oplus y) \geq min\{\mu(x), \mu(y)\}$;
(FI6) for any $x, y \in L$, $\mu(x \wedge y) \geq \mu(x)$.

Proof Assume that μ is a fuzzy ideal and $x, y \in L$. Since $x \wedge y \leq x$, we have $\mu(x \wedge y) \geq \mu(x)\}$.

Conversely, suppose that μ satisfies (FI2) and (FI6). Let $x, y \in L$ such that $y \leq x$, then $x \wedge y = y$ and $\mu(y) = \mu(x \wedge y) \geq \mu(x)$. Hence μ is a fuzzy ideal of $\mathcal{L}$.

Let I be a nonempty subset of L and $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$. Now we define fuzzy set μ_I by

$$\mu_I(x) = \begin{cases} \alpha, & \text{if } \mathrm{x \in I}, \\ \beta, & otherwise. \end{cases}$$

Particularly, μ_I is χ_I on I at $\alpha = 1, \beta = 0$. □

Theorem 3.17 Let I be a non-empty subset of $\mathcal{L}$. Then μ_I is a fuzzy ideal of $\mathcal{L}$ if and only if I is an ideal of $\mathcal{L}$.

Proof Assume that μ_I is a fuzzy ideal of $\mathcal{L}$. For any $x, y \in L$, if $x, y \in I$, then $\mu_I(x) = \mu_I(y) = \alpha$. So $\mu_I(x \oplus y) \geq min\{\mu_I(x), \mu_I(y)\} = \alpha$, we have $x \oplus y \in I$.

Let $x, y \in I$ such that $x \leq y$ and $y \in I$, we have $\mu_I(x) \geq \mu_I(y)$ and $\mu_I(y) = \alpha$. And so $\mu_I(x) = \alpha$, that is, $x \in I$. Therefore I is an ideal of $\mathcal{L}$.

Conversely, Let F be an ideal of $\mathcal{L}$ and $x, y \in L$.

(*Case I*) If $x, y \in I$, then $x \oplus y \in I$. Thus $\mu_I(x \oplus y) = \alpha = \min\{\mu_\mathrm{I}(\mathrm{x}), \mu_\mathrm{I}(\mathrm{y})\}$.
(*Case II*) If $x \notin F$ or $y \notin F$. Then $\mu_I(x) = \beta$ or $\mu_I(y) = \beta$. Thus $\mu_I(x \oplus y) \geq \beta = \min\{\mu_\mathrm{I}(\mathrm{x}), \mu_\mathrm{I}(\mathrm{y})\}$.

From Case I to Case II, we arrive at $\mu_I(x \oplus y) \geq \min\{\mu_\mathrm{I}(\mathrm{x}), \mu_\mathrm{I}(\mathrm{y})\}$ for any $x, y \in L$.

Let $x, y \in L$ and $x \leq y$.

(*Case I*) If $y \in I$, then $x \in I$then $\mu_I(y) = \alpha = \mu_I(x)$.
(*Case II*) If $y \notin I$, then $\mu_I(y) = \beta$. Thus $\mu_I(x) \geq \mu_I(y) = \beta$.

Therefore, for any $x, y \in L$ and $x \leq y$, we have $\mu_I(x) \leq \mu_I(y)$. So μ_I is a fuzzy ideal by Definition 3.4. □

Theorem 3.18 Let μ be an fuzzy ideal of $\mathcal{L}$. Then the set

$$I_0 = \{x \in L | \mu(x) = \mu(0)\}$$

is an ideal of $\mathcal{L}$.

Proof Let $x, y \in I_0$, then $\mu(x) = \mu(y) = \mu(0)$, and so $\mu(x \oplus y) \geq min\{\mu(x), \mu(y)\} = \mu(0)$. Since $\mu(0) \geq \mu(x)$ for any $x \in L$, we have $\mu(0) \geq \mu(x \oplus y)$, then $\mu(0) = \mu(x \oplus y)$, that is, $x \oplus y \in I_0$.

Let $x, y \in L$ such that $x \leq y$ and $y \in I_0$. Then $\mu(x) \geq \mu(y) = \mu(0)$, hence $\mu(x) = \mu(0)$. We have $x \in I_0$. Consequently, I_0 is an ideal of $\mathcal{L}$. □

Theorem 3.19 Let μ be a fuzzy set of $\mathcal{L}$. Define a fuzzy set ν as follows:

$$\nu(x) = \bigvee\{min\{\mu(x_1), \mu(x_2), \ldots, \mu(x_n)\} \\ | x \leq (\cdots((x_1 \oplus x_2) \oplus x_3)\cdots) \oplus \\ x_n \text{ for some } x_1, x_2, \ldots, x_n \in L\}.$$

Then ν is the smallest fuzzy ideal of $\mathcal{L}$ that contains μ.

Proof Obviously, $\nu(0) \geq \nu(x)$ for any $x \in L$. Let $x, y \in L$ such that

$$x \leq (\cdots((b_1 \oplus b_2) \oplus b_3)\cdots) \oplus b_m$$

and

$$x' \otimes y \leq (\cdots((a_1 \oplus a_2) \oplus a_3)\cdots) \oplus a_n.$$

Then

$$y \leq x \oplus (x' \otimes y) \\ = ((\cdots((b_1 \oplus b_2) \oplus b_3)\cdots) \oplus b_m) \\ \oplus ((\cdots((a_1 \oplus a_2) \oplus a_3)\cdots) \oplus a_n).$$

and so $\nu(y) \geq min\{\mu(a_1), \mu(a_2), \ldots, \mu(a_n), \mu(b_1), \mu(b_2), \ldots, \mu(b_m)\}$.

