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ORIGINAL ARTICLE

Interval-valued intuitionistic (*T*, *S*)-fuzzy filters theory on residuated lattices

Yi Liu · Xiaoyan Qin · Yang Xu

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Abstract The aim of this paper is further to develop the filter theory on residuated lattices. Firstly, the notion of interval valued intuitionistic (*T*, *S*)-fuzzy filter(IVI (*T*, *S*)-fuzzy filter for short) on residuated lattices is introduced by linking the interval valued intuitionistic fuzzy set, t-norm, s-norm and filter theory of residuated lattices; the properties and equivalent characterizations of interval valued intuitionistic (*T*, *S*)-fuzzy filter are investigated; the relation between IVI (*T*, *S*)-fuzzy filter and filter is studied. Secondly, the notions of interval valued intuitionistic (*T*, *S*)-fuzzy implicative filter and interval valued intuitionistic (*T*, *S*)-fuzzy Boolean filter are introduced; the properties and equivalent characterizations of them are investigated; the intuitionistic (*T*, *S*)-fuzzy implicative filter is proved to be equivalent to the intuitionistic (*T*, *S*)-fuzzy Boolean filter in residuated lattices. Finally, the intuitionistic (*T*, *S*)-fuzzy positive implicative filter and intuitionistic (*T*, *S*)-fuzzy *G* (MV) filter are introduced; some equivalent characterizations of them are obtained and the relations among these fuzzy filters are investigated.

Keywords Residuated lattices · Interval valued intuitionistic fuzzy set · t-norm · s-norm(t-conorm) · Interval valued intuitionistic (*T*, *S*)-fuzzy (implication, positive implicative, Boolean, *G*, *MV*) filters

1 Introduction

As is known to all, one significant function of artificial intelligence is to make computer simulate human being in dealing with uncertain information. And logic establishes the foundations for it. However, certain information process is based on the classic logic. Non-classical logics [4, 12] consist of these logics handling a wide variety of uncertainties (such as fuzziness, randomness, and so on) and fuzzy reasoning. Therefore, non-classical logic has been proved to be a formal and useful technique for computer science to deal with fuzzy and uncertain information. Many-valued logic, as the extension and development of classical logic, has always been a crucial direction in non-classical logic. Lattice-valued logic, an important many-valued logic, has two prominent roles: One is to extend the chain-type truth-valued field of the current logics to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by the general lattice can more effectively reflect the uncertainty of human being's thinking, judging and decision. Hence, lattice-valued logic has been becoming a research field and strongly influencing the development of algebraic logic, computer science and artificial intelligent technology. Various logical algebras have been proposed as the semantical systems of non-classical logic systems, such as residuated lattices [14], lattice implication algebras [16], BL-algebras,

Y. Liu (✉)
College of Mathematics and Information Sciences, Neijiang Normal University, Neijiang 641000, Sichuan, People's Republic of China
e-mail: liuyiyl@126.com

Y. Liu · Y. Xu
Intelligent Control Development Center, Southwest Jiaotong University, Chengdu 610031, Sichuan, People's Republic of China

X. Qin
College of Mathematics and Computer Sciences, Shanxi Norm University, Linfen 041000, Shanxi, People's Republic of China

MV-algebras, MTL-algebras, etc. Among these logical algebras, residuated lattices are very basic and important algebraic structure because the other logical algebras are all particular cases of residuated lattices.

The concept of fuzzy set was introduced by Zadeh [21]. Since then this idea has been applied to other algebraic structures such as groups, semigroups, rings, modules, vector spaces, topologies and filter theory of some logical algebraic structure [8–11, 16–19, 22, 24]. With the development of fuzzy set, it is widely used in many fields. The concept of intuitionistic fuzzy sets was first introduced by Atanassov [2] in 1986 which is a generalization of the fuzzy sets. Many authors applied the concept of intuitionistic fuzzy sets to other algebraic structure such as groups, fuzzy ideals of BCK-algebras, filter theory of lattice implication and BL-algebras,etc [1, 7, 13, 20, 23, 25]. A generalization of the notion of intuitionistic fuzzy set is given in the spirit of ordinary interval valued fuzzy sets. The new notion called interval valued intuitionistic fuzzy set (IVIFS) was introduced by Atanassov [3] in 1989.

As for lattice implication algebras, BL-algebras, R_0-algebras, MTL-algebras, MV-algebras, etc, they all are particular types of residuated lattices. Therefore, it is meaningful to establish the fuzzy filter theory of general residuated lattice for studying the common properties of the above-mentioned logical algebras. This paper, as a continuation of above work, we will apply the interval-valued intuitionistic fuzzy subset and t-norm T, s-norm S on $D[0,1]$ to filter theory of residuated lattices, proposed the concept interval-valued intuitionistic (T, S)-fuzzy (implication, positive implication, Boolean, G, MV)- filters of residuated lattices and some equivalent results are obtained. Meanwhile, The relations among these fuzzy filters are investigated in some special residuated lattices, such as BL-algebras and lattice implication algebras. We desperately hope that our work would serve as a foundation for enriching corresponding many-valued logical system.

The paper is organized as follows. In Sect. 2, we list some important concepts and results in residuated lattices, it is useful in other sections. In Sect. 3, we introduced the concept of IVI (T, S)-fuzzy filters, some important properties of it is also obtained; the relations between IVI (T, S)-fuzzy filters and the filters are also investigated. In Sect. 4, we introduced the notions of IVI (T, S)-fuzzy implicative filters and IVI (T, S)-fuzzy Boolean filters, some of their equivalent characterizations are derived, and the equivalence of two kinds fuzzy filters are obtained. In Sect. 5, we introduced the notions of IVI (T, S)-fuzzy positive implicative filters and IVI (T, S)-G filters, some necessary and sufficient condition of them are derived, and the equivalence of two kinds fuzzy filters are obtained. In Sect. 6, we introduced the notions of IVI (T, S)-fuzzy fantastic filters and IVI (T, S)-MV filters, some necessary and sufficient condition of them are derived; we also study the relations among these fuzzy filters.

2 Preliminaries

Definition 1 ([15]) A residuated lattice is an algebraic structure $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following axioms:

(C1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(C2) $(L, \otimes, 1)$ is a commutative semigroup (with the unit element 1).
(C3) $(\otimes, \rightarrow)$ is an adjoint pair, i.e., for any $x, y, z, w \in L$,

(R1) if $x \leq y$ and $z \leq w$, then $x \otimes z \leq y \otimes w$.
(R2) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$.
(R3) (adjointness condition) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$.

In what follows, let $\mathcal{L}$ denote a residuated lattice unless otherwise specified and $x' = x \rightarrow 0$ for any $x \in L$.

Proposition 1 ([15, 24]) *In each residuated lattice $\mathcal{L}$, the following properties hold for all $x, y, z \in L$:*

(P1) $(x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z)$.
(P2) $z \leq x \rightarrow y \Leftrightarrow z \otimes x \leq y$.
(P3) $x \leq y \Leftrightarrow z \otimes x \leq z \otimes y$.
(P4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
(P5) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y$.
(P6) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$ and $y' \leq x'$.
(P7) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$.
(P8) $y \rightarrow x \leq (x \rightarrow z) \rightarrow (y \rightarrow z)$.
(P9) $1 \rightarrow x = x, x \rightarrow x = 1$.
(P10) $x^m \leq x^n, m, n \in N, m \geq n$.
(P11) $x \leq y \Leftrightarrow x \rightarrow y = 1$.
(P12) $0' = 1$, $1' = 0$, $x' = x'''$, $x \leq x''$.
(P13) $x \vee y \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.
(P14) $x \otimes x' = 0$.
(P15) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Let $\mathcal{L}$ be a residuated lattice, $F \subseteq L$, and $x, y, z \in L$. We list some conditions which will be used in the following study:

(F1) $x, y \in F \Rightarrow x \otimes y \in F$.
(F2) $x \in F, x \leq y \Rightarrow y \in F$.
(F3) $1 \in F$.
(F4) $x \in F, x \rightarrow y \in F \Rightarrow y \in F$.
(F5) $z, z \rightarrow ((x \rightarrow y) \rightarrow x) \in F \Rightarrow x \in F$.
(F6) $z \rightarrow (x \rightarrow y), z \rightarrow x \in F \Rightarrow z \rightarrow y \in F$.
(F7) $x \otimes x' = 1$.
(F8) $y \rightarrow x \in F \Rightarrow ((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.

Definition 2 ([24])

1. A non-empty subset A of a residuated lattice $\mathcal{L}$ is called a **filter** of $\mathcal{L}$ if it satisfies (F1) and (F2).
2. A non-empty subset A of a residuated lattice $\mathcal{L}$ is called an **implicative filter** of $\mathcal{L}$ if it satisfies (F3) and (F5).
3. A non-empty subset A of a residuated lattice $\mathcal{L}$ is called a **positive implicative filter** of $\mathcal{L}$ if it satisfies (F3) and (F6).
4. A filter A of a residuated lattice $\mathcal{L}$ is called a **Boolean filter** if it satisfies the condition (F7).
5. A filter F of a residuated lattice $\mathcal{L}$ is called a **MV-filter** if it satisfies the condition (F8).

Proposition 2 ([24]) *A non-empty subset A of a residuated lattice $\mathcal{L}$ is called a* **filter** *of $\mathcal{L}$ if it satisfies* (F3) *and* (F4).

By an interval $\tilde{a}$ we mean an interval $[a^-, a^+]$, where $0 \le a^- \le a^+ \le 1$. The set of all intervals is denoted by $D[0,1]$. The interval $[a, a]$ is identified with the number a.

For interval $\tilde{a_i} = [a_i^-, a_i^+], \tilde{b_i} = [b_i^-, b_i^+]$, where $i \in I, I$ is an index set, we define

$\text{rmax}\{\tilde{a_i}, \tilde{b_i}\} = [max\{a_i^-, b_i^-\}, max\{a_i^+, b_i^+\}]$,
$\text{rmin}\{\tilde{a_i}, \tilde{b_i}\} = [min\{a_i^-, b_i^-\}, min\{a_i^+, b_i^+\}]$,
$\wedge_{i\in I}\tilde{a_i} = [\wedge_{i\in I}a_i^-, \wedge_{i\in I}a_i^+]$,
$\vee_{i\in I}\tilde{a_i} = [\vee_{i\in I}a_i^-, \vee_{i\in I}a_i^+]$;
Furthermore, we have

(i) $\tilde{a_i} \le \tilde{b_i}$ if and only if $a_i^- \le b_i^-$ and $a_i^+ \le b_i^+$,
(ii) $\tilde{a_i} = \tilde{b_i}$ if and only if $a_i^- = b_i^-$ and $a_i^+ = b_i^+$,
(iii) $k\tilde{a} = [ka_i^-, ka_i^+]$, where $0 \le k \le 1$.

Then, it can be shown that $(D[0, 1], \le, \vee, \wedge)$ is a complete lattice, $\tilde{0} = [0, 0]$ as its least element and $\tilde{1} = [1, 1]$ as its greatest element.

Let X be an nonempty set, by an interval valued fuzzy set on X we mean the set

$$F = \{(x, [A_F^-(x), A_F^+(x)])|x \in X\},$$

where A_F^- and A_F^+ are two fuzzy sets of X such that $A_F^-(x) \le A_F^+(x)$ for all $x \in X$.

Putting $\tilde{A}_F(x) = [A_F^-(x), A_F^+(x)]$, we see that $F = \{(x, \tilde{A}_F(x)|x \in X\}$, where $\tilde{A}_F : X \to D[0, 1]$.

