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ORIGINAL ARTICLE

Independence of axiom sets characterizing formal concepts

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Abstract The theory of concept lattice proposed by Wille has been generalized in three different ways based on binary formal contexts, and substantive properties with respect to these formal concepts have been derived. In this paper, we study a reverse problem, that is, how to characterize the notions of formal concepts in terms of their properties. Axiomatic characterizations for the theory of formal concept analysis are presented. By this approach, four types of conceptual knowledge system are defined, and axiom sets that must be satisfied by the conceptual knowledge system are stated. It is proved that axioms of the conceptual knowledge system guarantee the existence of certain types of binary relations producing the same formal concepts. The independence of axiom sets characterizing the conceptual knowledge system is examined.

Keywords Concept lattices · Axioms · Independence · Conceptual knowledge systems

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1 Introduction

The theory of concept lattice [formal concept analysis (FCA)], proposed by Wille in 1982, is an effective method for data analysis [5, 21]. The central notions of the FCA are formal concepts and concept lattices. A concept lattice is an ordered hierarchical structure of formal concepts that are defined by a binary relation between a set of objects and a set of attributes. Each formal concept is an object-attribute pair, which consists of two parts: the extension (objects covered by the concept) and intension (attributes describing the concept). As an effective tool for data analysis and knowledge processing, the FCA has been successfully applied to various fields, such as decision making, information retrieval, data mining, and knowledge discovery.

The FCA by Wille has been expanded widely with the development of science and technology in the modern society. For example, Burusco [3] and Belohlavek [2] generalized a model of the FCA in the fuzzy environment. Alcalde et al. [1] discussed a L-Fuzzy context with some unknown values. Zhang et al. [29] introduced the definition of variable threshold concept lattices to reduce the number of fuzzy concepts in a fuzzy formal context.

The theory of rough sets (RST) which was originated by Pawlak [12] is another important method for data analysis and knowledge processing. The basic structure of the RST is that an approximation space consists of a universe of discourse and a binary relation imposed on it. Using the concepts of lower and upper approximations, the RST argues that knowledge hidden in information systems may be unraveled and expressed in the form of decision rules [13, 19, 20, 23, 32].

In recent years, more and more research has been conducted to combine the FCA and RST together, which provides new, integrated approaches for data analysis. On one hand, rough set approximation operators can be introduced into the FCA by considering different types of definability. For example, some authors introduced concept approximations into the FCA [6, 7, 14, 15, 16], and some others introduced rough reduction into the FCA [8, 31]. On the other hand, the notions of formal concepts and concept lattices are also introduced into the RST by considering different types of formal concepts, such as the property and object oriented formal concept lattices defined by Düntsch and Gediga [4] and Yao [25–27], respectively, based on approximation operators.

It is well-known that the axiomatic characterization of rough approximation operators is an important approach for studying mathematical structures of rough set algebras [9, 11, 17, 18, 22, 24, 28]. The axiomatic approach takes the lower and upper approximations as primitive notions and a set of axioms are used to characterize approximation operators that are the same as the ones produced by using the constructive definitions. However, compared with the studies on the axiomatic systems of rough sets, less effort has been made for axiomatic characterizations in the FCA. Zhang et al. [30] presented an axiomatic approach for the formal concept introduced by Wille. Ma [10] discussed the axiomatic characteristics of four types of formal concepts using dualities. By comparing with the constructive approach, the axiomatic approach aims to investigating the mathematical characters of formal concepts rather than developing methods for applications. In this paper, we devote to the axiomatic approaches of the FCA.

In the following section, we first review the basic notions of formal concepts and formal concept lattices proposed by Wille. In Sect. 3, basic properties for the other three types of formal concepts are illustrated. In Sect. 4, the axiomatic characterizations of the four types of formal concepts are presented, and then the independence of these axiomatic systems is discussed. Section 5 concludes the paper.

2 Preliminaries

A formal context is a triplet (U, A, I), where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects, $A = \{a_1, a_2, \ldots, a_m\}$ a non-empty finite set of attributes, and I a binary relation between U and A, which is a subset of the Cartesian product $U \times A$. For a pair of elements $x \in U$ and $a \in A$, if $(x, a) \in I$, also written as $x\ I\ a$, we say that the object x has the attribute a, or the attribute a is possessed by the object x; if $(x, a) \notin I$, we say that the object x has not the attribute a, or the attribute a is not possessed by the object x.

Denote $(x, a) \in I$ by 1 and $(x, a) \notin I$ by 0, a formal context can be denoted as a table only with the value 0 and 1.

The complement of a formal context (U, A, I) is defined as $(U, A, I^{\sim})$, where $I^{\sim} = \{(x, a) | \neg(xIa), x \in U, a \in A\}$.

For a formal context $(U, A, I), \forall X \in P(U)$ and $B \in P(A)$, where $P(\cdot)$ is the power set of $\cdot$, a pair of set-theoretic operators *, * are defined by

$$X^* = \{a \in A | \forall x \in X, (x, a) \in I\},$$
$$B^* = \{x \in U | \forall a \in B, (x, a) \in I\}. \tag{1}$$

X^* denotes the set of attributes possessed by all objects in X, B^* denotes the set of objects which possess all attributes in B. For simplicity, $\forall x \in U$, we denote $\{x\}^*$ as x^*; and $\forall a \in A, \{a\}^*$ as a^*.

A pair $(X, B), X \in P(U), B \in P(A)$, is called a formal concept if $X^* = B$ and $B^* = X$. X is referred to as the extension of the concept (X, B) and B the intension of the concept (X, B).

Let (U, A, I) be a formal context, $\forall X, X_1, X_2 \in P(U)$ and $B, B_1, B_2 \in P(A)$, the pair of set-theoretic operators satisfies the following properties [5]:

(1) $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*, B_1 \subseteq B_2 \Rightarrow B_2^* \subseteq B_1^*$;
(2) $X \subseteq X^{**}, B \subseteq B^{**}$;
(3) $X^* = X^{***}$, $B^* = B^{***}$;
(4) $(X_1 \cup X_2)^* = X_1^* \cap X_2^*$, $(B_1 \cup B_2)^* = B_1^* \cap B_2^*$;
(5) $X \subseteq B^* \Leftrightarrow B \subseteq X^*$.

The set of all formal concepts of (U, A, I) forms a complete lattice which is called the formal concept lattice of (U, A, I).

