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ORIGINAL ARTICLE

Implication operators on the set of ∨-irreducible element in the linguistic truth-valued intuitionistic fuzzy lattice

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Abstract We construct a kind of linguistic truth-valued intuitionistic fuzzy lattice based on linguistic truth-valued lattice implication algebras to deal with linguistic truth values. We get some properties of implication operators on the set of ∨-irreducible elements. And furthermore the implication operators on the linguistic truth-valued intuitionistic fuzzy lattice are discussed. The proposed system can better express both comparable and incomparable information. Also it can deal with both positive and negative evidences which are represented by linguistic truth values at the same time during the information processing system.

Keywords Lattice implication algebra · Linguistic truth-valued intuitionistic fuzzy lattice · ∨-Irreducible element · Intuitionistic fuzzy implication operator

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1 Introduction

We often do some decision making under uncertain environments with vague and imprecise information. In many decision makings under uncertain environments, linguistic terms rather than probabilistic values are taken into account the managing of uncertainty in decision processes, fuzzy linguistic approaches provide a direct way to represent the linguistic information by means of linguistic variables and process linguistic information. The use of linguistic information thus enhances the reliability and flexibility of classical decision models.

In our real life, we often use knowledge gained from our experience to understand our surroundings, to learn new things, and to make plans for the future. On the one hand, limited by our capability to perceive the world and how profoundly we infer, we find ourselves everywhere confronted with uncertainty about the adequacy of our information and inferences. On the other hand, we almost always use natural languages to describe and communicate our gained knowledge recognition, decision and execution processes [23]. In Zadeh's linguistic variables [31], a linguistic value is consisted of atomic linguistic value and linguistic hedge, e.g., very true (true is the atomic linguistic value and very is linguistic hedge). In computing with words (CWW), semantic of very true is expressed by a fuzzy set on [0,1]. Information processing corresponding linguistic values is translated to their semantics, and fuzzy sets theory becomes main tool for CWW. Due to some drawbacks in linguistic approaches based on fuzzy sets, there exist many alternative methods for processing linguistic information [10, 11], e.g., Huynh proposed a new model for parametric representation of linguistic truth-values [9, 14]. Turksen studied the formalization and inference of descriptive words, substantive words and declarative sentence based on type-2 fuzzy sets [20]. Ho discussed the ordering structure of linguistic hedges, and proposed hedge algebra to deal with CWW [8, 19, 12]. Espinilla et al. presented several computational approaches to manage multigranular linguistic scales in decision making problems [4, 13, 16, 26]. There are many numeric aggregation operators and linguistic aggregation operators to aggregating them [5, 6, 7, 21, 24, 22]. Xu et al. proposed linguistic truth-valued lattice implication algebra to deal with linguistic truth inference [15, 17, 18, 27]. Zou et al. [32–35] proposed a framework of linguistic truth-valued propositional logic and developed the reasoning method of six-element linguistic truth-valued logic system.

In fuzzy set theory, a degree of membership is assigned to each element, where the degree of non-membership is just automatically equal to 1 minus the degree of membership. However, human being who expresses the degree of membership of a fuzzy set very often does not express the corresponding degree of non-membership as the complement to 1. Intuitionistic fuzzy sets (A-IFSs) introduced by Atanassov is a powerful tool to deal with uncertainty [2] which emerge from the simultaneous consideration of membership and non-membership of the elements of a set to the set itself. A-IFSs concentrate on expressing advantages and disadvantages, pros and cons [3, 30, 25] and so on. Formally, intuitionistic fuzzy set that is meant to reflect the fact that the degree of non-membership is not always equal to 1 minus degree of membership, but there may be some hesitation degree is defined as follows [1]:

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in U\}, \quad (1)$$

where U is a discourse, $\mu_A(x)$ and $\nu_A(x)$ are the membership degree and nonmembership degree of the object $x \in U$ belonging to $A \subseteq U$ which satisfied with $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for any $x \in U$. In the intuitionisitic fuzzy set $A, \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) (\forall x \in U)$ is called the degree of indeterminacy of x to A. In Zadeh's fuzzy set, if $\mu_A(x)$ is the membership degree of x to A, then $1 - \mu_A(x)$ is non-membership degree, *i.e.*,$\mu_A(x) + 1 - \mu_A(x) = 1$. Hence, the intuitionisitic fuzzy set is an extension of fuzzy set. For any intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) | x \in U\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) | x \in U\}$, the operations of union ($\cup$), joint($\cap$) and complement ($'$) are defined as follows:

$$\begin{aligned} A \cup B =& \{(x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x)) | x \in U\}, \\ A \cap B =& \{(x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x)) | x \in U\}, \\ A' =& \{(x, \nu_A(x), \mu_A(x)) | x \in U\}. \end{aligned}$$

All the intuitionistic fuzzy sets of U are denoted as $IFS(U)$, and the intuitionistic fuzzy sets have the following order relations: $\forall A, B \in IFS(U), A \leq B$ if and only if $\forall x \in U, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. Naturally, $A = B$ if and only if $A \leq B$ and $B \leq A$.