For any $\tilde{t} \in D[0, 1]$, the set $U(F; \tilde{t}) = \{x \in X|\tilde{A_F}(x) \ge \tilde{t}\}$ is called the **interval-valued level subset** of A.

Definition 3 ([15, 24]) Let δ be a mapping from $D[0, 1] \times D[0, 1]$ to $D[0,1]$. δ is called a t-norm (resp. s-norm) on $D[0, 1]$, if it satisfies the following conditions: for any $\tilde{x}, \tilde{y}, \tilde{z} \in D[0, 1]$,

1. $\delta(\tilde{x}, \tilde{1}) = \tilde{x}$(resp. $\delta(\tilde{x}, \tilde{0}) = \tilde{x}$),
2. $\delta(\tilde{x}, \tilde{y}) = \delta(\tilde{y}, \tilde{x})$,
3. $\delta(\delta(\tilde{x}, \tilde{y}), \tilde{z}) = \delta(\tilde{x}, \delta(\tilde{y}, \tilde{z}))$,
4. if $\tilde{x} \le \tilde{y}$, then $\delta(\tilde{x}, \tilde{z}) \le \delta(\tilde{y}, \tilde{z})$.

The set of all δ-idempotent elements $D_\delta = \{\tilde{x} \in D[0, 1]|\delta(\tilde{x}, \tilde{x}) = \tilde{x}\}$.

In [5, 6], there are some important examples on interval-valued t-norm. An interval-valued fuzzy subset A is said to satisfy **imaginable property** if $ImF \subseteq D_T$.

An interval valued intuitionistic fuzzy set on X is defined as an object of the form

$$A = \{(x, \tilde{M}_A(x), \tilde{N}_A(x))|x \in X\},$$

where $\tilde{M}_A, \tilde{N}_A$ are interval valued fuzzy sets on X such that $[0, 0] \le \tilde{M}_A(x) + \tilde{N}_A(x) \le [1, 1]$. For the sake of simplicity, in the following, such interval valued intuitionistic fuzzy sets will be denoted by $A = (\tilde{M}_A, \tilde{N}_A)$.

In this paper, all theorems are discussed under the condition that t-norm, s-norm are all idempotent.

3 IVI (*T*, *S*)-fuzzy filters

Definition 4 An interval valued intuitionistic fuzzy set A of $\mathcal{L}$ is called an interval valued intuitionistic (T, S)-fuzzy filter (IVI (T, S)-fuzzy filter for short) of $\mathcal{L}$, if for any $x, y, z \in L$:

(V1) $\tilde{M}_A(1) \ge \tilde{M}_A(x)$ and $\tilde{N}_A(1) \le \tilde{N}_A(x)$;
(V2) $\tilde{M}_A(y) \ge T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) \le S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$.

Example 1 Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$ with the operations $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$ and $\otimes$, $\to$ as Table 1.

Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. we define an interval-valued intuitionistic fuzzy subset A of $\mathcal{L}$ by

Table 1 $\otimes$, $\to$ in L

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1
$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\tilde{M}_A(0) = [0.3, 0.4], \tilde{M}_A(a) = \tilde{M}_A(b) = [0.7, 0.8], \tilde{M}_A(1) = \tilde{1}.$

$\tilde{N}_A(0) = [0.5, 0.6], \tilde{N}_A(a) = \tilde{N}_A(b) = [0.1, 0.2], \tilde{N}_A(1) = \tilde{0}.$

It is routine to verify that A is an interval-valued intuitionistic (T, S)-fuzzy filter of $\mathcal{L}$, where $T = T_P, T_P(\tilde{x}, \tilde{y}) = [x^- y^-, max\{x^- y^+, x^+ y^-\}], S = max.$

Lemma 1 *Let A be an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*. Then, for any* $x, y \in L:$

(V3) if $x \leq y$*, then* $\tilde{M}_A(x) \leq \tilde{M}_A(y)$ *and* $\tilde{N}_A(y) \leq \tilde{N}_A(x)$.

Proof Since $x \leq y$, it follows that $x \rightarrow y = I$. As A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, we have $\tilde{M}_A(y) \geq T(\tilde{M}_A(x \rightarrow y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) \leq S(\tilde{N}_A(x \rightarrow y), \tilde{N}_A(x))$. by (V1)we have, $\tilde{M}_A(1) \geq \tilde{M}_A(x), \tilde{N}_A(1) \leq \tilde{N}_A(x)$ for any $x \in L$, therefore,

$$\tilde{M}_A(y) \geq T(\tilde{M}_A(x \rightarrow y), \tilde{M}_A(x)) = T(\tilde{M}_A(1), \tilde{M}_A(x)) \geq \tilde{M}_A(x),$$

$$\tilde{N}_A(y) \leq S(\tilde{N}_A(x \rightarrow y), \tilde{N}_A(x)) \leq S(\tilde{N}_A(1), \tilde{N}_A(x)) \leq S(\tilde{N}_A(x), \tilde{N}_A(x)) = \tilde{N}_A(x).$$

And so V(3) is valid.

Proposition 3 *Let A be an interval valued intuitionistic fuzzy set on* $\mathcal{L}$*. Then A is an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*, if and only if, for any* $x, y, z \in L$*, (V1) holds and (V4)* $\tilde{M}_A(x \rightarrow z) \geq T(\tilde{M}_A(y \rightarrow (x \rightarrow z)), \tilde{M}_A(y))$ *and* $\tilde{N}_A(x \rightarrow z) \leq S(\tilde{N}_A(y \rightarrow (x \rightarrow z)), \tilde{N}_A(y))$.

Proof Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$, obviously, (V1)and (V4) hold. Conversely, assume that (V4) hold, taking $x = 1$ in (V4), we have $\tilde{M}_A(z) = \tilde{M}_A(1 \rightarrow z) \geq T(\tilde{M}_A(y \rightarrow (1 \rightarrow z)), \tilde{M}_A(y)) = T(\tilde{M}_A(y \rightarrow z), \tilde{M}_A(y))$ and $\tilde{N}_A(z) = \tilde{N}_A(1 \rightarrow z) \leq S(\tilde{N}_A(y \rightarrow (1 \rightarrow z)), \tilde{N}_A(y)) = S(\tilde{N}_A(y \rightarrow z), \tilde{N}_A(y))$. Hence (V2) holds, and so A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$.

Proposition 4 *Let A be an interval valued intuitionistic fuzzy set on* $\mathcal{L}$*. Then A is an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*, if and only if, for any* $x, y, z \in L, A$ *satisfies (V3) and*

(V5) $\tilde{M}_A(x \otimes y) \geq T(\tilde{M}_A(x), \tilde{M}_A(y))$ *and* $\tilde{N}_A(x \otimes y) \leq S(\tilde{N}_A(x), \tilde{N}_A(y))$.

Proof Assume that A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, obviously (V3)holds. Since $x \leq y \rightarrow (x \otimes y)$, we have $\tilde{M}_A(y \rightarrow (x \otimes y)) \geq \tilde{M}_A(x)$ and $\tilde{N}_A(y \rightarrow (x \otimes y)) \leq \tilde{N}_A(x)$. By (V2), it follows that $\tilde{M}_A(x \otimes y) \geq T(\tilde{M}_A(y), \tilde{M}_A(y \rightarrow (x \otimes y))) \geq T(\tilde{M}_A(y), \tilde{M}_A(x))$ and

$$\tilde{N}_A(x \otimes y) \leq S(\tilde{N}_A(y), \tilde{N}_A(y \rightarrow (x \otimes y))) \leq S(\tilde{N}_A(y), \tilde{N}_A(x)).$$

Conversely, assume (V3) and (V5) holds. Taking $y = I$ in (V3), then (V1)holds. As $x \otimes (x \rightarrow y) \leq y$, thus $\tilde{M}_A(y) \geq \tilde{M}_A(x \otimes (x \rightarrow y))$ and $\tilde{N}_A(y) \leq \tilde{N}_A(x \otimes (x \rightarrow y))$. By (V5), we have $\tilde{M}_A(y) \geq T(\tilde{M}_A(x), \tilde{M}_A(x \rightarrow y))$ and $\tilde{N}_A(y) \leq S(\tilde{N}_A(x), \tilde{N}_A(x \rightarrow y))$. Therefore (V2) is valid, so A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$.

Corollary 1 *An interval valued intuitionistic fuzzy set on* $\mathcal{L}$ *is an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*, if and only if, for any* $x, y, z \in L:$

(V6) if $x \rightarrow (y \rightarrow z) = 1$*, then*

$$\tilde{M}_A(z) \geq T(\tilde{M}_A(x), \tilde{M}_A(y))$$

and

$$\tilde{N}_A(z) \leq S(\tilde{N}_A(x), \tilde{N}_A(y)).$$

Corollary 2 *An interval valued intuitionistic fuzzy set on* $\mathcal{L}$ *is an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*, if and only if, for any* $x, y, z \in L:$

(V7) If $a_n \rightarrow (a_{n-1} \rightarrow \cdots \rightarrow (a_1 \rightarrow x) \cdots) = 1$*, then* $\tilde{M}_A(x) \geq T(\tilde{M}_A(a_n), \ldots, \tilde{M}_A(a_1))$ *and* $\tilde{N}_A(x) \leq S(\tilde{N}_A(a_n), \ldots, \tilde{N}_A(a_1))$.

Proposition 5 *An interval valued intuitionistic fuzzy set on* $\mathcal{L}$ *is an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*, if and only if, for any* $x, y, z \in L, A$ *satisfies (V1) and*

(V8) $\tilde{M}_A((x \rightarrow (y \rightarrow z)) \rightarrow z) \geq T(\tilde{M}_A(x), \tilde{M}_A(y))$ *and* $\tilde{N}_A((x \rightarrow (y \rightarrow z)) \rightarrow z) \leq S(\tilde{N}_A(x), \tilde{N}_A(y))$.

Proof If A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, (V1) is obvious. Since $\tilde{M}_A((x \rightarrow (y \rightarrow z)) \rightarrow z) \geq T(\tilde{M}_A((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)), \tilde{M}_A(y))$ and $\tilde{N}_A((x \rightarrow (y \rightarrow z)) \rightarrow z) \leq S(\tilde{N}_A((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)), \tilde{N}_A(y))$. As $(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z) \geq x$, by(V3), we have $\tilde{M}_A((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)) \geq \tilde{M}_A(x)$ and $\tilde{N}_A((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)) \leq \tilde{N}_A(x)$. Therefore, $\tilde{M}_A((x \rightarrow (y \rightarrow z)) \rightarrow z) \geq T(\tilde{M}_A(x), \tilde{M}_A(y))$ and $\tilde{N}_A((x \rightarrow (y \rightarrow z)) \rightarrow z) \leq S(\tilde{N}_A(x), \tilde{N}_A(y))$.

Conversely, suppose (V8) is valid. Since $\tilde{M}_A(y) = \tilde{M}_A(1 \rightarrow y) = \tilde{M}_A(((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow y) \geq T(\tilde{M}_A(x \rightarrow y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) = \tilde{N}_A(1 \rightarrow y) = \tilde{N}_A(((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow y) \leq S(\tilde{N}_A(x \rightarrow y), \tilde{N}_A(x))$. we have (V2). By(V1), A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$.

Proposition 6 *Let A be an interval valued intuitionistic fuzzy set on* $\mathcal{L}$*. Then A is an IVI (T, S)-fuzzy filter of* $\mathcal{L}$*, for any* $x, y, z \in L, A$ *satisfies (V1) and (V9)* $\tilde{M}_A(x \rightarrow z) \geq T(\tilde{M}_A(x \rightarrow y), \tilde{M}_A(y \rightarrow z))$ *and* $\tilde{N}_A(x \rightarrow z) \leq S(\tilde{N}_A(x \rightarrow y), \tilde{N}_A(y \rightarrow z))$.