3 Three other types of formal concepts and concept lattices

In this section, we discuss three other types of formal concepts which are complementary to the definition of formal concepts introduced by Wille.

Let (U, A, I) be a formal context, a pair of dual operators $^\sharp$,$^\sharp$ is defined as follows [25]: $\forall X \in P(U)$ and $B \in P(A)$,

$$X^\sharp = \{a \in A | \exists x \in U((x, a) \notin I \wedge x \notin X)\},$$
$$B^\sharp = \{x \in U | \exists a \in A((x, a) \notin I \wedge a \notin B)\}. \tag{2}$$

The operators $^\sharp$,$^\sharp$ and *, * are in fact dual operators related by:

$$X^\sharp = X^{\sim*\sim}, X^* = X^{\sim\sharp\sim}, B^\sharp = B^{\sim*\sim}, B^* = B^{\sim\sharp\sim},$$

where $X^{\sim}$ denotes the complement of the set X.

Theorem 3.1 [25] *Let (U, A, I) be a formal context, then:* $\forall X, X_1, X_2 \in P(U), B, B_1, B_2 \in P(A)$,

(1) $X_1 \subseteq X_2 \Rightarrow X_2^{\sharp} \subseteq X_1^{\sharp}, B_1 \subseteq B_2 \Rightarrow B_2^{\sharp} \subseteq B_1^{\sharp}$;
(2) $X \supseteq X^{\sharp\sharp}, B \supseteq B^{\sharp\sharp}$;
(3) $X^{\sharp} = X^{\sharp\sharp\sharp}, B^{\sharp} = B^{\sharp\sharp\sharp}$;
(4) $(X_1 \cap X_2)^{\sharp} = X_1^{\sharp} \cup X_2^{\sharp}, (B_1 \cap B_2)^{\sharp} = B_1^{\sharp} \cup B_2^{\sharp}$;
(5) $X \supseteq B^{\sharp} \Leftrightarrow B \supseteq X^{\sharp}$.

Let $X \in P(U), B \in P(A)$, a pair (X, B) is called a dual concept if $X = B^{\sharp}$ and $B = X^{\sharp}$ [25]. For a set of objects $X \in P(U)$ and a set of attributes $B \in P(A)$, from Theorem 3.1 we can obtain that $(X^{\sharp\sharp}, X^{\sharp})$ and $(B^{\sharp}, B^{\sharp\sharp})$ are dual concepts.

Analogously, all dual concepts of (U, A, I) form a complete lattice which is called the dual concept lattice of (U, A, I).

Let (U, A, I) be a formal context, the other two operators $^{\Box}$, $^{\diamond}$ are defined as follows [4, 25]: $\forall X \in P(U), B \in P(A)$,

$$X^{\Box} = \{a \in A | \forall x \in U((x, a) \in I \Rightarrow x \in X)\},$$
$$B^{\diamond} = \{x \in U | \exists a \in A((x, a) \in I \wedge a \in B)\}. \quad (3)$$

$$X^{\diamond} = \{a \in A | \exists x \in U((x, a) \in I \wedge x \in X)\},$$
$$B^{\Box} = \{x \in U | \forall a \in A((x, a) \in I \Rightarrow a \in B)\}. \quad (4)$$

They are also dual operators related by: $X^{\Box} = X^{\sim\diamond\sim}$, $X^{\diamond} = X^{\sim\Box\sim}; B^{\Box} = B^{\sim\diamond\sim}, B^{\diamond} = B^{\sim\Box\sim}$.

Theorem 3.2 [4, 25] *The operators* $^{\Box}$, $^{\diamond}$ *satisfy the following properties: for* $\forall X, X_1, X_2 \in P(U), B, B_1, B_2 \in P(A)$,

(1) $X_1 \subseteq X_2 \Rightarrow X_1^{\diamond} \subseteq X_2^{\diamond}, X_1^{\Box} \subseteq X_2^{\Box}$,
$B_1 \subseteq B_2 \Rightarrow B_1^{\Box} \subseteq B_2^{\Box}, B_1^{\diamond} \subseteq B_2^{\diamond}$;
(2) $X^{\Box\diamond} \subseteq X \subseteq X^{\diamond\Box}, B^{\Box\diamond} \subseteq B \subseteq B^{\diamond\Box}$;
(3) $X^{\Box} = X^{\Box\diamond\Box}, X^{\diamond} = X^{\diamond\Box\diamond}$,
$B^{\diamond} = B^{\diamond\Box\diamond}, B^{\Box} = B^{\Box\diamond\Box}$;
(4) $(X_1 \cup X_2)^{\diamond} = X_1^{\diamond} \cup X_2^{\diamond}, (X_1 \cap X_2)^{\Box} = X_1^{\Box} \cap X_2^{\Box}$,
$(B_1 \cup B_2)^{\diamond} = B_1^{\diamond} \cup B_2^{\diamond}, (B_1 \cap B_2)^{\Box} = B_1^{\Box} \cap B_2^{\Box}$;
(5) $X \subseteq B^{\Box} \Leftrightarrow X^{\diamond} \subseteq B$,
$B^{\diamond} \subseteq X \Leftrightarrow B \subseteq X^{\Box}$.

For $X \in P(U), B \in P(A)$, a pair (X, B) is called an object-oriented concept if $X = B^{\diamond}, B = X^{\Box}$. From Theorem 3.2, we can verify that $(X^{\Box\diamond}, X^{\Box})$ and $(B^{\diamond}, B^{\diamond\Box})$ are object-oriented concepts.

All object-oriented concepts of (U, A, I) form a complete lattice which is called the object-oriented concept lattice of (U, A, I).

By exchanging objects and attributes in the definition of an object-oriented concept, we can define an attribute-oriented concept, that is, a pair $(X, B), X \in P(U), B \in P(A)$, is called an attribute-oriented concept if $X = B^{\Box}, B = X^{\diamond}$. From Theorem 3.2, we can see that $(X^{\diamond\Box}, X^{\diamond})$ and $(B^{\Box}, B^{\Box\diamond})$ are attribute-oriented concepts.

All attribute-oriented concepts of (U, A, I) form a complete lattice which is called the attribute-oriented concept lattice of (U, A, I)

Example 3.1 Table 1 gives a formal context (U, A, I) with $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{a, b, c, d, e, f\}$. The concept lattice, the dual concept lattice, the object-oriented concept lattice and the attribute-oriented concept lattice of (U, A, I) are shown in Figs. 1, 2, 3, and 4, respectively.