Inspired by A-IFSs, we discuss the linguistic truth-valued intuitionistic fuzzy lattice in linguistic truth-valued lattice implication algebra, intuitively, we use linguistic truth-valued intuitionistic fuzzy set instead of classical linguistic truth of propositions to express degrees of "true" and "false" of uncertain propositions in practice. This paper is organized as follows: in Sect. 2, we construct linguistic truth-valued intuitionistic fuzzy lattice $\mathcal{LI}_{2n}$. In Sect. 3, we discuss $\vee$-irreducible elements of $\mathcal{LI}_{2n}$ and their properties. In Sect. 4, we discuss implication operators on $\mathcal{LI}_{2n}$. We conclude in Sect. 5.

2 Linguistic truth-valued intuitionistic fuzzy lattice $\mathcal{LI}_{2n}$

Definition 1 [26] Let (L, $\vee$, $\wedge$, O, I) be a bounded lattice with an order-reversing involution "$'$", I and O the greatest and the smallest element of L, respectively, and

$\rightarrow: L \times L \longrightarrow L$

be a mapping. $(L, \vee, \wedge, ', \rightarrow, O, I)$ is called a lattice implication algebra (LIA) if the following conditions hold for any $x, y, z \in L$:

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(I_2) $x \rightarrow x = I$;
(I_3) $x \rightarrow y = y' \rightarrow x'$;
(I_4) $x \rightarrow y = y \rightarrow x = I$ implies $x = y$;
(I_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;
(I_6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
(I_7) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

Definition 2 [28, 29] Let $L_n = \{d_1, d_2, \ldots, d_n\}$, $d_1 < d_2 < \cdots < d_n, L_2 = \{b_1, b_2\}, b_1 < b_2, (L_n, \vee_{(L_n)}, \wedge_{(L_n)}, '^{(L_n)}, \rightarrow_{(L_n)}, d_1, d_n)$ and $(L_2, \vee_{(L_2)}, \wedge_{(L_2)}, '^{(L_2)}, \rightarrow_{(L_2)}, b_1, b_2)$ be two Lukasiewicz implication algebra. For any $(d_i, b_j), (d_k, b_m) \in L_n \times L_2$, if

$$(d_i, b_j) \vee (d_k, b_m) = (d_i \vee_{(L_n)} d_k, b_j \vee_{(L_2)} b_m), \quad (2)$$

$$(d_i, b_j) \wedge (d_k, b_m) = (d_i \wedge_{(L_n)} d_k, b_j \wedge_{(L_2)} b_m), \quad (3)$$

$$(d_i, b_j)' = (d_i'^{(L_n)}, b_j'^{(L_2)}), \quad (4)$$

$$(d_i, b_j) \rightarrow (d_k, b_m) = (d_i \rightarrow_{(L_n)} d_k, b_j \rightarrow_{(L_2)} b_m), \quad (5)$$

then $(L_n \times L_2, \vee, \wedge, ', \rightarrow, (d_1, b_1), (d_n, b_2))$ is a lattice implication algebra, denote as $\mathcal{L}_n \times \mathcal{L}_2$ (Fig. 1).

Let $AD_n = \{h_1, h_2, \ldots, h_n\}$ be a set of n hedge operators and $h_1 < h_2 < \cdots < h_n, MT = \{f, t\}$ be "false (f)" and "true (t)", denote $f < t$ and $L_{V(n \times 2)} = AD_n \times MT$. Define the mapping $g : L_{V(n\times2)} \longrightarrow \mathcal{L}_n \times \mathcal{L}_2$ as follows:

$$g((h_i, mt)) = \begin{cases} (d_i', b_1), & mt = f, \\ (d_i, b_2), & mt = t. \end{cases} \quad (6)$$

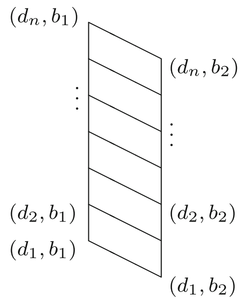

Fig. 1 Hasse Diagrams of $\mathcal{L}_n \times \mathcal{L}_2$

Then g is bejuction, its inverse mapping is g^{-1}. For any $x, y \in L_{V(n\times 2)}$, define

$$x \vee y = g^{-1}(g(x) \vee g(y)), \tag{7}$$

$$x \wedge y = g^{-1}(g(x) \wedge g(y)), \tag{8}$$

$$x' = g^{-1}((g(x))'), \tag{9}$$

$$x \rightarrow y = g^{-1}(g(x) \rightarrow g(y)). \tag{10}$$

Then $\mathcal{L}_{V(n\times 2)} = (L_{V(n\times 2)}, \vee, \wedge, ', \rightarrow, (h_n, f), (h_n, t))$ is called linguistic truth-valued lattice implication algebra from AD_n and MT (Fig. 2). g is an isomorphic mapping from $(L_{V(n\times 2)}, \vee, \wedge, ', \rightarrow, (h_n, f), (h_n, t))$ to $\mathcal{L}_n \times \mathcal{L}_2$.

Definition 3 In linguistic truth-valued lattice implication algebra $\mathcal{L}_{V(n\times 2)}$, for any $(h_i, t), (h_j, f) \in \mathcal{L}_{V(n\times 2)}, ((h_i, t), (h_j, f))$ is called as linguistic truth-valued intuitionistic fuzzy pair if $(h_i, t)' \geq (h_j, f)$.

Theorem 1 *For any* $(h_i, t), (h_j, f) \in \mathcal{L}_{V(9\times 2)}, ((h_i, t), (h_j, f))$ *is a linguistic truth-valued intuitionistic fuzzy pair if and only if* $i \leq j$.