Proof Assume that A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Since $(x \to y) \le (y \to z) \to (x \to z)$, it follows from Theorem 1 that $\tilde{M}_A((y \to z) \to (x \to z)) \ge \tilde{M}_A(x \to y)$ and $\tilde{N}_A((y \to z) \to (x \to z)) \le \tilde{N}_A(x \to y)$. As A is an IVI (T, S)-fuzzy filter, so $\tilde{M}_A(x \to z) \ge T(\tilde{M}_A(y \to z), \tilde{M}_A((y \to z) \to (x \to z)))$ and $\tilde{N}_A(x \to z) \le S(\tilde{N}_A(y \to z), \tilde{N}_A((y \to z) \to (x \to z)))$. We have

$$\tilde{M}_A(x \to z) \ge T(\tilde{M}_A(y \to z), \tilde{M}_A(x \to z))$$

and

$$\tilde{N}_A(x \to z) \le S(\tilde{N}_A(y \to z), \tilde{N}_A(x \to z)).$$

Conversely, if $\tilde{M}_A(x \to z) \ge T(\tilde{M}_A(x \to y), \tilde{M}_A(y \to z))$ and $\tilde{N}_A(x \to z) \le S(\tilde{N}_A(x \to y), \tilde{N}_A(y \to z))$ for any $x, y, z \in L$, then

$$\tilde{M}_A(I \to z) \ge T(\tilde{M}_A(I \to y), \tilde{M}_A(y \to z))$$

and

$$\tilde{N}_A(I \to z) \le S(\tilde{N}_A(I \to y), \tilde{N}_A(y \to z)),$$

that is $\tilde{M}_A(z) \ge T(\tilde{M}_A(y), \tilde{M}_A(y \to z))$ and $\tilde{N}_A(z) \le S(\tilde{N}_A(y), \tilde{N}_A(y \to z))$. By (V1), we have A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$.

Theorem 1 *Let A be an interval valued intuitionistic fuzzy set on $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, if and only if, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$ and $\tilde{\alpha} + \tilde{\beta} \le \tilde{1}$, the sets $U(\tilde{M}_A; \tilde{\alpha})(\ne \emptyset)$ and $L(\tilde{N}_A; \tilde{\beta})(\ne \emptyset)$ are filters of $\mathcal{L}$, where $U(\tilde{M}_A; \tilde{\alpha}) = \{x \in L | \tilde{M}_A(x) \ge \tilde{\alpha}\}, L(\tilde{N}_A; \tilde{\beta}) = \{x \in L | \tilde{N}_A(x) \le \tilde{\beta}\}$.*

Proof Assume A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, then $\tilde{M}_A(1) \ge \tilde{M}_A(x)$. By the condition $U(\tilde{M}_A, \tilde{\alpha}) \ne \emptyset$, it follows that there exists $a \in L$ such that $\tilde{M}_A(a) \ge \tilde{\alpha}$, and so $\tilde{M}_A(1) \ge \tilde{\alpha}$, hence $1 \in U(\tilde{M}_A; \tilde{\alpha})$.

Let $x, x \to y \in U(\tilde{M}_A; \tilde{\alpha})$, then $\tilde{M}_A(x) \ge \tilde{\alpha}, \tilde{M}_A(x \to y) \ge \tilde{\alpha}$. Since A is an IVI (T,S)-filter of $\mathcal{L}$, then $\tilde{M}_A(y) \ge T(\tilde{M}_A(x), \tilde{M}_A(x \to y)) \ge T(\tilde{\alpha}, \tilde{\alpha}) = \tilde{\alpha}$. Hence $y \in U(\tilde{M}_A; \tilde{\alpha})$. Therefore $U(\tilde{M}_A; \tilde{\alpha})$ is a filter of $\mathcal{L}$.

We will show that $L(\tilde{N}_A; \tilde{\beta})$ is a filter of $\mathcal{L}$.

Since A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, then $\tilde{N}_A(1) \le \tilde{N}_A(x)$. By the condition $L(\tilde{N}_A, \tilde{\beta}) \ne \emptyset$, it follows that there exists $a \in L$ such that $\tilde{N}_A(a) \le \tilde{\beta}$, and so $\tilde{N}_A(a) \le \tilde{\beta}$, we have $\tilde{N}_A(1) \le \tilde{N}_A(a) \le \tilde{\beta}$, hence $1 \in L(\tilde{N}_A; \tilde{\beta})$.

Let $x, x \to y \in L(\tilde{N}_A; \tilde{\beta})$, then $\tilde{N}_A(x) \le \tilde{\beta}; \tilde{N}_A(x \to y) \le \tilde{\beta}$. Since A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, then $\tilde{N}_A(y) \le S(\tilde{N}_A(x), \tilde{N}_A(x \to y)\} \le S(\tilde{\beta}, \tilde{\beta}) = \tilde{\beta}$. It follows that $\tilde{N}_A(y) \le \tilde{\beta}$, hence $y \in L(\tilde{N}_A; \tilde{\beta})$. Therefore $L(\tilde{N}_A; \tilde{\beta})$ is a filter of $\mathcal{L}$.

Conversely, suppose that $U(\tilde{M}_A; \tilde{\alpha})(\ne \emptyset)$ and $L(\tilde{N}_A; \tilde{\beta})(\ne \emptyset)$ are filters of $\mathcal{L}$, then, for any $x \in L, x \in U(\tilde{M}_A; \tilde{M}_A(x))$ and $x \in L(\tilde{N}_A; \tilde{N}_A(x))$. Since $U(\tilde{M}_A, \tilde{M}_A(x))(\ne \emptyset)$ and $L(\tilde{N}_A, \tilde{N}_A(x))(\ne \emptyset)$ are filters of $\mathcal{L}$, it follows that $I \in U(\tilde{M}_A, \tilde{M}_A(x))$ and $1 \in L(\tilde{N}_A, \tilde{N}_A(x))$, and so $q\tilde{M}_A(1) \ge \tilde{M}_A(x)$ and $\tilde{N}_A(1) \le \tilde{N}_A(x)$.

For any $x, y \in L$, let $\tilde{\alpha} = T(\tilde{M}_A(x), \tilde{M}_A(x \to y))$ and $\tilde{\beta} = S(\tilde{N}_A(x), \tilde{N}_A(x \to y))$, then $x, x \to y \in U(\tilde{M}_A; \tilde{\alpha})$ and $x, x \to y \in L(\tilde{N}_A; \tilde{\beta})$. And so $y \in U(\tilde{M}_A; \tilde{\alpha})$ and $y \in L(\tilde{N}_A; \tilde{\beta})$. Therefore

$$\tilde{M}_A(y) \ge \tilde{\alpha} = T(\tilde{M}_A(x), \tilde{M}_A(x \to y))$$

and

$$\tilde{N}_A(y) \le \tilde{\beta} = S(\tilde{N}_A(x), \tilde{N}_A(x \to y)\}.$$

We have A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$.

Let A, B be two interval valued intuitionistic fuzzy sets on $\mathcal{L}$, denote by C the intersection of A and B, i.e. $C = A \cap B$, where

$$\tilde{M}_C(x) = T(\tilde{M}_A(x), \tilde{M}_B(x)),$$
$$\tilde{N}_C(x) = S(\tilde{N}_A(x), \tilde{N}_B(x))$$

for any $x \in L$.

Proposition 7 *Let A, B be two IVI (T, S)-fuzzy filters of $\mathcal{L}$, then $A \cap B$ is also an IVI (T, S)-fuzzy filter of $\mathcal{L}$.*

Proof Let $x, y, z \in L$ such that $z \le x \to y$, then $z \to (x \to y) = 1$. Since A, B are two IVI (T, S)-fuzzy filters of $\mathcal{L}$, we have $\tilde{M}_A(y) \ge T(\tilde{M}_A(z), \tilde{M}_A(x)), \tilde{N}_A(y) \le S(\tilde{N}_A(z), \tilde{N}_A(x))$ and

$$\tilde{M}_B(y) \ge T(\tilde{M}_B(z), \tilde{M}_B(x)\},$$

$$\tilde{N}_B(y) \le S(\tilde{N}_B(z), \tilde{N}_B(x)\}.$$

Since

$$\begin{aligned} \tilde{M}_{A\cap B}(y) &= T(\tilde{M}_A(y), \tilde{M}_B(y)\} \\ &\ge T(T(\tilde{M}_A(z), \tilde{M}_A(x)), T(\tilde{M}_B(z), \tilde{M}_B(x))) \\ &= T(T(\tilde{M}_A(z), \tilde{M}_B(z)), T(\tilde{M}_A(x), \tilde{M}_B(x))) \\ &= T(\tilde{M}_{A\cap B}(z), \tilde{M}_{A\cap B}(x)) \end{aligned}$$

and

$$\begin{aligned} \tilde{N}_{A\cap B}(y) &= S(\tilde{N}_A(y), \tilde{N}_B(y)\} \\ &\le S(S(\tilde{N}_A(z), \tilde{N}_A(x)), S(\tilde{N}_B(z), \tilde{N}_B(x))) \\ &= S(S(\tilde{N}_A(z), \tilde{N}_B(z)), S(\tilde{N}_A(x), \tilde{N}_B(x))) \\ &= S(\tilde{N}_{A\cap B}(z), \tilde{N}_{A\cap B}(x)). \end{aligned}$$

Since A, B be two IVI (T, S)-fuzzy filters of $\mathcal{L}$, we have $\tilde{M}_A(1) \ge \tilde{M}_A(x)), \tilde{N}_A(1) \le \tilde{N}_A(x))$ and $\tilde{M}_B(1) \ge \tilde{M}_B(x)$, $\tilde{N}_B(1) \le \tilde{N}_B(x)$. Hence

$$\begin{aligned}\tilde{M}_{A\cap B}(1) &= T(\tilde{M}_A(1), \tilde{M}_B(1))\\ &\geq T(\tilde{M}_A(x), \tilde{M}_B(x))\\ &= \tilde{M}_{A\cap B}(x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\tilde{N}_{A\cap B}(1) &= S(\tilde{N}_A(1), \tilde{N}_B(1))\\ &\leq S(\tilde{N}_A(x), \tilde{N}_B(x)) = \tilde{N}_{A\cap B}(x).\end{aligned}$$

Then $A\cap B$ is an IVI (T, S)-fuzzy filters of $\mathcal{L}$.

Let A_i be a family interval valued intuitionistic fuzzy sets on $\mathcal{L}$, where I is an index set. Denoting by C the intersection of A_i, i.e. $\cap_{i\in I}A_i$, where

$$\tilde{M}_C(x) = T(\tilde{M}_{A_1}(x), \tilde{M}_{A_2}(x), \ldots),$$
$$\tilde{N}_C(x) = S(\tilde{N}_{A_1}(x), \tilde{N}_{A_2}(x), \ldots)$$

for any $x \in L$.

Corollary 3 *Let A_i be a family IVI (T, S)-fuzzy filters of $\mathcal{L}$, where $i \in I, I$ an index set. then $\cap_{i\in I}A_i$ is also an IVI (T, S)-fuzzy filter of $\mathcal{L}$.*

Suppose A is an interval valued intuitionistic fuzzy set on $\mathcal{L}$ and $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$. Denoting $A_{(\tilde{\alpha},\tilde{\beta})}$ by the set $\{x \in L | \tilde{M}_A(x) \geq \tilde{\alpha}, \tilde{N}_A(x) \leq \tilde{\beta}\}$.