Denote by $A = \{min\{\mu(b_1), \mu(b_2), \ldots, \mu(b_m)\} | x \leq (\cdots((b_1 \oplus b_2) \oplus b_3) \cdots) \oplus b_m$ for some $b_1, b_2, \cdots, b_m \in L\}$ and $B = \{min\{\mu(a_1), \mu(a_2), \ldots, \mu(a_n)\} | x' \otimes y \leq (\cdots((a_1 \oplus a_2) \oplus a_3)\cdots) \oplus a_n$ for some $a_1, a_2, \ldots, a_m \in L\}$.

We have $min\{\nu(x), \nu(x' \otimes y)\} = min\{\bigvee A, \bigvee B\} = \bigvee \{min\{\mu(a_1), \mu(a_2), \cdots, \mu(a_n), \mu(b_1), \mu(b_2), \cdots, \mu(b_m)\} | x' \otimes y \leq (\cdots((a_1 \oplus a_2) \oplus a_3) \cdots) \oplus a_n, x \leq (\cdots((b_1 \oplus b_2) \oplus b_3)\cdots) \oplus b_m$ for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in L\}$, and so

$$\nu(y) \geq min\{\nu(x), \nu(x' \otimes y)\}.$$

Hence ν is a fuzzy ideal of $\mathcal{L}$. Since $x \leq x \oplus x$ for any $x \in L$, we have $\nu(x) \geq min\{\mu(x), \mu(x)\} = \mu(x)$, that is, ν contains μ.

Let ω be a fuzzy ideal of $\mathcal{L}$ that contains μ. For any $x \in L$, $\nu(x) = \bigvee\{min\{\mu(x_1), \mu(x_2), \ldots, \mu(x_n)\} | x \leq (\cdots((x_1 \oplus x_2) \oplus x_3)\cdots) \oplus x_n$ for some $x_1, x_2, \cdots, x_n \in L\} \leq \bigvee\{min\{\omega(x_1), \omega(x_2), \cdots, \omega(x_n)\} | x \leq (\cdots((x_1 \oplus x_2) \oplus x_3)\cdots) \oplus x_n$ for some $x_1, x_2, \ldots, x_n \in L\} \leq \omega(x)$.

Therefore ν is the smallest fuzzy ideal of $\mathcal{L}$ that contains μ. □

Remark 3.4 The smallest fuzzy ideal containing μ is said to be generated by μ. It is also the intersection of all fuzzy ideals of $\mathcal{L}$ containing μ.

4 Fuzzy prime ideals

4.1 Prime ideals

In the paper [6, 11], various types of filters are defined and their properties are investigated. In particular, it is focused on three kinds of prime filters of residuated lattices, prime filters, prime filter of the second kind and prime filters of the third kind.

A prime filter (PF) of $\mathcal{L}$ is a filter F satisfying, for all $x, y \in L$: $x \to y \in F$ or $y \to x \in F$.

A prime filter of the second kind ($PF2$) is a filter F satisfying, for any $x, y \in L$, if $x \vee y \in F$, then $x \in F$ or $y \in F$.

A prime filter of the third kind ($PF3$) of a residuated lattice is a filter of $\mathcal{L}$ satisfying, for all $x, y \in L$, $(x \to y) \vee (y \to x) \in F$.

In [6], authors point out:

A filter is a prime filter iff $\mathcal{L}$ is linearly ordered. In this case, all filters are prime filters.

A filter is a prime filter of the second kind iff 1 is $\vee$-irreducible (i.e. if $1 = x \vee y$ for $x, y \in L$, then $x = 1$ or $y = 1$) in $\mathcal{L}$. Remark that in general, this does not imply that all filters are prime filters of the second kind.

A filter is a prime filter of the third kind iff $\mathcal{L}$ is an *MTL*-algebra. In this case, all filters are prime filters of the third kind.

The classes of such prime filters of a residuated lattice $\mathcal{L}$ are denoted by $PF(\mathcal{L})$, $PF_2(\mathcal{L})$ and $PF_3(\mathcal{L})$, respectively. It is proved in [6] that $PF(\mathcal{L}) \subseteq PF_2(\mathcal{L})$, $PF(\mathcal{L}) \subseteq PF_3(\mathcal{L})$ and that $PF(\mathcal{L}) = PF_2(\mathcal{L})$ implies the prelinearity of $\mathcal{L}$ if $\mathcal{L}$ is finite or 1 is $\vee$-irreducible. In the general case, it is left an open problem, that is, it is conjectured that if $PF(\mathcal{L}) = PF_2(\mathcal{L})$ then $\mathcal{L}$ is an MTL-algebra. In [11], Kondo and Turunen give an answer to the open problem, that is, For every residuated lattice $\mathcal{L}$, $PF(\mathcal{L}) = PF_2(\mathcal{L})$ then $\mathcal{L}$ is an MTL-algebra.

We give the notions of two types prime ideals of a residuated lattice $\mathcal{L}$, and the relations among them are given.

Definition 4.1 Let I be a proper ideal of a residuated lattice $\mathcal{L}$. I is said to be a prime ideal, if for any $x, y \in L$, $(x \to y)' \in I$ or $(y \to x)' \in I$.

Definition 4.2 Let I be a proper ideal of a residuated lattice $\mathcal{L}$. I is said to be a prime ideal of the second kind, if for any $x, y \in L$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

In a residuated lattice $\mathcal{L}$, denote $R_l = \{l \in L | l \oplus l = l\}$. Then $I_l = \{x \in L | x \leq l, l \in R_l\}$ is an ideal of $\mathcal{L}$. In fact, if $x, y \in L$ such that $x \leq y$ and $y \in I_l$, we have $x \leq y \leq l$, and so $x \in I_l$; At the meantime, if $x, y \in I_l$, then $x \leq l$ and $y \leq l$, and so $x \oplus y \leq l \oplus l = l$, therefore $x \oplus y \in I_l$.

