Int. J. Mach. Learn. & Cyber. (2013) 4:659–669
DOI 10.1007/s13042-012-0130-8

ORIGINAL ARTICLE

Intuitionistic fuzzy filter theory of *BL*-algebras

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Received: 25 April 2012 / Accepted: 31 August 2012 / Published online: 27 November 2012

Abstract In this paper, the intuitionistic fuzzy filter theory of *BL*-algebras is researched. The basic knowledge of *BL*-algebras and intuitionistic fuzzy sets is firstly reviewed. The notions of intuitionistic fuzzy filters, lattice filters, prime filters, Boolean filters, implicative filters, positive implicative filters, ultra filters and obstinate filters are introduced, respectively. Their important properties are investigated. In intuitionistic fuzzy sets, intuitionistic fuzzy filters, Boolean filters, ultra filters are proved to be equivalent to lattice filters, implicative filters, obstinate filters, respectively. Each intuitionistic fuzzy Boolean filter is an intuitionistic fuzzy positive implicative filter, but the converse may not be true in *BL*-algebras. The conditions under an intuitionistic fuzzy positive implicative filter being an intuitionistic fuzzy Boolean filter are constructed. Finally, the concepts of the intuitionistic fuzzy ultra and obstinate filters are introduced, and the intuitionistic fuzzy ultra filter is proved to be equivalent to the intuitionistic fuzzy obstinate filter in *BL*-algebras.

Keywords *BL*-algebras · Intuitionistic fuzzy filters · Intuitionistic fuzzy Boolean and implicative filters · Intuitionistic fuzzy positive implicative filters · Intuitionistic fuzzy ultra and obstinate filters

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1 Introduction

Fuzzy sets [1] are an important basic theory of machine learning and cybernetics, and provide a basic theory for fuzzy logical reasoning. Wu researched positive and negative fuzzy rule system, extreme machine learning and image classification [2]. Pearl presented a hierarchial multilevel thresholding method for edge information extraction using fuzzy entropy [3]. Wang researched particle swarm optimization for determining fuzzy measures from data [4] and studied maximum ambiguity based on sample selection in fuzzy decision tree induction [5].

The logical foundations of processes handling uncertainty in information use some classes of algebras as algebraic semantics [6]. *BL*-algebras are the important logical algebras, and have been widely applied to many fields [7–13]. The filter theory plays an important role in the study of logical algebras, and the sets of provable formulae in corresponding inference systems from the point of view of uncertain information can be described by fuzzy filters of those algebraic semantics [6], and it has been researched by many scholars [6–14, 17–22]. From the perspective of logic, different fields correspond to different sets of provable formulae. In 2010, the filter theory of residuated lattices was researched by Zhu and Xu [14]. Thus far, the filter theory of *BL*-algebras has been widely studied, and some important results have been obtained. In particular, Michiro studied the filter theory of *BL*-algebras [7], and Jiri researched fuzzy filters and fuzzy prime filters of bounded R*l*-monoids and pseudo *BL*-algebras [6]. The notions of fuzzy filters and fuzzy Boolean filters (implicative filters, positive implicative filters, prime and ultra filters, etc.) in *BL*-algebras were introduced, and their relative properties were obtained [8–11]. Moreover, based on the concept of interval valued fuzzy sets introduced by

Zadeh [15], Xu generalized the fuzzy filter theory of BL-algebras [12, 13], and so on.

However, based on the notion of intuitionistic fuzzy sets (IFS) proposed by Atanassov [16], the concept of the intuitionistic fuzzy filter of BCI-algebras was introduced and some important results were given [17]. Pei studied the intuitionistic fuzzy filter of lattice implication algebras and investigated their properties [18]. The intuitionistic fuzzy filter in Heyting algebras was presented by Wang and Guo [19]. Recently, Peng introduced the intuitionistic fuzzy filter of effect algebras [20]. At present, the researches about intuitionistic fuzzy filters in BL-algebras are less frequent.

In this paper, the intuitionistic fuzzy filter theory of BL-algebras is developed. This paper is organized as follows. In Sect. 2, we review related basic knowledge of BL-algebras and intuitionistic fuzzy sets. In Sect. 3, we introduce an intuitionistic fuzzy filter and a lattice filter of BL-algebras and discuss their properties. We show that the intuitionistic fuzzy filter is the intuitionistic fuzzy lattice filter. In Sect. 4, we give the notion of intuitionistic fuzzy filters of BL-algebras. In Sect. 5, intuitionistic fuzzy Boolean filters and implicative filters are presented and their important properties are investigated. The intuitionistic fuzzy Boolean filter is proved to be equivalent to the intuitionistic fuzzy implicative filter. In Sect. 6, we introduce the notion of intuitionistic fuzzy positive implicative filters in BL-algebras and investigate their properties. Furthermore, the conditions under an intuitionistic fuzzy positive implicative filter being an intuitionistic fuzzy Boolean filter are constructed. In Sect. 7, we give the concepts of intuitionistic fuzzy ultra filters and obstinate filters, and study their properties. Meanwhile, we prove that the intuitionistic fuzzy ultra filter is equivalent to the intuitionistic fuzzy obstinate filter in BL-algebras.

2 Preliminaries

For the sake of convenience for statement, we firstly summarize some related definitions and results which will be used in the following.

Definition 2.1 ([8, 9, 21–23]) A BL-algebra is a structure $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that the following conditions are satisfied:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(L, \otimes, 1)$ is an abelian monoid, i.e. $\otimes$ is commutative and associative and $x \otimes 1 = x$,
(3) $\otimes$ and $\rightarrow$ form an adjoint pair, i.e. $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$ (residuation),
(4) $x \wedge y = x \otimes (x \rightarrow y)$ (divisibility),
(5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

Lemma 2.1 ([8, 20–24]) *Let L be a BL-algebra. The following properties hold, for all $x, y, z \in L$ (where $\neg x = x \rightarrow 0$).*

(1) $x \otimes (x \rightarrow y) \leq y$,
(2) $x \leq y \rightarrow (x \otimes y)$,
(3) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(4) $x = 1 \rightarrow x$, $x \rightarrow x = 1$, $x \rightarrow (y \rightarrow x) = 1$, $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$,
(5) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(6) $x \otimes y \leq x \wedge y$,
(7) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
(8) *If* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$,
(9) $x \leq y \rightarrow x$,
(10) $y \leq (y \rightarrow x) \rightarrow x$,
(11) $x \otimes 0 = 0$,
(12) $x \leq y$ *implies* $x \otimes z \leq y \otimes z$,
(13) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(14) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
(15) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
(16) $x \rightarrow y \leq (x \otimes z) \rightarrow (y \otimes z)$,
(17) $x \otimes (y \rightarrow z) \leq y \rightarrow (x \otimes z)$,
(18) $(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$,
(19) $x \otimes \neg x = 0$,
(20) $x \otimes y = 0$ *if and only if* $x \leq \neg y$ *and* $x \leq y$ *implies* $\neg y \leq \neg x$,
(21) $x \vee y = 1$ *implies* $x \otimes y = x \wedge y$,
(22) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
(23) $x \leq \neg\neg x$, $\neg 1 = 0$, $\neg 0 = 1$, $\neg\neg\neg x = \neg x$, $\neg\neg x \leq \neg x \rightarrow x$,
(24) $\neg\neg(x \otimes y) = \neg\neg x \otimes \neg\neg y$,
(25) $\neg(x \vee y) = \neg x \wedge \neg y$,
(26) *If* $\neg\neg x \leq \neg\neg x \rightarrow x$, *then* $\neg\neg x = x$,
(27) $x = \neg\neg x \otimes (\neg\neg x \rightarrow x)$,
(28) $x \rightarrow \neg y = y \rightarrow \neg x = \neg\neg x \rightarrow \neg y = \neg(x \otimes y)$,
(29) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$,
(30) $\neg(\neg\neg x \rightarrow x) = 0$.

Definition 2.2 ([16]) Let a set U be the fixed domain. An intuitionistic fuzzy set A in U is an object, having the form

$$A(x) = \{<x, \mu_A(x), \nu_A(x)> | x \in U\}$$

Where $\mu_A(x)$: $U \rightarrow [0, 1]$ and $\nu_A(x)$: $U \rightarrow [0, 1]$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in U$, and $\mu_A(x)$ and $\nu_A(x)$ are the degree of membership and the degree of non-membership of the element $x \in U$ to A, respectively.

Definition 2.3 ([16]) Let A and B be arbitrary intuitionistic fuzzy sets, then inclusion relation of IFS is defined by

$$A \subseteq B \text{ if and only if } \mu_A(x) \leq \mu_B(x),\ \nu_A(x) \geq \nu_B(x).$$

For convenience, for all $x, y \in [0, 1]$, we denote max $\{x, y\} = x \vee y$ and min $\{x, y\} = x \wedge y$, respectively.

3 Intuitionistic fuzzy filters and lattice filters of *BL*-algebras

Definition 3.1 Let L be a BL-algebra, and an intuitionistic fuzzy set A is called an intuitionistic fuzzy filter of L if it satisfies:

(1) $\mu_A(1) \geq \mu_A(x)$, $\nu_A(1) \leq \nu_A(x)$ for all $x \in L$,
(2) $\mu_A(y) \geq \min\{\mu_A(x), \mu_A(x \to y)\} = \mu_A(x) \wedge \mu_A(x \to y)$ for all $x, y \in L$,
(3) $\nu_A(y) \leq \max\{\nu_A(x), \nu_A(x \to y)\} = \nu_A(x) \vee \nu_A(x \to y)$ for all $x, y \in L$.