Theorem 2 *Let A be an interval valued intuitionistic fuzzy set on $\mathcal{L}$. Then*

1. *for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$, if $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$. Then, for any $x, y, z \in L$,*
   (V10) $\tilde{M}_A(z) \leq T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(z) \geq S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$ imply $\tilde{M}_A(z) \leq \tilde{M}_A(y)$ and $\tilde{N}_A(z) \geq \tilde{N}_A(y)$.
2. *If A satisfy (V1)and (V10), then, for any $\alpha, \beta \in D[0,1], A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$.*

Proof

(1) Assume that $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$ for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$. Since $\tilde{M}_A(z) \leq T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(z) \geq S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$, it follows that

$$\tilde{M}_A(z) \leq \tilde{M}_A(x \to y), \tilde{M}_A(z) \leq \tilde{M}_A(x)$$

and

$$\tilde{N}_A(z) \geq \tilde{N}_A(x \to y), \tilde{N}_A(z) \geq \tilde{N}_A(x).$$

Therefore,

$$x \to y \in A_{(\tilde{M}_A(z),\tilde{N}_A(z))}, x \in A_{(\tilde{M}_A(z),\tilde{N}_A(z))}.$$

As $\tilde{M}_A(z), \tilde{N}_A(z) \in [0,1]$, and $A_{(\tilde{M}_A(z),\tilde{N}_A(z))}$ is a filter of $\mathcal{L}$, so $y \in A_{(\tilde{M}_A(z),\tilde{N}_A(z))}$.

Thus $\tilde{M}_A(z) \leq \tilde{M}_A(y)$ and$\tilde{N}_A(z) \geq \tilde{N}_A(y)$.

(2) Assume A satisfy (V1) and (V10). For any $x, y \in L, \tilde{\alpha}, \tilde{\beta} \in D[0,1]$, we have $x \to y \in A_{(\tilde{\alpha},\tilde{\beta})}, x \in A_{(\tilde{\alpha},\tilde{\beta})}$, therefore $\tilde{M}_A(x \to y) \geq \tilde{\alpha}, \tilde{N}_A(x \to y) \leq \tilde{\beta}$ and $\tilde{M}_A(x) \geq \tilde{\alpha}, \tilde{N}_A(x) \leq \tilde{\beta}$, and so

$$T(\tilde{M}_A(x \to y), \tilde{M}_A(x)) \geq T(\tilde{\alpha}, \tilde{\alpha}) = \tilde{\alpha},$$
$$S(\tilde{N}_A(x \to y), \tilde{N}_A(x)) \leq S(\tilde{\beta}, \tilde{\beta}) = \tilde{\beta}.$$

By (V10), we have $\tilde{M}_A(y) \geq \tilde{\alpha}$ and $\tilde{N}_A(y) \leq \tilde{\beta}$, that is, $y \in A_{(\tilde{\alpha},\tilde{\beta})}$.

Since $\tilde{M}_A(1) \geq \tilde{M}_A(x)$ and $\tilde{N}_A(1) \leq \tilde{N}_A(x)$ for any $x \in L$, it follows that $\tilde{M}_A(1) \geq \tilde{\alpha}$ and $\tilde{N}_A(1) \leq \tilde{\beta}$, that is, $1 \in A_{(\tilde{\alpha},\tilde{\beta})}$. Then, for any $\tilde{\alpha}, \tilde{\beta} \in [0,1], A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$.

Proposition 8 *Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$, then, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1], A_{(\tilde{\alpha},\tilde{\beta})}(\neq \phi)$ is a filter of $\mathcal{L}$.*

Proof Since $A_{(\tilde{\alpha},\tilde{\beta})} \neq \phi$, there exist $x \in L$ such that $\tilde{M}_A(x) \geq \tilde{\alpha}, \tilde{N}_A(x) \leq \tilde{\beta}$. And A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, we have $\tilde{M}_A(1) \geq \tilde{M}_A(x) \geq \tilde{\alpha}, N_A(1) \leq \tilde{N}_A(x) \leq \tilde{\beta}$, therefore $1 \in A_{(\tilde{\alpha},\tilde{\beta})}$.

Let $x, y \in L$ and $x \in A_{(\tilde{\alpha},\tilde{\beta})}, x \to y \in A_{(\tilde{\alpha},\tilde{\beta})}$, therefore $\tilde{M}_A(x) \geq \tilde{\alpha}, \tilde{N}_A(x) \leq \tilde{\beta}, \tilde{M}_A(x \to y) \geq \tilde{\alpha}, \tilde{M}_A(x \to y) \leq \tilde{\beta}$. Since A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, thus $\tilde{M}_A(y) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x)) \geq \tilde{\alpha}$ and $\tilde{N}_A(y) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x)) \leq \tilde{\beta}$, it follows that $y \in A_{(\tilde{\alpha},\tilde{\beta})}$. Therefore, $A_{(\tilde{\alpha},\tilde{\beta})}$ is a filter of $\mathcal{L}$.

In Proposition 8, the filter $A_{(\tilde{\alpha},\tilde{\beta})}$ is also called an **IVI-cut** filter of $\mathcal{L}$.

Theorem 3 *Any filter A of $\mathcal{L}$ is an IVI-cut filter of some IVI (T, S)-fuzzy filter of $\mathcal{L}$.*

Proof Consider the interval valued intuitionistic fuzzy set A of $\mathcal{L} : A = \{(x, M_A\tilde{(}x), N_A\tilde{(}x)) | x \in L\}$, where

If $x \in F$,

$$\tilde{M}_A(x) = \tilde{\alpha}, \tilde{N}_A(x) = \tilde{1} - \tilde{\alpha}. \tag{1}$$

If $x \notin F$,

$$\tilde{M}_A(x) = \tilde{0}, \tilde{N}_A(x) = \tilde{1}. \tag{2}$$

where $\tilde{\alpha} \in D[0,1]$. Since F is a filter of $\mathcal{L}$, we have $I \in F$. Therefore $\tilde{M}_A(1) = \tilde{\alpha} \geq \tilde{M}_A(x)$ and $\tilde{N}_A(1) = \tilde{1} - \tilde{\alpha} \leq \tilde{N}_A(x)$.

For any $x, y \in L$, if $y \in F$, then $\tilde{M}_A(y) = \tilde{\alpha} = T(\tilde{\alpha}, \tilde{\alpha}) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) = \tilde{1} - \tilde{\alpha} = S(\tilde{1} - \tilde{\alpha}, \tilde{1} - \tilde{\alpha}) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$.

If $y \notin F$, hence $x \notin F$ or $x \to y \notin F$. And so $\tilde{M}_A(y) = \tilde{0} = T(\tilde{0}, \tilde{0}) = T(\tilde{M}_A(x \to y), \tilde{M}_A(x))$ and $\tilde{N}_A(y) = \tilde{1} = S(\tilde{1}, \tilde{1}) = S(\tilde{N}_A(x \to y), \tilde{N}_A(x))$. Therefore A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$.

Proposition 9 *Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Then $F = \{x \in L | \tilde{M}_A(x) = \tilde{M}_A(1), \tilde{N}_A(x) = \tilde{N}_A(1)\}$ is a filter of $\mathcal{L}$.*

Proof Since $F = \{x \in L | \tilde{M}_A(x) = \tilde{M}_A(1), \tilde{N}_A(x) = \tilde{N}_A(1)\}$, obviously $1 \in F$. Let $x \to y \in F, x \in F$, so $\tilde{M}_A(x \to y) = \tilde{M}_A(x) = \tilde{M}_A(1)$ and $\tilde{N}_A(x \to y) = \tilde{N}_A(x) = \tilde{N}_A(1)$, Therefore

$$\tilde{M}_A(y) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x)\} = \tilde{M}_A(1).$$

And $\tilde{M}_A(1) \geq \tilde{M}_A(y)$, then $\tilde{M}_A(y) = \tilde{M}_A(1)$. Similarly, we have $\tilde{N}_A(y) = \tilde{N}_A(1)$. Thus $y \in F$. It follows that A is a filter of $\mathcal{L}$.

4 IVI (T, S)-fuzzy implicative (Boolean) filters

Definition 5 Let A be an interval-valued intuitionistic fuzzy set of L. Then A is an IVI (T, S)-fuzzy implicative filter if A satisfies (V1) and

(V11) $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(x \to (z' \to y)), \tilde{M}_A(y \to z))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(x \to (z' \to y)), \tilde{N}_A(y \to z))$.

Example 2 Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in Example 1, and define an interval-valued intuitionistic fuzzy subset A of $\mathcal{L}$ by $\tilde{M}_A(0) = [\frac{1}{2}, \frac{2}{3}], \tilde{M}_A(a) = \tilde{M}_A(b) = [\frac{7}{10}, \frac{4}{5}], \tilde{M}_A(1) = [\frac{5}{6}, \frac{6}{7}]$.
$\tilde{N}_A(0) = [\frac{2}{15}, \frac{3}{10}], \tilde{N}_A(a) = \tilde{N}_A(b) = [0, \frac{1}{10}], \tilde{N}_A(1) = [0, 0]$.

It is easy to verify that A is an interval-valued intuitionistic (T, S)-fuzzy filter of $\mathcal{L}$, where $T = T_P$, $T_P(\tilde{x}, \tilde{y}) = [x^- y^-, max\{x^- y^+, x^+ y^-\}], S = max$.

Proposition 10 *Let A be an IVI (T, S)-fuzzy filter of L. Then A is an IVI (T, S)-fuzzy implicative filters if and only if it satisfies $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A(z \to (x \to (y' \to y))), \tilde{M}_A(z))$ and $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A(z \to (x \to (y' \to y))), \tilde{N}_A(z))$ for any $x, y, z \in L$.*

Proof Assume that A is an IVI (T, S)-fuzzy implicative filter. Then $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A(x \to (y' \to y)), \tilde{M}_A(y \to y)) = T(\tilde{M}_A(x \to (y' \to y)), \tilde{M}_A(1)) \geq \tilde{M}_A(x \to (y' \to y))$. Since A is an IVI (T, S)-fuzzy filter of L, we have $\tilde{M}_A(x \to (y' \to y)) \geq T(\tilde{M}_A(z \to (x \to (y' \to y))), \tilde{M}_A(z))$. And so,

$$\begin{aligned}\tilde{M}_A(x \to y) &\geq \tilde{M}_A(x \to (y' \to y)) \\ &\geq T(\tilde{M}_A(z \to (x \to (y' \to y))), \tilde{M}_A(z)).\end{aligned}$$

Similarly, we have $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A(z \to (x \to (y' \to y))), \tilde{N}_A(z))$.