Proof For any $(h_i, t) \in \mathcal{L}_{V(9\times 2)}$, we have $(h_i, t)' = (h_i, f)$. Hence $(h_i, t)' \geq (h_j, f)$ if and only if $(h_i, f) \geq (h_j, f)$. In $\mathcal{L}_{V(9\times 2)}, (h_i, f) \geq (h_j, f)$ if and only if $i \leq j$.

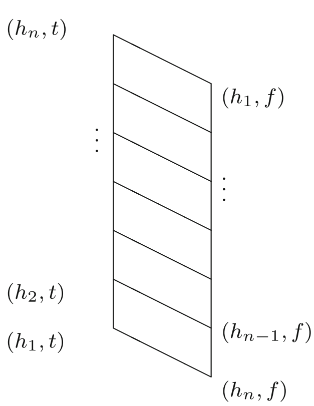

Fig. 2 Hasse Diagrams of $\mathcal{L}_{V(n\times 2)}$

From the theorem 1, for any $(h_i, t) \in \mathcal{L}_{V(n\times 2)}$, the number of (h_j, f) which can compose linguistic truth-valued intuitionisitic fuzzy pairs with (h_i, t) is $n - i + 1$. Hence, the number of linguistic truth-valued intuitionisitic fuzzy pairs in $\mathcal{L}_{V(n\times 2)}$ are

$$\sum_{i=1}^{n}(n - i + 1) = \frac{n \times (n+1)}{2}.$$

Denote all the linguistic truth-valued intuitionisitic fuzzy pairs based on $\mathcal{L}_{V(n\times 2)}$ as:

$$LI_{2n} = \{((h_i, t), (h_j, f)) | (h_i, t), (h_j, f) \in \mathcal{L}_{V(n\times 2)}, i \leq j\}. \tag{11}$$

For any $((h_i, t), (h_j, f)), ((h_k, t), (h_l, f)) \in LI_{18}$, define the operation "∪", "∩" and "¬" as follows:

$$((h_i, t), (h_j, f)) \cup ((h_k, t), (h_l, f)) = ((h_i, t) \vee (h_k, t), (h_j, f) \wedge (h_l, f)), \tag{12}$$

$$((h_i, t), (h_j, f)) \cap ((h_k, t), (h_l, f)) = ((h_i, t) \wedge (h_k, t), (h_j, f) \vee (h_l, f)). \tag{13}$$

Where "∨" and "∧" are operations of $\mathcal{L}_{V(n\times 2)}$.

Based on $2n$ linguistic truth-valued lattice implication algebra $\mathcal{L}_{V(n\times 2)}$, we can construct linguistic truth-valued intuitionistic fuzzy lattice. Formally, denote

$$\mathcal{LI}_{2n} = (LI_{2n}, \cup, \cap)$$

as a linguistic truth-valued intuitionistic fuzzy lattice where $((h_n, t), (h_n, f))$ and $((h_1, t), (h_1, f))$ are the greatest element and the least element of $\mathcal{LI}_{2n}$, respectively.

Definition 4 In the linguistic truth-valued intuitionistic fuzzy lattice $\mathcal{LI}_{2n} = (LI_{2n}, \cup, \cap)$ (Fig. 3), for any $((h_i, t), (h_j, f)), ((h_k, t), (h_l, f)) \in LI_{2n}$, $((h_i, t), (h_j, f)) \leq ((h_k, t), (h_l, f))$ if and only if $i \leq k$ and $j \leq l$, also

$$((h_i, t), (h_j, f)) \cup ((h_k, t), (h_l, f)) = ((h_{max(i,k)}, t), (h_{max(j,l)}, f)), \tag{14}$$

$$((h_i, t), (h_j, f)) \cap ((h_k, t), (h_l, f)) = ((h_{min(i,k)}, t), (h_{min(j,l)}, f)). \tag{15}$$

3 ∨-Irreducible element set in $\mathcal{LI}_{2n}$

Theorem 2 *In linguistic truth-valued intuitionistic fuzzy lattice* $\mathcal{LI}_{2n}$,

1. $((h_i, t), (h_i, f))(i \in \{2, 3, \ldots, n\})$ are ∨-irreducible elements of $\mathcal{LI}_{2n}$, denote as $J_1 = \{((h_i, t), (h_i, f)) | i = 2, 3, \ldots, n\}$;
2. $((h_1, t), (h_i, f))(i \in \{2, 3, \ldots, n\})$ are ∨-irreducible elements of $\mathcal{LI}_{2n}$, denote as $J_2 = \{((h_1, t), (h_i, f)) | i = 2, 3, \ldots, n\}$.