4 Axiomatic characterization of formal concepts

According to Sect. 3, from a formal context, four types of formal concepts can be derived. In this section, we use the axiomatic approach to characterize the essential properties of the four types of formal concepts. At the same time, we prove the independence of the axioms by exemplifications.

Definition 4.1 Let U be a finite set of objects, A a finite set of attributes, $L_c : P(U) \to P(A)$, and $H_c : P(A) \to P(U)$. Then (U, A, L_c, H_c) is called a conceptual knowledge system if the following axioms are satisfied: $\forall X_1, X_2 \in P(U), B_1, B_2 \in P(A)$,

Table 1 A formal context (U, A, I)

U	a	b	c	d	e	f
1	1	0	0	1	0	1
2	1	1	0	1	1	1
3	0	0	0	1	0	0
4	0	0	1	0	1	0
5	1	1	0	1	1	1
6	0	0	1	0	1	0

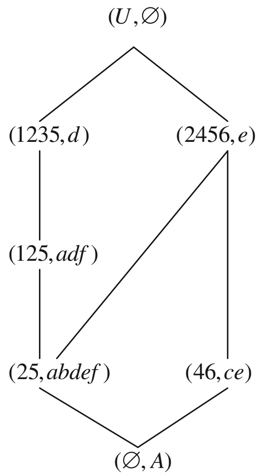

Fig. 1 The formal concept lattice

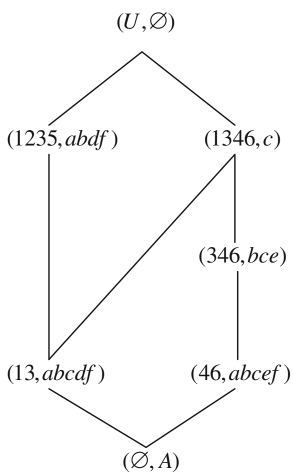

Fig. 2 The dualconcept lattice

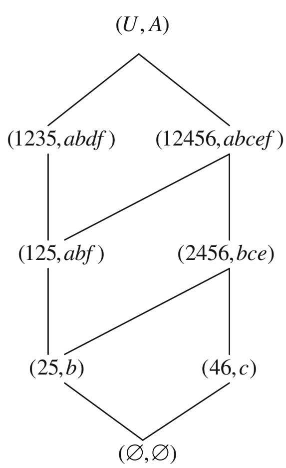

Fig. 3 The object-oriented concept lattice

$$
\begin{array}{ll}
(LC1) & L_c(X_1 \cup X_2) = L_c(X_1) \cap L_c(X_2),\\
(HC1) & H_c(B_1 \cup B_2) = H_c(B_1) \cap H_c(B_2),\\
(LHC) & a \in L_c(x) \Longleftrightarrow x \in H_c(a), \forall x \in U, a \in A.
\end{array}
$$

For simplicity, we denote $L_c(\{x\})$ as $L_c(x)$, $H_c(\{a\})$ as $H_c(a)$, $H_c(L_c(X))$ as $H_cL_c(X)$ and $L_c(H_c(B))$ as $L_cH_c(B)$.

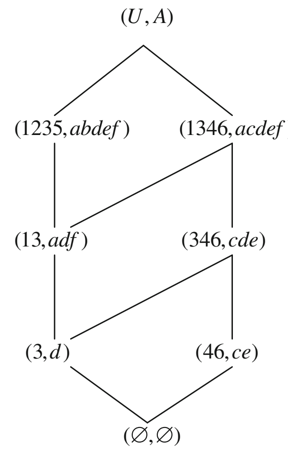

Fig. 4 The attribute-oriented concept lattice

Theorem 4.1 *Let U be a finite set of objects, A a finite set of attributes,* $L_c : P(U) \to P(A)$ *and* $H_c : P(A) \to P(U)$. *Then there exists a binary relation I between U and A such that* $L_c(X) = X^*$, $H_c(B) = B^*$ *for all* $X \in P(U)$ *and* $B \in P(A)$ iff (U, A, L_c, H_c) *is a conceptual knowledge system, where* X^* *and* B^* *are defined by Eq.* (1).

Proof "$\Rightarrow$" It follows immediately from the properties of X^* and B^*. "$\Leftarrow$" Define a binary relation $I \subseteq U \times A$ as follows:

$I = \{(x,a)|a \in L_c(x)\}$.

For any $X \in P(U)$, we have

$$
\begin{aligned}
X^* &= \{a \in A|\forall x \in X, (x,a) \in I\}\\
&= \{a \in A|\forall x \in X, a \in L_c(x)\}\\
&= \left\{a \in A|a \in \bigcap_{x\in X} L_c(x) = L_c(X)\right\} = L_c(X).
\end{aligned}
$$

Thus $X^* = L_c(X)$. Analogously, we can prove that $H_c(B) = B^*$. □

The following example shows that the three axioms for the conceptual knowledge system are independent, i.e., any two of the axioms can not deduce the third one.

Example 4.1 Let $U = \{1, 2, 3\}$, $A = \{a, b, c\}$.

(1) Define $L_c(X) = A$ for any $X \subseteq U$ and $H_c(B) = \emptyset$ for any $B \subseteq A$. It is easy to see that axioms $(LC1)$ and $(HC1)$ are satisfied, and $a \in L_c(1)$, but $1 \notin H_c(a)$, i.e., (LHC) is not satisfied. Thus L_c and H_c do not imply (LHC).

(2) Define $L_c(1) = \{a\}, L_c(2) = \{b\}, L_c(3) = \{c\}, L_c(\emptyset) = A, L_c(X) = \emptyset$ for all other $X \subseteq U$. $H_c(a) = \{1\}, H_c(b) = \{2\}, H_c(c) = \{3\}, H_c(\emptyset) = U, H_c(B) = U$ for all other $B \subseteq A$. Since $H_c(A\cup \{a\}) = H_c(A) = U \neq H_c(A) \cap H_c(\{a\})$, i.e., $(HC1)$ is not satisfied, we conclude that $(LC1)$, $(LHC) \not\Rightarrow (HC1)$.