Theorem 4.1 Let $\mathcal{L}$ be a residuated lattice. If $l \in R_l$ and l be $\wedge$-irreducible element of $\mathcal{L}$, then I_l is a prime ideal of the second kind of $\mathcal{L}$.

Proof Suppose $a \wedge b \in I_l$. Then $l \geq a \wedge b$ and therefore, $l = l \oplus l \geq l \oplus (a \wedge b) = l' \to (a \wedge b) = (l' \to a) \wedge (l' \to b) = (l \oplus a) \wedge (l \oplus b) \geq l \wedge l = l$. So $l = (l \oplus a) \wedge (l \oplus b)$, which implies $l = l \oplus a$ or $l = l \oplus b$. So $l \geq a$ or $l \geq b$, which means exactly that $a \in I_l$ or $b \in I_l$. □

Theorem 4.2 Let $\mathcal{L}$ be a residuated lattice. Every prime ideal of $\mathcal{L}$ is also a prime ideal of the second kind. If $\mathcal{L}$ is an *MTL*-algebra, then every prime ideal of the second kind of $\mathcal{L}$ is also a prime ideal.

Proof Suppose F is a prime ideal of the residuated lattice $\mathcal{L}$, and $a \wedge b \in I$. We know that $(a \to b)' \in I$ or $(b \to a)' \in I$. Without loss of generality, we assume $(a \to b)' \in I$. It follows that $(a \wedge b) \oplus (a \to b)' \in I$ because I is an ideal of $\mathcal{L}$. Since $(a \wedge b) \oplus (a \to b)' = (a \wedge b)' \to (a \to b)' = (a \to b) \to (a \wedge b)'' \geq (a \to b) \to (a \wedge b) = ((a \to b) \to a) \wedge ((a \to b) \to b) \geq a \wedge a = a$. This implies $a \in I$, because I is a prime ideal of the second kind of $\mathcal{L}$.

Now suppose I is a prime ideal of the second kind of the *MTL*-algebra $\mathcal{L}$, and $a, b \in L$. Because $(a \to b)' \wedge (b \to a)' = ((a \to b) \vee (b \to a))' = 1' = 0 \in I$, either $(a \to b)' \in I$ or $(b \to a)' \in I$. □

Remark 4.1 For a residuated lattice which satisfies the prelinearity, then the prime ideal of the second kind is a prime ideal. Such as, lattice implication algebras, MTL-algebras, BL-algebras, MV-algebras.

In residuated lattices that are not MTL-algebras, prime ideals of the second kind are in general not prime ideals. The counterexample as follows:

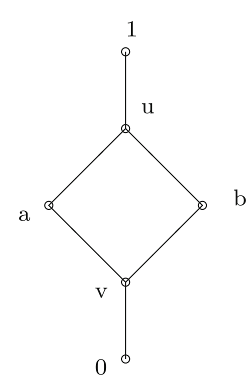

Fig. 3 Hasse diagram of L

Example 4.1 Three residuated lattices exist on the lattice in Fig. 3 (Example 7 in [6]). If we consider the Heyting-algebra, Also note that prelinearity does not hold: $(b \to a) \vee (a \to b) = b \vee a = u$. Remark that $\{0\}$ is a prime ideal of the second kind, but $\{0\}$ is not a prime ideal.

4.2 Fuzzy prime ideals

In this section, we mainly introduce two types fuzzy prime ideal, named fuzzy prime ideal and fuzzy prime ideal of the second kind, respectively. The relation between the fuzzy prime ideal and fuzzy prime ideal of the second kind are revealed. We mainly focus on the fuzzy prime ideal of the second kind. Its some important properties are investigated.

Definition 4.3 A fuzzy ideal μ of $\mathcal{L}$ is said to be fuzzy prime if it is non-constant and $\mu((x \to y)') = \mu(0)$ or $\mu((y \to x)') = \mu(0)$ for any $x, y \in L$.

Definition 4.4 A fuzzy ideal μ of $\mathcal{L}$ is said to be fuzzy prime of the second kind of $\mathcal{L}$ if it is non-constant and $\mu(x \wedge y) \leq max\{\mu(x), \mu(y)\}$ for any $x, y \in L$.

Lemma 4.1 Let μ be a fuzzy ideal of $\mathcal{L}$. Then μ is a constant fuzzy set if and only if $\mu(1) = \mu(0)$.

Proof Necessity is obvious and we need to prove the sufficiency:

Assume that μ satisfies $\mu(1) = \mu(0)$. Since μ is a fuzzy ideal, for any $x \in L$, $0 \leq x \leq 1$, it follows that $\mu(0) \geq \mu(x) \geq \mu(1)$. Hence $\mu(1) = \mu(0) = \mu(x)$ for any $x \in L$. Hence μ is constant. □

Example 4.2 In Example 3.2, we define a fuzzy set μ on $\mathcal{L}$ as follows :

$$\mu(0) = 0.9, \mu(d) = 0.6, \mu(a) = \mu(b) = \mu(c) = \mu(e) = \mu(f) = \mu(1) = 0.2.$$

It is routine to verify μ is both a fuzzy prime ideal and fuzzy prime ideal of the second kind of $\mathcal{L}$.

Remark 4.2 Let μ be a non constant fuzzy ideal of $\mathcal{L}$. Then μ is a fuzzy prime ideal of the second kind if and only if $max\{\mu(x), \mu(y)\} = \mu(x \wedge y)$.