Example 1 Let $L = \{0, a, b, 1\}$. $\otimes$ and $\to$ are defined by Tables 1 and 2, where $0 < a < b < 1$. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is a BL-algebra. Define an intuitionistic fuzzy set A in L.

$$\mu_A(1) = 0.7, \quad \mu_A(b) = 0.5, \quad \mu_A(0) = \mu_A(a) = 0.3$$
$$\nu_A(1) = 0.2, \quad \nu_A(b) = 0.4, \quad \nu_A(0) = \nu_A(a) = 0.6$$

It can be easily checked that A satisfies the conditions of the intuitionistic fuzzy filter. Therefore, A is an intuitionistic fuzzy filter of L.

Theorem 3.1 *Let the intuitionistic fuzzy set A be the intuitionistic fuzzy filter of L. For all $x, y \in L$, if $x \leq y$, then $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$.*

Proof Suppose that the intuitionistic fuzzy set A is the intuitionistic fuzzy filter. If $x \leq y$, then $x \to y = 1$. By Definition 3.1, we have

$$\mu_A(y) \geq \min\{\mu_A(x), \quad \mu_A(x \to y)\} = \min\{\mu_A(x), \mu_A(1)\} = \mu_A(x)$$
$$\nu_A(y) \leq \max\{\nu_A(x), \quad \nu_A(x \to y)\} = \max\{\nu_A(x), \nu_A(1)\} = \nu_A(x)$$

Hence, $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$. □

Table 1 "$\otimes$" operator table in L

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 2 "$\to$" operator table in L

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Theorem 3.2 *Let A be an intuitionistic fuzzy set in L. A is an intuitionistic fuzzy filter if and only if for all $x, y, z \in L$, $x \to (y \to z) = 1$ implies $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$.*

Proof Suppose that the intuitionistic fuzzy set A is an intuitionistic fuzzy filter. In view of Definition 3.1, then $\mu_A(z) \geq \mu_A(y) \wedge \mu_A(y \to z)$, $\nu_A(z) \leq \nu_A(y) \vee \nu_A(y \to z)$ and $\mu_A(y \to z) \geq \mu_A(x) \wedge \mu_A(x \to (y \to z))$, $\nu_A(y \to z) \leq \nu_A(x) \vee \nu_A(x \to (y \to z))$. If $x \to (y \to z) = 1$, then $\mu_A(y \to z) \geq \mu_A(x) \wedge \mu_A(1) = \mu_A(x)$, $\nu_A(y \to z) \leq \nu_A(x) \vee \nu_A(1) = \nu_A(x)$. So $\mu_A(z) \geq \mu_A(y) \wedge \mu_A(y \to z) \geq \mu_A(y) \wedge \mu_A(x)$, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y \to z) \leq \nu_A(y) \vee \nu_A(x)$. Hence, $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$.

Conversely, since $x \to (x \to 1) = 1$ for all $x \in L$, then $\mu_A(1) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$, $\nu_A(1) \leq \nu_A(x) \vee \nu_A(x) = \nu_A(x)$. On the other hand, from $(x \to y) \to (x \to y) = 1$, it follows that $\mu_A(y) \geq \mu_A(x) \wedge \mu_A(x \to y)$, $\nu_A(y) \leq \nu_A(x) \vee \nu_A(x \to y)$. By Definition 3.1, we get that A is an intuitionistic fuzzy filter. □

From $x \to (y \to z) = (x \otimes y) \to z$ and Theorem 3.2, we have the following.

Corollary 3.1 *Let A be an intuitionistic fuzzy set in L. A is an intuitionistic fuzzy filter if and only if for all $x, y, z \in L$, $x \otimes y \leq z$ or $y \otimes x \leq z$ implies $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$.*

Theorem 3.3 *Let A be an intuitionistic fuzzy set in L. A is an intuitionistic fuzzy filter if and only if* (1) *If $x \leq y$, then $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$, for all $x, y \in L$,* (2) *$\mu_A(x \otimes y) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \otimes y) \leq \nu_A(x) \vee \nu_A(y)$, for all $x, y \in L$.*

Proof Let A be an intuitionistic fuzzy filter of L. According to Theorem 3.1, we get, if $x \leq y$, then $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$. Since $x \otimes y \leq x \otimes y$ and Corollary 3.1, then $\mu_A(x \otimes y) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \otimes y) \geq \nu_A(x) \vee \nu_A(y)$.

Conversely, let A be an intuitionistic fuzzy set which satisfies (1) and (2). For all $x, y, z \in L$, if $x \otimes y \leq z$, then by (1) and (2), we have $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$. By Corollary 3.1, we get that A is an intuitionistic fuzzy filter. □

Corollary 3.2 *Let an intuitionistic fuzzy set A be an intuitionistic fuzzy filter of L. For all $x, y, z \in L$, the following hold.*

(1) *If $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$, then $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$,*
(2) $\mu_A(x \wedge y) = \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \wedge y) = \nu_A(x) \vee \nu_A(y)$,

(3) $\mu_A(x \otimes y) = \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \otimes y) = \nu_A(x) \vee \nu_A(y)$,
(4) $\mu_A(0) = \mu_A(x) \wedge \mu_A(\neg x)$, $\nu_A(0) = \nu_A(x) \vee \nu_A(\neg x)$,
(5) $\mu_A(x \to z) \geq \mu_A(x \to y) \wedge \mu_A(y \to z)$, $\nu_A(x \to z) \leq \nu_A(x \to y) \vee \nu_A(y \to z)$,
(6) $\mu_A(x \to y) \leq \mu_A(x \otimes z \to y \otimes z)$, $\nu_A(x \to y) \geq \nu_A(x \otimes z \to y \otimes z)$,
(7) $\mu_A(x \to y) \leq \mu_A((y \to z) \to (x \to z))$, $\nu_A(x \to y) \geq \nu_A((y \to z) \to (x \to z))$,
(8) $\mu_A(x \to y) \leq \mu_A((z \to x) \to (z \to y))$, $\nu_A(x \to y) \geq \nu_A((z \to x) \to (z \to y))$.

Proof

(1) In view of Definition 3.1, and since $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$, we have $\mu_A(y) \geq \mu_A(x) \wedge \mu_A(x \to y) = \mu_A(x) \wedge \mu_A(1) = \mu_A(x)$, $\nu_A(y) \leq \nu_A(x) \vee \nu_A(x \to y) = \nu_A(x) \vee \nu_A(1) = \nu_A(x)$. So, $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$.

(2) Since $x \wedge y \leq x$, $x \wedge y \leq y$ and Theorem 3.1, we get $\mu_A(x \wedge y) \leq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \wedge y) \geq \nu_A(x) \vee \nu_A(y)$. By Definition 3.1, we have

$$\begin{aligned}\mu_A(x\wedge y) &\geq \min\{\mu_A(x\to(x\wedge y)),\, \mu_A(x)\}\\ &= \min\{\mu_A((x\to x)\wedge(x\to y)),\, \mu_A(x)\}\\ &= \min\{\mu_A(x\to y),\, \mu_A(x)\}\\ &\geq \min\{\min\{\mu_A(y\to(x\to y)),\mu_A(y)\},\, \mu_A(x)\}\\ &= \min\{\min\{\mu_A(1),\mu_A(y)\},\, \mu_A(x)\}\\ &= \min\{\mu_A(y),\, \mu_A(x)\} = \mu_A(x)\wedge\mu_A(y)\end{aligned}$$

$$\begin{aligned}\nu_A(x\wedge y) &\leq \max\{\nu_A(x\to(x\wedge y)),\, \nu_A(x)\}\\ &= \max\{\nu_A((x\to x)\wedge(x\to y)),\, \nu_A(x)\}\\ &= \max\{\nu_A(x\to y),\, \nu_A(x)\}\\ &\leq \max\{\max\{\nu_A(y\to(x\to y)),\nu_A(y)\},\, \nu_A(x)\}\\ &= \max\{\max\{\nu_A(1),\nu_A(y)\},\, \nu_A(x)\}\\ &= \max\{\nu_A(y),\, \nu_A(x)\} = \nu_A(x)\vee\nu_A(y)\end{aligned}$$

Therefore, $\mu_A(x \wedge y) = \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \wedge y) = \nu_A(x) \vee \nu_A(y)$.

(3) According to Theorem 3.3(2), we have $\mu_A(x \otimes y) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \otimes y) \leq \nu_A(x) \vee \nu_A(y)$. Since $x \otimes y \leq x \wedge y$ and Theorem 3.1, we get $\mu_A(x \otimes y) \leq \mu_A(x \wedge y) = \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \otimes y) \geq \nu_A(x \wedge y) = \nu_A(x) \vee \nu_A(y)$ by (2). Hence, $\mu_A(x \otimes y) = \mu_A(x \wedge y)$, $\nu_A(x \otimes y) = \nu_A(x \wedge y)$.

(4) According to (3), we have $\mu_A(x) \wedge \mu_A(\neg x) = \mu_A(x \otimes \neg x) = \mu_A(0)$, $\nu_A(x) \vee \nu_A(\neg x) = \nu_A(x \otimes \neg x) = \nu_A(0)$. Thus, $\mu_A(0) = \mu_A(x) \wedge \mu_A(\neg x)$, $\nu_A(0) = \nu_A(x) \vee \nu_A(\neg x)$.

