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ORIGINAL RESEARCH

Linear Diophantine fuzzy algebraic structures

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Abstract
The main objective of this paper is to introduce some algebraic properties of finite linear Diophantine fuzzy subsets of group, ring and field. Relatedly, we define the concepts of linear Diophantine fuzzy subgroup and normal subgroup of a group, linear Diophantine fuzzy subring and ideal of a ring, and linear Diophantine fuzzy subfield of a field. We investigate their basic properties, relations and characterizations in detail. Furthermore, we establish the homomorphic images and preimages of the emerged linear Diophantine fuzzy algebraic structures. Finally, we describe linear Diophantine fuzzy code and investigate the relationships between this code and some linear Diophantine fuzzy algebraic structures.

Keywords Fuzzy set · Linear Diophantine fuzzy set · Group · Ring · Ideal · Field · Coding theory

1 Introduction

The membership of elements in classical set theory is considered in binary terms according to a bivalent condition whether an element belongs to the set. This is insufficient to deal with many real world problems. To eradicate these restrictions, Zadeh (1965) proposed the fuzzy set (FS), in which the membership degrees of elements range over the interval [0, 1]. The membership degree 0 implies that an element does not belong to the related fuzzy set, and the membership degree 1 implies that an element completely belongs to the related fuzzy set. The membership degrees on the interval (0, 1) mean the partial membership to the fuzzy set. Since that seminal publication, the fuzzy set theory was widely studied in various directions like operational research and decision making (Ekel 2002; Liu et al. 2019; Petchimuthu et al. 2020; Zimmermann 2001; Zhou et al. 2015). As an extension of fuzzy set, Atanassov (1986) generalized this notion and introduced a new concept called intuitionistic fuzzy set (IFS). For further work related to IFSs and their drawbacks, we may refer to Aydın and Enginoğlu (2020), Kamacı (2019), Karaaslan (2016), Karaaslan and Karataş (2016), Kumar and Garg (2018) and Uluçay et al. (2019). Yager (2013) highlighted that in some practical implementations, the sum of degrees of membership and non-membership to which an alternative satisfying attribute may be greater than 1, but their square sum is less than or equal to 1. Therefore, he familiarized the model of Pythagorean fuzzy set (PyFS). This paradigm was studied by many authors in various aspects (Peng 2019; Peng and Garg 2019; Wei and Wei 2018; Zhang et al. 2019). Yager (2017) proposed the intuitionistic fuzzy set of type q (where $q \geq 1$), called q-rung orthopair fuzzy set (q-ROFS), extending the spaces of FS, IFS and PyFS. Subsequently, significant advances were made with academic research related to q-ROFSs (Ali 2018; Liu and Wang 2018; Wang et al. 2019).

On the other hand, many authors studied the fuzzy set theoretic approaches to the algebraic structures. Rosenfeld (1971) formulated the notion of fuzzy subgroup of a group. Subsequently, many researchers were engaged in extending the notions of abstract algebra to the broader framework of the fuzzy environment. Naturally, there exist an interest in generalizing other types of algebraic structures (e.g. rings, ideals, fields) as fuzzy algebraic structures. Liu (1982) discussed fuzzy set in the realm of ring theory. immediately afterwards, among others, Aktaş and Çağman (2007), Dixit et al. (1992), Mukherjee and Sen (1987), Öztürk et al. (2010) studied on fuzzy ring/ideal and proposed certain ring theoretic analogues. Malik and Mordeson (1990) developed the fuzzy subfields of a field and characterized the properties of field extensions in terms of fuzzy subfields. In (Aygünoğlu et al. 2012; Biswas 1997; Marashdeh and Salleh 2011; Yu

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and Yuan 2011; Zhang et al. 2010), the intuitionistic fuzzy subalgebras of groups, rings and fields were explored and their characterizations were discussed. Several authors carried out further studies on the algebraic properties of Pythagorean fuzzy set and q-rung orthopair fuzzy set [see (Adak and Salokolaei 2019; Jansi and Mohana 2020; Luqman et al. 2019)]. Nowadays, research and improvement of new algebraic structures in fuzzy set theory is still an important issue.

In theory and practice, the fuzzy sets, intuitionistic fuzzy set, Pythagorean fuzzy set and q-rung orthopair fuzzy set have their own limitations related to the functions of membership and non-membership. Riaz and Hashmi (2019) detailed these limitations and supported them with illustrative examples. To eliminate the restrictions of existing methodologies, they proposed a new generalized form of fuzzy set called linear Diophantine fuzzy set. The algebraic properties of linear Diophantine fuzzy sets are issues that have not been considered deeply until now. In this paper, we aim to fill this research gap by revealing the structural characteristics of linear Diophantine fuzzy set of some algebraic structures. Accordingly, we define the prevailing concepts of linear Diophantine fuzzy subgroup/normal subgroup/subring/ideal/subfield. Thus, we are tempted to generalise as long as it is possible some of the results known in the fuzzy set theory. We get many interesting results related to the linear Diophantine fuzzy relationships and linear Diophantine fuzzy subalgebras of given algebraic structures. Also, we focus on the construction of codes by linear Diophantine fuzzy sets and their applications in practice.

The paper is organised according to the following. Section 2 addresses the requisite preliminary concepts and introduces the cartesian product of linear Diophantine fuzzy sets. Sections 3 and 4 describe the linear Diophantine fuzzy subgroup and normal subgroup of a classical group and discuss their characterizations. Sections 5 and 6, introduce the algebraic properties of linear Diophantine fuzzy set of a ring and provide some theoretical results related to them. Section 7 offers the way for revealing the structural characteristics of linear Diophantine fuzzy subfield of a field. Section 8 proposes an linear Diophantine fuzzy-based approach to coding theory. Section 9 summarizes the conclusion of this research.

2 Preliminaries

In this section, some essential concepts that are useful for discussions in the next sections are explained.

Suppose that $\mathcal{G}$ is a set and " $\cdot$ " is a binary operation on $\mathcal{G}$. Then, the algebraic structure $(\mathcal{G}, \cdot)$ is said to be a *group* if and only if the following properties are satisfied:

(i) $g_j \cdot g_k \in \mathcal{G}\ \forall g_j, g_k \in \mathcal{G}$.
(ii) $g_j \cdot (g_k \cdot g_l) = (g_j \cdot g_k) \cdot g_l\ \forall g_j, g_k, g_l \in \mathcal{G}$.
(iii) There exists an element $e_{\mathcal{G}} \in \mathcal{G}$ such that $g_j \cdot e_{\mathcal{G}} = e_{\mathcal{G}} \cdot g_j = g_j\ \forall g_j \in \mathcal{G}$.
(iv) For each $g_j \in \mathcal{G}$, there exists an element $g_j^{-1} \in \mathcal{G}$ such that $g_j \cdot g_j^{-1} = g_j^{-1} \cdot g_j = e_{\mathcal{G}}$.

If the properties (i) and (ii) are satisfied then $(\mathcal{G}, \cdot)$ is called a *semigroup*.

In addition to these properties, the following properties is satisfied then $(\mathcal{G}, \cdot)$ is termed to be a commutative group.

(v) $g_j \cdot g_k = g_k \cdot g_j\ \forall g_j, g_k \in \mathcal{G}$.

A subgroup $\mathcal{S}$ of a group $\mathcal{G}$ is a subset of $\mathcal{G}$ which is a group under the same operation as $\mathcal{G}$. That is, $\mathcal{S}$ is a subgroup of $\mathcal{G}$ if $g_j \cdot q_k^{-1} \in \mathcal{S}$ for all $g_j, g_k \in \mathcal{S} \subseteq \mathcal{G}$. A subgroup $\mathcal{N}$ of $\mathcal{G}$ is called a normal subgroup if $g \cdot g_j \cdot g^{-1} \in \mathcal{N}$ for all $g_j \in \mathcal{N}$, $q \in \mathcal{G}$ (or $g_j \cdot g_k \in \mathcal{N} \Leftrightarrow g_k \cdot g_j \in \mathcal{N}\ \forall g_j, g_k \in \mathcal{G}$). A function $\Psi_{grp} : \mathcal{G}_1 \to \mathcal{G}_2$ between groups $\mathcal{G}_1$ and $\mathcal{G}_2$ is called a group homomorphism if $\Psi_{grp}(g_j^1 \cdot_{\mathcal{G}_1} g_k^1) = \Psi_{grp}(g_j^1) \cdot_{\mathcal{G}_2} \Psi_{grp}(g_k^1)$ for all $g_j^1, g_k^1 \in \mathcal{G}_1$.

Assume that $\mathcal{R}$ is a set and " $+$ " and " $\cdot$ " is two binary operation on $\mathcal{R}$. Then, the algebraic structure $(\mathcal{R}, +, \cdot)$ is said to be a *ring* if and only if the following properties are satisfied:

(i) $(\mathcal{R}, +)$ is a commutative group.
(ii) $(\mathcal{R}, \cdot)$ is a semigroup.
(iii) $r_j \cdot (r_k + r_l) = r_j \cdot r_k + r_j \cdot r_l$ and $(r_j + r_k) \cdot r_l = r_j \cdot r_l + r_k \cdot r_l$ $\forall r_j, r_k, r_l \in \mathcal{R}$.

A subring $\mathcal{H}$ of a ring $\mathcal{R}$ is a subset of $\mathcal{R}$ which is a ring under the same operations as $\mathcal{R}$. That is, $\mathcal{H}$ is a subring of $\mathcal{R}$ if $r_j - r_k \in \mathcal{H}$ and $r_j \cdot r_k \in \mathcal{H}$ for all $r_j, r_k \in \mathcal{H} \subseteq \mathcal{R}$. A subring $\mathcal{I}$ of $\mathcal{R}$ is a left ideal if $r \cdot r_j \in \mathcal{I}$ for all $r_j \in \mathcal{I}$, $r \in \mathcal{R}$. It is a right ideal if $r_j \cdot r \in \mathcal{I}$ for all $r_j \in \mathcal{I}, r \in \mathcal{R}$. If $\mathcal{I}$ is both a left and right ideal, then it is called an ideal. A function $\Psi_{rng} : \mathcal{R}_1 \to \mathcal{R}_2$ between rings $\mathcal{R}_1$ and $\mathcal{R}_2$ is called a ring homomorphism if $\Psi_{rng}(r_j^1 +_{\mathcal{R}_1} r_k^1) = \Psi_{rng}(r_j^1) +_{\mathcal{R}_2} \Psi_{rng}(r_k^1)$ and $\Psi_{rng}(r_j^1 \cdot_{\mathcal{R}_1} r_k^1) = \Psi_{rng}(r_j^1) \cdot_{\mathcal{R}_2} \Psi_{rng}(r_k^1)$ for all $r_j^1, r_k^1 \in \mathcal{R}_1$.

A ring $(\mathcal{F}, +, \cdot)$ is called a *field* if $(\mathcal{F} - \{0_{\mathcal{F}}\}, \cdot)$ is an commutative group, where $0_{\mathcal{F}}$ indicates the identity element of the group $(\mathcal{F}, +)$. A subfield $\mathcal{K}$ of a field $\mathcal{F}$ is a subset of $\mathcal{F}$ which is a field under the same operations as $\mathcal{F}$. That is, $\mathcal{K}$ is a subfield of $\mathcal{F}$ if $f_j - f_k \in \mathcal{K}\ \forall f_j, f_k \in \mathcal{K}$ and $f_j \cdot f_k^{-1} \in \mathcal{K}\ \forall f_j, (0_{\mathcal{F}} \neq) f_k \in \mathcal{K}$. A function $\Psi_{fld} : \mathcal{F}_1 \to \mathcal{F}_2$ between fields $\mathcal{F}_1$ and $\mathcal{F}_2$ is called a field homomorphism if $\Psi_{fld}(f_j^1 +_{\mathcal{F}_1} f_k^1) = \Psi_{fld}(f_j^1) +_{\mathcal{F}_2} \Psi_{fld}(f_k^1)$ and $\Psi_{fld}(f_j^1 \cdot_{\mathcal{F}_1} f_k^1) = \Psi_{fld}(f_j^1) \cdot_{\mathcal{F}_2} \Psi_{fld}(f_k^1)$ for all $f_j^1, f_k^1 \in \mathcal{F}_1$.

In 1965, Zadeh introduced the fuzzy set which is characterized by a membership function that assigns a degree of membership ranging between zero and one for each object in a universal set. Simply, a *fuzzy set* is denoted and defined by $\mathfrak{F} = \{(q_j, \langle \mathfrak{M}(q_j)\rangle) : q_j \in \mathcal{Q}\}$, where $\mathcal{Q}$ is a universal set and $\mathfrak{M} : \mathcal{Q} \to [0, 1]$ a membership function. Since this seminal publication, the fuzzy set theory has been widely studied and expanded. In 1986, Atanassov described intuitionistic fuzzy set by including the non-membership function to the construction of fuzzy set. In the following years, it was defined the concepts of Pyhthagorean fuzzy set (Yager 2013) and q-rung orthopair fuzzy set (Yager 2017) which deal with the squares and q-rungs of the degrees of membership and non-membership in the intuitionistic fuzzy set, respectively. In 2019, Riaz and Hashmi pointed out that these sets have their own restrictions related to the membership and non-membership degrees (see Table 18 in (Riaz and Hashmi 2019)). To eradicate the restrictions related to membership and non-membership functions in the structures of these sets, Riaz and Hashmi (2019) developed the linear Diophantine fuzzy set by adding reference parameters into them. This set which was brought to the literature in 2019, is described as follows.