(3) Define $L_c(1) = \{a\}, L_c(2) = \{b\}, L_c(3) = \{c\}, L_c(\emptyset) = A, L_c(X) = A$ for all other $X \subseteq U$. $H_c(a) = \{1\}, H_c(b) = \{2\}, H_c(c) = \{3\}$, $H_c(\emptyset) = U, H_c(B) = \emptyset$ for all other $B \subseteq A$. It is easy to see that (*HC*1), (*LHC*) are satisfied, and $L_c(U \cup \{1\}) = L_c(U) = A \neq L_c(U) \cap L_c(\{1\})$, i.e., (*LC*1) is not satisfied. This implies that (*HC*1), (*LHC*)$\not\Rightarrow$(*LC*1).

Therefore, axioms (*LC*1), (*HC*1) and (*LHC*) are independent. □

Theorem 4.2. *Let* (U, A, L_c, H_c) *be a conceptual knowledge system, then* $\forall X \in P(U), B \in P(A)$, $X \subseteq H_c L_c(X)$ *and* $B \subseteq L_c H_c(B)$.

Proof $\forall x \in X, \forall a \in L_c(X) = \bigcap_{x \in X} L_c(x)$, by (*LHC*), we can obtain $x \in H_c(a)$. *So* $x \in \bigcap_{a \in L_c(X)} H_c(a) = H_c\left(\bigcup_{a \in L_c(X)} a\right) = H_c L_c(X)$, that is $X \subseteq H_c L_c(X)$. Similarly, we can conclude $B \subseteq L_c H_c(B)$. □

Let (U, A, L_c, H_c) be a conceptual knowledge system. A pair $(X, B), X \in P(U), B \in P(A)$, is called a concept of (U, A, L_c, H_c) if $X = H_c(B)$ and $B = L_c(X)$. We denote the set of all concepts as $L(U, A, L_c, H_c)$.

The partial ordered relation $\leq$ in $L(U, A, L_c, H_c)$ is defined as follows:

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \supseteq B_2).$$

It allows us to conclude the following:

Theorem 4.3 $(L(U, A, L_c, H_c), \leq)$ *forms a complete lattice, where the meet and the joint are, respectively, given by*:

$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \cap X_2, L_c H_c(B_1 \cup B_2));$$
$$(X_1, B_1) \vee (X_2, B_2) = (H_c L_c(X_1 \cup X_2), B_1 \cap B_2).$$

Proof It follows immediately from Theorem 4.1. □

Definition 4.2 Let U be a finite set of objects, A a finite set of attributes, $L_d : P(U) \to P(A)$, and $H_d : P(A) \to P(U)$. Then (U, A, L_d, H_d) is called a dual conceptual knowledge system if the following axioms are satisfied: $\forall X_1, X_2 \in P(U), B_1, B_2 \in P(A), (x, a) \in U \times A$,

(*LD*1) $L_d(X_1 \cap X_2) = L_d(X_1) \cup L_d(X_2)$,
(*HD*1) $H_d(B_1 \cap B_2) = H_d(B_1) \cup H_d(B_2)$,
(*LHD*) $a \notin L_d(U - \{x\}) \Leftrightarrow x \notin H_d(A - \{a\})$

Theorem 4.4 *Let U be a finite set of objects and A a finite set of attributes,* $L_d : P(U) \to P(A)$ *and* $H_d : P(A) \to P(U)$. *Then there exists a binary relation I between U and A such that* $L_d(X) = X^\sharp$ *and* $H_d(B) = B^\sharp$ *for all* $X \in P(U)$ *and* $B \in P(A)$ *if and only if* (U, A, L_d, H_d) *is a dual conceptual knowledge system, where* $X^\sharp$ *and* $B^\sharp$ *are defined by Eq.* (2).

Proof "$\Rightarrow$" It follows immediately from Theorem 3.1. "$\Leftarrow$" Define a binary relation I between U and A by $I = \{(x, a) | a \notin L_d(U - \{x\})\}$.

Note that for all $X \in P(U), X = \bigcap_{x \in X^\sim} (U - \{x\})$, we have

$$\begin{aligned} X^\sharp &= \{a \in A | \exists x \in U((x, a) \notin I \wedge x \notin X)\} \\ &= \{a \in A | \exists x \in X^\sim, (x, a) \notin I\} \\ &= \{a \in A | \exists x \in X^\sim, a \in L_d(U - \{x\})\} \\ &= \left\{a \in A | a \in \bigcup_{x \in X^\sim} L_d(U - \{x\}) = L_d(X)\right\} = L_d(X). \end{aligned}$$

Hence $X^\sharp = L_d(X)$. On the other hand, by axiom (*LHD*), if $I = \{(x, a) | a \notin L_d(U - \{x\})\}$, then $I = \{(x, a) | x \notin H_d(A - \{a\})\}$. Hence

$$\begin{aligned} B^\sharp &= \{x \in U | \exists a \in A((x, a) \notin I \wedge a \notin B)\} \\ &= \{x \in U | \exists a \in B^\sim, (x, a) \notin I\} \\ &= \left\{x \in U | x \in \bigcup_{a \in B^c} H_d(A - \{a\}) = H_d(B)\right\} = H_d(B), \end{aligned}$$

which implies that $B^\sharp = H_d(B)$. □

The following example shows that the three axioms in Definition 4.2 are independent.

Example 4.2 Let $U = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

(1) Define $L_d(X) = A$ for any $X \subseteq U$ and $H_d(B) = \emptyset$ for any $B \subseteq A$. It is easy to verify that L_d and H_d satisfy axioms (*LD*1) and (*HD*1), but they do not obey axiom (*LHD*).
(2) Define $L_d(\{1,2\}) = \{a,b\}, L_d(\{1,3\}) = \{a,c\}, L_d(\{2,3\}) = \{b,c\}, L_d(U) = \emptyset, L_d(X) = A$ for all other $X \subseteq U, H_d(\{a,b\}) = \{1,2\}, H_d(\{a,c\}) = \{1,3\}, H_d(\{b,c\}) = \{2,3\}, H_d(B) = \emptyset$ for all other $B \subseteq A$. Then L_d and H_d satisfy axioms (*LD*1) and (*LHD*). Since $H_d(\{a,b\} \cap \{a,c\}) = H_d(a) = \emptyset \neq H_d(\{a,b\}) \cup H_d(\{a,c\})$, we see that L_d and H_d do not obey axiom (*HD*1).
(3) Define $L_d(\{1,2\}) = \{a,b\}, L_d(\{1,3\}) = \{a,c\}, L_d(\{2,3\}) = \{b,c\}, L_d(X) = \emptyset$ for all other $X \subseteq U. H_d(\{a,b\}) = \{1,2\}, H_d(\{a,c\}) = \{1,3\}, H_d(\{b,c\}) = \{2,3\}, H_d(A) = \emptyset$, and $H_d(B) = U$ for all other $B \subseteq A$. Thus L_d and H_d satisfy axioms (*HD*1) and (*LHD*), but they do not obey axiom (*LD*1).