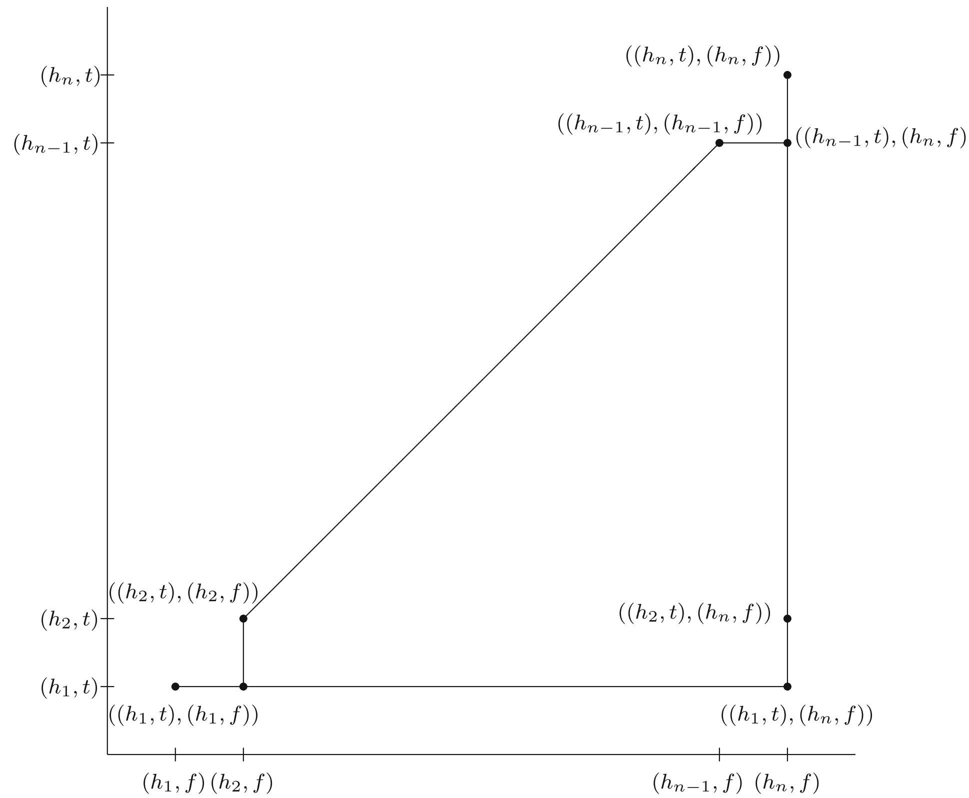

Fig. 3 Structure Diagrams of $\mathcal{LI}_{2n}$

Proof 1. For any $((h_k,t),(h_l,f)),((h_m,t),(h_s,f)) \in LI_{2n}$, assume $((h_k,t),(h_l,f)) \cup ((h_m,t),(h_s,f)) = ((h_i,t),(h_i,f))$, i.e.,

$((h_{max(k,m)},t),(h_{max(l,s)},f)) = ((h_i,t),(h_i,f))$.

Hence, $max(k,m) = i$ and $max(l,s) = i$. We discuss the following three cases:

Case 1: If $k = i$ and $s = i$, then $m \le k$ and $l \le s$. Since $k \le l$, hence $i = k \le l \le s = i$, i.e., $l = i$. We obtain $((h_k,t),(h_l,f)) = ((h_i,t),(h_i,f))$.
Case 2: If $m = i$ and $l = i$, then $k \le m$ and $s \le l$. Since $m \le s$, hence $i = m \le s \le l = i$, i.e. $s = i$. We obtain $((h_m,t),(h_s,f)) = ((h_i,t),(h_i,f))$.
Case 3: If $k = i$ and $l = i$, or $m = i$ and $s = i$, then $((h_k,t),(h_l,f)) = ((h_i,t),(h_i,f))$ or $((h_m,t),(h_s,f)) = ((h_i,t),(h_i,f))$.

From the above three cases, for any $((h_k,t),(h_l,f)),((h_m,t),(h_s,f)) \in LI_{2n}$, if $((h_k,t),(h_l,f)) \cup ((h_m,t),(h_s,f)) = ((h_i,t),(h_i,f))$, then $((h_k,t),(h_l,f)) = ((h_i,t),(h_i,f))$ or $((h_m,t),(h_s,f)) = ((h_i,t),(h_i,f))$. Hence, $((h_i,t),(h_i,f))$ $(i \in \{2,3,\ldots,n\})$ are $\vee$-irreducible elements of $\mathcal{LI}_{2n}$.

2. For any $((h_k,t),(h_l,f)),((h_m,t),(h_s,f)) \in LI_{2n}$, Assume $((h_k,t),(h_l,f)) \cup ((h_m,t),(h_s,f)) = ((h_1,t),(h_i,f))$, i.e.,

$((h_{max(k,m)},t),(h_{max(l,s)},f)) = ((h_1,t),(h_i,f))$,

clearly, $max(k,m) = 1$ and $max(l,s) = i$ and $k = m = 1$.

We discuss the two cases:

Case 1: If $l = i$, then $((h_k,t),(h_l,f)) = ((h_1,t),(h_i,f))$.
Case 2: If $s = i$, then $((h_m,t),(h_s,f)) = ((h_1,t),(h_i,f))$.

From the cases 1 and 2, for any $((h_k,t),(h_l,f)),((h_m,t),(h_s,f)) \in LI_{2n}$, if $((h_k,t),(h_l,f)) \cup ((h_m,t),(h_s,f)) = ((h_1,t),(h_i,f))$, then $((h_k,t),(h_l,f)) = ((h_1,t),(h_i,f))$ or $((h_m,t),(h_s,f)) = ((h_1,t),(h_i,f))$.

Hence, $((h_1,t),(h_i,f))(i \in \{2,3,\ldots,n\})$ are $\vee$-irreducible elements of $\mathcal{LI}_{2n}$.