(5) By (3) and Theorem 3.1, since $(x \to y) \otimes (y \to z) \leq x \to z$, we have $\mu_A((x \to y) \otimes (y \to z)) \leq \mu_A(x \to z)$, $\nu_A((x \to y) \otimes (y \to z)) \geq \nu_A(x \to z)$, and $\mu_A(x \to z) \geq \mu_A(x \to y) \wedge \mu_A(y \to z)$, $\nu_A(x \to z) \leq \nu_A(x \to y) \vee \nu_A(y \to z)$. Hence, (5) holds.

(6), (7), (8) can be easily proved by Lemma 2.1 and Theorem 3.1(1). □

Theorem 3.4 *Let A and B be intuitionistic fuzzy filters. Then $A \cap B$ is an intuitionistic fuzzy filter.*

Proof Suppose that $x \to (y \to z) = 1$ for all $x, y, z \in L$. Since A and B are intuitionistic fuzzy filters, then $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$ and $\mu_B(z) \geq \mu_B(x) \wedge \mu_B(y)$, $\nu_B(z) \leq \nu_B(x) \vee \nu_B(y)$. So $(\mu_A \wedge \mu_B)(z) \geq \mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \wedge \mu_B(y) = (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y)) = (\mu_A \wedge \mu_B)(x) \wedge (\mu_A \wedge \mu_B)(y)$, $(\nu_A \vee \nu_B)(z) \leq \nu_A(x) \vee \nu_A(y) \vee \nu_B(x) \vee \nu_B(y) = (\nu_A(x) \vee \nu_B(x)) \vee (\nu_A(y) \vee \nu_B(y)) = (\nu_A \vee \nu_B)(x) \vee (\nu_A \vee \nu_B)(y)$. By Theorem 3.2, $A \cap B$ is an intuitionistic fuzzy filter. □

Definition 3.2 Let L be a BL-algebra, and an intuitionistic fuzzy set A is called an intuitionistic fuzzy lattice filter of L, if it satisfies $\mu_A(x \wedge y) = \mu_A(x) \wedge \mu_A(y)$, $\nu_A(x \wedge y) = \nu_A(x) \vee \nu_A(y)$, for all $x, y \in L$.

Example 2 It can be easily checked that an intuitionistic fuzzy set A in Example 1 is an intuitionistic fuzzy lattice filter.

Theorem 3.5 *Each intuitionistic fuzzy filter is an intuitionistic fuzzy lattice filter.*

Proof It can be easily proved by Corollary 3.2(2). □

4 Intuitionistic fuzzy prime filters of *BL*-algebras

Definition 4.1 Let an intuitionistic fuzzy set A be a non-constant intuitionistic fuzzy filter of L. A is called an intuitionistic fuzzy prime filter, if it satisfies $\mu_A(x) \geq \mu_A(x \vee y)$, $\nu_A(x) \leq \nu_A(x \vee y)$ or $\mu_A(y) \geq \mu_A(x \vee y)$, $\nu_A(y) \leq \nu_A(x \vee y)$, for all $x, y \in L$.

Example 3 Let L and A be defined as in Example 1. It is easily verified that A is an intuitionistic fuzzy prime filter of BL-algebra L.

Theorem 4.1 *Let an intuitionistic fuzzy set A be a non-constant intuitionistic fuzzy filter in BL-algebra L. A is an intuitionistic fuzzy prime filter if and only if $\mu_A(x \vee y) \leq \mu_A(x) \vee \mu_A(y)$, $\nu_A(x \vee y) \geq \nu_A(x) \wedge \nu_A(y)$, for all $x, y \in L$.*

Proof Let A be an intuitionistic fuzzy prime filter, by Definition 4.1, we have $\mu_A(x) \geq \mu_A(x \vee y)$, $\nu_A(x) \leq \nu_A(x \vee y)$, and $\mu_A(y) \geq \mu_A(x \vee y)$, $\nu_A(y) \leq \nu_A(x \vee y)$. Therefore $\mu_A(x \vee y) \leq \mu_A(x) \vee \mu_A(y)$, $\nu_A(x \vee y) \geq \nu_A(x) \wedge \nu_A(y)$.

Conversely, we easily prove, $\mu_A(x) \geq \mu_A(x \vee y)$, $\nu_A(x) \leq \nu_A(x \vee y)$ or $\mu_A(y) \geq \mu_A(x \vee y)$, $\nu_A(y) \leq \nu_A(x \vee y)$. Hence, A is an intuitionistic fuzzy prime filter. □

Theorem 4.2 *Let an intuitionistic fuzzy set A be a non-constant intuitionistic fuzzy filter in BL-algebra L. A is an intuitionistic fuzzy prime filter if and only if* $(\mu_A)_{\mu_A(1)} = \{x|x \in L, \mu_A(x) \geq \mu_A(1)\}$ *and* $(\nu_A)_{\nu_A(1)} = \{x|x \in L, \nu_A(x) \leq \nu_A(1)\}$ *are prime filters.*

Proof Obviously, $(\mu_A)_{\mu_A(1)} = \{x|x \in L, \mu_A(x) = \mu_A(1)\}$, $(\nu_A)_{\nu_A(1)} = \{x|x \in L, \nu_A(x) = \nu_A(1)\}$. Since A is a non-constant intuitionistic fuzzy filter, then $\mu_A(0) \leq \mu_A(1)$, $\nu_A(0) \geq \nu_A(1)$, i.e., $0 \notin (\mu_A)_{\mu_A(1)}, 0 \notin (\nu_A)_{\nu_A(1)}$, then $(\mu_A)_{\mu_A(1)}$ and $(\nu_A)_{\nu_A(1)}$ are prime filters.

Conversely, suppose $(\mu_A)_{\mu_A(1)}$ and $(\nu_A)_{\nu_A(1)}$ are prime filters. Then $x \to y \in (\mu_A)_{\mu_A(1)}, x \to y \in (\nu_A)_{\nu_A(1)}$ or $y \to x \in (\mu_A)_{\mu_A(1)}$, $y \to x \in (\nu_A)_{\nu_A(1)}$ for all $x, y \in L$. This means that $(x \vee y) \to y = x \to y \in (\mu_A)_{\mu_A(1)}$, $(x \vee y) \to y = x \to y \in (\nu_A)_{\nu_A(1)}$ or $(x \vee y) \to x = y \to x \in (\mu_A)_{\mu_A(1)}$, $(x \vee y) \to x = y \to x \in (\nu_A)_{\nu_A(1)}$. So $\mu_A((x \vee y) \to y) = \mu_A(1)$, $\nu_A((x \vee y) \to y) = \nu_A(1)$. By Definition 3.1, we get $\mu_A(y) \geq \mu_A((x \vee y) \to y) \wedge \mu_A(x \vee y) = \mu_A(x \vee y)$, $\nu_A(y) \leq \nu_A((x \vee y) \to y) \vee \nu_A(x \vee y) = \nu_A(x \vee y)$ or $\mu_A(x) \geq \mu_A((x \vee y) \to x) \wedge \mu_A(x \vee y) = \mu_A(x \vee y)$, $\nu_A(x) \leq \nu_A((x \vee y) \to x) \vee \nu_A(x \vee y) = \nu_A(x \vee y)$. Thus, $\mu_A(x) \vee \mu_A(y) \geq \mu_A(x \vee y)$, $\nu_A(x) \wedge \nu_A(y) \leq \nu_A(x \vee y)$. Therefore, A is an intuitionistic fuzzy prime filter. □

Theorem 4.3 *Let A be a non-constant intuitionistic fuzzy filter in BL-algebra L. A is an intuitionistic fuzzy prime filter if and only if* $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$ *or* $\mu_A(y \to x) = \mu_A(1)$, $\nu_A(y \to x) = \nu_A(1)$.

Proof By Theorem 4.2, A is an intuitionistic fuzzy prime filter if and only if $(\mu_A)_{\mu_A(1)}$, $(\nu_A)_{\nu_A(1)}$ are prime filters if and only if $x \to y \in (\mu_A)_{\mu_A(1)}, x \to y \in (\nu_A)_{\nu_A(1)}$ or $y \to x \in (\mu_A)_{\mu_A(1)}, y \to x \in (\nu_A)_{\nu_A(1)}$ if and only if $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$ or $\mu_A(y \to x) = \mu_A(1)$, $\nu_A(y \to x) = \nu_A(1)$. □

Theorem 4.4 *Let A be a non-constant intuitionistic fuzzy prime filter of BL-algebra L and B be a non-constant intuitionistic fuzzy filter of BL-algebra L. If* $A \subseteq B, \mu_A(1) = \mu_B(1), \nu_A(1) = \nu_B(1)$, *then B is also an intuitionistic fuzzy prime filter.*

Proof Since A is an intuitionistic fuzzy prime filter, then $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$ or $\mu_A(y \to x) = \mu_A(1)$, $\nu_A(y \to x) = \nu_A(1)$, for all $x, y \in L$. If $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$, by $A \subseteq B$ and $\mu_A(1) = \mu_B(1)$, $\nu_A(1) = \nu_B(1)$, we have $\mu_B(x \to y) = \mu_B(1)$, $\nu_B(x \to y) = \nu_B(1)$. Similarly, if $\mu_A(y \to x) = \mu_A(1)$, $\nu_A(y \to x) = \nu_A(1)$, then $\mu_B(y \to x) = \mu_B(1)$, $\nu_B(y \to x) = \nu_B(1)$. Using Theorem 4.3, we have that B is an intuitionistic fuzzy prime filter. □

5 Intuitionistic fuzzy Boolean filters and implicative filters of *BL*-algebras

In this section, we introduce the notions of intuitionistic fuzzy Boolean filters and implicative filters of *BL*-algebras and investigate their properties.