Therefore, we have examined that axioms (*LD*1), (*HD*1) and (*LHD*) are independent. □

The following theorem shows that if (U, A, L_c, H_c) is a conceptual knowledge system, then a dual conceptual knowledge system can be constructed, that is, the dual operators of L_c and H_c satisfy the three axioms in Definition 4.2.

Theorem 4.5 *Let (U, A, L_c, H_c) be a conceptual knowledge system. If $\forall X \in P(U), B \in P(A), L_{d'}(X) = (L_c(X^\sim))^\sim$, $H_{d'}(B) = (H_c(B^\sim))^\sim$, then $(U, A, L_{d'}, H_{d'})$ is a dual conceptual knowledge system.*

Proof We only need to prove that $L_{d'}$ and $H_{d'}$ satisfy axioms $(LD1)$, $(HD1)$, and (LHD). By the properties of the duality and the complement, we have

$$L_{d'}(X_1 \cap X_2) = (L_c(X_1^\sim \cup X_2^\sim))^\sim = (L_c(X_1^\sim) \cap L_c(X_2^\sim))^\sim = (L_c(X_1^\sim))^\sim \cup (L_c(X_2^\sim))^\sim = L_{d'}(X_1) \cup L_{d'}(X_2),$$

thus $L_{d'}$ satisfies axiom $(LD1)$.

Similarly, we can prove that $H_{d'}$ satisfies axioms $(HD1)$. Finally, for $(x, a) \in U \times A$, we have

$$a \notin L_{d'}(U - \{x\}) \Longleftrightarrow a \notin (L_c(\{x\}))^\sim \Longleftrightarrow a \in L_c(\{x\}) \Longleftrightarrow x \in H_c(\{a\}) \Longleftrightarrow x \notin H_{d'}(A - \{a\}).$$

Thus $(U, A, L_{d'}, H_{d'})$ is a dual conceptual knowledge system. □

However, if (U, A, L_c, H_c) be a conceptual knowledge system, (U, A, L_d, H_d) a dual conceptual knowledge system, then for $X \in P(U)$ and $B \in P(A)$, the equations $L_d(X) = (L_c(X^\sim))^\sim$ and $H_d(B) = (H_c(B^\sim))^\sim$ may be not hold.

Example 4.3 Let $U = \{1, 2, 3\}$, $A = \{a, b, c\}$. L_c, H_c, L_d, H_d are defined as Table 2. It is easy to verify that (U, A, L_c, H_c) is a conceptual knowledge system, (U, A, L_d, H_d) is a dual conceptual knowledge system. But $L_d(\{2, 3\}) = \{c\}$, $(L_c(\{2, 3\})^\sim)^\sim = \{b, c\}$, so $L_d(\{2, 3\}) \neq (L_c(\{2, 3\})^\sim)^\sim$. $H_d(\{a,c\}) = \{3\}$ and $(H_c(\{a, c\})^\sim)^\sim = \{1, 3\}$, hence $H_d(\{a, c\}) = \{3\} \neq (H_c(\{a, c\})^\sim)^\sim$.

Theorem 4.6 *Let (U, A, L_d, H_d) be a dual conceptual knowledge system, then $\forall X \in P(U), B \in P(A), X \supseteq H_dL_d(X), B \supseteq L_dH_d(B)$.*

Proof $\forall X \subseteq U$, note that $X = \bigcap_{x \in X^\sim}(U - \{x\})$, then $\forall x \in X^\sim, \forall a \in (L_d(X))^\sim = \bigcap_{x \in X^\sim}(L_d(U - \{x\}))^\sim$, by (LHD) we have $x \in (H_d(A - \{a\}))^\sim$. Hence

$$x \in \bigcap_{a \in (L_d(X))^\sim}(H_d(A - \{a\}))^\sim = \left(\bigcup_{a \in (L_d(X))^\sim} H_d(A - \{a\})\right)^\sim = (H_dL_d(X))^\sim,$$

which implies that $X^\sim \subseteq (H_dL_d(X))^\sim$. Thus $X \supseteq H_dL_d(X)$.

On the other hand, $\forall B \subseteq A$, note that $B = \bigcap_{a \in B^\sim}(A - \{a\})$, then $\forall a \in B^\sim, \forall x \in (H_d(B))^\sim = \bigcap_{a \in B^\sim}(H_d(A - \{a\}))^\sim$, by (LHD) we have $a \in (L_d(U - \{x\}))^\sim$. Hence

$$a \in \bigcap_{x \in (H_d(B))^\sim}(L_d(U - \{x\}))^\sim = \left(\bigcup_{x \in (H_d(B))^\sim} L_d(U - \{x\})\right)^\sim = (L_dH_d(B))^\sim,$$

which implies that $B^\sim \subseteq (L_dH_d(B))^\sim$. Thus $B \supseteq L_dH_d(B)$. □

Table 2 An example to check that $L_d(X) \neq (L_c(X^\sim))^\sim$, $H_d(B) \neq (H_c(B^\sim))^\sim$

X	B	$L_c(X)$	$H_c(B)$	$L_d(X)$	$H_d(B)$
U	A	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1, 2\}$	$\{a, b\}$	$\emptyset$	$\emptyset$	$\{a, b\}$	$\{1, 2\}$
$\{1, 3\}$	$\{a, c\}$	$\emptyset$	$\emptyset$	$\{a, c\}$	$\{3\}$
$\{2, 3\}$	$\{b, c\}$	$\emptyset$	$\emptyset$	$\{c\}$	$\{2, 3\}$
$\{1\}$	$\{a\}$	$\{a\}$	$\{1\}$	A	U
$\{2\}$	$\{b\}$	$\{b\}$	$\{2\}$	A	U
$\{3\}$	$\{c\}$	$\{c\}$	$\{3\}$	$\{a, c\}$	$\{2, 3\}$
$\emptyset$	$\emptyset$	A	U	A	U

Let (U, A, L_d, H_d) be a dual conceptual knowledge system. A pair (X, B), $X \in P(U)$, $B \in P(A)$, is called a dual formal concept of (U, A, L_d, H_d) if $X = H_d(B)$ and $B = L_d(X)$. The set of all dual concepts is denoted as $L(U, A, L_d, H_d)$.