Theorem 4.3 Let μ be a non constant fuzzy set of $\mathcal{L}$. Then μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$ if and only if μ_t is a prime ideal of the second kind of $\mathcal{L}$, where $\mu_t = \{x \in L | \mu(x) \geq t\}$ for any $t \in [0, 1]$.

Proof By Theorem 3.10, we have μ is a fuzzy ideal of $\mathcal{L}$ if and only is μ_t is a ideal of $\mathcal{L}$. Now, we need to prove μ is fuzzy prime if and only if μ_t if is prime.

Let μ_t is prime and $x, y \in L$. Setting $t = \mu(x \wedge y)$, we have $x \wedge y \in \mu_t$. It follows that $x \in \mu_t$ or $y \in \mu_t$. Then $\mu(x) \geq t$ or $\mu(y) \geq t$. Therefore, $max\{\mu(x), \mu(y)\} \geq t = \mu(x \wedge y)$.

Conversely, assume that μ is non constant fuzzy ideal and $\mu(x \wedge y) \leq max\{\mu(x), \mu(y)\}$ for any $x, y \in L$. The there exists $t \in [0, 1]$ such that μ_t is proper. Let $x \wedge y \in \mu_t$, then $\mu(x \wedge y) \geq t$, and so $max\{\mu(x), \mu(y)\} \geq \mu(x \wedge y) \geq t$. Hence $\mu(x) \geq t$ or $\mu(y) \geq t$, which implies $x \in \mu_t$ or $y \in \mu_t$. □

Corollary 4.1 Let I be a proper ideal of $\mathcal{L}$. Then I is a prime ideal of the second kind if and only if its characteristic function χ_I is a fuzzy prime ideal of the second kind of $\mathcal{L}$.

Corollary 4.2 Let μ be a non constant fuzzy ideal of $\mathcal{L}$. Then μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$ if and only if $\mu_{\mu(0)}$ is a prime ideal of the second kind of $\mathcal{L}$.

Theorem 4.4 Let μ be a fuzzy prime ideal of $\mathcal{L}$. Then μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$. If $\mathcal{L}$ is a *MTL*-algebras, then every fuzzy prime ideal of the second kind of $\mathcal{L}$ is also a fuzzy prime ideal.

Proof The proof is straightforward from Theorems 4.2 and 4.3. □

Remark 4.3 For a residuated lattice which satisfies the prelinearity, then the fuzzy prime ideal of the second kind is the fuzzy prime ideal. Such as, lattice implication algebras, MTL-algebras, BL-algebras. In this section. We mainly focus on the fuzzy prime ideal of the second kind of $\mathcal{L}$.

Theorem 4.5 Let I be an ideal of $\mathcal{L}$ and μ be a fuzzy set in $\mathcal{L}$. Then I is a prime ideal of the second ideal of $\mathcal{L}$ if and only if μ_I is a fuzzy prime ideal of the second kind of $\mathcal{L}$.

Proof Assume that I is a prime ideal of the second kind of $\mathcal{L}$, we have μ_I is nonconstant. Let $x, y \in L$, if $x \wedge y \in I$, it follows that $x \in I$ or $y \in I$, hence $\mu_I(x \wedge y) = \alpha = max\{\mu_I(x), \mu_I(y)\}$. If $x \wedge y \notin I$, then $x \notin I$ and $y \notin I$ (in fact, if $x \in I$ or $y \in I$, since $x \wedge y \leq x, y$ and I is an ideal, we have $x \wedge y \in I$, contradiction). Hence $\mu_I(x \wedge y) = \beta = max\{\mu_I(x), \mu_I(y)\}$. Therefore, μ_I is a fuzzy prime ideal of the second kind of $\mathcal{L}$.

Conversely, assume that μ_I is a fuzzy ideal of $\mathcal{L}$. Then $\mu_I(x \wedge y) = max\{\mu_I(x), \mu_I(y)\}$. Let $x \wedge y \in I$, then $\mu_I(x \wedge y) = \alpha$, we have $\mu_I(x) = \alpha$ or $\mu_I(y) = \alpha$. That is, when $x \wedge y \in I$, we have $x \in I$ or $y \in I$. Therefore I is a prime ideal of the second kind of $\mathcal{L}$. □

Theorem 4.6 Let μ be a fuzzy ideal of $\mathcal{L}$. Then the following conditions are equivalent:

(1) μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$;

(2) for any $x, y \in L$, $\mu(x \wedge y) = \mu(0)$ implies $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$.

Proof Assume that μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$. Let $x, y \in L$ such that $\mu(x \wedge y) = \mu(0)$. We have $max\{\mu(x), \mu(y)\} = \mu(0)$. Hence $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$. Hence (2) holds.

In order to prove μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$, we only need to prove that $\mu_t (\neq \emptyset)$ is a prime ideal of the second kind of $\mathcal{L}$ for any $t \in [0, 1]$. Let $x, y \in L$ such that $x \wedge y \in \mu_t$, then $\mu(x \wedge y) \geq t$ for any $t \in [0, 1]$. Taking $t_0 = \mu(0)$, we have $\mu(x \wedge y) = \mu(0)$, it follows from (2) that $\mu(x) = \mu(0) \geq t$ or $\mu(y) = \mu(0) \geq t$. That is, $x \in \mu_t$ or $y \in \mu_t$. Hence μ_t is a prime ideal, it follows from Theorem 4.1 that μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$. □

Theorem 4.7 Let μ be a fuzzy ideal of $\mathcal{L}$ and $\mu(0) = 1$. Then μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$ if and only if

$$I_0 = \{x \in L | \mu(x) = \mu(0)\}$$

is a prime ideal of the second kind of $\mathcal{L}$.