Definition 2.1 (Riaz and Hashmi 2019) Let $\mathcal{Q}$ be a universal set. Then, a *linear Diophantine fuzzy set* (LDFS) $\mathfrak{D}$ on the universal set $\mathcal{Q}$ is described in the following form

$$\mathfrak{D} = \{(q_j, \langle \mathfrak{U}_{\mathfrak{D}}(q_j), \mathfrak{S}_{\mathfrak{D}}(q_j)\rangle, \langle \alpha, \beta\rangle) : q_j \in \mathcal{Q}\} \tag{2.1}$$

where $\mathfrak{U}_{\mathfrak{D}}(q_j), \mathfrak{S}_{\mathfrak{D}}(q_j), \alpha, \beta \in [0, 1]$ are the degrees of membership, non-membership and reference parameters, respectively. These degrees satisfy the the conditions $0 \le \alpha + \beta \le 1$ and $0 \le \alpha\mathfrak{U}_{\mathfrak{D}}(q_j) + \beta\mathfrak{S}_{\mathfrak{D}}(q_j) \le 1$ for all $q_j \in \mathcal{Q}$.

The reference parameters in the LDFS are specified attributes, but their degrees vary for each object in the universal set. The above definition evokes that the degrees of α and β are fixed for all objects. To emphasize that the degrees of reference parameters α and β for objects may vary (i.e. may not be fixed) and to better elucidate the concepts in the next sections, we revisit LDFS as follows.

$$\begin{aligned}\mathfrak{D} = \{(q_j, \langle \mathfrak{U}_{\mathfrak{D}}(q_j), \mathfrak{S}_{\mathfrak{D}}(q_j)\rangle,\\ \langle \alpha^{\mathfrak{D}}(q_j), \beta^{\mathfrak{D}}(q_j)\rangle) : q_j \in \mathcal{Q}\}\end{aligned} \tag{2.2}$$

where $\mathfrak{U}_{\mathfrak{D}}(q_j), \mathfrak{S}_{\mathfrak{D}}(q_j), \alpha^{\mathfrak{D}}(q_j), \beta^{\mathfrak{D}}(q_j) \in [0, 1]$ respectively represent the degrees of membership, non-membership and reference parameters for $q_j \in \mathcal{Q}$ with the conditions $0 \le \alpha^{\mathfrak{D}}(q_j) + \beta^{\mathfrak{D}}(q_j) \le 1$ and $0 \le \alpha^{\mathfrak{D}}(q_j)\mathfrak{U}_{\mathfrak{D}}(q_j) + \beta^{\mathfrak{D}}(q_j)\mathfrak{S}_{\mathfrak{D}}(q_j) \le 1$. These reference parameters can help in describing and classifying a particular system. The hesitancy degree can be considered as $\xi^{\mathfrak{D}}(q_j) \longmapsto_{\mathfrak{D}}(q_j) = 1 - (\alpha^{\mathfrak{D}}(q_j)\mathfrak{U}_{\mathfrak{D}}(q_j) + \beta^{\mathfrak{D}}(q_j)\mathfrak{S}_{\mathfrak{D}}(q_j))$, where $\xi^{\mathfrak{D}}$ is the reference parameter related to degree of indeterminacy in the LDFS $\mathfrak{D}$. Simply, a linear Diophantine fuzzy element (LDFE) is denoted by $\mathfrak{D}(q_j) = (\langle \mathfrak{U}_{\mathfrak{D}}(q_j), \mathfrak{S}_{\mathfrak{D}}(q_j)\rangle, \langle \alpha^{\mathfrak{D}}(q_j), \beta^{\mathfrak{D}}(q_j)\rangle)$. The set of all LDFSs on the universal set $\mathcal{Q}$ is symbolized by $LDFS(\mathcal{Q})$.

Example 2.2 Selection criteria are used to identify the most qualified candidate among all candidates who meet minimum qualifications and are selected for an interview for a particular position. The selection criteria go beyond minimum qualifications and look at the quality, quantity and relevance of the experience, education, knowledge and other skills each applicant has. Assume that it is desired to determine the best-qualified candidate who meets the specified selection criteria and is also young. Let $\mathcal{Q} = \{q_1, q_2, q_3, q_4\}$ be a set of candidates selected for an interview for a particular position. For the construction of LDFS, the reference parameters are considered as $\alpha = young$ and $\beta = not\ young\ (or\ old)$. Thus, the following LDFS is created.

$$\mathfrak{D} = \left\{ \begin{array}{l} (q_1, \langle 0.7, 0.6\rangle, \langle 0.8, 0.2\rangle), (q_2, \langle 0.4, 1\rangle, \langle 0.5, 0.4\rangle), \\ (q_3, \langle 0.9, 0.4\rangle, \langle 0.3, 0.7\rangle), (q_4, \langle 0.8, 0.3\rangle, \langle 0.6, 0.4\rangle) \end{array} \right\}.$$

In the structure of LDFS $\mathfrak{D}$, the LDFE $\mathfrak{D}(q_1) = (\langle 0.7, 0.6\rangle, \langle 0.8, 0.2\rangle)$ implies that for the candidate q_1, the degrees of membership and non-membership with respect to the selection criteria are 0.7 and 0.6, and the degrees of reference parameters: young and not young (or old) are 0.8 and 0.2. Others can be expounded similarly. As an explanatory to our proposal of Eq. (2.2) instead of Eq. (2.1), it is sufficient to highlight that the grades of reference parameters "young" and "not young" change for each candidate.

Definition 2.3 (Riaz and Hashmi 2019) An LDFS on the set $\mathcal{Q}$ of the form $\widetilde{\emptyset} = \{(q_j, \langle 0, 1\rangle, \langle 0, 1\rangle) : q_j \in \mathcal{Q}\}$ is termed to be an empty (or null) LDFS, and the form $\widetilde{\mathfrak{D}} = \{(q_j, \langle 1, 0\rangle, \langle 1, 0\rangle) : q_j \in \mathcal{Q}\}$ is termed to be an absolute LDFS.

Note 1 From now on, $I = \{1, 2, \dots, n\}$ unless otherwise specified.

Definition 2.4 (Riaz and Hashmi 2019) Let $\mathfrak{D}_i = \{(q_j, \langle \mathfrak{U}_{\mathfrak{D}_i}(q_j), \mathfrak{S}_{\mathfrak{D}_i}(q_j)\rangle, \langle \alpha^{\mathfrak{D}_i}(q_j), \beta^{\mathfrak{D}_i}(q_j)\rangle) : q_j \in \mathcal{Q}\}$ for $i \in I$ be the LDFSs on the universal set $\mathcal{Q}$.

(1) The complement of $\mathfrak{D}_i$, denoted by $\mathfrak{D}_i^c$, is defined as $\mathfrak{D}_i^c = \{(q_j, \langle \mathfrak{S}_{\mathfrak{D}_i}(q_j), \mathfrak{U}_{\mathfrak{D}_i}(q_j)\rangle, \langle \beta^{\mathfrak{D}_i}(q_j), \alpha^{\mathfrak{D}_i}(q_j)\rangle) : q_j \in \mathcal{Q}\}$. Also, it is clear that $(\mathfrak{D}_i^c)^c = \mathfrak{D}_i$.

(2) $\mathfrak{D}_i$ is a subset of $\mathfrak{D}_{i'}$ for $i, i' \in I$, denoted by $\mathfrak{D}_i \subseteq \mathfrak{D}_{i'}$, if and only if $\mathfrak{D}_i(q_j) \le \mathfrak{D}_{i'}(q_j)$ (i.e., $\mathfrak{U}_{\mathfrak{D}_i}(q_j) \le \mathfrak{U}_{\mathfrak{D}_{i'}}(q_j)$,

$\mathfrak{S}_{\mathfrak{D}_i}(q_j) \geq \mathfrak{S}_{\mathfrak{D}_{i'}}(q_j)$, $\alpha^{\mathfrak{D}_i}(q_j) \leq \alpha^{\mathfrak{D}_{i'}}(q_j)$ and $\beta^{\mathfrak{D}_i}(q_j) \geq \beta^{\mathfrak{D}_{i'}}(q_j)$) for all $q_j \in \mathcal{Q}$.

(3) $\mathfrak{D}_i$ and $\mathfrak{D}_{i'}$ for $i, i' \in I$ are equal, denoted by $\mathfrak{D}_i = \mathfrak{D}_{i'}$, if and only if $\mathfrak{D}_i(q_j) = \mathfrak{D}_{i'}(q_j)$ (i.e., $\mathfrak{U}_{\mathfrak{D}_i}(q_j) = \mathfrak{U}_{\mathfrak{D}_{i'}}(q_j)$, $\mathfrak{S}_{\mathfrak{D}_i}(q_j) = \mathfrak{S}_{\mathfrak{D}_{i'}}(q_j)$, $\alpha^{\mathfrak{D}_i}(q_j) = \alpha^{\mathfrak{D}_{i'}}(q_j)$ and $\beta^{\mathfrak{D}_i}(q_j) = \beta^{\mathfrak{D}_{i'}}(q_j)$) for all $q_j \in \mathcal{Q}$.

(4) The union of $\mathfrak{D}_i$ for all $i \in I$, denoted by $\bigcup_{i\in I} \mathfrak{D}_i$, is defined as

$$\bigcup_{i\in I} \mathfrak{D}_i = \{(q_j, \langle \mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(q_j), \mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(q_j)\rangle, \langle \alpha^{\tilde{\cup}\mathfrak{D}_i}(q_j), \beta^{\tilde{\cup}\mathfrak{D}_i}(q_j)\rangle) : q_j \in \mathcal{Q}\} \tag{2.3}$$

where for each $q_j \in \mathcal{Q}$, $\mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(q_j) = \bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(q_j)$, $= \sup_{i\in I}\{\mathfrak{U}_{\mathfrak{D}_i}(q_j)\}$, $\alpha^{\tilde{\cup}\mathfrak{D}_i}(q_j) = \bigvee_{i\in I} \alpha^{\mathfrak{D}_i}(q_j) = \sup_{i\in I}\{\alpha^{\mathfrak{D}_i}(q_j)\}$ and $\beta^{\tilde{\cup}\mathfrak{D}_i}(q_j) = \bigwedge_{i\in I} \beta^{\mathfrak{D}_i}(q_j) = \inf_{i\in I}\{\beta^{\mathfrak{D}_i}(q_j)\}$.

(5) The intersection of $\mathfrak{D}_i$ for all $i \in I$, denoted by $\bigcap_{i\in I} \mathfrak{D}_i$, is defined as

$$\bigcap_{i\in I} \mathfrak{D}_i = \{(q_j, \langle \mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}(q_j), \mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}(q_j)\rangle, \langle \alpha^{\tilde{\cap}\mathfrak{D}_i}(q_j), \beta^{\tilde{\cap}\mathfrak{D}_i}(q_j)\rangle) : q_j \in \mathcal{Q}\} \tag{2.4}$$

where for each $q_j \in \mathcal{Q}$, $\mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}(q_j) = \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(q_j)$ $= \inf_{i\in I}\{\mathfrak{U}_{\mathfrak{D}_i}(q_j)\}$, $\mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}(q_j) = \bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(q_j) = \sup_{i\in I}\{\mathfrak{S}_{\mathfrak{D}_i}(q_j)\}$, $\alpha^{\tilde{\cap}\mathfrak{D}_i}(q_j) = \bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(q_j) = \inf_{i\in I}\{\alpha^{\mathfrak{D}_i}(q_j)\}$ and $\beta^{\tilde{\cap}\mathfrak{D}_i}(q_j) = \bigvee_{i\in I} \beta^{\mathfrak{D}_i}(q_j) = \sup_{i\in I}\{\beta^{\mathfrak{D}_i}(q_j)\}$.