For two dual concepts (X_1, B_1) and (X_2, B_2), we define

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \supseteq B_2).$$

Then we get the following

Theorem 4.7 *$(L(U, A, L_d, H_d), \leq)$ forms a complete lattice in which the meet and the joint of the dual concepts are defined as follows*:

$$(X_1, B_1) \wedge (X_2, B_2) = (H_dL_d(X_1 \cap X_2), B_1 \cup B_2);$$
$$(X_1, B_1) \vee (X_2, B_2) = (X_1 \cup X_2, L_dH_d(B_1 \cap B_2)).$$

Proof It follows immediately from Theorem 4.4. □

In the following, we present the axiomatic characterizations of the other two types of concept lattices.

Definition 4.3 Let U be a finite set of objects, A a finite set of attributes, $L_o : P(U) \to P(A)$, and $H_o : P(A) \to P(U)$. Then (U, A, L_o, H_o) is called an object-oriented conceptual knowledge system if the following axioms are satisfied: $\forall X_1, X_2 \in P(U), B_1, B_2 \in P(A), (x, a) \in U \times A$,

$(LO1)$ $L_o(X_1 \cap X_2) = L_o(X_1) \cap L_o(X_2)$,
$(HO1)$ $H_o(B_1 \cup B_2) = H_o(B_1) \cup H_o(B_2)$,
(LHO) $a \in L_o(U - \{x\}) \Longleftrightarrow x \notin H_o(a)$.

Theorem 4.8 *Let U be a finite set of objects, A a finite set of attributes, $L_o : P(U) \to P(A)$, $H_o : P(A) \to P(U)$. There exists a binary relation I between U and A such that*

$L_o(X) = X^{\square}$ and $H_o(B) = B^{\diamond}$ for all $X \in P(U)$ and $B \in P(A)$ if and only if (U, A, L_o, H_o) is an object-oriented conceptual knowledge system, where $X^{\square}$ and $B^{\diamond}$ are defined by Eq. (3).

Proof "$\Rightarrow$" It follows immediately from Theorem 3.2. "$\Leftarrow$" Define a binary relation I between U and A by $I^{\sim} = \{(x,a) | x \notin H_o(a)\}$. Then, for $B \subseteq A$, we have

$$\begin{aligned} B^{\diamond} &= \{x \in U | \exists a \in A(a \in B \wedge (x,a) \in I)\} \\ &= \{x \in U | \exists a \in A(a \in B \wedge x \in H_o(a))\} \\ &= \left\{ x \in U | x \in \bigcup_{a \in B} H_o(a) = H_o(B) \right\} = H_o(B). \end{aligned}$$

So $H_o(B) = B^{\diamond}$. on the other hand, by axiom (*LHO*), we have

$$\begin{aligned} X^{\square} &= \{a \in A | \forall x \in U((x,a) \in I \Rightarrow x \in X)\} \\ &= \{a \in A | \forall x \in U(x \in X^{\sim} \Rightarrow (x,a) \in I^{\sim}\} \\ &= \{a \in A | \forall x \in X^{\sim}, a \in L_o(U - \{x\})\} \\ &= \left\{ a \in A | a \in \bigcap_{x \in X^{\sim}} L_o(U - \{x\}) = L_o(X) \right\} = L_o(X). \end{aligned}$$

Therefore, $L_o(X) = X^{\square}$. □

The next example shows that the three axioms in Definition 4.3 are independent.

Example 4.4 Let $U = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

(1) Define $L_o(X) = A$ for any $X \subseteq U, H_o(B) = U$ for any $B \subseteq A$. Then it is clear that L_o and H_o satisfy axioms (*LO*1) and (*HO*1), but they do not obey axiom (*LHO*).

(2) Define $L_o(\{1,2\}) = \{a\}$, $L_o(\{1,3\}) = \{b\}, L_o(\{2,3\}) = \{c\}, L_o(U) = A, L_o(X) = \emptyset$ for all other $X \subseteq U$. $H_o(\{a,b\}) = \{3\}, H_o(\{a\}) = \{1,2\}, H_o(\{b\}) = \{1,3\}$, $H_o(\{c\}) = \{2,3\}, H_o(\emptyset) = \emptyset$, and $H_o(B) = U$ for all other $B \subseteq A$. Since $H_o(\{a, b\} \cup \{a\}) = H_o(\{a, b\}) = \{3\} \neq H_o(\{a, b\}) \cup H_o(\{a\})$, we conclude that L_o and H_o satisfy axioms (*LO*1) and (*LHO*), but they do not obey axiom (*HO*1).

(3) Define $L_o(\{1,2\}) = \{a\}, L_o(\{1,3\}) = \{b\}, L_o(\{2,3\}) = \{c\}, L_o(U) = A, L_o(3) = \{a,b\}, L_o(X) = \emptyset$ for all other $X \subseteq U$. $H_o(\{a\}) = \{1,2\}$, $H_o(\{b\}) = \{1,3\}$, $H_o(\{c\}) = \{2,3\}, H_o(\emptyset) = \emptyset$, and $H_o(B) = U$ for all other $B \subseteq A$. Then it is easy to check that axioms (*HO*1) and (*LHO*) are satisfied, however, $L_o(\{1,2\} \cap \{3\}) = L_o(\emptyset) = \emptyset \neq L_o(\{1,2\}) \cap L_o(\{3\})$, that is, axiom (*LO*1) is not satisfied. Hence (*HO*1), (*LHO*)$\not\Rightarrow$(*LO*1).

Thus, axioms (*LO*1), (*HO*1) and (*LHO*) are independent. □

Theorem 4.9 *Let (U, A, L_o, H_o) be an object-oriented conceptual knowledge system, then $\forall X \in P(U), B \in P(A), H_oL_o(X) \subseteq X$ and $L_oH_o(B) \supseteq B$.*

Proof $\forall X \subseteq U$, note that $X = \bigcap_{x \in X^{\sim}} (U - \{x\})$, then $\forall x \in X^{\sim}, \forall a \in L_o(X) = \bigcap_{x \in X^{\sim}} L_o(U - \{x\})$, by (*LHO*), we have $x \in (H(a))^{\sim}$. Hence

$$\begin{aligned} x &\in \bigcap_{a \in L_o(X)} (H_o(a))^{\sim} \\ &= \left(\bigcup_{a \in L_o(X)} H_o(a) \right)^{\sim} \\ &= (H_oL_o(X))^{\sim}. \end{aligned}$$

So $X^{\sim} \subseteq (H_oL_o(X))^{\sim}$. It follows that $X \supseteq H_oL_o(X)$.