Definition 5.1 Let an intuitionistic fuzzy set A be an intuitionistic fuzzy filter of L. A is called an intuitionistic fuzzy Boolean filter if $\mu_A(x \vee \neg x) = \mu_A(1), \nu_A(x \vee \neg x) = \nu_A(1)$, for all $x \in L$.

The following example shows that intuitionistic fuzzy Boolean filters exist.

Example 4 Let $L = \{0, a, b, 1\}$. $\otimes$ and $\to$ are defined by Tables 3 and 4 on L as follows.

Define $\wedge$ and $\vee$ operations on L as *min* and *max*, respectively. Then $(L, \wedge\ \vee\ \otimes\ \to, 0, 1)$ is a *BL*-algebra. Define an intuitionistic fuzzy set A in L by

$$\mu_A(0) = t_1, \quad \mu_A(1) = \mu_A(a) = \mu_A(b) = t_3$$
$$\nu_A(0) = t_2, \quad \nu_A(1) = \nu_A(a) = \nu_A(b) = t_4$$

Where $0 \leq t_1 < t_3 \leq 1$, $0 \leq t_4 < t_2 \leq 1$, $0 \leq t_1 + t_2 \leq 1$, $0 \leq t_3 + t_4 \leq 1$.

It can be easily verified that A is an intuitionistic fuzzy Boolean filter.

Theorem 5.1 *Let an intuitionistic fuzzy set A be an intuitionistic fuzzy filter. A is an intuitionistic fuzzy Boolean filter if and only if* $\mu_A((x \to \neg x) \to \neg x) = \mu_A((\neg x \to x) \to x) = \mu_A(1), \nu_A((x \to \neg x) \to \neg x) = \nu_A((\neg x \to x) \to x) = \nu_A(1)$, *for all* $x \in L$.

Table 3 "$\otimes$" operator table in L

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

Table 4 "$\to$" operator table in L

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Proof Suppose A is an intuitionistic fuzzy Boolean filter, by Definition 5.1, Lemma 2.1(29) and Corollary 3.2(2), we have $\mu_A(x \vee \neg x) = \mu_A(((x \to \neg x) \to \neg x) \wedge ((\neg x \to x) \to x)) = \mu_A((x \to \neg x) \to \neg x) \wedge \mu_A((\neg x \to x) \to x) = \mu_A(1)$, $\nu_A(x \vee \neg x) = \nu_A(((x \to \neg x) \to \neg x) \wedge ((\neg x \to x) \to x)) = \nu_A((x \to \neg x) \to \neg x) \vee \nu_A((\neg x \to x) \to x) = \nu_A(1)$. So, $\mu_A((x \to \neg x) \to \neg x) = \mu_A((\neg x \to x) \to \dot{x}) = \mu_A(1)$, $\nu_A((x \to \neg x) \to \neg x) = \nu_A((\neg x \to x) \to x) = \nu_A(1)$.

Conversely, it can be easily proved that A is an intuitionistic fuzzy Boolean filter. □

Theorem 5.2 *Let A and B be two intuitionistic fuzzy filters of L, which satisfy $A \subseteq B, \mu_A(1) = \mu_B(1), \nu_A(1) = \nu_B(1)$. If A is an intuitionistic fuzzy Boolean filter, so is B.*

Proof Let B be an intuitionistic fuzzy filter of L. If A is an intuitionistic fuzzy Boolean filter, then $\mu_A(x \vee \neg x) = \mu_A(1), \nu_A(x \vee \neg x) = \nu_A(1)$ for all $x \in L$. From $A \subseteq B$ and $\mu_A(1) = \mu_B(1)$, $\nu_A(1) = \nu_B(1)$, it follows that $\mu_B(x \vee \neg x) \geq \mu_B(1), \nu_B(x \vee \neg x) \leq \nu_B(1)$. Taking Definition 3.1(1) into account, we have $\mu_B(x \vee \neg x) = \mu_B(1), \nu_B(x \vee \neg x) = \nu_B(1)$. This implies that B is an intuitionistic fuzzy Boolean filter. □

In [25], Turunen introduced the notion of implicative filters in BL-algebras and proved that implicative filters are equivalent to Boolean filters. Motivated by this result, in the following, we introduce the notion of intuitionistic fuzzy implicative filters in BL-algebras and want to obtain similar result regarding intuitionistic fuzzy Boolean filters and intuitionistic fuzzy implicative filters.

Definition 5.2 Let A be an intuitionistic fuzzy filter of L. A is called an intuitionistic fuzzy implicative filter if it satisfies for all $x, y, z \in L$

(1) $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to y)) \wedge \mu_A(y \to z)$,
(2) $\nu_A(x \to z) \leq \nu_A(x \to (\neg z \to y)) \vee \nu_A(y \to z)$.

Theorem 5.3 *Let A be an intuitionistic fuzzy filter of L. The following are equivalent.*

(1) *A is an intuitionistic fuzzy implicative filter*,
(2) $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z))$, $\nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z))$,
(3) $\mu_A(x \to z) = \mu_A(x \to (\neg z \to z))$, $\nu_A(x \to z) = \nu_A(x \to (\neg z \to z))$,
(4) $\mu_A(x \to z) \geq \mu_A(y \to (x \to (\neg z \to z))) \wedge \mu_A(y)$, $\nu_A(x \to z) \leq \nu_A(y \to (x \to (\neg z \to z))) \vee \nu_A(y)$.

Proof (1) ⇒ (2) Suppose that A is an intuitionistic fuzzy implicative filter, then $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z)) \wedge \mu_A(z \to z), \nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z)), \nu_A(z \to z)$ follows from Definition 5.2, i.e. $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z)) \wedge \mu_A(1), \nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z)) \vee \nu_A(1)$. Taking Definition 3.1(1) into account, we have $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z)), \nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z))$. Thus, (2) holds.

(2) ⇒ (3) Since $x \to z = 1 \to (x \to z) \leq \neg z \to (x \to z) = x \to (\neg z \to z)$, by Theorem 3.3, we have $\mu_A(x \to z) \leq \mu_A(x \to (\neg z \to z))$, $\nu_A(x \to z) \geq \nu_A(x \to (\neg z \to z))$. Considering (2), we get $\mu_A(x \to z) = \mu_A(x \to (\neg z \to z))$, $\nu_A(x \to z) = \nu_A(x \to (\neg z \to z))$. Thus, (3) holds.

(3) ⇒ (4) Since A is an intuitionistic fuzzy filter, then $\mu_A(x \to (\neg z \to z)) \geq \mu_A(y \to (x \to (\neg z \to z))) \wedge \mu_A(y)$, $\nu_A(x \to (\neg z \to z)) \leq \nu_A(y \to (x \to (\neg z \to z))) \vee \nu_A(y)$. Using (3), we obtain $\mu_A(x \to z) \geq \mu_A(y \to (x \to (\neg z \to z))) \wedge \mu_A(y), \nu_A(x \to z) \leq \nu_A(y \to (x \to (\neg z \to z))) \vee \nu_A(y)$. Thus, (4) holds.

(4) ⇒ (1) Let A be an intuitionistic fuzzy filter, which satisfies $\mu_A(x \to z) \geq \mu_A(y \to (x \to (\neg z \to z))) \wedge \mu_A(y)$, $\nu_A(x \to z) \leq \nu_A(y \to (x \to (\neg z \to z))) \vee \nu_A(y)$ for all $x, y, z \in L$. According to Corollary 3.2(5), we have $\mu_A(x \otimes \neg z \to z) \geq \mu_A(x \otimes \neg z \to y) \wedge \mu_A(y \to z)$, $\nu_A(x \otimes \neg z \to z) \leq \nu_A(x \otimes \neg z \to y) \vee \nu_A(y \to z)$, i.e., $\mu_A(x \to (\neg z \to z)) \geq \mu_A(x \to (\neg z \to y)) \wedge \mu_A(y \to z)$, $\nu_A(x \to (\neg z \to z)) \leq \nu_A(x \to (\neg z \to y)) \vee \nu_A(y \to z)$. On the other hand, by (4), we have $\mu_A(x \to z) \geq \mu_A(1 \to (x \to (\neg z \to z))) \wedge \mu_A(1), \nu_A(x \to z) \leq \nu_A(1 \to (x \to (\neg z \to z))) \vee \nu_A(1)$, i.e., $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z))$, $\nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z))$. Hence, $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to y)) \wedge \mu_A(y \to z)$, $\nu_A(x \to z) \leq \nu_A(x \to (\neg z \to y)) \vee \nu_A(y \to z)$.

Therefore, A is an intuitionistic fuzzy implicative filter. □

Next, we investigate the relationship between intuitionistic fuzzy implicative filters and intuitionistic fuzzy Boolean filters.