Definition 2.6 Let $\mathcal{Q}_i$ $(i \in I)$ be the universal sets. Also, let $\mathfrak{D}_i$ be the LDFS on the universal set $\mathcal{Q}_i$ for each $i \in I$. Then, the cartesian product of LDFSs $\mathfrak{D}_i$ $(i \in I)$, denoted by $\prod_{i\in I} \mathfrak{D}_i$, is defined as

$$\prod_{i\in I} \mathfrak{D}_i = \left\{ \left((q_j^i)_{i\in I}, \langle \mathfrak{U}_{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}), \mathfrak{S}_{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I})\rangle, \left\langle \alpha^{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}), \beta^{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}) \right\rangle \right) : (q_j^i)_{i\in I} \in \prod_{i\in I} \mathcal{Q}_i \right\} \tag{2.5}$$

where for each $(q_j^i)_{i\in I} \in \prod_{i\in I} \mathcal{Q}_i$, $\mathfrak{U}_{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}) = \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(q_j^i) = \inf_{i\in I}\{\mathfrak{U}_{\mathfrak{D}_i}(q_j^i)\}$, $\mathfrak{S}_{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}) = \bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(q_j^i) = \sup_{i\in I}\{\mathfrak{S}_{\mathfrak{D}_i}(q_j^i)\}$, $\alpha^{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}) = \bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(q_j^i) = \inf_{i\in I}\{\alpha^{\mathfrak{D}_i}(q_j^i)\}$ and $\beta^{\tilde{\Pi}\mathfrak{D}_i}((q_j^i)_{i\in I}) = \bigvee_{i\in I} \beta^{\mathfrak{D}_i}(q_j^i) = \sup_{i\in I}\{\beta^{\mathfrak{D}_i}(q_j^i)\}$.

Proposition 2.7 *Let $\mathfrak{D}_i$ be the LDFS on the universal set $\mathcal{Q}_i$ for each $i \in I$. Then, $\prod_{i\in I} \mathfrak{D}_i$ is also LDFS.*

Proof This can be easily proved by using the Definition 2.6, therefore omitted. □

Example 2.8 Let $\mathcal{Q}_1 = \{q_1^1\}$, $\mathcal{Q}_2 = \{q_1^2, q_2^2\}$ and $\mathcal{Q}_3 = \{q_1^3, q_2^3\}$ be three universal sets. We consider the following LDFSs $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$ on the universal sets $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, respectively. $\mathfrak{D}_1 = \{(q_1^1, \langle 0.5, 0.9\rangle, \langle 0.2, 0.5\rangle)\}$, $\mathfrak{D}_2 = \{(q_1^2, \langle 0.9, 0.8\rangle, \langle 0.3, 0.7\rangle), (q_2^2, \langle 0.8, 0.7\rangle, \langle 0.5, 0.3\rangle)\}$ and $\mathfrak{D}_3 = \{(q_1^2, \langle 0.4, 0.5\rangle, \langle 0.8, 0.1\rangle), (q_2^2, \langle 0.3, 0.4\rangle, \langle 0.3, 0.6\rangle)\}$.

Then, the cartesian product of $\mathfrak{D}_1$, $\mathfrak{D}_2$ and $\mathfrak{D}_3$ is obtained as follows.

$$\prod_{i\in I=\{1,2,3\}} \mathfrak{D}_i = \left\{ \begin{array}{l} ((q_1^1, q_1^2, q_1^3), \langle 0.4, 0.9\rangle, \langle 0.2, 0.7\rangle), ((q_1^1, q_1^2, q_2^3), \langle 0.3, 0.9\rangle, \langle 0.2, 0.7\rangle), \\ ((q_1^1, q_2^2, q_1^3), \langle 0.4, 0.9\rangle, \langle 0.2, 0.5\rangle), ((q_2^1, q_1^2, q_2^3), \langle 0.3, 0.9\rangle, \langle 0.2, 0.6\rangle) \end{array} \right\}.$$

Proposition 2.5 *Let $\mathfrak{D}_i \in LDFS(\mathcal{Q})$ for $i = 1, 2, 3$. Then, the following properties are valid.*

(i) $\mathfrak{D}_i \cap \widetilde{\emptyset} = \widetilde{\emptyset}$ and $\mathfrak{D}_i \cup \widetilde{\emptyset} = \mathfrak{D}_i$.
(ii) $\mathfrak{D}_i \cap \widetilde{\mathfrak{D}_i} = \mathfrak{D}_i$ and $\mathfrak{D}_i \cup \widetilde{\mathfrak{D}_i} = \widetilde{\mathfrak{D}_i}$.
(iii) $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \mathfrak{D}_2 \cap \mathfrak{D}_1$ and $\mathfrak{D}_1 \cup \mathfrak{D}_2 = \mathfrak{D}_2 \cup \mathfrak{D}_1$.
(iv) $\mathfrak{D}_1 \cap (\mathfrak{D}_2 \cap \mathfrak{D}_3) = (\mathfrak{D}_1 \cap \mathfrak{D}_2) \cap \mathfrak{D}_3$ and $\mathfrak{D}_1 \cup (\mathfrak{D}_2 \cup \mathfrak{D}_3) = (\mathfrak{D}_1 \cup \mathfrak{D}_2) \cup \mathfrak{D}_3$.
(v) $\mathfrak{D}_1 \cap (\mathfrak{D}_2 \cup \mathfrak{D}_3) = (\mathfrak{D}_1 \cap \mathfrak{D}_2) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_3)$ and $\mathfrak{D}_1 \cup (\mathfrak{D}_2 \cap \mathfrak{D}_3) = (\mathfrak{D}_1 \cup \mathfrak{D}_2) \cap (\mathfrak{D}_1 \cup \mathfrak{D}_3)$.
(vi) $(\mathfrak{D}_1 \cap \mathfrak{D}_2)^c = \mathfrak{D}_1^c \cup \mathfrak{D}_2^c$ and $(\mathfrak{D}_1 \cup \mathfrak{D}_2)^c = \mathfrak{D}_1^c \cap \mathfrak{D}_2^c$

Proof The proofs are clear from Proposition 3.7 in Riaz and Hashmi (2019) and Definition 2.3. □

Definition 2.9 Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two universal sets and $\Psi : \mathcal{Q}_1 \to \mathcal{Q}_2$ be a function.

(1) If $\mathfrak{D}_1 = \{(q_j^1, \langle \mathfrak{U}_{\mathfrak{D}_1}(q_j^1), \mathfrak{S}_{\mathfrak{D}_1}(q_j^1)\rangle, \langle \alpha^{\mathfrak{D}_1}(q_j^1), \beta^{\mathfrak{D}_1}(q_j^1)\rangle) : q_j^1 \in \mathcal{Q}_1\}$ is an LDFS on $\mathcal{Q}_1$, then the image $\Psi(\mathfrak{D}_1)$ of LDFS $\mathfrak{D}_1$ is an LDFS on $\mathcal{Q}_2$ and it is defined as follows:

$$\begin{aligned} \Psi(\mathfrak{D}_1) &= \{(q_j^2, \Psi(\mathfrak{D}_1)(q_j^2)) : q_j^2 \in \mathcal{Q}_2\} \\ &= \{(q_j^2, \langle \mathfrak{U}_{\Psi(\mathfrak{D}_1)}(q_j^2), \mathfrak{S}_{\Psi(\mathfrak{D}_1)}(q_j^2)\rangle, \langle \alpha^{\Psi(\mathfrak{D}_1)}(q_j^2), \beta^{\Psi(\mathfrak{D}_1)}(q_j^2)\rangle) : q_j^2 \in \mathcal{Q}_2\} \\ &= \{(q_j^2, \langle \Psi(\mathfrak{U}_{\mathfrak{D}_1})(q_j^2), \Psi(\mathfrak{S}_{\mathfrak{D}_1})(q_j^2)\rangle, \langle \Psi(\alpha^{\mathfrak{D}_1})(q_j^2), \Psi(\beta^{\mathfrak{D}_1})(q_j^2)\rangle) : q_j^2 \in \mathcal{Q}_2\} \end{aligned} \tag{2.6}$$

where for all $q_j^2 \in \mathcal{Q}_2$,

$$\Psi(\mathfrak{U}_{\mathfrak{D}_1})(q_j^2) = \begin{cases} \bigvee \mathfrak{U}_{\mathfrak{D}_1}(q_j^1), & \text{if } q_j^1 \in \Psi^{-1}(q_j^2) \\ 0, & \text{otherwise} \end{cases},$$

$$\Psi(\mathfrak{S}_{\mathfrak{D}_1})(q_j^2) = \begin{cases} \bigwedge \mathfrak{S}_{\mathfrak{D}_1}(q_j^1), & \text{if } q_j^1 \in \Psi^{-1}(q_j^2) \\ 1, & \text{otherwise} \end{cases},$$

$$\Psi(\alpha^{\mathfrak{D}_1}(q_j^2)) = \begin{cases} \bigvee \alpha^{\mathfrak{D}_1}(q_j^1), & \text{if } q_j^1 \in \Psi^{-1}(q_j^2) \\ 0, & \text{otherwise} \end{cases},$$

$$\Psi(\beta^{\mathfrak{D}_1}(q_j^2)) = \begin{cases} \bigwedge \beta^{\mathfrak{D}_1}(q_j^1), & \text{if } q_j^1 \in \Psi^{-1}(q_j^2) \\ 1, & \text{otherwise} \end{cases}.$$

(2) If $\mathfrak{D}_2 = \{(q_j^2, \langle \mathfrak{U}_{\mathfrak{D}_2}(q_j^2), \mathfrak{S}_{\mathfrak{D}_2}(q_j^2)\rangle, \langle \alpha^{\mathfrak{D}_2}(q_j^2), \beta^{\mathfrak{D}_2}(q_j^2)\rangle) : q_j^2 \in \mathcal{Q}_2\}$ is an LDFS on $\mathcal{Q}_2$, then the preimage $\Psi^{-1}(\mathfrak{D}_2)$ of LDFS $\mathfrak{D}_2$ is an LDFS on $\mathcal{Q}_1$ and it is defined as follows:

$$\begin{aligned}
\Psi^{-1}(\mathfrak{D}_2) &= \left\{ \left(q_j^1, \Psi^{-1}(\mathfrak{D}_2)\left(q_j^1\right)\right) : q_j^1 \in \mathcal{Q}_1 \right\} \\
&= \left\{ \left(q_j^1, \left\langle \mathfrak{U}_{\Psi^{-1}(\mathfrak{D}_2)}\left(q_j^1\right), \mathfrak{S}_{\Psi^{-1}(\mathfrak{D}_2)}\left(q_j^1\right)\right\rangle, \right.\right. \\
&\quad \left.\left. \left\langle \alpha^{\Psi^{-1}(\mathfrak{D}_2)}\left(q_j^1\right), \beta^{\Psi^{-1}(\mathfrak{D}_2)}\left(q_j^1\right)\right\rangle\right) : q_j^1 \in \mathcal{Q}_1 \right\} \\
&= \left\{ \left(q_j^2, \left\langle \mathfrak{U}_{\mathfrak{D}_2}\left(\Psi\left(q_j^1\right)\right), \mathfrak{S}_{\mathfrak{D}_2}\left(\Psi\left(q_j^1\right)\right)\right\rangle, \right.\right. \\
&\quad \left.\left. \left\langle \alpha^{\mathfrak{D}_2}\left(\Psi\left(q_j^1\right)\right), \beta^{\mathfrak{D}_2}\left(\Psi\left(q_j^1\right)\right)\right\rangle\right) : q_j^1 \in \mathcal{Q}_1 \right\} \\
&= \left\{ \left(q_j^1, \mathfrak{D}_2\left(\Psi\left(q_j^1\right)\right)\right) : q_j^1 \in \mathcal{Q}_1 \right\}.
\end{aligned} \tag{2.7}$$

3 Linear Diophantine fuzzy subgroup

In this section, we introduce the notion of linear Diophantine fuzzy subgroup and discuss its crucial properties.

Definition 3.1 Let $(\mathcal{G}, \cdot)$ be a classical group and $\mathfrak{D}$ be an LDFS on $\mathcal{G}$. Then, $\mathfrak{D}$ is called a linear Diophantine fuzzy subgroupoid (LDF-subgroupoid) of $\mathcal{G}$ iff the following condition is satisfied

(G1) $\mathfrak{D}(g_j \cdot g_k) \geq \mathfrak{D}(g_j) \wedge \mathfrak{D}(g_k)$ for all $g_j, g_k \in \mathcal{G}$, i.e.,

$$\begin{aligned}
&\mathfrak{U}_{\mathfrak{D}}(g_j \cdot g_k) \geq \mathfrak{U}_{\mathfrak{D}}(g_j) \wedge \mathfrak{U}_{\mathfrak{D}}(g_k), \\
&\quad \mathfrak{S}_{\mathfrak{D}}(g_j \cdot g_k) \leq \mathfrak{S}_{\mathfrak{D}}(g_j) \vee \mathfrak{S}_{\mathfrak{D}}(g_k), \\
&\quad \alpha^{\mathfrak{D}}(g_j \cdot g_k) \geq \alpha^{\mathfrak{D}}(g_j) \wedge \alpha^{\mathfrak{D}}(g_k), \\
&\quad \beta^{\mathfrak{D}}(g_j \cdot g_k) \leq \beta^{\mathfrak{D}}(g_j) \vee \beta^{\mathfrak{D}}(g_k).
\end{aligned}$$

The linear Diophantine fuzzy subgroupoid $\mathfrak{D}$ is called a linear Diophantine fuzzy subgroup (LDF-sugroup) of $\mathcal{G}$ iff the following condition is provided:

(G2) $\mathfrak{D}(g_j^{-1}) = \mathfrak{D}(g_j)$ for all $g_j \in \mathcal{G}$, i.e.,

$$\begin{aligned}
&\mathfrak{U}_{\mathfrak{D}}\left(g_j^{-1}\right) = \mathfrak{U}_{\mathfrak{D}}(g_j),\ \mathfrak{S}_{\mathfrak{D}}\left(g_j^{-1}\right) = \mathfrak{S}_{\mathfrak{D}}(g_j), \\
&\quad \alpha^{\mathfrak{D}}\left(g_j^{-1}\right) = \alpha^{\mathfrak{D}}(g_j),\ \beta^{\mathfrak{D}}\left(g_j^{-1}\right) = \beta^{\mathfrak{D}}(g_j).
\end{aligned}$$

The collection of all LDF-subgroups of $\mathcal{G}$ is denoted by $LDFsG(\mathcal{G})$.