On the other hand, $\forall B \subseteq A$ and $\forall a \in B, \forall x \in (H_o(B))^{\sim} = \bigcap_{a \in B} (H_o(a))^{\sim}$, by (*LHO*), we have $a \in L_o(U - \{x\})$, then

$$\begin{aligned} a &\in \bigcap_{x \in (H_o(B))^{\sim}} L_o(U - \{x\}) \\ &= L_o\left(\bigcap_{x \in (H_o(B))^{\sim}} (U - \{x\}) \right) \\ &= L_oH_o(B). \end{aligned}$$

Thus $B \subseteq L_oH_o(B)$. □

Let (U, A, L_o, H_o) be an object-oriented conceptual knowledge system. A pair (X, B), $X \in P(U)$, $B \in P(A)$, is called an object-oriented concept of (U, A, L_o, H_o) if $X = H_o(B)$ and $B = L_o(X)$. The family of all object-oriented concepts is denoted as $L(U, A, L_o, H_o)$.

For two object-oriented concepts (X_1, B_1) and (X_2, B_2), we define

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \subseteq B_2).$$

Then we conclude

Theorem 4.10 *$(L(U, A, L_o, H_o), \leq)$ is a complete lattice, where the meet and the joint of the object-oriented concepts are defined as follows*:

$$(X_1, B_1) \wedge (X_2, B_2) = (H_oL_o(X_1 \cap X_2), B_1 \cap B_2);$$
$$(X_1, B_1) \vee (X_2, B_2) = (X_1 \cup X_2, L_oH_o(B_1 \cup B_2)).$$

Definition 4.4 Let U be a finite set of objects, A a finite set of attributes, $L_p : P(U) \to P(A)$, and $H_p : P(A) \to P(U)$. Then (U, A, L_p, H_p) is called an attribute-oriented conceptual knowledge system if the following axioms are satisfied:

$\forall X_1, X_2 \in P(U), \forall B_1, B_2 \in P(A), \forall (x,a) \in U \times A$,

(*LP*1) $L_p(X_1 \cup X_2) = L_p(X_1) \cup L_p(X_2)$,
(*HP*1) $H_p(B_1 \cap B_2) = H_p(B_1) \cap H_p(B_2)$,
(*LHP*) $a \notin L_p(x) \Leftrightarrow x \in H_p(A - \{a\})$.

Theorem 4.11 *Let U be a finite set of objects and A finite set of attributes, $L_p : P(U) \to P(A)$ and $H_p : P(A) \to P(U)$. Then there exists a binary relation I between*

U and A such that $L_p(X) = X^{\diamond}$ and $H_p(B) = B^{\square}$ for all $X \in P(U)$ and $B \in P(A)$ if and only if (U, A, L_p, H_p) is an attribute-oriented conceptual knowledge system, where $X^{\diamond}$ and $B^{\square}$ are defined by Eq. (4).

Proof "⇒" It follows immediately from Theorem 3.2. "⇐" Define a binary relation I between U and A by

$I^{\sim} = \{(x,a) | a \notin L_p(x)\}$.

Then, for $X \subseteq U$, we have

$$\begin{aligned} X^{\diamond} &= \{a \in A | \exists x \in U((x,a) \in I \wedge x \in X)\} \\ &= \{a \in A | \exists x \in U(x \in X \cap a \in L_p(x))\} \\ &= \left\{ a \in A | a \in \bigcup_{x \in X} L_p(x) = L_p(X) \right\} = L_p(X). \end{aligned}$$

So $L_p(X) = X^{\diamond}$. On the other hand, by axiom (*LHP*), note that $I^{\sim} = \{(x,a) | a \notin L_p(x)\}$, then $I^{\sim} = \{(x,a) | x \in H_p(A - \{a\})\}$. Hence

$$\begin{aligned} B^{\square} &= \{x \in U | \forall a \in A((x,a) \in I \Rightarrow a \in B)\} \\ &= \{x \in U | \forall a \in A(a \in B^{\sim} \Rightarrow (x,a) \in I^{\sim}\} \\ &= \{x \in U | \forall a \in B^{\sim}, x \in H_p(A - \{a\})\} \\ &= \left\{ x \in U | x \in \bigcap_{a \in B^{\sim}} H_p(A - \{a\}) = H_p(B) \right\} = H_p(B). \end{aligned}$$

□

The following example shows that the three axioms in Definition 4.4 are independent.

Example 4.5 Let $U = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

(1) Define $L_p(X) = A$ for any $X \subseteq U, H_p(B) = U$ for any $B \subseteq A$. Thus L_p and H_p satisfy axioms (*LP*1) and (*HP*1), respectively, but they do not obey axiom (*LHO*).
(2) Define $L_p(\{1\}) = \{a,b\}, L_p(\{2\}) = \{a,c\}, L_p(\{3\}) = \{b,c\}, L_p(\emptyset) = \emptyset, L_p(X) = A$ for all other $X \subseteq U$. $H_p(A) = U$, $H_p(\{a,b\}) = \{1\}, H_p(\{a,c\}) = \{2\}, H_p(\{b,c\}) = \{3\}$, $H_p(\{c\}) = \{1,2\}$, and $H_p(B) = \emptyset$ for all other $B \subseteq A$. Then L_p and H_p satisfy axioms (*LP*1) and (*LHP*). Note that $H_p(\{a,b\} \cap \{c\}) = H_p(\emptyset) = \emptyset \neq H_p(\{a,b\}) \cap H_p(\{c\})$, we see that H_p does not obey axiom (*HP*1).
(3) Define $L_p(\{1\}) = \{a,b\}, L_p(\{2\}) = \{a,c\}, L_p(\{3\}) = \{b,c\}, L_p(\emptyset) = \emptyset$, $L_p(\{1,2\}) = \{c\}, L_p(X) = A$ for all other $X \subseteq U$. $H_p(\{a,b\}) = \{1\}$, $H_p(\{a,c\}) = \{2\}, H_p(\{b,c\}) = \{3\}, H_p(A) = U$, and $H_p(B) = \emptyset$ for all other $B \subseteq A$. It is easy to see that axioms (*HP*1) and (*LHP*) are satisfied. Note that $L_p(\{1, 2\} \cup \{1\}) = L_p(\{1, 2\}) = \{c\} \neq L_p(\{1, 2\}) \cup L_p(\{1\})$, thus axiom (*LP*1) is not satisfied.