Conversely, assume that A is an IVI (T, S)-fuzzy filter of L and $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A(z \to (x \to (y' \to y))), \tilde{M}_A(z))$, $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A(z \to (x \to (y' \to y)), \tilde{N}_A(z)))$. Since $x \to (z' \to y) = (x \otimes z') \to y$ and $(x \to y) \otimes (y \to z) \leq x \to z$, we have

$$\begin{aligned}(x \to (z' \to y)) \otimes (y \to z) &= ((x \otimes z') \to y) \otimes (y \to z) \\ &\leq x \otimes z' \to z = x \to (z' \to z).\end{aligned}$$

It follows that $\tilde{M}_A(x \to (z' \to z)) \geq \tilde{M}_A(((x \otimes z') \to y) \otimes (y \to z)), \tilde{N}_A(x \to (z' \to z)) \leq \tilde{N}_A(((x \otimes z') \to y) \otimes (y \to z))$ and $\tilde{M}_A((((x \otimes z') \to y) \otimes (y \to z)) \geq T(\tilde{M}_A(x \to (z' \to y)), \tilde{M}_A(y \to z)), \tilde{N}_A(((x \otimes z') \to y) \otimes (y \to z)) \leq S(\tilde{N}_A(x \to (z' \to y)), \tilde{N}_A(y \to z))$, hence $\tilde{M}_A(x \to (z' \to z)) \geq T(\tilde{M}_A(x \to (z' \to y)), \tilde{M}_A(y \to z))$ and $\tilde{N}_A(x \to (z' \to z)) \leq S(\tilde{N}_A(x \to (z' \to y)), \tilde{N}_A(y \to z))$. By the hypothesis, we have

$$\begin{aligned}\tilde{M}_A(x \to z) &\geq T(\tilde{M}_A(1 \to (x \to (z' \to z))), \tilde{M}_A(1)) \\ &\geq \tilde{M}_A(x \to (z' \to z)) \\ &\geq T(\tilde{M}_A(x \to (z' \to y)), \tilde{M}_A(y \to z)).\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(x \to z) &\leq S(\tilde{N}_A(1 \to (x \to (z' \to z))), \tilde{N}_A(1)) \\ &\leq \tilde{M}_A(x \to (z' \to z)) \\ &\leq S(\tilde{N}_A(x \to (z' \to y)), \tilde{N}_A(y \to z)).\end{aligned}$$

Therefore A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$.

Proposition 11 *Let A be an IVI (T, S)-fuzzy filter of L. Then A is an IVI (T, S)-fuzzy implicative filter if and only if it satisfies $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A(z \to (x \to (x \to y)))$, $\tilde{M}_A(z))$ and $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A(z \to (x \to (x \to y))), \tilde{N}_A(z))$ for any $x, y, z \in L$.*

Proof It is similar to Proposition 10.

Proposition 12 *Let A be an IVI (T, S)-fuzzy filter of L. Then A is an IVI (T, S)-fuzzy implicative filter if and only if it satisfies $\tilde{M}_A(x \to z) \geq \tilde{M}_A(x \to (z' \to z))$ and $\tilde{N}_A(x \to z) \leq \tilde{N}_A(x \to (z' \to z))$ for any $x, y, z \in L$.*

Proof Assume that A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$, we have $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(x \to (z' \to y)$, $\tilde{M}_A(y \to z)))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(x \to (z' \to y), \tilde{N}_A(y \to z)))$. Taking $y = z$, we have $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A\ (x \to$

$(z' \to z)), \tilde{M}_A(1)) \geq \tilde{M}_A(x \to (z' \to z))$ and $\tilde{N}_A(x \to z) \leq T(\tilde{N}_A(x \to (z' \to z)), \tilde{N}_A(1)) \leq \tilde{N}_A(x \to (z' \to z))$.

Conversely, assume that A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Then $\tilde{M}_A(x \to (z' \to z)) \geq T(\tilde{M}_A(y \to (x \to (z' \to z))), \tilde{M}_A(y))$ and $\tilde{N}_A(x \to (z' \to z)) \leq S(\tilde{N}_A(y \to (x \to (z' \to z))), \tilde{N}_A(y))$, it follows that $\tilde{M}_A(x \to z) \geq \tilde{M}_A(x \to (z' \to z)) \geq T(\tilde{M}_A(y \to (x \to (z' \to z))), \tilde{M}_A(y))$ and $\tilde{N}_A(x \to z) \leq \tilde{N}_A(x \to (z' \to z)) \leq S(\tilde{N}_A(y \to (x \to (z' \to z))), \tilde{N}_A(y))$. By Proposition 10, we have A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$.

Proposition 13 *Let A be an IVI (T, S)-fuzzy filter of L. Then A is an IVI (T, S)-fuzzy implicative filter if and only if it satisfies* $\tilde{M}_A(x) \geq T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ *and* $\tilde{N}_A(x) \leq S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$ *for any* $x, y, z \in L$.

Proof From Proposition 12, we have $\tilde{M}_A(x) = \tilde{M}_A(1 \to x) \geq \tilde{M}_A(1 \to (x' \to x)) = \tilde{M}_A(x' \to x), \tilde{N}_A(x) = \tilde{N}_A(1 \to x) \leq \tilde{N}_A(1 \to (x' \to x)) = \tilde{N}_A(x' \to x)$. Since A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$ and $x' \leq x \to y$, we have $(x \to y) \to x \geq x' \to x$, and so $\tilde{M}_A(x' \to x) \leq \tilde{M}_A((x \to y) \to x), \tilde{N}_A(x' \to x) \leq \tilde{N}_A((x \to y) \to x)$. Therefore $\tilde{M}_A(x) \geq \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(x) \leq \tilde{N}_A((x \to y) \to x)$. Since $\tilde{M}_A((x \to y) \to x) \geq T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ and $\tilde{N}_A((x \to y) \to x) \leq S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$, it follows that $\tilde{M}_A(x) \geq T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ and $\tilde{N}_A(x) \geq S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$.

Conversely, Since $(x \to z)' \leq z'$, we have $z' \to (x \to z) \leq (x \to z)' \to (x \to z)$ and so $\tilde{M}_A((x \to z)' \to (x \to z)) \geq \tilde{M}_A(z' \to (x \to z)), \tilde{N}_A((x \to z)' \to (x \to z)) \leq \tilde{N}_A(z' \to (x \to z))$. It follows that

$$\begin{aligned}\tilde{M}_A(x \to z) &\geq T(\tilde{M}_A(1 \to ((x \to z) \to 0) \to (x \to z)), \tilde{M}_A(1)) \\ &\geq \tilde{M}_A((x \to z)' \to (x \to z)) \\ &\geq \tilde{M}_A(z' \to (x \to z)) \\ &= \tilde{M}_A(x \to (z' \to z)).\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(x \to z) &\leq S(\tilde{N}_A(1 \to ((x \to z) \to 0) \to (x \to z)), \tilde{N}_A(1)) \\ &\leq \tilde{N}_A((x \to z)' \to (x \to z)) \\ &\leq \tilde{N}_A(z' \to (x \to z)) \\ &= \tilde{N}_A(x \to (z' \to z)).\end{aligned}$$

It follows from Proposition 12 that A is an an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$.

Definition 6 Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$.A is called an IVI (T, S)-fuzzy Boolean filter if $\tilde{M}_A(x \vee x') = \tilde{M}_A(1)$ and $\tilde{N}_A(x \vee x') = \tilde{N}_A(1)$ for any $x \in L$.

Theorem 4 *Let A be an IVI (T, S)-fuzzy filter of* $\mathcal{L}$. *Then A is an IVI (T, S)-fuzzy Boolean filter of* $\mathcal{L}$ *if and only if A is an IVI (T, S)-fuzzy implicative filter of* $\mathcal{L}$.

Proof Suppose that A is an IVI (T, S)-fuzzy Boolean filter, then $\tilde{M}_A(x \vee x') = \tilde{M}_A(1), \tilde{N}_A(x \vee x') = \tilde{N}_A(1)$ for any $x \in L$, it follows that $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A((z \vee z') \to (x \to z)), \tilde{M}_A(z \vee z')) = T(\tilde{M}_A((z \vee z') \to (x \to z)), \tilde{M}_A(1)) \geq \tilde{M}_A((z \vee z') \to (x \to z))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A((z \vee z') \to (x \to z)), \tilde{N}_A(z \vee z')) = S(\tilde{N}_A((z \vee z') \to (x \to z)), \tilde{N}_A(1)) \leq \tilde{N}_A((z \vee z') \to (x \to z))$. Since $(z \vee z') \to (x \to z) = (z \to (x \to z)) \wedge (z' \to (x \to z)) = z' \to (x \to z) = x \to (z' \to z)$, it follows that $\tilde{M}_A((z \vee z') \to (x \to z)) = \tilde{M}_A(x \to (z' \to z))$ and $\tilde{N}_A((z \vee z') \to (x \to z)) = \tilde{N}_A(x \to (z' \to z))$, thus $\tilde{M}_A(x \to z) \geq \tilde{M}_A(x \to (z' \to z))$ and $\tilde{N}_A(x \to z) \leq \tilde{N}_A(x \to (z' \to z))$. It follows from Proposition 12 that A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$.

Conversely, suppose that A is an IVI (T, S)-fuzzy implicative filter. Since $x' \to (((x' \to x) \to x) \to (x' \to x)') = 1$ and $x' \to ((x' \to x) \to x) = 1$, we have $\tilde{M}_A((x' \to x) \to x) = \tilde{M}_A(x' \to (x' \to x)') \geq T(\tilde{M}_A(x' \to (((x' \to x) \to x) \to (x' \to x)')), \tilde{M}_A(x' \to ((x' \to x) \to x))) = T(\tilde{M}_A(1), \tilde{M}_A(1)) = \tilde{M}_A(1)$ and $\tilde{N}_A((x' \to x) \to x) = \tilde{N}_A(x' \to (x' \to x)') \leq S(\tilde{N}_A(x' \to (((x' \to x) \to x) \to (x' \to x)')), \tilde{N}_A(x' \to ((x' \to x) \to x))) = S(\tilde{N}_A(1), \tilde{N}_A(1)) = \tilde{N}_A(1)$. Hence $\tilde{M}_A((x' \to x) \to x) = \tilde{M}_A(1)$ and $\tilde{N}_A((x' \to x) \to x) = \tilde{N}_A(1)$. Similarly, we can prove $\tilde{M}_A((x \to x') \to x') = \tilde{M}_A(1)$ and $\tilde{N}_A((x \to x') \to x') = \tilde{N}_A(1)$. Therefore

$$\begin{aligned}\tilde{M}_A(x \vee x') &= \tilde{M}_A(((x' \to x) \to x) \wedge ((x' \to x) \to x)) \\ &\geq T(\tilde{M}_A((x \to x') \to x'), \tilde{M}_A((x' \to x) \to x)) \\ &= T(\tilde{M}_A(1), \tilde{M}_A(1)) = \tilde{M}_A(1)\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(x \vee x') &= \tilde{N}_A(((x' \to x) \to x) \wedge ((x' \to x) \to x)) \\ &\leq S(\tilde{N}_A((x \to x') \to x'), \tilde{N}_A((x' \to x) \to x)) \\ &= S(\tilde{N}_A(1), \tilde{N}_A(1)) = \tilde{N}_A(1).\end{aligned}$$

Thus $\tilde{M}_A(x \vee x') = \tilde{M}_A(1)$ and $\tilde{N}_A(x \vee x') = \tilde{N}_A(1)$. Therefore A is an IVI (T, S)-fuzzy Boolean filter of $\mathcal{L}$.

Proposition 14 *Every IVI (T, S)-fuzzy implicative (Boolean) filter is an IVI (T, S)-fuzzy filter.*

Proof It is easy to obtain the theorem by taking $y = 1$ in Proposition 13.

Proposition 15 *Let A be an IVI (T, S)-fuzzy filter of* $\mathcal{L}$. *Then the following conditions are equivalent:*

1. *A is an IVI (T, S)-fuzzy Boolean filter*;
2. $\tilde{M}_A(x) = \tilde{M}_A(x' \to x)$ *and* $\tilde{N}_A(x) = \tilde{N}_A(x' \to x)$ *for any* $x \in L$;

3. $\tilde{M}_A(x) = \tilde{M}_A((x \to y) \to x)$ *and* $\tilde{N}_A(x) = \tilde{N}_A((x \to y) \to x)$ *for any* $x, y \in L$.