Example 3.2 Let us consider the Klein's 4-group $\mathcal{G} = \{e, a, b, c\}$ with the multiplication table:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Then, the LDFS

$$\mathfrak{D}_1 = \left\{ \begin{array}{l} (e, \langle 0.7, 0.5\rangle, \langle 0.5, 0.4\rangle), (a, \langle 0.5, 0.7\rangle, \langle 0.4, 0.5\rangle), \\ (b, \langle 0.3, 0.8\rangle, \langle 0.2, 0.6\rangle), (c, \langle 0.3, 0.8\rangle, \langle 0.2, 0.6\rangle) \end{array} \right\}$$

is an LDF-subgroup of $\mathcal{G}$. However, the LDFS

$$\mathfrak{D}_2 = \left\{ \begin{array}{l} (e, \langle 0.3, 0.5\rangle, \langle 0.5, 0.4\rangle), (a, \langle 0.4, 0.7\rangle, \langle 0.4, 0.5\rangle), \\ (b, \langle 0.3, 0.8\rangle, \langle 0.2, 0.6\rangle), (c, \langle 0.3, 0.8\rangle, \langle 0.2, 0.9\rangle) \end{array} \right\}$$

is not an LDF-subgroup of $\mathcal{G}$ since $\mathfrak{U}_{\mathfrak{D}}(e) \ngeq \mathfrak{U}_{\mathfrak{D}_2}(a) \wedge \mathfrak{U}_{\mathfrak{D}_2}(a)$ and $\beta^{\mathfrak{D}_2}(c) \nleq \beta^{\mathfrak{D}_2}(a) \vee \beta^{\mathfrak{D}_2}(b)$.

Proposition 3.3 *Let $\mathcal{G}$ be a classical group and $\mathfrak{D}$ be an LDF-subgroup of $\mathcal{G}$. Then,*

(i) $\mathfrak{D}(e_{\mathcal{G}}) \geq \mathfrak{D}(g_j)$ for all $g_j \in \mathcal{G}$, where $e_{\mathcal{G}}$ is the unit element of $\mathcal{G}$.

(ii) $\mathfrak{D}({g_j}^{\lambda}) \geq \mathfrak{D}(g_j)$ for all $g_j \in \mathcal{G}$, where $\lambda \in \mathbb{N}$.

Proof Let $\mathcal{G}$ be a classical group and $\mathfrak{D}$ be an LDF-subgroup of $\mathcal{G}$.

(i) Let $e_{\mathcal{G}}$ be the unit element of $\mathcal{G}$. From Definition 3.1, we obtain that for any $g_j \in \mathcal{G}$

$$\begin{aligned}
\mathfrak{U}_{\mathfrak{D}}(e_{\mathcal{G}}) =&\mathfrak{U}_{\mathfrak{D}}\left(g_j \cdot g_j^{-1}\right) \geq \mathfrak{U}_{\mathfrak{D}}(g_j) \wedge \mathfrak{U}_{\mathfrak{D}}(g_j^{-1}) \\
=&\mathfrak{U}_{\mathfrak{D}}(g_j) \wedge \mathfrak{U}_{\mathfrak{D}}(g_j) = \mathfrak{U}_{\mathfrak{D}}(g_j), \\
\mathfrak{S}_{\mathfrak{D}}(e_{\mathcal{G}}) =&\mathfrak{S}_{\mathfrak{D}}\left(g_j \cdot g_j^{-1}\right) \leq \mathfrak{S}_{\mathfrak{D}}(g_j) \vee \mathfrak{S}_{\mathfrak{D}}\left(g_j^{-1}\right) \\
=&\mathfrak{S}_{\mathfrak{D}}(g_j) \vee \mathfrak{S}_{\mathfrak{D}}(g_j) = \mathfrak{S}_{\mathfrak{D}}(g_j), \\
\alpha^{\mathfrak{D}}(e_{\mathcal{G}}) =&\alpha^{\mathfrak{D}}\left(g_j \cdot g_j^{-1}\right) \geq \alpha^{\mathfrak{D}}(g_j) \wedge \alpha^{\mathfrak{D}}\left(g_j^{-1}\right) \\
=&\alpha^{\mathfrak{D}}(g_j) \wedge \alpha^{\mathfrak{D}}(g_j) = \alpha^{\mathfrak{D}}(g_j), \\
\beta^{\mathfrak{D}}(e_{\mathcal{G}}) =&\beta^{\mathfrak{D}}\left(g_j \cdot g_j^{-1}\right) \leq \beta^{\mathfrak{D}}(g_j) \vee \beta^{\mathfrak{D}}\left(g_j^{-1}\right) \\
=&\beta^{\mathfrak{D}}(g_j) \vee \beta^{\mathfrak{D}}(g_j) = \beta^{\mathfrak{D}}(g_j).
\end{aligned}$$

Thus, by Definition 2.4 (2), we have $\mathfrak{D}(e_{\mathcal{G}}) \geq \mathfrak{D}(g_j)$ for all $g_j \in \mathcal{G}$.

(ii) It can be verified with the similar discussion, so it is omitted.

□

Theorem 3.4 *Let $\mathcal{G}$ be a classical group and $\mathfrak{D}$ be an LDFS on $\mathcal{G}$. Then, $\mathfrak{D}$ is an LDF-subgroup of $\mathcal{G}$ if and only if $\mathfrak{D}(g_j \cdot g_k{}^{-1}) \geq \mathfrak{D}(g_j) \wedge \mathfrak{D}(g_k)$ for all $g_j, g_k \in \mathcal{G}$.*

Proof Let $\mathfrak{D}$ be an LDF-subgroup of the classical group $\mathcal{G}$. By Definition 3.1, we obtain that for all $g_j, g_k \in \mathcal{G}$

$$\begin{aligned}
\mathfrak{U}_{\mathfrak{D}}\left(g_j \cdot g_k{}^{-1}\right) \geq&\mathfrak{U}_{\mathfrak{D}}(g_j) \wedge \mathfrak{U}_{\mathfrak{D}}\left(g_k^{-1}\right) \\
=&\mathfrak{U}_{\mathfrak{D}}(g_j) \wedge \mathfrak{U}_{\mathfrak{D}}(g_k), \\
\mathfrak{S}_{\mathfrak{D}}\left(g_j \cdot g_k^{-1}\right) \leq&\mathfrak{S}_{\mathfrak{D}}(g_j) \vee \mathfrak{S}_{\mathfrak{D}}\left(g_k^{-1}\right) \\
=&\mathfrak{S}_{\mathfrak{D}}(g_j) \vee \mathfrak{S}_{\mathfrak{D}}(g_k), \\
\alpha^{\mathfrak{D}}\left(g_j \cdot g_k^{-1}\right) \geq&\alpha^{\mathfrak{D}}(g_j) \wedge \alpha^{\mathfrak{D}}\left(g_k^{-1}\right) \\
=&\alpha^{\mathfrak{D}}(g_j) \wedge \alpha^{\mathfrak{D}}(g_k), \\
\beta^{\mathfrak{D}}\left(g_j \cdot g_k^{-1}\right) \leq&\beta^{\mathfrak{D}}(g_j) \vee \beta^{\mathfrak{D}}\left(g_k^{-1}\right) \\
=&\beta^{\mathfrak{D}}(g_j) \vee \beta^{\mathfrak{D}}(g_k).
\end{aligned}$$

Thus, the desired inequality $\mathfrak{D}(g_j \cdot g_k{}^{-1}) \geq \mathfrak{D}(g_j) \wedge \mathfrak{D}(g_k)$ is satisfied.

Conversely, let $\mathfrak{D}(g_j \cdot g_k{}^{-1}) \geq \mathfrak{D}(g_j) \wedge \mathfrak{D}(g_k)$ for all $g_j, g_k \in \mathcal{G}$. If it is taken $g_j = e_{\mathcal{G}}$, then we have $\mathfrak{D}(e_{\mathcal{G}} \cdot g_k{}^{-1}) \geq \mathfrak{D}(e_{\mathcal{G}}) \wedge \mathfrak{D}(g_k) \Rightarrow \mathfrak{D}(g_k{}^{-1}) \geq \mathfrak{D}(g_k)$ (using Proposition 3.3 (i)). Then, we can write $\mathfrak{D}(g_k) = \mathfrak{D}((g_k{}^{-1})^{-1}) \geq \mathfrak{D}(g_k{}^{-1})$ and so $\mathfrak{D}(g_k{}^{-1}) = \mathfrak{D}(g_k)$. Considering the arbitrary property of g_k, this equality meets the condition (G2) of being the LDF-subgroup given in Definition 3.1. Taking into account the assumption and the last equality, we obtain $\mathfrak{D}(g_j \cdot g_k) = \mathfrak{D}(g_j \cdot (g_k{}^{-1})^{-1}) \geq \mathfrak{D}(g_j) \wedge \mathfrak{D}(g_k{}^{-1}) = \mathfrak{D}(g_j) \wedge \mathfrak{D}(g_k)$ for all $g_j, g_k \in \mathcal{G}$. This inequality meets the condition (G1) of being the LDF-subgroup given in Definition 3.1. Consequently, the LDFS $\mathfrak{D}$ is an LDF-subgroup of $\mathcal{G}$. □

Theorem 3.5 *Let $\mathcal{G}$ be a classical group and $\mathfrak{D}_i\,(i \in I)$ be the LDF-subgroups of $\mathcal{G}$. Then,*

(i) $\bigcap_{i\in I} \mathfrak{D}_i$ is an LDF-subgroup of $\mathcal{G}$.
(ii) $\bigcup_{i\in I} \mathfrak{D}_i$ is an LDF-subgroup of $\mathcal{G}$.

Proof Let $\mathfrak{D}_i\,(i \in I)$ be the LDF-subgroups of the group $\mathcal{G}$.

(i) To complete the proof, we should demonstrate that $\bigcap_{i\in I} \mathfrak{D}_i(g_j \cdot g_k{}^{-1}) \geq \bigcap_{i\in I} \mathfrak{D}_i(g_j) \wedge \bigcap_{i\in I} \mathfrak{D}_i(g_k)$ for all $g_j, g_k \in \mathcal{G}$.

By Definition 2.4 (5), we have

$$\begin{aligned}
&\bigcap_{i\in I} \mathfrak{D}_i\left(g_j \cdot g_k{}^{-1}\right) \\
&= \left(\left\langle \mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right), \mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right)\right\rangle,\right. \\
&\qquad \left.\left\langle \alpha^{\tilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right), \beta^{\tilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right)\right\rangle\right)
\end{aligned} \tag{3.1}$$

for all $g_j, g_k \in \mathcal{G}$. Since $\mathfrak{D}_i\ (i \in I)$ are the LDF-subgroups of $\mathcal{G}$, we obtain

$$\begin{aligned}
\mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right) &= \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right) \\
&\geq \bigwedge_{i\in I} (\mathfrak{U}_{\mathfrak{D}_i}(g_j) \wedge \mathfrak{U}_{\mathfrak{D}_i}(g_k) \\
&=\left(\bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(g_j)\right) \wedge \left(\bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(g_k)\right) \\
&=\mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}(g_j) \wedge \mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}(g_k),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right) &= \bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right) \\
&\leq \bigvee_{i\in I} (\mathfrak{S}_{\mathfrak{D}_i}(g_j) \vee \mathfrak{S}_{\mathfrak{D}_i}(g_k) \\
&=\left(\bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(g_j)\right) \vee \left(\bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(g_k)\right) \\
&=\mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}(g_j) \vee \mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}(g_k),
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\alpha^{\widetilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right) &= \bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right)\\
&\geq \bigwedge_{i\in I}(\alpha^{\mathfrak{D}_i}(g_j) \wedge \alpha^{\mathfrak{D}_i}(g_k))\\
&=\left(\bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(g_j)\right) \wedge \left(\bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(g_k)\right)\\
&=\alpha^{\widetilde{\cap}\mathfrak{D}_i}(g_j) \wedge \alpha^{\widetilde{\cap}\mathfrak{D}_i}(g_k),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\beta^{\widetilde{\cap}\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right) &= \bigvee_{i\in I} \beta^{\mathfrak{D}_i}\left(g_j \cdot g_k{}^{-1}\right)\\
&\leq \bigvee_{i\in I}(\beta^{\mathfrak{D}_i}(g_j) \vee \beta^{\mathfrak{D}_i}(g_k))\\
&=\left(\bigvee_{i\in I} \beta^{\mathfrak{D}_i}(g_j)\right) \vee \left(\bigvee_{i\in I} \beta^{\mathfrak{D}_i}(g_k)\right)\\
&=\beta^{\widetilde{\cap}\mathfrak{D}_i}(g_j) \vee \beta^{\widetilde{\cap}\mathfrak{D}_i}(g_k).
\end{aligned} \tag{3.5}
$$

Form Eqs. (3.2)–(3.5), the desired inequality (Eq. (3.1)) is satisfied. Thus, the proof of (i) ends.