Theorem 5.4 *Let A be an intuitionistic fuzzy filter of L. A is an intuitionistic fuzzy Boolean filter if and only if A is an intuitionistic fuzzy implicative filter.*

Proof Suppose that A is an intuitionistic fuzzy Boolean filter. According to Definition 3.1,

$$\begin{aligned}
\mu_A(x \to z) &\geq \mu_A((z \vee \neg z) \to (x \to z)) \wedge \mu_A(z \vee \neg z) \\
&= \mu_A((z \vee \neg z) \to (x \to z)) \wedge \mu_A(1) \\
&= \mu_A((z \vee \neg z) \to (x \to z)) \\
&= \mu_A((z \to (x \to z)) \wedge (\neg z \to (x \to z))) \\
&= \mu_A(\neg z \to (x \to z)) = \mu_A(x \to (\neg z \to z)) \\
\nu_A(x \to z) &\leq \nu_A((z \vee \neg z) \to (x \to z)) \vee \nu_A(z \vee \neg z) \\
&= \nu_A((z \vee \neg z) \to (x \to z)) \vee \nu_A(1) \\
&= \nu_A((z \vee \neg z) \to (x \to z)) \\
&= \nu_A((z \to (x \to z)) \wedge (\neg z \to (x \to z))) \\
&= \nu_A(\neg z \to (x \to z)) = \nu_A(x \to (\neg z \to z))
\end{aligned}$$

It follows that $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z))$, $\nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z))$. By Theorem 5.3, A is an intuitionistic fuzzy implicative filter.

Conversely, suppose that A is an intuitionistic fuzzy implicative filter. By Theorem 5.3(3) and Lemma 2.1(28), we have $\mu_A((\neg x \to x) \to x) = \mu_A((\neg x \to x) \to (\neg x \to x)) = \mu_A(1), \nu_A((\neg x \to x) \to x) = \nu_A((\neg x \to x) \to (\neg x \to x)) = \nu_A(1)$ and $\mu_A(x \to \neg x) \to \neg x) = \mu_A((x \to \neg x) \to (\neg\neg x \to \neg x)) = \mu_A((\neg x \to x) \to (\neg x \to x)) = \mu_A(1), \nu_A(x \to \neg x) \to \neg x) = \nu_A((x \to \neg x) \to (\neg\neg x \to \neg x)) = \nu_A((\neg x \to x) \to (\neg x \to x)) = \nu_A(1)$. By Lemma 2.1(29) and Corollary 3.2, we get $\mu_A(x \vee \neg x) = \mu_A((x \to \neg x) \to \neg x) \wedge \mu_A((\neg x \to x) \to x) = \mu_A(1), \nu_A(x \vee \neg x) = \nu_A((x \to \neg x) \to \neg x) \wedge \nu_A((\neg x \to x) \to x) = \nu_A(1)$. Thus, A is an intuitionistic fuzzy Boolean filter. □

Obviously, Theorem 5.4 shows that intuitionistic fuzzy implicative filters are equivalent to intuitionistic fuzzy Boolean filters of BL-algebras.

Theorem 5.5 *Let A be an intuitionistic fuzzy filter of L. The following are equivalent.*

(1) *A is an intuitionistic fuzzy Boolean filter,*
(2) $\mu_A(x) = \mu_A(\neg x \to x),\ \nu_A(x) = \nu_A(\neg x \to x)$,
(3) $\mu_A((x \to y) \to x) \leq \mu_A(x),\ \nu_A((x \to y) \to x) \geq \nu_A(x)$,
(4) $\mu_A((x \to y) \to x) = \mu_A(x)$,
$\nu_A((x \to y) \to x) = \nu_A(x)$,
(5) $\mu_A(x) \geq \mu_A(z \to ((x \to y) \to x)) \wedge \mu_A(z)$,
$\nu_A(x) \leq \nu_A(z \to ((x \to y) \to x)) \vee \nu_A(z)$.

Proof (1) ⇒ (2) Suppose that A is an intuitionistic fuzzy Boolean filter, according to Theorem 5.4, so A is also an intuitionistic fuzzy implicative filter. By Theorem 5.3(3), this implies that $\mu_A(x) = \mu_A(1 \to x) = \mu_A(1 \to (\neg x \to x)) = \mu_A(\neg x \to x)$, $\nu_A(x) = \nu_A(1 \to x) = \nu_A(1 \to (\neg x \to x)) = \nu_A(\neg x \to x)$. Hence, (2) holds.

(2) ⇒ (3) Since $\neg x = x \to 0$, according to Lemma 2.1(8), we have $\neg x \leq x \to y$, $(x \to y) \to x \leq \neg x \to x$, which imply $\mu_A((x \to y) \to x) \leq \mu_A(\neg x \to x)$, $\nu_A((x \to y) \to x) \geq \nu_A(\neg x \to x)$, thus A is an intuitionistic fuzzy filter. From (2), we deduce that $\mu_A((x \to y) \to x) \leq \mu_A(x)$, $\nu_A((x \to y) \to x) \geq \nu_A(x)$.

(3) ⇒ (4) Since $x \leq (x \to y) \to x$, it follows that $\mu_A(x) \leq \mu_A((x \to y) \to x)$, $\nu_A(x) \geq \nu_A((x \to y) \to x)$. Taking (3) into account, we get $\mu_A((x \to y) \to x) = \mu_A(x)$, $\nu_A((x \to y) \to x) = \nu_A(x)$.

(4) ⇒ (5) Clearly, $\mu_A((x \to y) \to x) \geq \mu_A(z \to ((x \to y) \to x)) \wedge \mu_A(z)$, $\nu_A((x \to y) \to x) \leq \nu_A(z \to ((x \to y) \to x)) \vee \nu_A(z)$. Taking (4) into account, this implies that $\mu_A(x) \geq \mu_A(z \to ((x \to y) \to x)) \wedge \mu_A(z), \nu_A(x) \leq \nu_A(z \to ((x \to y) \to x)) \vee \nu_A(z)$. This shows that (5) holds.

(5) ⇒ (1) Let A be an intuitionistic fuzzy filter. In order to prove that A is an intuitionistic fuzzy Boolean filter, from Theorems 5.3 and 5.4, we only prove that $\mu_A(x \to z) \geq \mu_A(x \to (\neg z \to z)), \nu_A(x \to z) \leq \nu_A(x \to (\neg z \to z))$ for all $x, y, z \in L$. Since $z \leq x \to z$, it follows that $\neg(x \to z) \leq \neg z$ and $\neg z \to (x \to z) \leq \neg(x \to z) \to (x \to z)$. By (5), this implies that $\mu_A(\neg z \to (x \to z)) \leq \mu_A(\neg(x \to z) \to (x \to z)) = \mu_A(1 \to (((x \to z) \to 0) \to (x \to z))) \wedge \mu_A(1) \leq \mu_A(x \to z)$, $\nu_A(\neg z \to (x \to z)) \geq \nu_A(\neg(x \to z) \to (x \to z)) = \nu_A(1 \to (((x \to z) \to 0) \to (x \to z))) \vee \nu_A(1) \geq \nu_A(x \to z)$, i.e. $\mu_A(x \to z) \geq \mu_A(\neg z \to (x \to z)), \nu_A(x \to z) \leq \nu_A(\neg z \to (x \to z))$. Therefore, A is an intuitionistic fuzzy Boolean filter. So, (1) holds. □

6 Intuitionistic fuzzy positive implicative filters of *BL*-algebras

In this section, we introduce the notion of intuitionistic fuzzy positive implicative filters of *BL*-algebras and investigate their properties.

Definition 6.1 Let A be an intuitionistic fuzzy set in L. A is called an intuitionistic fuzzy positive implicative filter, for all $x, y, z \in L$, if it satisfies

(1) $\mu_A(1) \geq \mu_A(x),\ \nu_A(1) \leq \nu_A(x)$,
(2) $\mu_A(x \to z) \geq \mu_A(x \to (y \to z)) \wedge \mu_A(x \to y)$,
$\nu_A(x \to z) \leq \nu_A(x \to (y \to z)) \vee \nu_A(x \to y)$.

The following example shows that intuitionistic fuzzy positive implicative filters exist.

Example 5 Let $L = \{0, a, b, 1\}$. ⊗ and → are defined by Tables 5 and 6 on L as follows.

Define ∧ and ∨ operations on L as *min* and *max*, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is a *BL*-algebra. Define an intuitionistic fuzzy set A in L by

$$\mu_A(a) = \mu_A(b) = \mu_A(0) = t_1, \quad \mu_A(1) = t_3$$
$$\nu_A(a) = \nu_A(b) = \nu_A(0) = t_2, \quad \nu_A(1) = t_4$$

Where $0 \leq t_1 + t_2 \leq 1$, $0 \leq t_3 + t_4 \leq 1$, $0 \leq t_1 < t_3 \leq 1$, $0 \leq t_4 < t_2 \leq 1$.

It can be easily verified that A is an intuitionistic fuzzy positive implicative filter.

Table 5 "⊗"operator table in L

⊗	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 6 "→" operator table in L

→	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

The relationship between intuitionistic fuzzy positive implicative filters and intuitionistic fuzzy filters is as follows.