Proof

(1)⇒(2) Suppose that A is an IVI (T, S)-fuzzy Boolean filter. It follows from Proposition 12 and Proposition 14 that $\tilde{M}_A(x) = \tilde{M}_A(1 \to x) = \tilde{M}_A(1 \to (x' \to x)) = \tilde{M}_A(x' \to x), \tilde{N}_A(x) = \tilde{N}_A(1 \to x) = \tilde{N}_A(1 \to (x' \to x)) = \tilde{N}_A(x' \to x)$.

(2)⇒(3) From $x' \le x \to y$, it follows that $(x \to y) \to x \le x' \to x$, which implies $\tilde{M}_A(x' \to x) \ge \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(x' \to x) \le \tilde{N}_A((x \to y) \to x)$. From (2), we have $\tilde{M}_A(x) \ge \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(x) \le \tilde{N}_A((x \to y) \to x)$. Since $(x \to y) \to x \ge x$, we have $\tilde{M}_A(x) \le \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(x) \ge \tilde{N}_A((x \to y) \to x)$, thus $\tilde{M}_A(x) = \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(x) = \tilde{N}_A((x \to y) \to x)$.

(3)⇒(1) As A is an IVI (T, S)-fuzzy filter, we can obtain $\tilde{M}_A((x \to y) \to x) \ge T(\tilde{M}_A(z \to (x \to y) \to x), \tilde{M}_A(z))$ and $\tilde{N}_A((x \to y) \to x) \le S(\tilde{N}_A(z \to (x \to y) \to x), \tilde{N}_A(z))$, which together with (3), we have $\tilde{M}_A(x) \ge T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ and $\tilde{N}_A(x) \le S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$. Since $z \le x \to z$, it follows that $(x \to z)' \le z'$ and $z' \to (x \to z) \le (x \to z)' \to (x \to z)$, this implies that $\tilde{M}_A(z' \to (x \to z)) \le \tilde{M}_A((x \to z)' \to (x \to z))$ and $\tilde{N}_A(z' \to (x \to z)) \ge \tilde{N}_A((x \to z)' \to (x \to z))$. And so $\tilde{M}_A((x \to z)' \to (x \to z)) \ge T(\tilde{M}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{M}_A(1))$ and $\tilde{N}_A((x \to z)' \to (x \to z)) \le S(\tilde{N}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{N}_A(1))$. Since $\tilde{M}_A((x \to z)' \to (x \to z)) = T(\tilde{M}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{M}_A((x \to z)' \to (x \to z))) \le T(\tilde{M}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{M}_A(1))$ and $\tilde{N}_A((x \to z)' \to (x \to z)) = S(\tilde{N}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{N}_A((x \to z)' \to (x \to z))) \ge S(\tilde{N}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{N}_A(1))$, we have $\tilde{M}_A((x \to z)' \to (x \to z)) = T(\tilde{M}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{M}_A(1))$ and $\tilde{N}_A((x \to z)' \to (x \to z)) = S(\tilde{N}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{N}_A(1))$. Therefore $\tilde{M}_A(z' \to (x \to z)) \le \tilde{M}_A((x \to z)' \to (x \to z)) = T(\tilde{M}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{M}_A(1)) = T(\tilde{M}_A(1 \to (((x \to z) \to 0) \to (x \to z))), \tilde{M}_A(1)) \le \tilde{M}_A(((x \to z) \to 0) \to (x \to z)) = \tilde{M}_A(x \to z)$ and $\tilde{N}_A(z' \to (x \to z)) \ge \tilde{N}_A((x \to z)' \to (x \to z)) = S(\tilde{N}_A(1 \to ((x \to z)' \to (x \to z))), \tilde{N}_A(1)) = S(\tilde{N}_A(1 \to (((x \to z) \to 0) \to (x \to z))), \tilde{N}_A(1)) \ge \tilde{N}_A(((x \to z) \to 0) \to (x \to z)) = \tilde{N}_A(x \to z)$, i.e. $\tilde{M}_A(x \to (z' \to z)) \le \tilde{M}_A(x \to z)$ and $\tilde{N}_A(x \to (z' \to z)) \ge \tilde{N}_A(x \to z)$for any $x \in L$. It follows from Proposition 12 that A is an IVI (T, S)-fuzzy implicative filter. Furthermore, we have A is an IVI (T, S)-fuzzy Boolean filter from Theorem 4.

From Propositions 12 to 15 and Theorem 4, we have the following:

Theorem 5 *Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Then the following are equivalent:*

1. *A is an IVI (T, S)-fuzzy implicative (Boolean) filter*;
2. $\tilde{M}_A(x \to y) \ge T(\tilde{M}_A(z \to (x \to (y' \to y))), \tilde{M}_A(z))$ *and* $\tilde{N}_A(x \to y) \le S(\tilde{N}_A(z \to (x \to (y' \to y))), \tilde{N}_A(z))$ *for any* $x, y, z \in L$;
3. $\tilde{M}_A(x) \ge T(\tilde{M}_A(z \to (x \to (x \to y))), \tilde{M}_A(z))$ *and* $\tilde{N}_A(x) \le S(\tilde{N}_A(z \to (x \to (x \to y))), \tilde{N}_A(z))$ *for any* $x, y, z \in L$;
4. $\tilde{M}_A(x \to y) \ge \tilde{M}_A(x \to (y' \to y))$ *and* $\tilde{N}_A(x \to y) \le \tilde{N}_A(x \to (y' \to y))$ *for any* $x, y \in L$;
5. $\tilde{M}_A(x) \ge T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ *and* $\tilde{N}_A(x) \ge S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$ *for any* $x, y, z \in L$;
6. $\tilde{M}_A(x) = \tilde{M}_A(x' \to x)$ *and* $\tilde{N}_A(x) = \tilde{N}_A(x' \to x)$ *for any* $x \in L$;
7. $\tilde{M}_A(x) = \tilde{M}_A((x \to y) \to x)$ *and* $\tilde{N}_A(x) = \tilde{N}_A((x \to y) \to x)$ *for any* $x, y \in L$.

Proposition 16 *Let A be an interval valued intuitionistic fuzzy set on $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy implicative (Boolean) filter of $\mathcal{L}$, if and only if, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$ and $\tilde{\alpha} + \tilde{\beta} \le \tilde{1}$, the sets $U(\tilde{M}_A; \tilde{\alpha})(\ne \emptyset)$ and $L(\tilde{N}_A; \tilde{\beta})(\ne \emptyset)$ are implicative (Boolean) filters of $\mathcal{L}$, where $U(\tilde{M}_A; \tilde{\alpha}) = \{x \in L | \tilde{M}_A(x) \ge \tilde{\alpha}\}, L(\tilde{N}_A; \tilde{\beta}) = \{x \in L | \tilde{N}_A(x) \le \tilde{\beta}\}$.*

Proof It similar to Theorem 1, the details is omitted.

5 IVI (T, S)-fuzzy positive implicative $(G\text{-})$ filters

Definition 7 Let A be an interval-valued intuitionistic fuzzy subset. A is called an IVI (T, S)-fuzzy G filter if it satisfies (V1) and

(V12) $\tilde{M}_A(x \to (x \to y)) \le \tilde{M}_A(x \to y)$ and $\tilde{N}_A(x \to (x \to y)) \ge \tilde{N}_A(x \to y)$ for any $x, y \in L$.

Remark 1 Obviously, in Definition 7, the condition (V12) could equivalently be replaced by the following condition: $\tilde{M}_A(x \to (x \to y)) = \tilde{M}_A(x \to y)$ and $\tilde{N}_A(x \to (x \to y)) = \tilde{N}_A(x \to y)$.

Definition 8 Let A be an interval-valued intuitionistic fuzzy subset. A is called an IVI (T, S)-fuzzy positive implicative filter of $\mathcal{L}$ if it satisfies (V1) and

(V13)$\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(x \to (y \to z)), \tilde{M}_A(x \to y))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(x \to (y \to z)), \tilde{N}_A(x \to y))$ for any $x, y, z \in L$.

Example 3 Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in example 1, and define an interval-valued intuitionistic fuzzy subset A of $\mathcal{L}$ by

$$\tilde{M}_A(0) = [0.3, 0.4] = \tilde{M}_A(a) = \tilde{M}_A(b), \tilde{M}_A(1) = [0.5, 0.6].$$

$$\tilde{N}_A(0) = [0.4, 0.5] = \tilde{N}_A(a) = \tilde{N}_A(b), \tilde{N}_A(1) = [0.2, 0.3].$$

It is easy to verify that A is an interval-valued intuitionistic $(\mathcal{T}_P, S)$ -fuzzy positive implicative filter of $\mathcal{L}$, and also an interval-valued intuitionistic $(\mathcal{T}_P, S)$-fuzzy G-filter. But it is not an imaginable interval-valued intuitionistic $(\mathcal{T}_P, S)$-fuzzy filter, where

$$T_P(\tilde{x}, \tilde{y}) = [x^- y^-, max\{x^- y^+, x^+ y^-\}], S = max.$$

Taking $x = 1$ in (V12), we can obtain the following Proposition.

Proposition 17 *Each IVI (T, S)-fuzzy positive implicative filter is an IVI (T, S)-fuzzy filter.*

Theorem 6 *Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Then the following are equivalent*:

1. *A is an IVI (T, S)-fuzzy positive implicative filter;*
2. $\tilde{M}_A(x \to y) \geq \tilde{M}_A(x \to (x \to y))$ *and* $\tilde{N}_A(x \to y) \leq \tilde{N}_A(x \to (x \to y))$ *for any* $x, y \in L$;
3. $\tilde{M}_A((x \to y) \to (x \to z)) \geq \tilde{M}_A(x \to (y \to z))$ *and* $\tilde{N}_A((x \to y) \to (x \to z)) \leq \tilde{N}_A(x \to (y \to z))$ *for any* $x, y, z \in L$.

Proof Assume that A is an IVI (T, S)-fuzzy positive implicative filter, we have $\tilde{M}_A(x \to y) \geq T(\tilde{M}_A(x \to (x \to y)), \tilde{M}_A(x \to x)) = T(\tilde{M}_A(x \to (x \to y)), \tilde{M}_A(1)) \geq \tilde{M}_A(x \to (x \to y))$ and $\tilde{N}_A(x \to y) \leq S(\tilde{N}_A(x \to (x \to y)), \tilde{N}_A(x \to x)) = S(\tilde{N}_A(x \to (x \to y)), \tilde{N}_A(1)) \leq \tilde{N}_A(x \to (x \to y))$. Thus (2) is valid.

Suppose that (2) holds. That is, assume that A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$ and $\tilde{M}_A(x \to y) \geq \tilde{M}_A(x \to (x \to y)), \tilde{N}_A(x \to y) \leq \tilde{N}_A(x \to (x \to y))$. Note that $x \to (y \to z) \leq x \to ((x \to y) \to (x \to z))$, it follows that

$$\begin{aligned}\tilde{M}_A((x \to y) \to (x \to z)) &= \tilde{M}_A(x \to ((x \to y) \to z)) \\ &\geq \tilde{M}_A(x \to (x \to ((x \to y) \to z))) \\ &= \tilde{M}_A(x \to ((x \to y) \to (x \to z))) \\ &\geq \tilde{M}_A(x \to (y \to z)).\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A((x \to y) \to (x \to z)) &= \tilde{N}_A(x \to ((x \to y) \to z)) \\ &\leq \tilde{N}_A(x \to (x \to ((x \to y) \to z))) \\ &= \tilde{N}_A(x \to ((x \to y) \to (x \to z))) \\ &\leq \tilde{N}_A(x \to (y \to z)).\end{aligned}$$

Assume that (3) holds, since A ia an IVI (T, S)-fuzzy filter, we have $\tilde{M}_A(x \to z) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A((x \to y) \to (x \to z))) \geq T(\tilde{M}_A(x \to y), \tilde{M}_A(x \to (y \to z)))$ and $\tilde{N}_A(x \to z) \leq S(\tilde{N}_A(x \to y), \quad \tilde{N}_A((x \to y) \to (x \to z))) \leq S(\tilde{N}_A(x \to y), \tilde{N}_A(x \to (y \to z)))$. Therefore, from Definition 8, A is an IVI (T, S)-fuzzy positive implicative filter of $\mathcal{L}$.