Corollary 1 *Assume $((h_i,t),(h_j,f)) \in LI_{2n}$ be a $\vee$-irreducible. Then we get $i = j$ or $i = 1$, i.e., $((h_i,t),(h_j,f)) \in J_1 \cup J_2$.*

Proof If $i = 1$, then $((h_i,t),(h_j,f)) = ((h_1,t),(h_j,f)) \in J_2 \subset J_1 \cup J_2$ can be obtained obviously. Assume $i \neq 1$ and $i < j$, Since

$$((h_1,t),(h_j,f)) \cup ((h_i,t),(h_i,f)) = ((h_{max(1,i)},t),(h_{max(j,i)},f)) = ((h_i,t),(h_j,f)),$$

and $((h_1,t),(h_j,f)) \neq ((h_i,t),(h_j,f))$, $((h_i,t),(h_i,f)) \neq ((h_i,t),(h_j,f))$, while $((h_i,t),(h_j,f))$ is $\vee$-irreducible element, we get contradiction. Hence $i = 1$ and $i = j$, i.e., $((h_i,t),(h_j,f)) \in J_1 \cup J_2$.

The Corollary 1 means that $J_1 \cup J_2$ are all the $\vee$-irreducible elements of $\mathcal{LI}_{2n}$. According to the Theorem 2 and the representation theorem of attributive lattice, we get the following corollary.

Corollary 2 For any $((h_k,t),(h_l,f)) \in LI_{2n} - \{((h_1,t),(h_1,f))\}$, $((h_k,t),(h_l,f))$ can be represented by the union of two elements in $J_1 \cup J_2$, i.e.,

$((h_k,t),(h_l,f)) = ((h_k,t),(h_k,f)) \cup ((h_1,t),(h_l,f))$.

According to the Corollary 2, we consider the following implication operators of $J_1 \cup J_2 \cup \{((h_1,t),(h_1,f))\}$ in which $\mathcal{J}_1 = J_1 \cup \{((h_1,t),(h_1,f))\}$ and $\mathcal{J}_2 = J_2 \cup \{((h_1,t),(h_1,f))\}$.

4 Implication operator on $\mathcal{LI}_{2n}$

Definition 5 For any $((h_i,t),(h_j,f)),((h_k,t),(h_l,f)) \in J_1 \cup J_2 \cup \{((h_1,t),(h_1,f))\}$,

1. If $((h_i,t),(h_i,f)),((h_k,t),(h_k,f)) \in \mathcal{J}_1$, then
$$((h_i,t),(h_i,f)) \to ((h_k,t),(h_k,f)) = ((h_{min(n,n-i+k)},t),(h_{min(n,n-i+k)},f)).$$
2. If $((h_1,t),(h_j,f)),((h_1,t),(h_l,f)) \in \mathcal{J}_2$, then
$$((h_1,t),(h_j,f)) \to ((h_1,t),(h_l,f)) = ((h_{min(n,n-j+l)},t),(h_n,f)).$$
3. If $((h_i,t),(h_i,f)) \in \mathcal{J}_1$ and $((h_1,t),(h_l,f)) \in \mathcal{J}_2$, then
$$((h_i,t),(h_i,f)) \to ((h_1,t),(h_l,f)) = ((h_{min(n,n-i+1)},t),(h_{min(n,n-i+l)},f)),\ ((h_1,t),(h_l,f)) \to ((h_i,t),(h_i,f)) = ((h_{min(n,n-l+i)},t),(h_n,f)).$$

According to the definitions of the operator "∪", "∩" and "→", we can prove the following properties.

Proposition 1 *For any* $((h_i,t),(h_i,f)),((h_k,t),(h_k,f)) \in \mathcal{J}_1$,

1. $((h_i,t),(h_i,f)) \cup ((h_k,t),(h_k,f)) \in \mathcal{J}_1$;
2. $((h_i,t),(h_i,f)) \cap ((h_k,t),(h_k,f)) \in \mathcal{J}_1$;
3. $((h_i,t),(h_i,f)) \to ((h_k,t),(h_k,f)) \in \mathcal{J}_1$.
That is, the operators "∪", "∩" and "→" are closed on $\mathcal{J}_1$.

Proposition 2 *For any* $((h_1,t),(h_j,f)),((h_1,t),(h_l,f)) \in \mathcal{J}_2$,

1. $((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f)) \in \mathcal{J}_2$;
2. $((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f)) \in \mathcal{J}_2$;
3. $((h_1,t),(h_j,f)) \to ((h_1,t),(h_l,f)) \in \mathcal{J}'_2 = J'_2 \cup \{((h_1,t),(h_1,f))\}$. *That is, the operators "∪" and "∩" are closed on* $\mathcal{J}_2$. *But the operator "→" is not closed* on $\mathcal{J}_2$.

Corollary 3 *For any* $((h_i,t),(h_i,f)),((h_k,t),(h_k,f)),((h_m,t),(h_m,f)) \in \mathcal{J}_1$

1. $(((h_i,t),(h_i,f)) \cup ((h_k,t),(h_k,f))) \to ((h_m,t),(h_m,f)) = (((h_i,t),(h_i,f)) \to ((h_m,t),(h_m,f))) \cap (((h_k,t),(h_k,f)) \to ((h_m,t),(h_m,f)))$;
2. $((h_i,t),(h_i,f)) \to (((h_k,t),(h_k,f)) \cup ((h_m,t),(h_m,f))) = (((h_i,t),(h_i,f)) \to ((h_m,t),(h_m,f))) \cup (((h_k,t),(h_k,f)) \to ((h_m,t),(h_m,f)))$;
3. $(((h_i,t),(h_i,f)) \cap ((h_k,t),(h_k,f))) \to ((h_m,t),(h_m,f))) = (((h_i,t),(h_i,f)) \to ((h_m,t),(h_m,f))) \cup (((h_k,t),(h_k,f)) \to ((h_m,t),(h_m,f)))$;
4. $((h_i,t),(h_i,f)) \to (((h_k,t),(h_k,f)) \cap ((h_m,t),(h_m,f)))) = (((h_i,t),(h_i,f)) \to ((h_m,t),(h_m,f))) \cap (((h_k,t),(h_k,f)) \to ((h_m,t),(h_m,f)))$.