Theorem 6.1 *Each intuitionistic fuzzy positive implicative filter is an intuitionistic fuzzy filter.*

Proof Suppose that A is an intuitionistic fuzzy positive implicative filter of L. Taking x=1 and Definition 6.1 (2) into account, we have $\mu_A(1 \to z) \geq \mu_A(1 \to (y \to z)) \wedge \mu_A(1 \to y)$, $\nu_A(1 \to z) \leq \nu_A(1 \to (y \to z)) \vee \nu_A(1 \to y)$, for all $y, z \in L$, i.e., $\mu_A(z) \geq \mu_A(y \to z) \wedge \mu_A(y)$, $\nu_A(z) \leq \nu_A(y \to z) \vee \nu_A(y)$. By Definition 6.1(1), we get that A is an intuitionistic fuzzy filter.

The converse of Theorem 6.1 may not be true. Actually, in Example 4, we define an intuitionistic fuzzy set A by

$$\mu_A(0) = t_1, \quad \mu_A(a) = \mu_A(b) = t_3, \quad \mu_A(1) = t_5$$
$$\nu_A(0) = t_2, \quad \nu_A(a) = \nu_A(b) = t_4, \quad \nu_A(1) = t_6$$

Where
$0 \leq t_1 + t_2 \leq 1$, $0 \leq t_3 + t_4 \leq 1$, $0 \leq t_5 + t_6 \leq 1$, $0 \leq t_1 < t_3 < t_5 \leq 1$, $0 \leq t_6 < t_4 < t_2 \leq 1$.

It can be easily checked that A is an intuitionistic fuzzy filter, but A is not an intuitionistic fuzzy positive implicative filter, because $\mu_A(b \to a) = \mu_A(b) = t_3 < \mu_A(b \to (b \to a)) \wedge \mu_A(b \to b) = t_5$, $\nu_A(b \to a) = \nu_A(b) = t_4 > \nu_A(b \to (b \to a)) \wedge \nu_A(b \to b) = t_6$. □

We investigate some characteristics of intuitionistic fuzzy positive implicative filters as follows.

Theorem 6.2 *Let A be an intuitionistic fuzzy filter of L. The following are equivalent, for all $x, y, z \in L$.*

(1) *A is an intuitionistic fuzzy positive implicative filter*,
(2) $\mu_A(x \to y) \geq \mu_A(x \to (x \to y))$,
$\nu_A(x \to y) \leq \nu_A(x \to (x \to y))$,
(3) $\mu_A(x \to y) = \mu_A(x \to (x \to y))$,
$\nu_A(x \to y) = \nu_A(x \to (x \to y))$,
(4) $\mu_A(x \to (y \to z)) \leq \mu_A((x \to y) \to (x \to z))$,
$\nu_A(x \to (y \to z)) \geq \nu_A((x \to y) \to (x \to z))$,
(5) $\mu_A(x \to (y \to z)) = \mu_A((x \to y) \to (x \to z))$,
$\nu_A(x \to (y \to z)) = \nu_A((x \to y) \to (x \to z))$,
(6) $\mu_A((x \otimes y) \to z) = \mu_A((x \wedge y) \to z)$,
$\nu_A((x \otimes y) \to z) = \nu_A((x \wedge y) \to z)$.

Proof (1) ⇒ (2) Let A be an intuitionistic fuzzy positive implicative filter. In view of Definition 6.1, then $\mu_A(x \to y) \geq \mu_A(x \to (x \to y)) \wedge \mu_A(x \to x)$, $\nu_A(x \to y) \leq \nu_A(x \to (x \to y)) \vee \nu_A(x \to x)$, i.e., $\mu_A(x \to y) \geq \mu_A(x \to (x \to y))$, $\nu_A(x \to y) \leq \nu_A(x \to (x \to y))$. So, (2) holds.

(2) ⇒ (3) Since $x \to y \leq x \to (x \to y)$, according to Theorem 3.3(1), we have $\mu_A(x \to y) \leq \mu_A(x \to (x \to y))$, $\nu_A(x \to y) \geq \nu_A(x \to (x \to y))$. Taking (2) into account, $\mu_A(x \to y) = \mu_A(x \to (x \to y))$, $\nu_A(x \to y) = \nu_A(x \to (x \to y))$. Hence, (3) holds.

(3) ⇒ (1) Suppose that A is an intuitionistic fuzzy filter. By Corollary 3.2(5) and Lemma 2.1(5), we get $\mu_A(x \to (x \to z)) \geq \mu_A(x \to y) \wedge \mu_A(y \to (x \to z)) = \mu_A(x \to y) \wedge \mu_A(x \to (y \to z))$, $\nu_A(x \to (x \to z)) \leq \nu_A(x \to y) \vee \nu_A(y \to (x \to z)) = \nu_A(x \to y) \vee \nu_A(x \to (y \to z))$. Taking (3) into account, we have $\mu_A(x \to z) \geq \mu_A(x \to y) \wedge \mu_A(x \to (y \to z))$, $\nu_A(x \to z)) \leq \nu_A(x \to y) \vee \nu_A(x \to (y \to z))$. By Definition 3.1(1), we get that A is an intuitionistic fuzzy positive implicative filter. Thus, (1) holds.

(1) ⇒ (4) Suppose that A is an intuitionistic fuzzy positive implicative filter. According to Definition 6.1, $\mu_A(x \to ((x \to y) \to z)) \geq \mu_A(x \to ((y \to z) \to ((x \to y) \to z))) \wedge \mu_A(x \to (y \to z))$, $\nu_A(x \to ((x \to y) \to z)) \leq \nu_A(x \to ((y \to z) \to ((x \to y) \to z))) \vee \nu_A(x \to (y \to z))$. By Lemma 2.1(5) and Lemma 2.1(7), we get $\mu_A(x \to ((x \to y) \to z)) = \mu_A((x \to y) \to (x \to z))$, $\nu_A(x \to ((x \to y) \to z)) = \nu_A((x \to y) \to (x \to z))$ and $\mu_A(x \to ((y \to z) \to ((x \to y) \to z))) = \mu_A((y \to z) \to ((x \to y) \to (x \to z))) = \mu_A(1)$, $\nu_A(x \to ((y \to z) \to ((x \to y) \to z))) = \nu_A((y \to z) \to ((x \to y) \to (x \to z))) = \nu_A(1)$. It follows that $\mu_A((x \to y) \to (x \to z)) \geq \mu_A(1) \wedge \mu_A(x \to (y \to z)) = \mu_A(x \to (y \to z))$, $\nu_A((x \to y) \to (x \to z)) \leq \nu_A(1) \vee \nu_A(x \to (y \to z)) = \nu_A(x \to (y \to z))$. Therefore, (4) holds.

(4) ⇒ (5) Since $x \to (y \to z) = y \to (x \to z) = (1 \to y) \to (x \to z) \geq (x \to y) \to (x \to z)$, according to Theorem 3.1, we have $\mu_A(x \to (y \to z)) \geq \mu_A((x \to y) \to (x \to z))$, $\nu_A(x \to (y \to z)) \leq \nu_A((x \to y) \to (x \to z))$. Taking (4) into account, we have $\mu_A(x \to (y \to z)) = \mu_A((x \to y) \to (x \to z))$, $\nu_A(x \to (y \to z)) = \nu_A((x \to y) \to (x \to z))$. Hence, (5) holds.

(5) ⇒ (6) Since $x \to (y \to z) = x \otimes y \to z$ and $(x \wedge y) \to z = (x \otimes (x \to y)) \to z = (x \to y) \to (x \to z)$, considering (5), we obtain $\mu_A((x \otimes y) \to z) = \mu_A((x \wedge y) \to z)$, $\nu_A((x \otimes y) \to z) = \nu_A((x \wedge y) \to z)$. So, (6) holds.

(6) ⇒ (1) Suppose that A is an intuitionistic fuzzy filter, then $\mu_A(1) \geq \mu_A(x)$, $\nu_A(1) \leq \nu_A(x)$. From Corollary 3.2(5) and Lemma 2.1(5), we obtain $\mu_A(x \to (x \to z)) \geq \mu_A(x \to y) \wedge \mu_A(y \to (x \to z)) = \mu_A(x \to y) \wedge \mu_A(x \to (y \to z))$, $\nu_A(x \to (x \to z)) \leq \nu_A(x \to y) \vee \nu_A(y \to (x \to z)) = \nu_A(x \to y) \vee \nu_A(x \to (y \to z))$. Since $\mu_A(x \to (x \to z)) = \mu_A(x \otimes x \to z)$, $\nu_A(x \to (x \to z)) = \nu_A(x \otimes x \to z)$, it follows from (6) that $\mu_A((x \otimes x) \to z) = \mu_A((x \wedge x) \to z) = \mu_A(x \to z)$, $\nu_A((x \otimes x) \to z) = \nu_A((x \wedge x) \to z) = \nu_A(x \to z)$. Hence, $\mu_A(x \to z) \geq \mu_A(x \to (y \to z)) \wedge \mu_A(x \to y)$, $\nu_A(x \to z) \leq \nu_A(x \to (y \to z)) \vee \nu_A(x \to y)$, it can be obtained that A is an intuitionistic fuzzy positive implicative filter. □

Theorem 6.3 *Let A and B be two intuitionistic fuzzy filters which satisfy $A \subseteq B$, $\mu_A(1) = \mu_B(1)$, $\nu_A(1) = \nu_B(1)$. If A is an intuitionistic fuzzy positive implicative filter, so is B.*