Corollary 4 *Let A be an interval-valued intuitionistic fuzzy subset of $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy G filter if and only if A is an IVI (T, S)-fuzzy positive implicative filter.*

Proposition 18 *Let A be an IVI (T, S)-fuzzy filter of $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy positive implicative filter if and only if* $\tilde{M}_A(y \to x) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ *and* $\tilde{N}_A(y \to x) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$ *for any* $x, y, z \in L$.

Proof Since A is an IVI (T, S)-fuzzy positive implicative filter, we have $\tilde{M}_A(y \to (y \to x)) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to (y \to x)) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$. Since $y \to x = 1 \to (y \to x) = (y \to y) \to (y \to x)$, it follows from that $\tilde{M}_A(y \to x) = \tilde{M}_A((y \to y) \to (y \to x)) \geq \tilde{M}_A(y \to (y \to x)) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to x) = \tilde{N}_A((y \to y) \to (y \to x)) \leq \tilde{N}_A(y \to (y \to x)) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$.

Conversely, let A be an IVI (T, S)-fuzzy filter and satisfies the condition: $\tilde{M}_A(y \to x) \geq T(\tilde{M}_A(z \to (y \to (y \to x))), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to x) \leq S(\tilde{N}_A(z \to (y \to (y \to x))), \tilde{N}_A(z))$ for any $x, y, z \in L$. And so

$$\tilde{M}_A(z \to x) \geq T(\tilde{M}_A((z \to y) \to (z \to (z \to x))), \tilde{M}_A(z \to y)).$$

and

$$\tilde{N}_A(z \to x) \leq S(\tilde{N}_A((z \to y) \to (z \to (z \to x))), \tilde{N}_A(z \to y)).$$

Since $z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x))$, we have $\tilde{M}_A((z \to y) \to (z \to (z \to x))) \geq \tilde{M}_A(z \to (y \to x))$ and $\tilde{N}_A((z \to y) \to (z \to (z \to x))) \leq \tilde{N}_A(z \to (y \to x))$. We have $\tilde{M}_A(z \to x) \geq T(\tilde{M}_A((z \to y) \to (z \to (z \to x))), \tilde{M}_A(z \to y)) \geq T(\tilde{M}_A(z \to (y \to x)), \tilde{M}_A(z \to y))$ and $\tilde{N}_A(z \to x) \leq S(\tilde{N}_A((z \to y) \to (z \to (z \to x))), \tilde{N}_A(z \to y)) \leq S(\tilde{N}_A(z \to (y \to x)), \tilde{N}_A(z \to y))$. Hence A is an IVI (T, S)-fuzzy positive implicative filter.

Proposition 19 *Let A be an IVI (T, S)-fuzzy positive implicative filter of $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$ if and only if $\tilde{M}_A((y \to x) \to x) = \tilde{M}_A((x \to y) \to y)$ and $\tilde{N}_A((y \to x) \to x) = \tilde{N}_A((x \to y) \to y)$ for any $x, y \in L$.*

Proof Assume that A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$. We have, for any $x, y, z \in L$,

$$\tilde{M}_A((y \to x) \to x) \geq T(\tilde{M}_A(z \to (((y \to x) \to x) \to y) \to ((y \to x) \to x)), \tilde{M}_A(z)). \tag{3}$$

and

$$\tilde{N}_A((y \to x) \to x) \leq S(\tilde{N}_A(z \to (((y \to x) \to x) \to y) \to ((y \to x) \to x)), \tilde{N}_A(z)). \tag{4}$$

Taking $z = 1$ in (3) and (4), we have

$$\tilde{M}_A((y \to x) \to x) \geq \tilde{M}_A(((y \to x) \to x) \to y) \to ((y \to x) \to x). \tag{5}$$

and

$$\tilde{N}_A((y \to x) \to x) \leq \tilde{N}_A(((y \to x) \to x) \to y) \to ((y \to x) \to x). \tag{6}$$

Since $(x \to y) \to y \leq (y \to x) \to ((x \to y) \to x) = (x \to y) \to ((y \to x) \to x) \leq (((y \to x) \to x) \to y) \to (y \to x) \to x$ and A is an IVI (T, S)-fuzzy filter of L, we have $\tilde{M}_A((((y \to x) \to x) \to y) \to (y \to x) \to x) \geq \tilde{M}_A((x \to y) \to y)$ and $\tilde{N}_A((((y \to x) \to x) \to y) \to (y \to x) \to x) \leq \tilde{N}_A((x \to y) \to y)$. From (5) and (6), $\tilde{M}_A((y \to x) \to x) \geq \tilde{M}_A((x \to y) \to y)$ and $\tilde{N}_A((y \to x) \to x) \leq \tilde{N}_A((x \to y) \to y)$ for any $x, y \in L$. Similarly, we can prove $\tilde{M}_A((x \to y) \to y) \geq \tilde{M}_A((y \to x) \to x)$ and $\tilde{N}_A((x \to y) \to y) \leq \tilde{N}_A((y \to x) \to x)$. Hence $\tilde{M}_A((y \to x) \to x) = \tilde{M}_A((x \to y) \to y)$ and $\tilde{N}_A((y \to x) \to x) = \tilde{N}_A((x \to y) \to y)$.

Conversely, since A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, we have, for any $x, y, z \in L$, $\tilde{M}_A((x \to y) \to x) \geq T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ and $\tilde{N}_A((x \to y) \to x) \leq S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$. Since $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$, we have $\tilde{M}_A((x \to y) \to ((x \to y) \to y)) \geq \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A((x \to y) \to ((x \to y) \to y)) \leq \tilde{N}_A((x \to y) \to x)$. By hypotheses, $\tilde{M}_A((y \to x) \to x) = \tilde{M}_A((x \to y) \to y)$ and $\tilde{N}_A((y \to x) \to x) = \tilde{N}_A((x \to y) \to y)$. Since A is IVI (T, S)-fuzzy filter of $\mathcal{L}$, $\tilde{M}_A((x \to y) \to y) \geq \tilde{M}_A((x \to y) \to ((x \to y) \to y))$ and $\tilde{N}_A((x \to y) \to y) \leq \tilde{N}_A((x \to y) \to ((x \to y) \to y))$. It follows that

$$\tilde{M}_A((y \to x) \to x) \geq \tilde{M}_A((x \to y) \to x). \tag{7}$$

and

$$\tilde{N}_A((y \to x) \to x) \leq \tilde{N}_A((x \to y) \to x). \tag{8}$$

Since $y \leq x \to y$ and $y \to x \leq z \to (y \to x)$, we get $(x \to y) \to x \leq y \to x \leq z \to (y \to x)$. It follows that

$$\tilde{M}_A(z \to (y \to x)) \geq \tilde{M}_A((x \to y) \to x). \tag{9}$$

and

$$\tilde{N}_A(z \to (y \to x)) \leq \tilde{N}_A((x \to y) \to x). \tag{10}$$

Hence $\tilde{M}_A(y \to x) \geq T(\tilde{M}_A(z \to (x \to y), \tilde{M}_A(z)) \geq T(\tilde{M}_A((x \to y) \to x), \tilde{M}_A(z))$ and $\tilde{N}_A(y \to x) \leq S(\tilde{N}_A(z \to (x \to y), \tilde{N}_A(z)) \leq S(\tilde{N}_A((x \to y) \to x), \tilde{N}_A(z))$. And so

$$\begin{aligned}\tilde{M}_A(x) &\geq T(\tilde{M}_A((y \to x) \to x), \tilde{M}_A(y \to x)) \\ &\geq T(\tilde{M}_A((x \to y) \to x), \tilde{M}_A(y \to x)) \\ &\geq T(\tilde{M}_A((x \to y) \to x), T(\tilde{M}_A((x \to y) \to x), \tilde{M}_A(z))) \\ &\geq T(T(\tilde{M}_A(z), \tilde{M}_A(z \to ((x \to y) \to x))), \tilde{M}_A(z)) \\ &= T(\tilde{M}_A(z), \tilde{M}_A(z \to ((x \to y) \to x))).\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(x) &\leq S(\tilde{N}_A((y \to x) \to x), \tilde{N}_A(y \to x)) \\ &\leq S(\tilde{M}_A((x \to y) \to x), \tilde{N}_A(y \to x)) \\ &\leq S(\tilde{N}_A((x \to y) \to x), S(\tilde{N}_A((x \to y) \to x), \tilde{N}_A(z))) \\ &\leq S(S(\tilde{N}_A(z), \tilde{N}_A(z \to ((x \to y) \to x))), \tilde{N}_A(z)) \\ &= S(\tilde{N}_A(z), \tilde{N}_A(z \to ((x \to y) \to x))).\end{aligned}$$

Therefore A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$ from Proposition 18.

Proposition 20 *Let A be an IVI (T, S)-fuzzy filter. Then A is an IVI (T, S)-fuzzy positive implicative filter if and only if $\tilde{M}_A(x \to x^2) = \tilde{M}_A(1)$ and $\tilde{N}_A(x \to x^2) = \tilde{N}_A(1)$ for any $x \in L$.*

Proof Suppose that A is an IVI (T, S)-fuzzy positive implicative filter. Since $x \rightarrow (x \rightarrow x^2) = 1$, we have $\tilde{M}_A(x \rightarrow (x \rightarrow x^2)) = \tilde{M}_A(1)$ and $\tilde{N}_A(x \rightarrow (x \rightarrow x^2)) = \tilde{N}_A(1)$. Therefore $\tilde{M}_A(x \rightarrow x^2) \geq T(\tilde{M}_A(x \rightarrow (x \rightarrow x^2)), \tilde{M}_A(x \rightarrow x)) = T(\tilde{M}_A(1), \tilde{M}_A(1)) = \tilde{M}_A(1)$ and $\tilde{N}_A(x \rightarrow x^2) \leq S(\tilde{N}_A(x \rightarrow (x \rightarrow x^2)), \tilde{N}_A(x \rightarrow x)) = S(\tilde{N}_A(1), \tilde{N}_A(1)) = \tilde{N}_A(1)$. Therefore $\tilde{M}_A(x \rightarrow x^2) = \tilde{M}_A(1)$ and $\tilde{N}_A(x \rightarrow x^2) = \tilde{N}_A(1)$ for any $x \in L$.