Proof Assume that μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$. Then I_0 is an ideal of $\mathcal{L}$ by Theorem 3.14. Since μ is a nonconstant, I_0 is proper. Let $x, y \in L$ such that $x \wedge y \in I_0$, then $\mu(x \wedge y) = \mu(0)$. Since μ is a fuzzy prime ideal of the second kind, it follows from Theorem 4.2 that $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$, that is, $x \in I_0$ or $y \in I_0$. Therefore I_0 is a prime ideal of the second kind of $\mathcal{L}$.

Conversely, assume that I_0 is a prime ideal of the second kind of $\mathcal{L}$. Let $t \in [0, 1]$ such that μ_t is nonempty, we have $I_0 \subseteq \mu_t$. Let $x \wedge y \in \mu_t$, we have $\mu(x \wedge y) \geq t$. Taking $t = \mu(0)$, by μ is a fuzzy ideal, we have $\mu(x \wedge y) = \mu(0)$. Hence $x \wedge y \in I_0$. Since I_0 is a prime ideal of the second kind of $\mathcal{L}$, it follows that $x \in I_0 \subseteq \mu_t$ or $y \in I_0 \subseteq \mu_t$. And so μ_t is a prime ideal of the second kind of $\mathcal{L}$. It follows from Theorem 4.1 that μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$. □

Theorem 4.8 Let μ be a fuzzy set of $\mathcal{L}$. Define a mapping $\mu^* : L \to \mathbf{R}$ as

$$\mu^*(x) = \mu(x) + 1 - \mu(0),$$

for any $x \in L$. Then μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$ if and only if μ^* is a fuzzy prime ideal of the second kind of $\mathcal{L}$.

Proof Suppose μ is a fuzzy prime ideal of the second kind of $\mathcal{L}$, then $\mu(x) \leq \mu(0)$ for any $x \in L$. Then μ^* is a fuzzy set of $\mathcal{L}$. Furthermore, for any $x, y \in L$,

$$\mu^*(0) = \mu(0) + 1 - \mu(0) = 1 \geq \mu^*(x)$$

and $\min\{\mu^*(\mathrm{x}), \mu^*(\mathrm{x}' \otimes \mathrm{y})\} = \min\{\mu(\mathrm{x}) + 1 - \mu(0), \mu(\mathrm{x}' \otimes \mathrm{y}) + 1 - \mu(0)\} = \min\{\mu(\mathrm{x}), \mu(\mathrm{x}' \otimes \mathrm{y})\} + 1 - \mu(0) \leq \mu(\mathrm{y}) + 1 - \mu(0) = \mu^*(\mathrm{y})$. Therefore, μ^* is a fuzzy ideal of $\mathcal{L}$. Now, we prove μ^* is prime of the second kind.

Since μ is prime of the second kind, it follows that

$$\mu(x \wedge y) = max\{\mu(x), \mu(y)\}$$

and

$$\mu(x \wedge y) + 1 - \mu(0) = max\{\mu(x), \mu(y))\} + 1 - \mu(0)$$

which implies $\mu(x \wedge y) + 1 - \mu(0) = max\{(\mu(x) + 1 - \mu(0)), (\mu(y) + 1 - \mu(0))\}$. Hence $\mu^*(x \wedge y) = max\{\mu^*(x), \mu^*(y)\}$ for any $x, y \in L$, and so μ^* is a fuzzy prime ideal of the second kind of $\mathcal{L}$.

Conversely, suppose μ^* is a fuzzy prime ideal of the second kind, then $\mu^*(x) \leq \mu^*(0)$, that is, $\mu(x) + 1 - \mu(0I) \leq \mu(0) + 1 - \mu(0)$, it follows that $\mu(x) \leq \mu(0)$.

Since $\min\{\mu^*(\mathrm{x}), \mu^*(\mathrm{x}' \otimes \mathrm{y})\} \leq \mu^*(\mathrm{y})$, so $\min\{\mu(\mathrm{x}), \mu(\mathrm{x}' \otimes \mathrm{y})\} \leq \mu(\mathrm{y})$.

As μ^* is prime of the second kind, it follows that $\mu^*(x \wedge y) = max\{\mu^*(x), \mu^*(y)\}$, we have $\mu(x' \otimes y) = max\{\mu(x), \mu(y)\}$. Therefore, μ is a fuzzy prime ideal of the second kind $\mathcal{L}$. □

Theorem 4.9 Let ν be a fuzzy prime ideal of the second kind of a residuated lattice $\mathcal{L}$ and $\alpha \in [0, \nu(0))$. Then $(\nu \vee \alpha)(x) = \nu(x) \vee \alpha$ is also a fuzzy prime ideal of the second kind of $\mathcal{L}$.

Proof Let ν be a fuzzy prime ideal of the second kind and $\alpha \in [0, \nu(1))$. Assume that there exist $x, y \in L$ such that $x \leq y$. Since ν is a fuzzy ideal, we have $\nu(x) \geq \nu(y)$, and so $\nu(x) \vee \alpha \geq \nu(y) \vee \alpha$, that is, $(\nu \vee \alpha)(x) \geq (\nu \vee \alpha)(y)$. Let $x, y \in L$, since ν is a fuzzy ideal, we have $\nu(x \oplus y) \min\{\nu(x), \nu(y)\}$. And $(\nu \vee \alpha)(x \oplus y) = \nu(x \oplus y) \vee \alpha \geq min\{\nu(x), \nu(y)\} \vee \alpha = min\{\nu(x) \vee \alpha, \nu(y) \vee \alpha\} = min\{(\nu \vee \alpha)(x), (\nu \vee \alpha)(y)\}$. Therefore $\nu \vee \alpha$ is a fuzzy ideal of $\mathcal{L}$. Since ν is nonconstant and $\alpha \leq \nu(0)$, we have $(\nu \vee \alpha)(0) = \nu(0) \vee \alpha = \nu(0) \neq \nu(1) \vee \alpha$. Hence $\nu \vee \alpha$ is nonconstant.