Therefore, we have examined that axioms (*LP*1), (*HP*1) and (*LHP*) are independent. □

The following theorem shows that if (U, A, L_o, H_o) is an object-conceptual knowledge system, then the dual operators of L_o and H_o obey the three axioms in Definition 4.4.

Theorem 4.12 *Let (U, A, L_o, H_o) be an object-conceptual knowledge system. If $L_{p'}(X) = (L_o(X^{\sim}))^{\sim}, H_{p'}(B) = (H_o(B^{\sim}))^{\sim}, \forall X \in P(U), \forall B \in P(A)$, then $(U, A, L_{p'}, H_{p'})$ is an attribute-oriented conceptual knowledge system.*

Proof It is similar to the proof of Theorem 4.5. □

However, if (U, A, L_o, H_o) is an object-conceptual knowledge system and (U, A, L_p, H_p) an attribute-conceptual knowledge system, then for $X \in P(U)$ and $B \in P(A)$, the equations $L_p(X) = (L_o(X^{\sim}))^{\sim}$ and $H_p(B) = (H_o(B^{\sim}))^{\sim}$ may not hold.

Example 4.6 Let $U = \{1, 2, 3\}$ and $A = \{a, b, c\}$. L_o, H_o, L_p, H_p are defined as Table 3. Then it is easy to verify that (U, A, L_o, H_o) is an object-conceptual knowledge system and (U, A, L_p, H_p) an attribute-conceptual knowledge system. But $L_p(\{2, 3\}) = A$, $(L_o(\{2, 3\})^{\sim})^{\sim} = \{b, c\}$, so $L_p(\{2, 3\}) \neq (L_o(\{2, 3\})^{\sim})^{\sim}$. Also, $H_p(\{b, c\}) = \{3\}$, $(H_o(\{b, c\})^{\sim})^{\sim} = \{2, 3\}$, thus $H_p(\{b, c\}) \neq (H_o(\{b, c\})^{\sim})^{\sim}$.

Theorem 4.13 *Let (U, A, L_p, H_p) be an attribute-oriented conceptual knowledge system, then, $\forall X \in P(U), \forall B \in P(A), X \subseteq H_pL_p(X), B \supseteq L_pH_p(B)$.*

Proof $\forall x \in X, \forall a \in (L_p(X))^{\sim} = \bigcap_{x \in X}(L_p(x))^{\sim}$, by (*LHP*) we have $x \in H_p(A - \{a\})$. Then

$$\begin{aligned} x &\in \bigcap_{a \in (L_p(X))^{\sim}} H_p(A - \{a\}) \\ &= H_p\left(\bigcap_{a \in (L_p(X))^{\sim}} (A - \{a\})\right) \\ &= H_pL_p(X). \end{aligned}$$

Hence $X \subseteq H_pL_p(X)$.

Table 3 An example to check that $L_o(X) \neq (L_p(X^{\sim}))^{\sim}$, $H_o(B) \neq (H_p(B^{\sim}))^{\sim}$

X	B	$L_o(X)$	$H_o(B)$	$L_p(X)$	$H_p(B)$
U	A	A	U	A	U
{1, 2}	{a, b}	{a}	{1,3}	A	{1}
{1, 3}	{a, c}	{a, b}	U	A	{2}
{2, 3}	{b, c}	{c}	U	A	{3}
{1}	{a}	{a}	{1}	{a, b}	∅
{2}	{b}	∅	{1, 3}	{a, c}	∅
{3}	{c}	∅	{2, 3}	{b, c}	∅
∅	∅	∅	∅	∅	∅

On the other hand, $\forall B \subseteq A$, note that $B = \bigcap_{a \in B^{\sim}} (A - \{a\})$, then $\forall a \in B^{\sim}, \forall x \in H_p(B) = \bigcap_{a \in B^{\sim}} H_p(A - \{a\})$, by (*LHP*) we have $a \in (L_p(x))^{\sim}$. Hence

$$a \in \bigcap_{x \in H_p(B)} (L_p(x))^{\sim} = \left(\bigcup_{x \in H_p(B)} L_p(x) \right)^{\sim} = (L_p H_p(B))^{\sim}.$$

Consequently, $B^{\sim} \subseteq (L_p H_p(B))^{\sim}$. Thus $B \supseteq L_p H_p(B)$. □

Let (U, A, L_p, H_p) be an attribute-oriented conceptual knowledge system. A pair (X, B), $X \in P(U)$, $B \in P(A)$, is called an attribute-oriented concept of (U, A, L_p, H_p) if $X = H_p(B)$ and $B = L_p(X)$. The set of all attribute-oriented concepts is denoted as $L(U, A, L_p, H_p)$.

For two attribute-oriented concepts (X_1, B_1) and (X_2, B_2), we define

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \subseteq B_2).$$

Then, by Theorem 4.11, we can conclude

Theorem 4.14 ($L(U, A, L_p, H_p)$, $\leq$) *forms a complete lattice, where the meet and the joint of the attribute-oriented concepts are, respectively, defined as follows*:

$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \cap X_2, L_p H_p(B_1 \cap B_2)),$$
$$(X_1, B_1) \vee (X_2, B_2) = (H_p L_p(X_1 \cup X_2), B_1 \cup B_2).$$

5 Conclusion

Generalizations of the FCA proposed by Wille have been studied in various ways. A majority of the studies examined formal concepts and concept lattices by using constructive approach; however, less effort has been made by using axiomatic approaches. In this paper, we discuss the axiomatic characterizations of formal concepts. Using the axiomatic approach, four types of conceptual knowledge systems from a formal context are defined. Axioms for characterizing conceptual knowledge systems guarantee the existence of certain types of binary crisp relations which produce the same formal concept operators. Furthermore, the independence of axiom set for characterizing each type of conceptual knowledge system is examined. The formal concept axiomatic systems will help us to understand the structural features of various formal concepts.

An important generalization of the FCA is fuzzy concept lattice. For may examine further research, the axiomatic characteristic of concept lattice theory in the fuzzy environment need to be investigated.

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