Conversely, suppose that $\tilde{M}_A(x \rightarrow x^2) = \tilde{M}_A(1)$ and $\tilde{N}_A(x \rightarrow x^2) = \tilde{N}_A(1)$ for any $x \in L$. Since $(x \rightarrow x^2) \rightarrow (x \rightarrow y) \geq x^2 \rightarrow y = x \rightarrow (x \rightarrow y)$ and A is an IVI (T, S)-fuzzy filter of $\mathcal{L}$, we have $\tilde{M}_A((x \rightarrow x^2) \rightarrow (x \rightarrow y)) \geq \tilde{M}_A(x \rightarrow (x \rightarrow y))$ and $\tilde{N}_A((x \rightarrow x^2) \rightarrow (x \rightarrow y)) \leq \tilde{N}_A(x \rightarrow (x \rightarrow y))$. Hence $\tilde{M}_A(x \rightarrow y) \geq T(\tilde{M}_A((x \rightarrow x^2) \rightarrow (x \rightarrow y)), \tilde{M}_A(x \rightarrow x^2)) \geq T(\tilde{M}_A(x \rightarrow (x \rightarrow y)), \tilde{M}_A(x \rightarrow x^2)) = T(\tilde{M}_A(x \rightarrow (x \rightarrow y)), \tilde{M}_A(1)) \geq \tilde{M}_A(x \rightarrow (x \rightarrow y))$ and $\tilde{N}_A(x \rightarrow y) \leq S(\tilde{N}_A((x \rightarrow x^2) \rightarrow (x \rightarrow y)), \tilde{N}_A(x \rightarrow x^2)) \leq S(\tilde{N}_A(x \rightarrow (x \rightarrow y)), \tilde{N}_A(x \rightarrow x^2)) = S(\tilde{N}_A(x \rightarrow (x \rightarrow y)), \tilde{N}_A(1)) \leq \tilde{N}_A(x \rightarrow (x \rightarrow y))$. It follows from Theorem 6 that A is an IVI (T, S)-fuzzy positive implicative filter of $\mathcal{L}$.

Proposition 21 *Let A be an interval valued intuitionistic fuzzy set on $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy positive implicative (G)-filter of $\mathcal{L}$, if and only if, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$ and $\tilde{\alpha} + \tilde{\beta} \leq \tilde{1}$, the sets $U(\tilde{M}_A; \tilde{\alpha})(\neq \emptyset)$ and $L(\tilde{N}_A; \tilde{\beta})(\neq \emptyset)$ are positive implicative (G)-filters of $\mathcal{L}$, where $U(\tilde{M}_A; \tilde{\alpha}) = \{x \in L | \tilde{M}_A(x) \geq \tilde{\alpha}\}, L(\tilde{N}_A; \tilde{\beta}) = \{x \in L | \tilde{N}_A(x) \leq \tilde{\beta}\}$.*

Proof It similar to Theorem 1, the details is omitted.

6 IVI (T, S)-fuzzy MV- (fantastic) filters

Definition 9 An interval-valued intuitionistic fuzzy set A is called an IVI (T, S)-fuzzy MV-filter of $\mathcal{L}$ if it is an IVI (T, S)-fuzzy filter and satisfies

(V14) $\tilde{M}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \tilde{M}_A(y \rightarrow x)$ and $\tilde{N}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) \leq \tilde{N}_A(y \rightarrow x)$ for any $x, y \in L$.

Remark 2 In lattice implication algebras, BL-algebras, R_0-algebras, the MV-filters are called fantastic filters.

Theorem 7 *Let A be an interval-valued intuitionistic fuzzy set $\mathcal{L}$. Then A be an interval-valued T-fuzzy MV-filter, if and only if, it satisfies (V1) and*

(V15) $\tilde{M}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq T(\tilde{M}_A(z), \tilde{M}_A(z \rightarrow (y \rightarrow x)))$ *and*

$$\tilde{N}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) \leq S(\tilde{N}_A(z), \tilde{N}_A(z \rightarrow (y \rightarrow x)))$$

for any $x, y, z \in L$.

Proof Suppose that A be an IVI (T, S)-fuzzy MV-filter, then (V1) is trivial. By Definition 9, we have

$$\begin{aligned}\tilde{M}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) &\geq \tilde{M}_A(y \rightarrow x)\\ &\geq T(\tilde{M}_A(z \rightarrow (y \rightarrow x)), \tilde{M}_A(z))\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) &\leq \tilde{N}_A(y \rightarrow x)\\ &\leq S(\tilde{N}_A(z \rightarrow (y \rightarrow x)), \tilde{N}_A(z)).\end{aligned}$$

Conversely, suppose that A satisfies the conditions (V15). Taking $z = 1$ in (V15), we have

$$\tilde{M}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \tilde{M}_A(y \rightarrow x)$$

and

$$\tilde{N}_A(((x \rightarrow y) \rightarrow y) \rightarrow x) \leq \tilde{N}_A(y \rightarrow x).$$

Taking $y = 1$ in (V15),

$$\begin{aligned}\tilde{M}_A(x) &= \tilde{M}_A(((x \rightarrow 1) \rightarrow 1) \rightarrow x)\\ &\geq T(\tilde{M}_A(z), \tilde{M}_A(z \rightarrow (1 \rightarrow x)))\\ &= T(\tilde{M}_A(z \rightarrow x), \tilde{M}_A(z))\end{aligned}$$

and

$$\begin{aligned}\tilde{N}_A(x) &= \tilde{N}_A(((x \rightarrow 1) \rightarrow 1) \rightarrow x)\\ &\leq S(\tilde{N}_A(z), \tilde{N}_A(z \rightarrow (1 \rightarrow x)))\\ &= S(\tilde{N}_A(z \rightarrow x), \tilde{N}_A(z)).\end{aligned}$$

Together with (V1), we have A is an IVI (T, S)-fuzzy filter of L. Therefore, A is an IVI (T, S)-fuzzy MV-filter.

In MV-algebras and lattice implication algebras, we have $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ and $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$. Therefore, we have the following Corollary.

Corollary 5 *In MV-algebras and lattice implication algebras, IVI (T, S)-fuzzy filters and IVI (T, S)-fuzzy MV-filters are equivalent.*

Proposition 22 *Let A be an interval valued intuitionistic fuzzy set on $\mathcal{L}$. Then A is an IVI (T, S)-fuzzy MV filter of $\mathcal{L}$, if and only if, for any $\tilde{\alpha}, \tilde{\beta} \in D[0,1]$ and $\tilde{\alpha} + \tilde{\beta} \leq \tilde{1}$, the sets $U(\tilde{M}_A; \tilde{\alpha})(\neq \emptyset)$ and $L(\tilde{N}_A; \tilde{\beta})(\neq \emptyset)$ are MV filter of $\mathcal{L}$, where $U(\tilde{M}_A; \tilde{\alpha}) = \{x \in L | \tilde{M}_A(x) \geq \tilde{\alpha}\}, L(\tilde{N}_A; \tilde{\beta}) = \{x \in L | \tilde{N}_A(x) \leq \tilde{\beta}\}$.*

Proof It similar to Theorem 1, the details is omitted.

Proposition 23 *In a residuated lattice, every IVI (T, S)-fuzzy implicative filter is an IVI (T, S)-fuzzy MV-filter.*

Proof Assume that A be an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$. Since $x \leq ((x \rightarrow y) \rightarrow y) \rightarrow x$, and so $(((x \rightarrow y) \rightarrow y) \rightarrow x) \rightarrow y \leq x \rightarrow y$. This implies that

$$((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)$$
$$\geq (x \to y) \to ((((x \to y) \to y) \to x))$$
$$= ((x \to y) \to y) \to ((x \to y) \to x)$$
$$\geq y \to x.$$

It follows that $\tilde{M}_A(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \geq \tilde{M}_A(y \to x)$ and $\tilde{N}_A(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \leq \tilde{N}_A(y \to x)$. Since A be an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$, we have $\tilde{M}_A((((x \to y) \to y) \to x)) \geq T(\tilde{M}_A(z \to (((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x))), \tilde{M}_A(z))$and $\tilde{N}_A((((x \to y) \to y) \to x)) \leq S(\tilde{N}_A(z \to (((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x))), \tilde{N}_A(z))$. Taking $z = 1$, we get $\tilde{M}_A((((x \to y) \to y) \to x)) \geq \tilde{M}_A(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \geq \tilde{M}_A(y \to x)$ and $\tilde{N}_A((((x \to y) \to y) \to x)) \leq \tilde{N}_A(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \leq \tilde{N}_A(y \to x)$. Hence, A is an IVI (T, S)-fuzzy MV-filter.

The following Theorem gives the relation among the IVI (T, S)-fuzzy implicative filter, IVI (T, S)-fuzzy positive implicative filter and IVI (T, S)-fuzzy MV filter.

Theorem 8 *An interval-valued intuitionistic fuzzy set A of $\mathcal{L}$ is an IVI (T, S)-fuzzy implicative (Boolean) filter of L if and only if A is both an IVI (T, S)-fuzzy positive implicative filter and an IVI (T, S)-fuzzy MV-filter of $\mathcal{L}$.*

Proof Assume that A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$. From Proposition 23, we know A is both an IVI (T, S)-fuzzy positive implicative filter and an IVI (T, S)-fuzzy MV filter of $\mathcal{L}$.

Conversely, suppose that A is both an IVI (T, S)-fuzzy positive implicative filter and an IVI (T, S)-fuzzy MV filter of $\mathcal{L}$, we have, $\tilde{M}_A((x \to y) \to y) \geq \tilde{M}_A((x \to y) \to ((x \to y) \to y))$ and $\tilde{N}_A((x \to y) \to y) \leq \tilde{N}_A((x \to y) \to ((x \to y) \to y))$. Since $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$, it follows that $\tilde{M}_A((x \to y) \to ((x \to y) \to y)) \geq \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A((x \to y) \to ((x \to y) \to y)) \leq \tilde{N}_A((x \to y) \to x)$. And so $\tilde{M}_A((x \to y) \to y) \geq \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A((x \to y) \to y) \leq \tilde{N}_A((x \to y) \to x)$. On the other hand, since A is an IVI (T, S)-fuzzy MV-filter of $\mathcal{L}$. So we have $\tilde{M}_A(((x \to y) \to y) \to x) \geq \tilde{M}_A(y \to x)$ and $\tilde{N}_A(((x \to y) \to y) \to x) \leq \tilde{N}_A(y \to x)$. Since $(x \to y) \to x \leq y \to x$, we have $\tilde{M}_A(y \to x) \geq \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(y \to x) \leq \tilde{N}_A((x \to y) \to x)$. Thus, $\tilde{M}_A(((x \to y) \to y) \to x) \geq \tilde{M}_A((x \to y) \to x)$ and $\tilde{N}_A(((x \to y) \to y) \to x) \leq \tilde{N}_A((x \to y) \to x)$, and so $\tilde{M}_A(x) \geq T(\tilde{M}_A(((x \to y) \to y) \to x), \tilde{M}_A((x \to y) \to y)) \geq \tilde{M}_A((x \to y) \to x), \tilde{M}_A((x \to y) \to x) \geq T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ and $\tilde{N}_A(x) \leq S(\tilde{N}_A(((x \to y) \to y) \to x), \tilde{N}_A((x \to y) \to y)) \leq \tilde{N}_A((x \to y) \to x), \tilde{N}_A((x \to y) \to x) \leq S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$. Hence $\tilde{M}_A(x) \geq T(\tilde{M}_A(z \to ((x \to y) \to x)), \tilde{M}_A(z))$ and $\tilde{N}_A(x) \leq S(\tilde{N}_A(z \to ((x \to y) \to x)), \tilde{N}_A(z))$. So A is an IVI (T, S)-fuzzy implicative filter of $\mathcal{L}$.

7 Conclusions

Filter theory plays an very important role in studying logical systems and the related algebraic structures. In this paper, we have combined the interval-valued intuitionistic fuzzy set, t-norm, s-norm and the filter theory to develop the interval-valued intuitionistic (T, S)-fuzzy (implicative, Boolean, positive implication, G) filter theory of residuated lattices. Mainly, we give some new characterizations of interval-valued intuitionistic (T, S)-fuzzy (implicative, Boolean, positive implication, G, MV) filters in residuated lattices. Meanwhile, the relations among these fuzzy filters are investigated, especially in some BL-algebras and lattice implication algebras. We desperately hope that our work would serve as a foundation for enriching corresponding many-valued logical system.

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