Proof In view of Theorem 6.2, we only prove that $\mu_B(x \to z) \geq \mu_B(x \to (x \to z))$, $\nu_B(x \to z) \leq \nu_B(x \to (x \to z))$, for all $x, z \in L$. Let $t = x \to (x \to z)$, then $x \to (x \to (t \to z)) = t \to (x \to (x \to z)) = t \to t = 1$. If A is an intuitionistic fuzzy positive implicative filter, and by Theorem 6.2(3), then $\mu_A(x \to (t \to z)) = \mu_A(x \to (x \to (t \to z))) = \mu_A(1)$, $\nu_A(x \to (t \to z)) = \nu_A(x \to (x \to (t \to z))) = \nu_A(1)$, i.e., $\mu_A(t \to (x \to z)) = \mu_A(1) = \mu_B(1)$, $\nu_A(t \to (x \to z)) = \nu_A(1) = \nu_B(1)$. From $A \subseteq B$, we get $\mu_B(t \to (x \to z)) \geq \mu_A(t \to (x \to z)) = \mu_B(1)$, $\nu_B(t \to (x \to z)) \leq \nu_A(t \to (x \to z)) = \nu_B(1)$, according to Definition 3.1(1), we have $\mu_B(t \to (x \to z)) = \mu_B(1)$, $\nu_B(t \to (x \to z)) = \nu_B(1)$. Since B is an intuitionistic fuzzy filter, then $\mu_B(x \to z) \geq \mu_B(t \to (x \to z)) \wedge \mu_B(t)$, $\nu_B(x \to z) \leq \nu_B(t \to (x \to z)) \vee \nu_B(t)$. Therefore, $\mu_B(x \to z) \geq \mu_B(1) \wedge \mu_B(t) = \mu_B(t) = \mu_B(x \to (x \to z))$, $\nu_B(x \to z) \leq \nu_B(1) \vee \nu_B(t) = \nu_B(t) = \nu_B(x \to (x \to z))$. Hence, B is an intuitionistic fuzzy positive implicative filter. □

In the next, we investigate the relationship between intuitionistic fuzzy Boolean filters and intuitionistic fuzzy positive implicative filters.

Theorem 6.4 *Each intuitionistic fuzzy Boolean filter is an intuitionistic fuzzy positive implicative filter, the converse may not be true.*

Proof Suppose that A is an intuitionistic fuzzy Boolean filter. Then $\mu_A(x \to z) \geq \mu_A((x \vee \neg x) \to (x \to z)) \wedge \mu_A(x \vee \neg x) = \mu_A((x \vee \neg x) \to (x \to z)) \wedge \mu_A(1) = \mu_A((x \vee \neg x) \to (x \to z))$, $\nu_A(x \to z) \leq \nu_A((x \vee \neg x) \to (x \to z)) \vee \nu_A(x \vee \neg x) = \nu_A((x \vee \neg x) \to (x \to z)) \vee \nu_A(1) = \nu_A((x \vee \neg x) \to (x \to z))$. Since $(x \vee \neg x) \to (x \to z) = (x \to (x \to z)) \wedge (\neg x \to (x \to z)) = x \to (x \to z)$, and by Lemma 2.1, we get $\mu_A((x \vee \neg x) \to (x \to z)) = \mu_A(x \to (x \to z))$, $\nu_A((x \vee \neg x) \to (x \to z)) = \nu_A(x \to (x \to z))$. Consequently, $\mu_A(x \to z) \geq \mu_A(x \to (x \to z))$, $\nu_A(x \to z) \leq \nu_A(x \to (x \to z))$. Taking Theorem 6.2 into account, we get that A is an intuitionistic fuzzy positive implicative filter. In Example 5, we know that A is an intuitionistic fuzzy positive implicative filter, but A is not an intuitionistic fuzzy Boolean filter because $\mu_A(b \vee \neg b) = \mu_A(b) = t_1 \neq t_3 = \mu_A(1)$, $\nu_A(b \vee \neg b) = \nu_A(b) = t_2 \neq t_4 = \nu_A(1)$. □

Theorem 6.5 *Let A be an intuitionistic fuzzy positive implicative filter. A is an intuitionistic fuzzy Boolean filter if and only if it satisfies $\mu_A((x \to y) \to y) = \mu_A((y \to x) \to x)$, $\nu_A((x \to y) \to y) = \nu_A((y \to x) \to x)$, for all $x, y \in L$.*

Proof Assume that A is an intuitionistic fuzzy Boolean filter. From $x = 1 \to x \leq (y \to x) \to x$ and $y \leq (y \to x) \to x$, it follows that $\neg((y \to x) \to x) \leq \neg x \leq x \to y$ and $(x \to y) \to y \leq \neg((y \to x) \to x) \to y \leq \neg((y \to x) \to x) \to ((y \to x) \to x)$. Then $\mu_A(\neg((y \to x) \to x) \to ((y \to x) \to x)) \geq \mu_A((x \to y) \to y)$, $\nu_A(\neg((y \to x) \to x) \to ((y \to x) \to x)) \leq \nu_A((x \to y) \to y)$. Since A is an intuitionistic fuzzy Boolean filter, and by Theorem 5.5(2), we obtain $\mu_A((y \to x) \to x) = \mu_A(\neg((y \to x) \to x) \to ((y \to x) \to x))$, $\nu_A((y \to x) \to x) = \nu_A(\neg((y \to x) \to x) \to ((y \to x) \to x))$. So, $\mu_A((y \to x) \to x) \geq \mu_A((x \to y) \to y)$, $\nu_A((y \to x) \to x) \leq \nu_A((x \to y) \to y)$. Similarly, we prove $\mu_A((y \to x) \to x) \leq \mu_A((x \to y) \to y)$, $\nu_A((y \to x) \to x) \geq \nu_A((x \to y) \to y)$. Consequently, $\mu_A((x \to y) \to y) = \mu_A((y \to x) \to x)$, $\nu_A((x \to y) \to y) = \nu_A((y \to x) \to x)$.

Conversely, suppose that A is an intuitionistic fuzzy positive implicative filter, which satisfies $\mu_A((x \to y) \to y) = \mu_A((y \to x) \to x)$, $\nu_A((x \to y) \to y) = \nu_A((y \to x) \to x)$. Then $\mu_A((x \to \neg x) \to \neg x) = \mu_A((\neg x \to x) \to x)$, $\nu_A((x \to \neg x) \to \neg x) = \nu_A((\neg x \to x) \to x)$. This implies $\mu_A(x \vee \neg x) = \mu_A((x \to \neg x) \to \neg x)$, $\nu_A(x \vee \neg x) = \nu_A((x \to \neg x) \to \neg x)$. In order to prove that A is an intuitionistic fuzzy Boolean filter, we only need to show $\mu_A((x \to \neg x) \to \neg x) = \mu_A(1)$, $\nu_A((x \to \neg x) \to \neg x) = \nu_A(1)$. Since A is an intuitionistic fuzzy positive implicative filter, from Theorem 6.2(5), it follows that $\mu_A((x \to \neg x) \to \neg x) = \mu_A((x \to \neg x) \to (x \to 0)) = \mu_A(x \to (\neg x \to 0)) = \mu_A(x \to \neg\neg x) = \mu_A(1)$, $\nu_A((x \to \neg x) \to \neg x) = \nu_A((x \to \neg x) \to (x \to 0)) = \nu_A(x \to (\neg x \to 0)) = \nu_A(x \to \neg\neg x) = \nu_A(1)$. Therefore, $\mu_A(x \vee \neg x) = \mu_A(1)$, $\nu_A(x \vee \neg x) = \nu_A(1)$. Hence, A is an intuitionistic fuzzy Boolean filter. □

7 Intuitionistic fuzzy ultra filters and intuitionistic fuzzy obstinate filters of *BL*-algebras

Definition 7.1 An intuitionistic fuzzy set A in L is called an intuitionistic fuzzy ultra filter of L, if it is an intuitionistic fuzzy filter of L, which satisfies the following conditions. $\mu_A(x) = \mu_A(1)$, $\nu_A(x) = \nu_A(1)$ or $\mu_A(\neg x) = \mu_A(1)$, $\nu_A(\neg x) = \nu_A(1)$.

Example 6 Let $L = \{0, a, b, c, d, 1\}$. $\to$ and $\otimes$ are defined by Tables 7 and 8.

Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is a *BL*-algebra. Let A be an intuitionistic fuzzy set in L given by

$$\mu_A(x) = \begin{cases} \alpha & x \in \{1, a, d\} \\ \beta & otherwise \end{cases} \quad \nu_A(x) = \begin{cases} \eta & x \in \{1, a, d\} \\ \lambda & otherwise \end{cases}$$

Table 7 "→"operator table in L

→	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

Table 8 "$\otimes$" operator table in L

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Where $0 \le \beta < \alpha \le 1$, $0 \le \eta < \lambda \le 1$, $0 \le \alpha + \eta \le 1$, $0 \le \beta + \lambda \le 1$.

It can be easily verified that A is an intuitionistic fuzzy ultra filter of L.