Since ν is fuzzy prime of the second kind, we have $\nu(x \wedge y) = max\{\nu(x), \nu(y)\}$ for any $x, y \in L$. Hence $(\nu \vee \alpha)(x \wedge y) = \nu(x \wedge y) \vee \alpha = max\{\nu(x), \nu(y)\} \vee \alpha = max\{\nu(x) \vee \alpha, \nu(y) \vee \alpha\} = max\{(\nu \vee \alpha)(x), (\nu \vee \alpha)(y)\}$. Therefore $\nu \vee \alpha$ is also a fuzzy prime ideal of the second kind of $\mathcal{L}$. □

5 Fuzzy congruence relation

Definition 5.1 Let θ be a fuzzy relation on a residuated lattice $\mathcal{L}$. θ is called an fuzzy congruence relation on $\mathcal{L}$, if it satisfies, for any $x, y, z \in L$:

(IFC1) $\theta(x,x) = Sup_{(y,z)\in L\times L}\theta(y,z)$;
(IFC2) $\theta(x,y) = \theta(y,x)$;
(IFC3) $\theta(x,y) \geq min\{\theta(x,y), \theta(y,z)\}$;
(IFC4) $\theta(x,y) \leq \theta(x\otimes z, y\otimes z)$;
(IFC5) $\theta(x,y) \leq min\{\theta(x\to z, y\to z), \theta(z\to x, z\to y)\}$.

Let θ be a fuzzy relation on $\mathcal{L}$, (IFC5) is equivalent with the following conditions hold: $\theta(x,y) \leq \theta(x\to z, y\to z)$ and $\theta(x,y) \leq \theta(z\to x, z\to y)$.

For a fuzzy congruence relation θ, the fuzzy subset $\theta^x : L \to [0,1]$, which is defined by $\theta^x(y) = \theta(x,y)$, is called the fuzzy congruence class containing x. Let $\mathcal{L}/\theta$ be the set of all fuzzy congruence classes θ^x, where $x \in L$.

Theorem 5.1 For any fuzzy congruence relation θ in $\mathcal{L}$. Then θ^0 is an fuzzy ideal of $\mathcal{L}$.

Proof Let $x \in L$, then $\theta^0(0) = \theta(0,0) = \theta(1,1) \geq \theta(0,x) = \theta^0(x)$. Let $x, y \in L$, by transitivity, we have $\theta^0(y) = \theta(0,y) \geq min\{\theta(0, x'\otimes y), \theta(x'\otimes y, y\}$. Since θ be a fuzzy congruence relation on $\mathcal{L}$, it follows that $\theta(x'\otimes y, y) = \theta(x'\otimes y, 1\otimes y) \geq \theta(x', 1) = \theta(x\to 0, x\to x) \geq \theta(0,x) = \theta^0(x)$. Therefore, $\theta^0(y) \geq min\{\theta^0(x'\otimes y), \theta^0(x)\}$. It follows from Theorem 3.14 that θ^0 is a fuzzy ideal of $\mathcal{L}$. □

Theorem 5.2 Let μ be a fuzzy ideal of $\mathcal{L}$ and a fuzzy relation θ on $\mathcal{L}$ by $\theta(x,y) = min\{\mu((x\to y)'), \mu((y\to x)')\}$. Then θ is a fuzzy congruence on $\mathcal{L}$.

Proof Let μ be a fuzzy ideal of $\mathcal{L}$, we have $\mu(0) \geq \mu(x)$ for any $x \in L$. Then $\theta(x,x) = min\{\mu((x\to x)'), \mu(((x\to x)')\} = \mu(0) \geq min\{\mu((x\to y)'), \mu((y\to x)')\} = \theta(x,y)$. Thus (IFC1) is valid. Obviously, (IFC2) is valid. Next, we prove (IFC3) holds. We observe $z\to y \leq (x\to z)\to(x\to y) \leq (x\to z)''\to(x\to y)''$, and $(z\to y)' \geq ((x\to z)''\to(x\to y)'')'$. By μ is a fuzzy ideal, we have $\mu((z\to y)') \leq \mu(((x\to z)''\to(x\to y)'')')$ and $\mu((x\to y)') \geq min\{\mu((x\to z)'), \mu(((x\to z)''\to(x\to y)'')')\} \geq min\{\mu((x\to z)'), \mu((z\to y)')\}$. Similarly, we can prove $\mu((y\to x)') \geq min\{\mu((z\to x)'), \mu((y\to z)')\}$. Therefore $min\{\theta(x,z), \theta(z,y)\} = min\{min\{\mu((x\to z)'), \mu((z\to x)')\}, min\{\mu((z\to y)'), \mu((y\to z)')\}\} = min\{min\{\mu((x\to z)'), \mu((z\to y)')\}, min\{\mu((z\to x)'), \mu((y\to z)')\}\} \leq min\{\mu((x\to y)'), \mu((y\to x)')\} = \theta(x,y)$, So (IFC3) is valid.