Corollary 4 For any $((h_1,t),(h_j,f)),((h_1,t),(h_l,f)),((h_1,t),(h_s,f)) \in \mathcal{J}_2$

1. $(((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f))) \to ((h_1,t),(h_s,f)) = (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))) \cap (((h_1,t),(h_l,f)) \to ((h_1,t),(h_s,f)))$;
2. $((h_1,t),(h_j,f)) \to (((h_1,t),(h_l,f)) \cup ((h_1,t),(h_s,f))) = (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))) \cup (((h_1,t),(h_l,f)) \to ((h_1,t),(h_s,f)))$;
3. $(((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f))) \to ((h_1,t),(h_s,f))) = (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))) \cup (((h_1,t),(h_l,f)) \to ((h_1,t),(h_s,f)))$;
4. $((h_1,t),(h_j,f)) \to (((h_1,t),(h_l,f)) \cap ((h_1,t),(h_s,f)))) = (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))) \cap (((h_1,t),(h_l,f)) \to ((h_1,t),(h_s,f)))$.

Proof From the Eqs. (14), (15) and the Definition 5,

$$\begin{aligned}
&(((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f))) \to ((h_1,t),(h_s,f)) \\
&= ((h_1,t),(h_{max(j,l)},f)) \to ((h_1,t),(h_s,f)) \\
&= ((h_{min(n,n-max(j,l)+s)},t),(h_n,f)) \\
&= ((h_{min(n,min(n-j+s,n-l+s))},t),(h_n,f)) \\
&= ((h_{min(min(n,n-j+s),min(n,n-l+s))},t),(h_n,f)) \\
&= ((h_{min(n,n-j+s)},t),(h_n,f)) \cap ((h_{min(n,n-l+s)},t),(h_n,f)) \\
&= (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))) \cap (((h_1,t),(h_l,f)) \to ((h_1,t),(h_s,f))).
\end{aligned}$$

$$((h_1,t),(h_j,f)) \to (((h_1,t),(h_l,f)) \cup ((h_1,t),(h_s,f)))$$
$$= ((h_1,t),(h_j,f)) \to ((h_1,t),(h_{max(l,s)},f))$$
$$= ((h_{min(n,n-j+max(l,s))},t),(h_n,f))$$
$$= ((h_{min(n,max(n-j+l,n-j+s))},t),(h_n,f))$$
$$= ((h_{max(min(n,n-j+l),min(n,n-j+s))},t),(h_n,f))$$
$$= ((h_{min(n,n-j+l)},t),(h_n,f)) \cup ((h_{min(n,n-j+s)},t),(h_n,f))$$
$$= (((h_1,t),(h_j,f)) \to ((h_1,t),(h_l,f))) \cup (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))).$$

$$(((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f))) \to ((h_1,t),(h_s,f))$$
$$= ((h_1,t),(h_{min(j,l)},f)) \to ((h_1,t),(h_s,f))$$
$$= ((h_{min(n,n-min(j,l)+s)},t),(h_n,f))$$
$$= ((h_{min(n,max(n-j+s,n-l+s))},t),(h_n,f))$$
$$= ((h_{max(min(n,n-j+s),min(n,n-l+s))},t),(h_n,f))$$
$$= ((h_{min(n,n-j+s)},t),(h_n,f)) \cup ((h_{min(n,n-l+s)},t),(h_n,f))$$
$$= (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))) \cup (((h_1,t),(h_l,f)) \to ((h_1,t),(h_s,f))).$$

$$((h_1,t),(h_j,f)) \to (((h_1,t),(h_l,f)) \cap ((h_1,t),(h_s,f)))$$
$$= ((h_1,t),(h_j,f)) \to ((h_1,t),(h_{min(l,s)},f))$$
$$= ((h_{min(n,n-j+min(l,s))},t),(h_n,f))$$
$$= ((h_{min(n,min(n-j+l,n-j+s))},t),(h_n,f))$$
$$= ((h_{min(min(n,n-j+l),min(n,n-j+s))},t),(h_n,f))$$
$$= ((h_{min(n,n-j+l)},t),(h_n,f)) \cap ((h_{min(n,n-j+s)},t),(h_n,f))$$
$$= (((h_1,t),(h_j,f)) \to ((h_1,t),(h_l,f))) \cap (((h_1,t),(h_j,f)) \to ((h_1,t),(h_s,f))).$$

Corollary 5 For any $((h_i,t),(h_i,f)),((h_k,t),(h_k,f)) \in \mathcal{J}_1((h_1,t),(h_s,f)) \in \mathcal{J}_2$,