(ii) The proof can be demonstrated similar to the proof of (i).

□

Theorem 3.6 *Let $\mathcal{G}_i$ $(i \in I)$ be the classical groups and $\mathfrak{D}_i$ be the LDF-subgroups of $\mathcal{G}_i$ for $i \in I$. Then, $\prod_{i\in I} \mathfrak{D}_i$ is an LDF-subgroup of the group $\prod_{i\in I} \mathcal{G}_i$.*

Proof Suppose that $\mathfrak{D}_i$ are the LDF-subgroups of $\mathcal{G}_i$ for $i \in I$. To conclude the proof, we have to verify that $\prod_{i\in I} \mathfrak{D}_i((g_j^i)_{i\in I} \cdot (g_k^i)^{-1}_{i\in I}) \geq \prod_{i\in I} \mathfrak{D}_i((g_j^i)_{i\in I}) \wedge \prod_{i\in I} \mathfrak{D}_i((g_k^i)_{i\in I})$ for all $(g_j^i)_{i\in I}, (g_k^i)_{i\in I} \in \prod_{i\in I} \mathcal{G}_i$.

By Definition 2.6, we have

$$
\begin{aligned}
&\prod_{i\in I} \mathfrak{D}_i\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right)\\
&= \Bigg(\Big\langle \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right),\\
&\quad \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right)\Big\rangle,\\
&\quad \Big\langle \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right),\\
&\quad \beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right)\Big\rangle\Bigg)
\end{aligned} \tag{3.6}
$$

for all $(g_j^i)_{i\in I}, (g_k^i)_{i\in I} \in \prod_{i\in I} \mathcal{G}_i$. Since $\mathfrak{D}_i$ is the LDF-subgroups of $\mathcal{G}_i$ for each $i \in I$, we obtain

$$
\begin{aligned}
&\mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right) = \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i \cdot \left(g_k^i\right)^{-1}\right)_{i\in I}\right)\\
&= \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}\left(g_j^i \cdot \left(g_k^i\right)^{-1}\right)\\
&\geq \bigwedge_{i\in I} \left(\mathfrak{U}_{\mathfrak{D}_i}\left(g_j^i\right) \wedge \mathfrak{U}_{\mathfrak{D}_i}\left(g_k^i\right)\right)\\
&= \left(\bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}\left(g_j^i\right)\right) \wedge \left(\bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}\left(g_k^i\right)\right)\\
&= \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I}\right) \wedge \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_k^i\right)_{i\in I}\right),
\end{aligned} \tag{3.7}
$$

According to the similar discussion, the following inequalities are also true.

$$
\begin{aligned}
&\mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right)\right.\\
&\leq \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I}\right) \vee \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_k^i\right)_{i\in I}\right),
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
&\alpha^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right)\\
&\geq \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I}\right) \wedge \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_k^i\right)_{i\in I}\right),
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&\beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I} \cdot \left(g_k^i\right)^{-1}_{i\in I}\right)\\
&\leq \beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_j^i\right)_{i\in I}\right) \vee \beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(g_k^i\right)_{i\in I}\right).
\end{aligned} \tag{3.10}
$$

Hence, $\prod_{i\in I} \mathfrak{D}_i$ is an LDF-subgroup of the group $\prod_{i\in I} \mathcal{G}_i$. □

Example 3.7 Let us take into consideration the classical groups $(\mathbb{Z}_2, +)$ and $(\mathbb{Z}_4, +)$. Also, we devised the following LDF-subgroups of $(\mathbb{Z}_2, +)$ and $(\mathbb{Z}_4, +)$, respectively.

$$\mathfrak{D}_1 = \{(\bar{0}, \langle 0.7, 0.5\rangle, \langle 0.3, 0.6\rangle), (\bar{1}, \langle 0.2, 0.6\rangle, \langle 0.2, 0.7\rangle)\},$$

$$\mathfrak{D}_2 = \left\{\begin{array}{l}(\bar{\bar{0}}, \langle 0.4, 0.4\rangle, \langle 0.2, 0.4\rangle), (\bar{\bar{1}}, \langle 0.3, 0.8\rangle, \langle 0.1, 0.5\rangle),\\ (\bar{\bar{2}}, \langle 0.3, 0.8\rangle, \langle 0.1, 0.5\rangle), (\bar{\bar{3}}, \langle 0.3, 0.8\rangle, \langle 0.1, 0.5\rangle)\end{array}\right\}.$$

Then, we obtain an LDFS on the group $\mathbb{Z}_2 \times \mathbb{Z}_4$ as follows:

$$\prod_{i\in I=\{1,2\}} \mathfrak{D}_i = \left\{\begin{array}{l}((\bar{0}, \bar{\bar{0}}), \langle 0.4, 0.5\rangle, \langle 0.2, 0.6\rangle), ((\bar{0}, \bar{\bar{1}}), \langle 0.3, 0.8\rangle, \langle 0.1, 0.6\rangle),\\ ((\bar{0}, \bar{\bar{2}}), \langle 0.3, 0.8\rangle, \langle 0.1, 0.6\rangle), ((\bar{0}, \bar{\bar{3}}), \langle 0.3, 0.8\rangle, \langle 0.1, 0.6\rangle),\\ ((\bar{1}, \bar{\bar{0}}), \langle 0.2, 0.6\rangle, \langle 0.2, 0.7\rangle), ((\bar{1}, \bar{\bar{1}}), \langle 0.2, 0.8\rangle, \langle 0.1, 0.7\rangle),\\ ((\bar{1}, \bar{\bar{2}}), \langle 0.2, 0.8\rangle, \langle 0.1, 0.7\rangle), ((\bar{1}, \bar{\bar{3}}), \langle 0.2, 0.8\rangle, \langle 0.1, 0.7\rangle)\end{array}\right\}.$$

It is clear that the LDFS $\prod_{i\in I=\{1,2\}}\mathfrak{D}_i$ is an LDF-subgroup of $\mathbb{Z}_2\times\mathbb{Z}_4$.

Theorem 3.8 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two classical groups and $\Psi_{grp}:\mathcal{G}_1\to\mathcal{G}_2$ be a group homomorphism. If $\mathfrak{D}_1$ is an LDF-subgroup of $\mathcal{G}_1$, then the image $\Psi_{grp}(\mathfrak{D}_1)$ of $\mathfrak{D}_1$ is also an LDF-subgroup of $\mathcal{G}_2$.*

Proof Let $\mathfrak{D}_1\in LDFsG(\mathcal{G}_1)$. If $\Psi^{-1}_{grp}(g_1^2)=\emptyset$ and $\Psi^{-1}_{grp}(g_2^2)=\emptyset$ for $g_1^2,g_2^2\in\mathcal{G}_2$, then it is clear that $\Psi_{grp}(\mathfrak{D}_1)\in LDFsG(\mathcal{G}_2)$. Suppose that there exist $g_1^1,g_2^1\in\mathcal{G}_1$ such that $\Psi_{grp}(g_1^1)=g_1^2$ and $\Psi_{grp}(g_2^1)=g_2^2$. To complete the proof, we must demonstrate that $\Psi_{grp}(\mathfrak{D}_1)(g_1^2\cdot(g_2^2)^{-1})\geq\Psi_{grp}(\mathfrak{D}_1)(g_1^2)\wedge\Psi_{grp}(\mathfrak{D}_1)(g_2^2)$. Since $\mathfrak{D}_1\in LDFsG(\mathcal{G}_1)$ and Ψ_{grp} is a group homomorphism (i.e., $\Psi_{grp}(g_1^1\cdot(g_2^1)^{-1})=\Psi_{grp}(g_1^1)\cdot\Psi_{grp}((g_2^1)^{-1})=g_1^2\cdot(g_2^2)^{-1}$)

$$\begin{aligned}
\Psi_{grp}(\mathfrak{U}_{\mathfrak{D}_1})\left(g_1^2\cdot(g_2^2)^{-1}\right)
&=\bigvee_{g_1^1\cdot(g_2^1)^{-1}\in\Psi^{-1}_{grp}\left(g_1^2\cdot(g_2^2)^{-1}\right)}\mathfrak{U}_{\mathfrak{D}_1}\left(g_1^1\cdot(g_2^1)^{-1}\right)\\
&\geq\bigvee_{g_1^1\in\Psi^{-1}_{grp}(g_1^2),\ g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\left(\mathfrak{U}_{\mathfrak{D}_1}(g_1^1)\wedge\mathfrak{U}_{\mathfrak{D}_1}(g_2^1)\right)\\
&=\left(\bigvee_{g_1^1\in\Psi^{-1}_{grp}(g_1^2)}\mathfrak{U}_{\mathfrak{D}_1}(g_1^1)\right)\wedge\left(\bigvee_{g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\mathfrak{U}_{\mathfrak{D}_1}(g_2^1)\right)\\
&=\Psi_{grp}(\mathfrak{U}_{\mathfrak{D}_1})(g_1^2)\wedge\Psi_{grp}(\mathfrak{U}_{\mathfrak{D}_1})(g_2^2),
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
\Psi_{grp}(\mathfrak{S}_{\mathfrak{D}_1})\left(g_1^2\cdot(g_2^2)^{-1}\right)
&=\bigwedge_{g_1^1\cdot(g_2^1)^{-1}\in\Psi^{-1}_{grp}\left(g_1^2\cdot(g_2^2)^{-1}\right)}\mathfrak{S}_{\mathfrak{D}_1}\left(g_1^1\cdot(g_2^1)^{-1}\right)\\
&\leq\bigwedge_{g_1^1\in\Psi^{-1}_{grp}(g_1^2),\ g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\left(\mathfrak{S}_{\mathfrak{D}_1}(g_1^1)\vee\mathfrak{S}_{\mathfrak{D}_1}(g_2^1)\right)\\
&=\left(\bigwedge_{g_1^1\in\Psi^{-1}_{grp}(g_1^2)}\mathfrak{S}_{\mathfrak{D}_1}(g_1^1)\right)\vee\left(\bigwedge_{g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\mathfrak{S}_{\mathfrak{D}_1}(g_2^1)\right)\\
&=\Psi_{grp}(\mathfrak{S}_{\mathfrak{D}_1})(g_1^2)\vee\Psi_{grp}(\mathfrak{S}_{\mathfrak{D}_1})(g_2^2),
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
\Psi_{grp}(\alpha^{\mathfrak{D}_1})\left(g_1^2\cdot(g_2^2)^{-1}\right)
&=\bigvee_{g_1^1\cdot(g_2^1)^{-1}\in\Psi^{-1}_{grp}\left(g_1^2\cdot(g_2^2)^{-1}\right)}\alpha^{\mathfrak{D}_1}\left(g_1^1\cdot(g_2^1)^{-1}\right)\\
&\geq\bigvee_{g_1^1\in\Psi^{-1}_{grp}(g_1^2),\ g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\left(\alpha^{\mathfrak{D}_1}(g_1^1)\wedge\alpha^{\mathfrak{D}_1}(g_2^1)\right)\\
&=\left(\bigvee_{g_1^1\in\Psi^{-1}_{grp}(g_1^2)}\alpha^{\mathfrak{D}_1}(g_1^1)\right)\wedge\left(\bigvee_{g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\alpha^{\mathfrak{D}_1}(g_2^1)\right)\\
&=\Psi_{grp}(\alpha^{\mathfrak{D}_1})(g_1^2)\wedge\Psi_{grp}(\alpha^{\mathfrak{D}_1})(g_2^2),
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
\Psi_{grp}(\beta^{\mathfrak{D}_1})\left(g_1^2\cdot(g_2^2)^{-1}\right)
&=\bigwedge_{g_1^1\cdot(g_2^1)^{-1}\in\Psi^{-1}_{grp}\left(g_1^2\cdot(g_2^2)^{-1}\right)}\beta^{\mathfrak{D}_1}\left(g_1^1\cdot(g_2^1)^{-1}\right)\\
&\leq\bigwedge_{g_1^1\in\Psi^{-1}_{grp}(g_1^2),\ g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\left(\beta^{\mathfrak{D}_1}(g_1^1)\vee\beta^{\mathfrak{D}_1}(g_2^1)\right)\\
&=\left(\bigwedge_{g_1^1\in\Psi^{-1}_{grp}(g_1^2)}\beta^{\mathfrak{D}_1}(g_1^1)\right)\vee\left(\bigwedge_{g_2^1\in\Psi^{-1}_{grp}(g_2^2)}\beta^{\mathfrak{D}_1}(g_2^1)\right)\\
&=\Psi_{grp}(\beta^{\mathfrak{D}_1})(g_1^2)\vee\Psi_{grp}(\beta^{\mathfrak{D}_1})(g_2^2).
\end{aligned}\tag{3.14}$$