Since $(x\otimes z)\to(y\otimes z) \geq x\to y$ and $((x\otimes z)\to(y\otimes z))' \leq (x\to y)'$, we have $\mu((x\to y)') \leq \mu(((x\otimes z)\to(y\otimes z))')$. Similarly, $\mu((y\to x)') \leq \mu(((y\otimes z)\to(x\otimes z))')$. Therefore $\theta(x\otimes z, y\otimes z) = min\{\mu(((x\otimes z)\to(y\otimes z))'), \mu(((y\otimes z)'\to(x\otimes z))')\} \geq min\{\mu((x\to y)'), \mu((y\to x)')\} = \theta(x,y)$. Then (IFC4) is valid.

We observe $(x\to z)\to(y\to z) \geq x\to y$, it follows that $((x\to z)\to(y\to z))' \leq (x\to y)'$, and so $\mu(((x\to z)\to(y\to z))') \geq \mu((x\to y)')$. Similarly, $\mu(((z\to x)\to(z\to y))') \geq \mu((y\to x)')$. It follows that $\theta(x\to z, y\to z) = min\{\mu(((x\to z)\to(y\to z))'), \mu(((y\to z)\to(x\to z))')\} \geq min\{\mu((x\to y)'), \mu((y\to x)')\} = \theta(x,y)$. Therefore, ρ is a fuzzy congruence relation on $\mathcal{L}$.

□

Remark 5.1 The fuzzy congruence relation θ in Theorem 5.2 is called an fuzzy congruence relation induced by fuzzy ideal μ and denoted by θ_μ.

Theorem 5.3 $\theta^x_\mu = \theta^y_\mu$ if and only if $\mu((x\to y)') = \mu((y\to x)') = \mu(0)$.

Proof Let $\theta^x_\mu = \theta^y_\mu$ for any $x, y \in L$, then $\theta^x_\mu(x) = \theta^y_\mu(x)$. Since $\theta^x_\mu(x) = \mu((x\to x)') = \theta^x_\mu(y) = min\{\mu((x\to y)'), \mu((y\to x)')\}$, that is, $\mu(0) = min\{\mu((x\to y)'), \mu((y\to x)')\}$. It follows that $\mu(0) \leq \mu((x\to y)')$ and $\mu(0) \leq \mu((y\to x)')$. Since μ is a fuzzy ideal, it follows that $\mu((x\to y)') = \mu((y\to x)') = \mu(0)$.

Conversely, for any $z \in L$, $\theta^x_\mu(z) = min\{(\mu((x\to z)'), \mu((z\to x)'))$. Since $((z\to y)''\to(z\to z)'')' \leq (y\to x)'$ and μ is a fuzzy ideal of $\mathcal{L}$, we have $\mu((z\to x)') \geq min\{\mu((z\to y)'), \mu(((z\to y)''\to(z\to z)'')') \geq min\{\mu((z\to y)'), \mu\ ((y\to x)')\}$. Similarly, we have $\mu((x\to z)') \geq min(\mu((x\to y)'), \mu((y\to z)'))$. Since $\mu((x\to y)') = \mu((y\to x)') = \mu(0)$, we have $\mu((z\to x)') \geq min\{\mu((z\to y)'), \mu((y\to x)')\} = min\{\mu((z\to y)'), \mu(0)\} \geq \mu((z\to y)')$ and $\mu((z\to y)') \geq min\{\mu((z\to x)'), \mu(0)\} \geq \mu((z\to x)')$. And so $\mu((z\to x)') = \mu((z\to y)')$. Similarly, we can prove $\mu((x\to z)') = \mu((y\to z)')$. Consequently, we have

$\theta^x_\mu(z) = min\{\mu((x \to z)'), \mu((z \to x)') = min\{\mu((z \to y)'), \mu((y \to z)')) = \theta^y_\mu(z)$. Hence $\theta^x_\mu = \theta^y_\mu$. □

Corollary 5.1 If μ is a fuzzy ideal of a residuated lattice $\mathcal{L}$, then $\mu^x = \mu^y$ if and only if $x \sim_{\mu_{\mu(0)}} y$, where $x \sim_{\mu_{\mu(0)}} y$ if and only if $(x \to y)' \in \mu_{\mu(0)}$ and $(y \to x)' \in \mu_{\mu(0)}$.

Theorem 5.4 Let θ, μ be a fuzzy congruence and a fuzzy ideal of $\mathcal{L}$, respectively. Then

(1) $\theta_{\mu_\theta} = \theta$;
(2) $\mu_{\theta_\mu} = \mu$.

Thus there is a bijection between the set $FI(\mathcal{L})$ and $FC(\mathcal{L})$.

Corollary 5.2 Let θ be a fuzzy ideal of $\mathcal{L}$. Then $min\{\theta(0,(x \to y)'), \theta(0,(y \to x)')\} = \theta(x,y)$ for any $x, y \in L$.

Theorem 5.5 Let μ be a fuzzy set of $\mathcal{L}$. Then μ is a fuzzy ideal of $\mathcal{L}$ if and only if $U(\mu;\mu(0))$ is an ideal of $\mathcal{L}$.

Proof The proof is straightforward from Theorem 3.5. □

Let μ be a fuzzy ideal of $\mathcal{L}$. For any $x \in L$, the fuzzy subset μ_x is called a fuzzy coset of μ such that $\mu_x(y) = min\{\mu((x \to y)'), \mu((y \to x)')\}$. Let $\mathcal{L}/\mu = \{\mu_x | x \in L\}$. Obviously, $\mu_x = \theta^x_\mu$ and so $\mathcal{L}/\mu = \mathcal{L}/\theta_\mu$.