1. $(((h_i,t),(h_i,f)) \cup ((h_k,t),(h_k,f))) \to ((h_1,t),(h_s,f)) = (((h_i,t),(h_i,f)) \to ((h_1,t),(h_s,f))) \cap (((h_k,t),(h_k,f)) \to ((h_1,t),(h_s,f)))$;
2. $((h_1,t),(h_s,f)) \to (((h_i,t),(h_i,f)) \cup ((h_k,t),(h_k,f))) = (((h_1,t),(h_s,f)) \to ((h_i,t),(h_i,f))) \cup (((h_1,t),(h_s,f)) \to ((h_k,t),(h_k,f)))$;
3. $(((h_i,t),(h_i,f)) \cap ((h_k,t),(h_k,f))) \to ((h_1,t),(h_s,f))) = (((h_i,t),(h_i,f)) \to ((h_1,t),(h_s,f))) \cup (((h_k,t),(h_k,f)) \to ((h_1,t),(h_s,f)))$;
4. $((h_1,t),(h_s,f)) \to (((h_i,t),(h_i,f)) \cap ((h_k,t),(h_k,f))) = (((h_1,t),(h_s,f)) \to ((h_i,t),(h_i,f))) \cap (((h_1,t),(h_s,f)) \to ((h_k,t),(h_k,f)))$.

Proof From the Eqs. (14), (15) and the Definition 5,

$$(((h_i,t),(h_i,f)) \cup ((h_k,t),(h_k,f))) \to ((h_1,t),(h_s,f))$$
$$= ((h_{max(i,k)},t),(h_{max(i,k)},f)) \to ((h_1,t),(h_s,f))$$
$$= ((h_{min(n,n-max(i,k)+1)},t),(h_{min(n,n-max(i,k)+l)},f))$$
$$= ((h_{min(n,min(n-i+1,n-k+1))},t),(h_{min(n,min(n-i+l,n-k+l))},f))$$
$$= ((h_{min(min(n,n-i+1),min(n,n-k+1))},t),(h_{min(min(n,n-i+l),min(n,n-k+l))},f))$$
$$= ((h_{min(n,n-i+1)},t),(h_{min(n,n-i+l)},f)) \cap ((h_{min(n,n-k+1)},t),(h_{min(n,n-k+l)},f))$$
$$= (((h_i,t),(h_i,f)) \to ((h_1,t),(h_s,f))) \cap (((h_k,t),(h_k,f)) \to ((h_1,t),(h_s,f))).$$

$$((h_1,t),(h_s,f)) \to (((h_i,t),(h_i,f)) \cup ((h_k,t),(h_k,f)))$$
$$= ((h_1,t),(h_s,f)) \to ((h_{max(i,k)},t),(h_{max(i,k)},f))$$
$$= ((h_{min(n,n-s+max(i,k))},t),(h_n,f))$$
$$= ((h_{min(n,max(n-s+i,n-s+k))},t),(h_n,f))$$
$$= ((h_{max(min(n,n-s+i),min(n,n-s+k))},t),(h_n,f))$$
$$= ((h_{min(n,n-s+i)},t),(h_n,f)) \cup ((h_{min(n,n-s+k)},t),(h_n,f))$$
$$= (((h_1,t),(h_s,f)) \to ((h_i,t),(h_i,f))) \cup (((h_1,t),(h_s,f)) \to ((h_k,t),(h_k,f))).$$

$$(((h_i,t),(h_i,f)) \cap ((h_k,t),(h_k,f))) \to ((h_1,t),(h_s,f))$$
$$= ((h_{min(i,k)},t),(h_{min(i,k)},f)) \to ((h_1,t),(h_s,f))$$
$$= ((h_{min(n,n-min(i,k)+1)},t),(h_{min(n,n-min(i,k)+l)},f))$$
$$= ((h_{min(n,max(n-i+1,n-k+1))},t),(h_{min(n,max(n-i+l,n-k+l))},f))$$
$$= ((h_{max(min(n,n-i+1),min(n,n-k+1))},t),(h_{max(min(n,n-i+l),min(n,n-k+l))},f))$$
$$= ((h_{min(n,n-i+1)},t),(h_{min(n,n-i+l)},f)) \cup ((h_{min(n,n-k+1)},t),(h_{min(n,n-k+l)},f))$$
$$= (((h_i,t),(h_i,f)) \to ((h_1,t),(h_s,f))) \cup (((h_k,t),(h_k,f)) \to ((h_1,t),(h_s,f))).$$

$$((h_1,t),(h_s,f)) \to (((h_i,t),(h_i,f)) \cap ((h_k,t),(h_k,f)))$$
$$= ((h_1,t),(h_s,f)) \to ((h_{min(i,k)},t),(h_{min(i,k)},f))$$
$$= ((h_{min(n,n-s+min(i,k))},t),(h_n,f))$$
$$= ((h_{min(n,min(n-s+i,n-s+k))},t),(h_n,f))$$
$$= ((h_{min(min(n,n-s+i),min(n,n-s+k))},t),(h_n,f))$$
$$= ((h_{min(n,n-s+i)},t),(h_n,f)) \cap ((h_{min(n,n-s+k)},t),(h_n,f)) = (((h_1,t),(h_s,f)) \to ((h_i,t),(h_i,f))) \cap (((h_1,t),(h_s,f)) \to ((h_k,t),(h_k,f))).$$