Theorem 7.1 *A non-constant intuitionistic fuzzy set A in L is an intuitionistic fuzzy ultra filter of L if and only if A is an intuitionistic fuzzy Boolean filter and intuitionistic fuzzy prime filter of L.*

Proof Assume that A is an intuitionistic fuzzy Boolean and intuitionistic fuzzy prime filter of L. According to Theorem 4.1, when its formulae are equalities, we have $\mu_A(x \vee \neg x) = \mu_A(1) = \mu_A(x) \vee \mu_A(\neg x), \nu_A(x \vee \neg x) = \nu_A(1) = \nu_A(x) \wedge \nu_A(\neg x)$, for all $x \in L$. If $\mu_A(x) \neq \mu_A(1)$, $\nu_A(x) \neq \nu_A(1)$, we know $\mu_A(x) \le \mu_A(1)$, $\nu_A(x) \ge \nu_A(1)$ and $\mu_A(\neg x) \le \mu_A(1), \nu_A(\neg x) \ge \nu_A(1)$. Since $\mu_A(1) = \mu_A(x) \vee \mu_A(\neg x), \nu_A(1) = \nu_A(x) \wedge \nu_A(\neg x)$, we get $\mu_A(\neg x) = \mu_A(1)$, $\nu_A(\neg x) = \nu_A(1)$. Thus, A is an intuitionistic fuzzy ultra filter of L.

Conversely, suppose that A is an intuitionistic fuzzy ultra filter of L. For all $x \in L$, since $x \le x \vee \neg x, \neg x \le x \vee \neg x$, so $\mu_A(x) \le \mu_A(x \vee \neg x), \nu_A(x) \ge \nu_A(x \vee \neg x), \mu_A(\neg x) \le \mu_A(x \vee \neg x), \nu_A(\neg x) \ge \nu_A(x \vee \neg x)$ by Theorem 3.3(1). In view of Definition 7.1, we obtain $\mu_A(x) = \mu_A(1)$, $\nu_A(x) = \nu_A(1)$ or $\mu_A(\neg x) = \mu_A(1), \nu_A(\neg x) = \nu_A(1)$. $\mu_A(1) \le \mu_A(x \vee \neg x), \nu_A(1) \ge \nu_A(x \vee \neg x)$. Thus, by Definition 3.1(1), we get $\mu_A(1) = \mu_A(x \vee \neg x), \nu_A(1) = \nu_A(x \vee \neg x)$. This means that A is an intuitionistic fuzzy Boolean filter of L. And, by Lemma 2.1(29) and Theorem 3.1, we have $\mu_A(x \vee y) = \mu_A(((x \to y) \to y) \wedge ((y \to x) \to x)) \le \mu_A((x \to y) \to y)$, $\nu_A(x \vee y) = \nu_A(((x \to y) \to y) \wedge ((y \to x) \to x)) \ge \nu_A((x \to y) \to y)$, for all $x, y \in L$. From $0 \le y$ and Lemma 2.1(8), we get $x \to 0 \le x \to y$ and $(x \to y) \to y \le \neg x \to y$. Thus, $\mu_A((x \to y) \to y) \le \mu_A(\neg x \to y), \nu_A((x \to y) \to y) \ge \nu_A(\neg x \to y)$ by Theorem 3.1. Hence, $\mu_A(x \vee y) \le \mu_A(\neg x \to y), \nu_A(x \vee y) \ge \nu_A(\neg x \to y)$. If $\mu_A(x) = \mu_A(1)$, $\nu_A(x) = \nu_A(1)$, then $\mu_A(x \vee y) \le \mu_A(1) = \mu_A(x) \le \mu_A(x) \vee \mu_A(y)$, $\nu_A(x \vee y) \ge \nu_A(1) = \nu_A(x) \ge \nu_A(x) \vee \nu_A(y)$. If $\mu_A(x) \neq \mu_A(1)$, $\nu_A(x) \neq \nu_A(1)$, then $\mu_A(\neg x) = \mu_A(1)$, $\nu_A(\neg x) = \nu_A(1)$ by Definition 7.1. So, $\mu_A(y) \ge \mu_A(\neg x \to y) \wedge \mu_A(\neg x) = \mu_A(\neg x \to y) \wedge \mu_A(1) = \mu_A(\neg x \to y), \nu_A(y) \le \nu_A(\neg x \to y) \vee \nu_A(\neg x) = \nu_A(\neg x \to y) \vee \nu_A(1) = \nu_A(\neg x \to y)$. Therefore, $\mu_A(x \vee y) \le \mu_A(y) \le \mu_A(x) \vee \mu_A(y)$, $\nu_A(x \vee y) \ge \nu_A(y) \ge \nu_A(x) \wedge \nu_A(y)$. This means that A is an intuitionistic fuzzy prime filter of L. □

Definition 7.2 An intuitionistic fuzzy set A in L is called an intuitionistic fuzzy obstinate filter of L if it is an intuitionistic fuzzy filter of L that satisfies the following conditions:

$\mu_A(x) \neq \mu_A(1)$, $\nu_A(x) \neq \nu_A(1)$ and $\mu_A(y) \neq \mu_A(1)$, $\nu_A(y) \neq \nu_A(1)$, which imply $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$ and $\mu_A(y \to x) = \mu_A(1)$, $\nu_A(y \to x) = \nu_A(1)$ for all $x, y \in L$.

Theorem 7.2 *Let A be a non-constant intuitionistic fuzzy filter of L. The following are equivalent*:

(1) *A is an intuitionistic fuzzy ultra filter,*
(2) *A is an intuitionistic fuzzy prime filter and intuitionistic fuzzy Boolean filter,*
(3) *A is an intuitionistic fuzzy prime filter and intuitionistic fuzzy implicative filter,*
(4) *A is an intuitionistic fuzzy obstinate filter.*

Proof (1) $\Rightarrow$ (2) It is proved by Theorem 7.1.

(2) $\Rightarrow$ (3) It is easily proved by Theorem 5.4.

(3) $\Rightarrow$ (1) Since the intuitionistic fuzzy Boolean filter is equivalent to the intuitionistic fuzzy implicative filter. In order to prove that an intuitionistic fuzzy prime filter or an intuitionistic fuzzy implicative filter is an intuitionistic fuzzy ultra filter, we only prove an intuitionistic fuzzy prime filter or an intuitionistic fuzzy Boolean filter is an intuitionistic fuzzy ultra filter. It is proved by Theorem 7.1.

(1) $\Rightarrow$ (4) Assume that A is an intuitionistic fuzzy ultra filter and $\mu_A(x) \neq \mu_A(1)$, $\nu_A(x) \neq \nu_A(1)$, $\mu_A(y) \neq \mu_A(1)$, $\nu_A(y) \neq \nu_A(1)$. Then $\mu_A(\neg x) = \mu_A(1), \nu_A(\neg x) = \nu_A(1)$ and $\mu_A(\neg y) = \mu_A(1), \nu_A(\neg y) = \nu_A(1)$ by Definition 7.1. Since $\neg x \le x \to y$, using Theorem 3.3(1), we get $\mu_A(x \to y) \ge \mu_A(\neg x) = \mu_A(1), \nu_A(x \to y) \le \nu_A(\neg x) = \nu_A(1)$. According to Definition 3.1(1), it follows that $\mu_A(x \to y) = \mu_A(1)$, $\nu_A(x \to y) = \nu_A(1)$. Similarly, we prove that $\mu_A(y \to x) = \mu_A(1)$, $\nu_A(y \to x) = \nu_A(1)$ from $\mu_A(\neg y) = \mu_A(1), \nu_A(\neg y) = \nu_A(1)$. This means that (4) holds.

(4) $\Rightarrow$ (1) Assume that A is an intuitionistic fuzzy obstinate filter and $\mu_A(x) \neq \mu_A(1)$, $\nu_A(x) \neq \nu_A(1)$, for all $x \in L$. Since A is a non-constant intuitionistic fuzzy filter in L, $\mu_A(0) \neq \mu_A(1)$, $\nu_A(0) \neq \nu_A(1)$. By Definition 7.2, we have $\mu_A(\neg x) = \mu_A(x \to 0) = \mu_A(1), \nu_A(\neg x) = \nu_A(x \to 0) = \nu_A(1)$. So, A is an intuitionistic fuzzy ultra filter by Definition 7.1. Therefore, (1) holds. □

8 Conclusions

The filter theory plays an important role in the study of logical algebras. Up to now, the filter theory and fuzzy filter theory of *BL*-algebras have been widely studied, and some important results have been obtained. In this paper, we develop the intuitionistic fuzzy filter theory of *BL*-algebras, which is important for researching logical algebras. We introduce the notions of intuitionistic fuzzy filters, lattice filters, prime filters, Boolean filters, implicative filters, positive implicative filters, ultra filters and obstinate filters in *BL*-algebras, and investigate their characteristics and their important properties, respectively. Meanwhile, we also give the relationship between an intuitionistic fuzzy filter and an intuitionistic fuzzy lattice filter. The intuitionistic fuzzy Boolean filter is proved to be equivalent to the intuitionistic fuzzy implicative filter. The intuitionistic fuzzy ultra filter is also equivalent to the intuitionistic fuzzy obstinate filter, and each intuitionistic fuzzy Boolean filter is an intuitionistic fuzzy positive implicative filter, but the converse may not be true in *BL*-algebras. Furthermore, the conditions under an intuitionistic fuzzy positive implicative filter being an intuitionistic fuzzy Boolean filter are constructed. Finally, we give the concepts of the intuitionistic fuzzy ultra and obstinate filters, and study their properties. Meanwhile, we prove that the intuitionistic fuzzy ultra filter is equivalent to the intuitionistic fuzzy obstinate filter in *BL*-algebras.

Acknowledgments This work is supported by the national natural science foundation of China under Grant No. 61273018 and the natural science research project of the Education Department of Henan Province of China under Grant No. 2009A520015.

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