By Eqs. (3.11)–(3.14), we obtain that $\Psi_{grp}(\mathfrak{D}_1)(g_1^2\cdot(g_2^2)^{-1})\geq\Psi_{grp}(\mathfrak{D}_1)(g_1^2)\wedge\Psi_{grp}(\mathfrak{D}_1)(g_2^2)$ for all $g_1^2,g_2^2\in\mathcal{G}_2$. Therefore, $\Psi_{grp}(\mathfrak{D}_1)$ is an LDF-subgroup of $\mathcal{G}_2$. □

Theorem 3.9 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two classical groups and $\Psi_{grp}:\mathcal{G}_1\to\mathcal{G}_2$ be a group homomorphism. If $\mathfrak{D}_2$ is an LDF-subgroup of $\mathcal{G}_2$, then the preimage $\Psi^{-1}_{grp}(\mathfrak{D}_2)$ of $\mathfrak{D}_2$ is also an LDF-subgroup of $\mathcal{G}_1$.*

Proof Let $\mathfrak{D}_2\in LDFsG(\mathcal{G}_2)$. To conclude the proof, we have to prove that $\Psi^{-1}_{grp}(\mathfrak{D}_2)(g_1^1\cdot(g_2^1)^{-1})\geq\Psi^{-1}_{grp}(\mathfrak{D}_2)(g_1^1)\wedge\Psi_{grp}(\mathfrak{D}_2)(g_2^1)$ for all $g_1^1,g_2^1\in\mathcal{G}_1$. Since $\mathfrak{D}_2\in LDFsG(\mathcal{G}_2)$ and Ψ_{grp} is a group homomorphism

$$\begin{aligned}
\mathfrak{U}_{\Psi^{-1}_{grp}(\mathfrak{D}_2)}\left(g_1^1\cdot(g_2^1)^{-1}\right)&=\mathfrak{U}_{\mathfrak{D}_2}\left(\Psi\left(g_1^1\cdot(g_2^1)^{-1}\right)\right)\\
&=\mathfrak{U}_{\mathfrak{D}_2}\left(\Psi(g_1^1)\cdot\Psi\left((g_2^1)^{-1}\right)\right)\\
&\geq\mathfrak{U}_{\mathfrak{D}_2}\left(\Psi(g_1^1)\right)\wedge\mathfrak{U}_{\mathfrak{D}_2}\left(\Psi(g_2^1)\right)\\
&=\mathfrak{U}_{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g_1^1)\wedge\mathfrak{U}_{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g_2^1),
\end{aligned}\tag{3.15}$$

$$\begin{aligned}\mathfrak{S}_{\Psi^{-1}_{grp}(\mathfrak{D}_2)}\left(g^1_1\cdot(g^1_2)^{-1}\right) &= \mathfrak{S}_{\mathfrak{D}_2}\left(\Psi\left(g^1_1\cdot(g^1_2)^{-1}\right)\right)\\ &= \mathfrak{S}_{\mathfrak{D}_2}\left(\Psi(g^1_1)\cdot\Psi\left((g^1_2)^{-1}\right)\right)\\ &\le \mathfrak{S}_{\mathfrak{D}_2}(\Psi(g^1_1))\vee\mathfrak{S}_{\mathfrak{D}_2}(\Psi(g^1_2))\\ &= \mathfrak{S}_{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g^1_1)\vee\mathfrak{S}_{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g^1_2),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\alpha^{\Psi^{-1}_{grp}(\mathfrak{D}_2)}\left(g^1_1\cdot(g^1_2)^{-1}\right) &= \mathfrak{U}_{\mathfrak{D}_2}\left(\Psi\left(g^1_1\cdot(g^1_2)^{-1}\right)\right)\\ &= \alpha^{\mathfrak{D}_2}\left(\Psi(g^1_1)\cdot\Psi\left((g^1_2)^{-1}\right)\right)\\ &\ge \alpha^{\mathfrak{D}_2}(\Psi(g^1_1))\wedge\alpha^{\mathfrak{D}_2}(\Psi(g^1_2))\\ &= \alpha^{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g^1_1)\wedge\alpha^{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g^1_2),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\beta^{\Psi^{-1}_{grp}(\mathfrak{D}_2)}\left(g^1_1\cdot(g^1_2)^{-1}\right) &= \beta^{\mathfrak{D}_2}\left(\Psi\left(g^1_1\cdot(g^1_2)^{-1}\right)\right)\\ &= \beta^{\mathfrak{D}_2}\left(\Psi(g^1_1)\cdot\Psi\left((g^1_2)^{-1}\right)\right)\\ &\le \beta^{\mathfrak{D}_2}(\Psi(g^1_1))\vee\beta^{\mathfrak{D}_2}(\Psi(g^1_2))\\ &= \beta^{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g^1_1)\vee\beta^{\Psi^{-1}_{grp}(\mathfrak{D}_2)}(g^1_2),\end{aligned} \tag{3.18}$$

Thus, we have $\Psi^{-1}_{grp}(\mathfrak{D}_2)(g^1_1\cdot(g^1_2)^{-1})\ge\Psi^{-1}_{grp}(\mathfrak{D}_2)(g^1_1)$ $\wedge\Psi^{-1}_{grp}(\mathfrak{D}_1)(g^1_2)$ for all $g^1_1,g^1_2\in\mathcal{G}_1$. Hence, $\Psi^{-1}_{grp}(\mathfrak{D}_2)$ is an LDF-subgroup of $\mathcal{G}_1$. □

Example 3.10 Consider the groups $(\mathbb{Z}_2,+)$ and $(\mathbb{Z}_4,+)$. Also, we take the function $\Psi_{grp}:\mathbb{Z}_4\to\mathbb{Z}_2$, $\Psi_{grp}(\bar{\bar{r}})=\bar{r}$. Then, it is obvious that Ψ_{grp} is a group homomorphism. If it is taken

$$\mathfrak{D}_1=\left\{\begin{array}{l}(\bar{\bar{0}},\langle0.7,0.5\rangle,\langle0.3,0.3\rangle),(\bar{\bar{1}},\langle0.4,0.7\rangle,\langle0.2,0.8\rangle),\\(\bar{\bar{2}},\langle0.4,0.7\rangle,\langle0.2,0.8\rangle),(\bar{\bar{3}},\langle0.4,0.7\rangle,\langle0.2,0.8\rangle)\end{array}\right\}$$

then $\mathfrak{D}_1\in LDFsG(\mathbb{Z}_4)$. By Definition 2.9, we have $\Psi_{grp}(\mathfrak{U}_{\mathfrak{D}_1})(\bar{0})=\mathfrak{U}_{\mathfrak{D}_1}(\bar{\bar{0}})\vee\mathfrak{U}_{\mathfrak{D}_1}(\bar{\bar{2}})=0.7$, $\Psi_{grp}(\mathfrak{U}_{\mathfrak{D}_1})(\bar{1})=\mathfrak{U}_{\mathfrak{D}_1}(\bar{\bar{1}})\vee\mathfrak{U}_{\mathfrak{D}_1}(\bar{\bar{3}})=0.4$ and others can be obtained similarly. Then, by Theorem 3.8, we can say that

$$\mathfrak{D}_2=\{(\bar{0},\langle0.7,0.5\rangle,\langle0.3,0.3\rangle),(\bar{1},\langle0.4,0.7\rangle,\langle0.2,0.8\rangle)\}$$

is an LDF-subgroup of $\mathbb{Z}_2$. Indeed, we obtain that $\mathfrak{D}_2\in LDFsG(\mathbb{Z}_2)$ from Definition 3.1.

4 Linear Diophantine fuzzy normal subgroup

This section focuses on the description of linear Diophantine fuzzy normal subgroup.

Definition 4.1 Let $\mathcal{G}$ be a classical group group and $\mathfrak{D}$ be an LDF-subgroup of $\mathcal{G}$. Then, $\mathfrak{D}$ is called a *linear Diophantine fuzzy normal subgroup* (LDF-Nsubgroup) of $\mathcal{G}$ if

(N1) $\mathfrak{D}(g_j\cdot g_k\cdot g_j{}^{-1})=\mathfrak{D}(g_k)\ \ \forall g_j,g_k\in\mathcal{G}$.

Note that the collection of all LDF-Nsubgroups of $\mathcal{G}$ is denoted by $LDFNsG(\mathcal{G})$.

Example 4.2 Consider the symmetric group $S_3=\{\sigma_0=e,\sigma_1=(123),\sigma_2=(132),\sigma_3=(23),\sigma_4=(13),\sigma_5=(12)\}$. We can create the following Cayley table for S_3.

$\circ$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	σ_1	σ_2	σ_0	σ_4	σ_5	σ_3
σ_2	σ_2	σ_0	σ_1	σ_5	σ_3	σ_4
σ_3	σ_3	σ_5	σ_4	σ_0	σ_2	σ_1
σ_4	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2
σ_5	σ_5	σ_4	σ_3	σ_2	σ_1	σ_0

Then, the LDFS

$$\mathfrak{D}=\left\{\begin{array}{l}(\sigma_0,\langle0.8,0.6\rangle,\langle0.4,0.6\rangle),(\sigma_1,\langle0.6,0.7\rangle,\langle0.2,0.7\rangle),\\(\sigma_2,\langle0.6,0.7\rangle,\langle0.2,0.7\rangle),(\sigma_3,\langle0.4,0.9\rangle,\langle0.1,0.7\rangle),\\(\sigma_4,\langle0.4,0.9\rangle,\langle0.1,0.7\rangle),(\sigma_5,\langle0.4,0.9\rangle,\langle0.1,0.7\rangle)\end{array}\right\}.$$

is an LDF-subgroup of S_3. It is clear from (N1) of Definition 4.1 that $\mathfrak{D}$ is an LDF-Nsubgroup of S_3.

Proposition 4.3 *Let $\mathfrak{D}$ be an LDF-Nsubgroup of $\mathcal{G}$. Then, the following are equivalent*:

(i) $\mathfrak{D}(g_j\cdot g_k\cdot g_j{}^{-1})=\mathfrak{D}(g_k)\ \ \forall g_j,g_k\in\mathcal{G}$.
(ii) $\mathfrak{D}(g_j\cdot g_k)=\mathfrak{D}(g_k\cdot g_j)\ \ \forall g_j,g_k\in\mathcal{G}$.

Proof

(i)⇒ (ii): Assume that $\mathfrak{D}(g_j \cdot g_k \cdot g_j^{-1}) = \mathfrak{D}(g_k)$ for all $g_j, g_k \in \mathcal{G}$. Taking the advantage of arbitrary property g_k, the equality in (ii) is shown easily.
(ii)⇒(i): Let $\mathfrak{D}(g_j \cdot g_k) = \mathfrak{D}(g_j \cdot g_k)$ for all $g_j, g_k \in \mathcal{G}$. Then, we obtain $\mathfrak{D}(g_j \cdot g_k \cdot g_j^{-1}) = \mathfrak{D}(g_k \cdot g_j \cdot g_j^{-1}) = \mathfrak{D}(g_k)$ for all $g_j, g_k \in \mathcal{G}$. This is sufficient for proof.

□

Lemma 4.4 *Let $\mathcal{G}$ be a classical group group and $\mathfrak{D}$ be an LDF-subgroup of $\mathcal{G}$. If the group $\mathcal{G}$ is a commutative then $\mathfrak{D}$ is an LDF-Nsubgroup of $\mathcal{G}$.*

Proof The proof is straightforward, hence omitted. □

Example 4.5 We consider the classical group $\mathcal{G} = \{1, -1, i, -i\}$ with the following natural multiplication:

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Since the group $\mathcal{G}$ is a commutative, each LDF-subgroup of $\mathcal{G}$ is also LDF-Nsubgroup.

Theorem 4.6 *Let $\mathcal{G}$ be a classical group and $\mathfrak{D}_i$ $(i \in I)$ be the LDF-Nsubgroups of $\mathcal{G}$. Then,*

(i) $\bigcap_{i\in I} \mathfrak{D}_i$ is an LDF-Nsubgroup of $\mathcal{G}$.
(ii) $\bigcup_{i\in I} \mathfrak{D}_i$ is an LDF-Nsubgroup of $\mathcal{G}$.

Proof Assume that $\mathcal{G}$ is a classical group and $\mathfrak{D}_i$ $(i \in I)$ are the LDF-Nsubgroups of $\mathcal{G}$.