Corollary 6 For any $((h_m,t),(h_m,f)) \in \mathcal{J}_1$ and $((h_1,t),(h_j,f)),((h_1,t),(h_l,f)) \in \mathcal{J}_2$,

1. $(((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f))) \to ((h_m,t),(h_m,f)) = (((h_1,t),(h_j,f)) \to ((h_m,t),(h_m,f))) \cap (((h_1,t),(h_l,f)) \to ((h_m,t),(h_m,f)))$;
2. $((h_m,t),(h_m,f)) \to (((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f))) = (((h_m,t),(h_m,f)) \to ((h_1,t),(h_j,f))) \cup (((h_m,t),(h_m,f)) \to ((h_1,t),(h_l,f)))$;
3. $(((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f))) \to ((h_m,t),(h_m,f))) = (((h_1,t),(h_j,f)) \to ((h_m,t),(h_m,f))) \cup (((h_1,t),(h_l,f)) \to ((h_m,t),(h_m,f)))$;
4. $((h_m,t),(h_m,f)) \to (((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f)))) = (((h_m,t),(h_m,f)) \to ((h_1,t),(h_j,f))) \cap (((h_m,t),(h_m,f)) \to ((h_1,t),(h_l,f)))$.

Proof From the Eqs. (14), (15) and the Definition 5,

$$\begin{aligned}
&(((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f))) \to ((h_m,t),(h_m,f)) \\
&= ((h_1,t),(h_{max(j,l)},f)) \to ((h_m,t),(h_m,f)) \\
&= ((h_{min(n,n-max(j,l)+m)},t),(h_n,f)) \\
&= ((h_{min(n,min(n-j+m,n-l+m))},t),(h_n,f)) \\
&= ((h_{min(min(n,n-j+m),min(n,n-l+m))},t),(h_n,f)) \\
&= ((h_{min(n,n-j+m)},t),(h_n,f)) \cap ((h_{min(n,n-l+m)},t),(h_n,f)) \\
&= (((h_1,t),(h_j,f)) \to ((h_m,t),(h_m,f))) \cap (((h_1,t),(h_l,f)) \to ((h_m,t),(h_m,f))).
\end{aligned}$$

$$\begin{aligned}
&((h_m,t),(h_m,f)) \to (((h_1,t),(h_j,f)) \cup ((h_1,t),(h_l,f))) \\
&= ((h_m,t),(h_m,f)) \to ((h_1,t),(h_{max(j,l)},f)) \\
&= ((h_{min(n,n-m+1)},t),(h_{min(n,n-m+max(j,l))},f)) \\
&= ((h_{min(n,n-m+1)},t),(h_{max(min(n,n-m+j),min(n,n-m+l))},f)) \\
&= ((h_{min(n,n-m+1)},t),(h_{min(n,n-m+j)},f)) \cup ((h_{min(n,n-m+1)},t),(h_{min(n,n-m+l)},f)) \\
&= (((h_m,t),(h_m,f)) \to ((h_1,t),(h_j,f))) \cup (((h_m,t),(h_m,f)) \to ((h_1,t),(h_l,f))).
\end{aligned}$$

$$\begin{aligned}
&(((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f))) \to ((h_m,t),(h_m,f)) \\
&= ((h_1,t),(h_{min(j,l)},f)) \to ((h_m,t),(h_m,f)) \\
&= ((h_{min(n,n-min(j,l)+m)},t),(h_n,f)) \\
&= ((h_{min(n,max(n-j+m,n-l+m))},t),(h_n,f)) \\
&= ((h_{max(min(n,n-j+m),min(n,n-l+m))},t),(h_n,f))
\end{aligned}$$

$$\begin{aligned}
&= ((h_{min(n,n-j+m)},t),(h_n,f)) \cup ((h_{min(n,n-l+m)},t),(h_n,f)) \\
&= (((h_1,t),(h_j,f)) \to ((h_m,t),(h_m,f))) \cup (((h_1,t),(h_l,f)) \to ((h_m,t),(h_m,f))).
\end{aligned}$$

$$\begin{aligned}
&((h_m,t),(h_m,f)) \to (((h_1,t),(h_j,f)) \cap ((h_1,t),(h_l,f))) \\
&= ((h_m,t),(h_m,f)) \to ((h_1,t),(h_{min(j,l)},f)) \\
&= ((h_{min(n,n-m+1)},t),(h_{min(n,n-m+min(j,l))},f)) \\
&= ((h_{min(n,n-m+1)},t),(h_{min(min(n,n-m+j),min(n,n-m+l))},f)) \\
&= ((h_{min(n,n-m+1)},t),(h_{min(n,n-m+j)},f)) \cap ((h_{min(n,n-m+1)},t),(h_{min(n,n-m+l)},f)) \\
&= (((h_m,t),(h_m,f)) \to ((h_1,t),(h_j,f))) \cup (((h_m,t),(h_m,f)) \to ((h_1,t),(h_l,f))).
\end{aligned}$$

5 Conclusion

In this paper, based on linguistic truth-valued lattice implication algebra and A-IFSs, we construct linguistic truth-valued intuitionistic fuzzy lattice, especially, we discuss $\vee$-irreducible elements and implication operator of the linguistic truth-valued intuitionistic fuzzy lattice, in which, both comparable and incomparable information as well as positive and negative evidences can be represented by linguistic truth values at the same time during the information processing system. Further work is to develop linguistic intuitionisitic fuzzy logic and its logic reasoning.

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