Defining the binary operations on $\mathcal{L}/\mu$ as follows:

$$\mu_x \sqcup \mu_y = \mu_{x\vee y}, \mu_x \sqcap \mu_y = \mu_{x\wedge y}, \mu_x \odot \mu_y = \mu_{x\otimes y}, \mu_x \to \mu_y = \mu_{x\to y}.$$

a partial ordering $'\preceq'$ on $\mathcal{L}/\mu$ defined by $\mu_x \preceq \mu_y$ if and only if $\mu_x \sqcup \mu_y = \mu_y$.

Theorem 5.6 Let μ be a fuzzy idealof $\mathcal{L}$. Then

$$(\mathcal{L}/\mu, \sqcup, \sqcap, \odot, \to, \mu_0, \mu_1)$$

is a residuated lattice.

Proof First, we prove that the operators on $\mathcal{L}/\mu$ are well defined. Indeed, if $\mu_x = \mu_s$, $\mu_y = \mu_t$. By corollary 4.1, we have $x \sim_{U(\mu,\mu(0))} s$, $y \sim_{U(\mu,\mu(0))} t$. Since $\sim_{U(\mu,\mu(0))}$ is a congruence relation on $\mathcal{L}$, we have $x \vee y \sim_{U(\mu,\mu(0))} s \vee t$, and so $\mu_{x\vee y} = \mu_{s\vee t}$. Similarly, we can prove $\mu_{x\wedge y} = \mu_{s\wedge t}$, $\mu_{x\otimes y} = \mu_{s\otimes t}$, $\mu_{x\to y} = \mu_{s\to t}$, respectively. Now, we prove that $\mathcal{L}/\mu$ is a residuated lattice. Clearly, $\mathcal{L}/\mu$ satisfies (C1) and (C2). We only to prove $(\odot, \to)$ is an adjoin pair. We note that the lattice order $\preceq$ on $\mathcal{L}/\mu$ is $\mu_x \preceq \mu_y$ if and only if $\mu_x \sqcup \mu_y = \mu_y$. $\mu_{x\vee y} = \mu_y \Leftrightarrow (x \vee y) \sim_{U(\mu,\mu(0))} y \Leftrightarrow \mu(((x \vee y) \to y)') = \mu(0) \Leftrightarrow \mu(((x \to y) \wedge (y \to y))') = \mu(0) \Leftrightarrow \mu((x \to y)') = \mu(0)$. Let $\mu_x, \mu_y, \mu_z \in \mathcal{L}/\mu$, $\mu_x \odot \mu_y \preceq \mu_z \Leftrightarrow \mu_{x\otimes y} \preceq \mu_z \Leftrightarrow \mu(((x \otimes y) \to z)') = \mu(0) \Leftrightarrow \mu((x \to (y \to z))') = \mu(0) \Leftrightarrow \mu_x \leq \mu_{y\to z} \Leftrightarrow \mu_x \leq \mu_y \to \mu_z$. It is easy to verify $\odot$ is isotone on $\mathcal{L}/\mu$, $\to$ is anti tone in the first and isotone in the second variable Therefore, (C3) holds. Consequently, $\mathcal{L}/\mu$ is a residuated lattice. □

Theorem 5.7 (Homomorphism Theorem) Let μ be a fuzzy ideal in $\mathcal{L}$. Define a mapping $\phi : \mathcal{L} \to \mathcal{L}/\mu$ by $\phi(x) = \mu_x$. Then $ker(\phi) = U(\mu, \mu(0))$ and $\mathcal{L}/\mu \cong \mathcal{L}/ker(\phi)$.

Proof For any $x \in ker(\phi)$ if and only if $\phi(x) = \mu_0$ if and only if $\mu_x = \mu_0$ if and only if $x \sim_{U(\mu,\mu(0))} 0$ if and only if $x \in U(\mu, \mu(0))$. Therefore, $ker(\phi) = U(\mu, \mu(0))$.

Clearly, ϕ is surjective. It is easy to verify that ϕ is a surjective homomorphism. And so $\mathcal{L}/\mu \cong \mathcal{L}/ker(\phi)$. □

6 Conclusions

Ideal theory and congruence theory play an very important role in studying logical systems and the related algebraic structures. In this paper, we develop the ideals theory of general residuated lattices which enables us to analyze some important algebraic properties of residuated lattices, especially MTL-algebras.

In our future work, we will continue investigating the relation among the prime ideal, prime ideal of the second kind and MTL-prime ideal(i.e., A MTL-prime ideal of a residuated lattice is an ideal of $\mathcal{L}$ satisfying, for all $x, y \in L$, $(x \to y)' \wedge (y \to x)' \in I$). Another direction is to investigate some types ideals of a residuated lattice. For more details, we shall give them out in the future paper. It is our hope that this work will settle once and for all the existence of ideals in residuated lattices.

Acknowledgments We would like to thank the anonymous reviewers' comments and suggestions improved both content and the presentation of this paper. One of reviewer point out many typing mistakes and grammar mistakes in the manuscripts, we gave him (her) heartfelt thanks. This work was supported by National Natural Science Foundation of P.R.China (Grant No. 61175055, 61305074), Sichuan Key Technology Research and Development Program (Grant No.2011FZ0051), Sichuan Province Science and technology plan project (No. 15JC0239), Radio Administration Bureau of MIIT of China (Grant No. [2011]146), China Institution of Communications (Grant No. [2011]051). The Speciality Comprehensive Reform of Mathematics and Applied Mathematics of Ministry of Education (ZG0464). The Speciality Comprehensive Reform of Mathematics and Applied Mathematics of Ministry of Education (01249). A Project Supported by Scientific Research Fund of Sichuan Provincial Education Department (14ZA0245). The Scientific Research Research Fund of Neijiang Normal University (No. 13ZB05).

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