(i) We have to prove that $\bigcap_{i\in I} \mathfrak{D}_i(g_j \cdot g_k^i \cdot g_j^{-1}) = \bigcap_{i\in I} \mathfrak{D}_i(g_k)$ for all $g_j, g_k \in \mathcal{G}$. Since $\mathfrak{D}_i \in LDFNsG(\mathcal{G})$ for each $i \in I$, we obtain by the definition of intersection, $\mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}(g_j \cdot g_k \cdot g_j^{-1}) = \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(g_j \cdot g_k \cdot g_j^{-1}) = \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(g_k) = \mathfrak{U}_{\tilde{\cap}\mathfrak{D}_i}(g_k)$. On the other hand, the following equalities are obtained in a similar way. $\mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}(g_j \cdot g_k \cdot g_j^{-1}) = \mathfrak{S}_{\tilde{\cap}\mathfrak{D}_i}(g_k)$, $\alpha^{\tilde{\cap}\mathfrak{D}_i}(g_j \cdot g_k \cdot g_j^{-1}) = \alpha^{\tilde{\cap}\mathfrak{D}_i}(g_k)$, $\beta^{\tilde{\cap}\mathfrak{D}_i}(g_j \cdot g_k \cdot g_j^{-1}) = \beta^{\tilde{\cap}\mathfrak{D}_i}(g_k)$. Hence, we say that $\bigcap_{i\in I} \mathfrak{D}_i$ is an LDF-Nsubgroup of $\mathcal{G}$.
(ii) It can be proved by using the similar techniques.

□

Theorem 4.7 *Let $\mathcal{G}_i$ be the classical groups and $\mathfrak{D}_i$ be the LDF-Nsubgroups of $\mathcal{G}_i$ for $i \in I$. Then, $\prod_{i\in I} \mathfrak{D}_i$ is an LDF-Nsubgroup of the group $\prod_{i\in I} \mathcal{G}_i$.*

Proof By considering the normality condition (N1) in Definition 4.1, it can be proved similar to that of Theorem 3.6. □

Theorem 4.8 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two classical groups and $\Psi_{grp} : \mathcal{G}_1 \to \mathcal{G}_2$ be a surjective homomorphism of groups. If $\mathfrak{D}_1$ is an LDF-Nsubgroup of $\mathcal{G}_1$, then the image $\Psi_{grp}(\mathfrak{D}_1)$ of $\mathfrak{D}_1$ is also an LDF-Nsubgroup of $\mathcal{G}_2$.*

Proof From Theorem 4.8, $\Psi_{grp}(\mathfrak{D}_1) \in LDFsG(\mathcal{G}_2)$ if $\mathfrak{D}_1 \in LDFsG(\mathcal{G}_1)$. Hence, it is sufficient to demonstrate the normality property of $\Psi_{grp}(\mathfrak{D}_1)$ (i.e., $\Psi_{grp}(\mathfrak{D}_1)(g_1^2 \cdot g_2^2) = \Psi_{grp}(\mathfrak{D}_1)(g_2^2 \cdot g_1^2)$). Assume that there exist $g_1^2, g_2^2 \in \mathcal{G}_2$ such that $\Psi_{grp}^{-1}(g_1^2) = g_1^1$, $\Psi_{grp}^{-1}(g_2^2) = g_2^1$ and so $\Psi_{grp}^{-1}(g_1^2 \cdot g_2^2) = g_1^1 \cdot g_2^1$. Since $\mathfrak{D}_1 \in LDFNsG(\mathcal{G}_1)$ and Ψ_{grp} is a group homomorphism (i.e., $\Psi_{grp}(g_1^1 \cdot (g_2^1)^{-1}) = \Psi_{grp}(g_1^1) \cdot \Psi_{grp}((g_2^1)^{-1}) = g_1^2 \cdot (g_2^2)^{-1}$)

$$\begin{aligned}\Psi_{grp}(\mathfrak{U}_{\mathfrak{D}_1})(g_1^2 \cdot g_2^2) &= \bigvee_{g_1^1 \cdot g_2^1 \in \Psi_{grp}^{-1}(g_1^2 \cdot g_2^2)} \mathfrak{U}_{\mathfrak{D}_1}(g_1^1 \cdot g_2^1) \\ &= \bigvee_{g_2^1 \cdot g_1^1 \in \Psi_{grp}^{-1}(g_2^2 \cdot g_1^2)} \mathfrak{U}_{\mathfrak{D}_1}(g_2^1 \cdot g_1^1) = \mathfrak{U}_{\mathfrak{D}_1}(g_2^1 \cdot g_1^1),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Psi_{grp}(\mathfrak{S}_{\mathfrak{D}_1})(g_1^2 \cdot g_2^2) &= \bigwedge_{g_1^1 \cdot g_2^1 \in \Psi_{grp}^{-1}(g_1^2 \cdot g_2^2)} \mathfrak{S}_{\mathfrak{D}_1}(g_1^1 \cdot g_2^1) \\ &= \bigwedge_{g_2^1 \cdot g_1^1 \in \Psi_{grp}^{-1}(g_2^2 \cdot g_1^2)} \mathfrak{S}_{\mathfrak{D}_1}(g_2^1 \cdot g_1^1) = \mathfrak{S}_{\mathfrak{D}_1}(g_2^1 \cdot g_1^1),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\Psi_{grp}(\alpha^{\mathfrak{D}_1})(g_1^2 \cdot g_2^2) &= \bigvee_{g_1^1 \cdot g_2^1 \in \Psi_{grp}^{-1}(g_1^2 \cdot g_2^2)} \alpha^{\mathfrak{D}_1}(g_1^1 \cdot g_2^1) \\ &= \bigvee_{g_2^1 \cdot g_1^1 \in \Psi_{grp}^{-1}(g_2^2 \cdot g_1^2)} \alpha^{\mathfrak{D}_1}(g_2^1 \cdot g_1^1) = \alpha^{\mathfrak{D}_1}(g_2^1 \cdot g_1^1),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Psi_{grp}(\beta^{\mathfrak{D}_1})(g_1^2 \cdot g_2^2) &= \bigwedge_{g_1^1 \cdot g_2^1 \in \Psi_{grp}^{-1}(g_1^2 \cdot g_2^2)} \beta^{\mathfrak{D}_1}(g_1^1 \cdot g_2^1) \\ &= \bigwedge_{g_2^1 \cdot g_1^1 \in \Psi_{grp}^{-1}(g_2^2 \cdot g_1^2)} \beta^{\mathfrak{D}_1}(g_2^1 \cdot g_1^1) = \beta^{\mathfrak{D}_1}(g_2^1 \cdot g_1^1).\end{aligned} \tag{4.4}$$

Thus, the desired equality $\Psi_{grp}(\mathfrak{D}_1)(g_1^2 \cdot g_2^2) = \Psi_{grp}(\mathfrak{D}_1)(g_2^2 \cdot g_1^2)$ is satisfied, and so $\Psi_{grp}(\mathfrak{D}_1) \in LDFNsG(\mathcal{G}_2)$. □

Theorem 4.9 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two classical groups and $\Psi_{grp} : \mathcal{G}_1 \to \mathcal{G}_2$ be a group homomorphism. If $\mathfrak{D}_2$ is an LDF-Nsubgroup of $\mathcal{G}_2$, then the preimage $\Psi_{grp}^{-1}(\mathfrak{D}_2)$ of $\mathfrak{D}_2$ is also an LDF-Nsubgroup of $\mathcal{G}_1$.*

Proof From Theorem 3.9, $\Psi_{grp}^{-1}(\mathfrak{D}_2) \in LDFsG(\mathcal{G}_1)$ if $\mathfrak{D}_2 \in LDFsG(\mathcal{G}_2)$. To complete the proof, it is enough to show that $\Psi_{grp}^{-1}(\mathfrak{D}_2)(g_1^1 \cdot g_2^1) = \Psi_{grp}^{-1}(\mathfrak{D}_2)(g_2^1 \cdot g_1^1)$. By the normality of $\mathfrak{D}_2$ and the homomorphism of Ψ_{grp}, we obtain

$$\begin{aligned}\Psi_{grp}^{-1}(\mathfrak{D}_2)(g_1^1 \cdot g_2^1) =& \mathfrak{D}_2(\Psi(g_1^1 \cdot g_2^1)) \\ =& \mathfrak{D}_2(\Psi(g_1^1) \cdot \Psi(g_1^1)) \\ =& \mathfrak{D}_2(\Psi(g_2^1) \cdot \Psi(g_1^1)) \\ =& \mathfrak{D}_2(\Psi(g_2^1 \cdot g_1^1)) \\ =& \Psi_{grp}^{-1}(\mathfrak{D}_2)(g_2^1 \cdot g_1^1)\end{aligned} \tag{4.5}$$

for each $g_1^1, g_2^1 \in \mathcal{G}_1$. So, we have $\Psi_{grp}^{-1}(\mathfrak{D}_2) \in LDFNsG(\mathcal{G}_1)$. □

Example 4.10 We consider the LDFSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ on the groups $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_2, +)$, and the group homomorphism $\Psi_{grp} : \mathbb{Z}_4 \to \mathbb{Z}_2$ given in Example 3.10. We know that $\mathfrak{D}_1 \in LDFsG(\mathbb{Z}_4)$ and $\mathfrak{D}_2 \in LDFsG(\mathbb{Z}_2)$. Since $\mathbb{Z}_4$ is a commutative (abelian) group, $\mathfrak{D}_1 \in LDFNsG(\mathbb{Z}_4)$ from Lemma 4.4. By Theorem 4.8, we can say that $\mathfrak{D}_2$ is an LDF-Nsubgroup of $\mathbb{Z}_2$. Indeed, this is obvious since $\mathfrak{D}_2 \in LDFNsG(\mathbb{Z}_2)$ and $\mathbb{Z}_2$ is a commutative (abelian) group.

5 Linear Diophantine fuzzy subring

In this section, the notion of linear Diophantine fuzzy subring is introduced and the related propositions are derived.

Definition 5.1 Let $(\mathcal{R}, +, \cdot)$ be a classical ring and $\mathfrak{D}$ be an LDFS on $\mathcal{R}$. Then, $\mathfrak{D}$ is said to be a *linear Diophantine fuzzy subring* (LDF-subring) of $\mathcal{R}$ if and only if the following properties are satisfied:

(R1) $\mathfrak{D}(r_j + r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.
(R2) $\mathfrak{D}(-r_j) = \mathfrak{D}(r_j) \forall r_j \in \mathcal{R}$.
(R3) $\mathfrak{D}(r_j \cdot r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.

Note that the collection of all LDF-subrings of $\mathcal{R}$ is denoted by $LDFsR(\mathcal{R})$.

Example 5.2 Let $(\mathcal{R}, +, \cdot)$ be a classical ring and $\mathcal{C} = \{c \in \mathcal{R} : rc = cr \text{ for all } r \in \mathcal{R}\}$. The set $\mathcal{C}$ denotes center of $\mathcal{R}$. Define an LDFS $\mathfrak{D}$ on $\mathcal{R}$ as follows:

$$\mathfrak{D}(r_j) = \begin{cases} (\langle 0.9, 0.5\rangle, \langle 0.7, 0.2\rangle), & \text{if } r_j \in \mathcal{C} \\ (\langle 0.4, 0.8\rangle, \langle 0.5, 0.4\rangle), & \text{otherwise} \end{cases}$$

This LDFS $\mathfrak{D}$ is an LDF-subring of $\mathcal{R}$.

Proposition 5.3 *Let $(\mathcal{R}, +, \cdot)$ be a classical ring and $\mathfrak{D}$ be an LDF-subring of $\mathcal{R}$. Then, $\mathfrak{D}(0_\mathcal{R}) \geq \mathfrak{D}(r_j)$ for all $r_j \in \mathcal{R}$, where $0_\mathcal{R}$ is the unit element of $\mathcal{R}$ related to the binary operation $+$.*

Proof Let $(\mathcal{R}, +, \cdot)$ be a classical ring. Obviously, $(\mathcal{R}, +)$ is a group. Also, $\mathfrak{D}$ is an LDF-subgroup of $(\mathcal{R}, +)$ since $\mathfrak{D}$ is an LDF-subring of $\mathcal{R}$. Hence, the proof is clear from Proposition 3.3. □

Theorem 5.4 *Let $\mathcal{R}$ be a classical ring and $\mathfrak{D}$ be an LDFS on $\mathcal{R}$. Then, $\mathfrak{D}$ is an LDF-subring of $\mathcal{R}$ iff the following axioms are provided.*

(i) $\mathfrak{D}(r_j - r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.
(ii) $\mathfrak{D}(r_j \cdot r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.

Proof If $\mathfrak{D}$ is an LDF-subring of $\mathcal{R}$, it has properties (R1), (R2) and (R3) in Definition 5.1. Then, we obtain $\mathfrak{D}(r_j - r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(-r_k) = \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k)$ from (R1) and (R2), and also $\mathfrak{D}(r_j \cdot r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k)$ from (R3).

Conversely, assume that the axioms (i) and (ii) are satisfied. Especially, if we take $r_j = 0_\mathcal{R}$ for the axiom (i), then we obtain $\mathfrak{D}(-r_k) \geq \mathfrak{D}(r_k)$, and so $\mathfrak{D}(-(-r_k)) \geq \mathfrak{D}(-r_k)$. Consequently, we have $\mathfrak{D}(-r_k) = \mathfrak{D}(r_k)$. Thus, we say that the axiom (i) corresponds to the properties (R1) and (R2) in Definition 5.1. Evidently, the axiom (ii) corresponds to the property (R3) in Definition 5.1. Hence, the proof is completed. □

Theorem 5.5 *Let $\mathcal{R}$ be a classical ring and $\mathfrak{D}_i$ $(i \in I)$ be the LDFN-subrings of $\mathcal{R}$. Then,*

(i) $\bigcap_{i\in I} \mathfrak{D}_i$ is an LDFN-subring of $\mathcal{R}$.
(ii) $\bigcup_{i\in I} \mathfrak{D}_i$ is an LDFN-subring of $\mathcal{R}$.

Proof Let $\mathfrak{D}_i$ $(i \in I)$ be the LDF-subrings of the classical ring $\mathcal{R}$.

(i) We know that $(\mathcal{R}, +)$ is a group since $(\mathcal{R}, +, \cdot)$ is a ring. Then, $\mathfrak{D}_i$ $(i \in I)$ are the LDF-subgroups of the group $(\mathcal{R}, +)$. From Theorem 3.5, we have $\bigcap_{i\in I} \mathfrak{D}_i(r_j - r_k) \geq \bigcap_{i\in I} \mathfrak{D}_i(r_j) \wedge \bigcap_{i\in I} \mathfrak{D}_i(r_k)$ for all $r_j, r_k \in \mathcal{R}$. To terminate the proof, we must show that $\bigcap_{i\in I} \mathfrak{D}_i(r_j \cdot r_k) \geq \bigcap_{i\in I} \mathfrak{D}_i(r_j) \wedge \bigcap_{i\in I} \mathfrak{D}_i(r_k)$ for all $r_j, r_k \in \mathcal{R}$.

Since $\mathfrak{D}_i$ for each $i \in I$ is the LDF-subring of $\mathcal{R}$, we obtain

$$\mathfrak{U}_{\widetilde{\cap}\mathfrak{D}_i}(r_j \cdot r_k) = \bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(r_j \cdot r_k) \geq \left(\bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(r_j)\right) \wedge \left(\bigwedge_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(r_k)\right) = \mathfrak{U}_{\widetilde{\cap}\mathfrak{D}_i}(r_j) \wedge \mathfrak{U}_{\widetilde{\cap}\mathfrak{D}_i}(r_k), \tag{5.1}$$

$$\mathfrak{S}_{\widetilde{\cap}\mathfrak{D}_i}(r_j \cdot r_k) = \bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(r_j \cdot r_k) \leq \left(\bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(r_j)\right) \vee \left(\bigvee_{i\in I} \mathfrak{S}_{\mathfrak{D}_i}(r_k)\right) = \mathfrak{S}_{\widetilde{\cap}\mathfrak{D}_i}(r_j) \vee \mathfrak{S}_{\widetilde{\cap}\mathfrak{D}_i}(r_k), \tag{5.2}$$

$$\alpha^{\widetilde{\cap}\mathfrak{D}_i}(r_j \cdot r_k) = \bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(r_j \cdot r_k) \geq \left(\bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(r_j)\right) \wedge \left(\bigwedge_{i\in I} \alpha^{\mathfrak{D}_i}(r_k)\right) = \alpha^{\widetilde{\cap}\mathfrak{D}_i}(r_j) \wedge \alpha^{\widetilde{\cap}\mathfrak{D}_i}(r_k), \tag{5.3}$$

$$\beta^{\widetilde{\cap}\mathfrak{D}_i}(r_j \cdot r_k) = \bigvee_{i\in I} \beta^{\mathfrak{D}_i}(r_j \cdot r_k) \leq \left(\bigvee_{i\in I} \beta^{\mathfrak{D}_i}(r_j)\right) \vee \left(\bigvee_{i\in I} \beta^{\mathfrak{D}_i}(r_k)\right) = \beta^{\widetilde{\cap}\mathfrak{D}_i}(r_j) \vee \beta^{\widetilde{\cap}\mathfrak{D}_i}(r_k). \tag{5.4}$$

Considering Definition 2.4 (2) and (5), by Eqs. (5.1)–(5.4), we have $\bigcap_{i\in I} \mathfrak{D}_i(r_j \cdot r_k) \geq \bigcap_{i\in I} \mathfrak{D}_i(r_j) \wedge \bigcap_{i\in I} \mathfrak{D}_i(r_k)$ for all $r_j, r_k \in \mathcal{R}$. This completes the proof of (i).

(ii) It can be proved similar to the proof of (i).

□

Theorem 5.6 *Let $\mathcal{R}_i$ be the classical rings and $\mathfrak{D}_i$ be the LDFN-subrings of $\mathcal{R}_i$ for $i \in I$. Then, $\prod_{i\in I} \mathfrak{D}_i$ is an LDFN-subring of the ring $\prod_{i\in I} \mathcal{R}_i$.*

Proof By considering the properties in Definition 5.1, the proof can be verified similar to that of Theorem 6.6. □

Theorem 5.7 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two classical rings and $\Psi_{rng} : \mathcal{R}_1 \to \mathcal{R}_2$ be a ring homomorphism. If $\mathfrak{D}_1$ is an LDF-subring of $\mathcal{R}_1$, then the image $\Psi_{rng}(\mathfrak{D}_1)$ of $\mathfrak{D}_1$ is also an LDF-subring of $\mathcal{R}_2$.*

Proof By the computations similar to the proof of Theorem 3.8, it can be demonstrated by considering Definition 5.1. □

Theorem 5.8 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two classical rings and $\Psi_{rng} : \mathcal{R}_1 \to \mathcal{R}_2$ be a ring homomorphism. If $\mathfrak{D}_2$ is an LDF-subring of $\mathcal{R}_2$, then the preimage $\Psi^{-1}_{rng}(\mathfrak{D}_2)$ of $\mathfrak{D}_2$ is also an LDF-subring of $\mathcal{R}_1$.*

Proof By the computations similar to the proof of Theorem 3.9, it can be demonstrated by considering Definition 5.1. □

6 Linear Diophantine fuzzy ideal

In this section, the linear Diophantine fuzzy ideal is described and it is supported by some illustrative examples.

Definition 6.1 Let $(\mathcal{R}, +, \cdot)$ be a classical ring and $\mathfrak{D}$ be an LDFS on $\mathcal{R}$. Then,

(1) $\mathfrak{D}$ is called a *linear Diophantine fuzzy left ideal* (LDF-leftideal) of $\mathcal{R}$ if and only if the following properties are satisfied:

(LI1) $\mathfrak{D}(r_j - r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.
(LI2) $\mathfrak{D}(r_j \cdot r_k) \geq \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.

(2) $\mathfrak{D}$ is called a *linear Diophantine fuzzy right ideal* (LDF-rightideal) of $\mathcal{R}$ if and only if the following properties are satisfied:

(RI1) $\mathfrak{D}(r_j - r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.
(RI2) $\mathfrak{D}(r_j \cdot r_k) \geq \mathfrak{D}(r_j) \forall r_j, r_k \in \mathcal{R}$.

Definition 6.2 Let $(\mathcal{R}, +, \cdot)$ be a classical ring and $\mathfrak{D}$ be an LDFS on $\mathcal{R}$. Then, $\mathfrak{D}$ is called a *linear Diophantine fuzzy ideal* (LDF-ideal) of $\mathcal{R}$ if and only if the following properties are satisfied:

(I1) $\mathfrak{D}(r_j - r_k) \geq \mathfrak{D}(r_j) \wedge \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.
(I2) $\mathfrak{D}(r_j \cdot r_k) \geq \mathfrak{D}(r_j) \vee \mathfrak{D}(r_k) \forall r_j, r_k \in \mathcal{R}$.

Note that the collection of all LDF-ideals of $\mathcal{R}$ is denoted by *LDFI*$(\mathcal{R})$.

Remark Each LDF-ideal of the classical ring $\mathcal{R}$ is an LDF-subring of $\mathcal{R}$, but this argument is conversely not true in general. For example, the LDFS $\mathfrak{D}$ on $\mathcal{R}$ given in Example 5.2 is an LDF-subring of $\mathcal{R}$ but may not be an LDF-ideal of $\mathcal{R}$.

Proposition 6.3 *Let $(\mathcal{R}, +, \cdot)$ be a classical ring and $\mathfrak{D}$ be an LDF-ideal of $\mathcal{R}$.*

(i) $\mathfrak{D}(0_{\mathcal{R}}) \geq \mathfrak{D}(r_j)$ for all $r_j \in \mathcal{R}$, where $0_{\mathcal{R}}$ is the unit element of $\mathcal{R}$ related to the binary operation $+$.
(ii) If $(\mathcal{R}, +, \cdot)$ is a classical ring with identity, then $\mathfrak{D}(r_j) \geq \mathfrak{D}(1_{\mathcal{R}})$ for all $r_j \in \mathcal{R}$, where $1_{\mathcal{R}}$ is the identity of $\mathcal{R}$.

Proof (i) It is similar to the proof of Proposition 5.3, therefore omitted.
(ii) By considering Definition 6.1 (2), we obtain $\mathfrak{D}(r_j) = \mathfrak{D}(1_{\mathcal{R}} \cdot r_j) \geq \mathfrak{D}(1_{\mathcal{R}})$ for all $r_j \in \mathcal{R}$, So the proof is completed. □

Example 6.4 We consider $\mathcal{R} = \{0, p, q, r\}$ with the following Cayley tables:

$+$	0	p	q	r
0	0	p	q	r
p	p	0	r	q
q	q	r	0	p
r	r	q	p	0

$\cdot$	0	p	q	r
0	0	0	0	0
p	0	p	0	p
q	0	0	q	q
r	0	p	q	r

It can be easily verified that $\mathcal{R}$ is a ring with identity. Also, we consider the following LDFS $\mathfrak{D}$ on the ring $\mathcal{R}$.

$$\mathfrak{D} = \left\{ \begin{array}{l} (0, \langle 0.6, 0.7\rangle, \langle 0.4, 0.5\rangle), (p, \langle 0.3, 0.9\rangle, \langle 0.1, 0.7\rangle), \\ (q, \langle 0.3, 0.9\rangle, \langle 0.1, 0.7\rangle), (r, \langle 0.3, 0.9\rangle, \langle 0.1, 0.7\rangle) \end{array} \right\}$$

Then, we say that $\mathfrak{D}$ is an LDF-ideal of $\mathcal{R}$ and it provides the axioms in Proposition 6.3.

Theorem 6.5 *Let $\mathcal{R}$ be a classical ring and $\mathfrak{D}_i$ $(i \in I)$ be the LDFN-ideals of $\mathcal{R}$. Then,*

(i) $\bigcap_{i \in I} \mathfrak{D}_i$ is an LDFN-ideal of $\mathcal{R}$.
(ii) $\bigcup_{i \in I} \mathfrak{D}_i$ is an LDFN-ideal of $\mathcal{R}$.

Proof By considering the conditions in Definition 6.2, it can be demonstrated similar to proof of Theorem 5.5. □

Theorem 6.6 *Let $\mathcal{R}_i$ be the classical rings and $\mathfrak{D}_i$ be the LDFN-ideals of $\mathcal{R}_i$ for $i \in I$. Then, $\prod_{i \in I} \mathfrak{D}_i$ is an LDFN-ideal of the ring $\prod_{i \in I} \mathcal{R}_i$.*

Proof Assume that $\mathfrak{D}_i$ are the LDF-ideals of $\mathcal{R}_i$ for $i \in I$. To complete the proof, we have to prove that for all $(g^i_j)_{i \in I}, (g^i_k)_{i \in I} \in \prod_{i \in I} \mathcal{G}_i$,

$$\prod_{i \in I} \mathfrak{D}_i\Big((r^i_j)_{i \in I} - (r^i_k)_{i \in I}\Big) \geq \prod_{i \in I} \mathfrak{D}_i((r^i_j)_{i \in I}) \wedge \prod_{i \in I} \mathfrak{D}_i\big((r^i_k)_{i \in I}\big),$$

$$\prod_{i \in I} \mathfrak{D}_i\Big((r^i_j)_{i \in I} \cdot (r^i_k)_{i \in I}\Big) \geq \prod_{i \in I} \mathfrak{D}_i((r^i_j)_{i \in I}) \vee \prod_{i \in I} \mathfrak{D}_i((r^i_k)_{i \in I}).$$

By Definition 2.6, we have

$$\prod_{i \in I} \mathfrak{D}_i\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big) = \Big(\Big\langle \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big),\ \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big)\Big\rangle, \Big\langle \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big),\ \beta^{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big)\Big\rangle\Big) \tag{6.1}$$

$$\prod_{i \in I} \mathfrak{D}_i\Big(\Big(r^i_j\Big)_{i \in I} \cdot \big(r^i_k\big)_{i \in I}\Big) = \Big(\Big\langle \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} \cdot \big(r^i_k\big)_{i \in I}\Big),\ \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} \cdot \big(r^i_k\big)_{i \in I}\Big)\Big\rangle, \Big\langle \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} \cdot \big(r^i_k\big)_{i \in I}\Big),\ \beta^{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} \cdot \big(r^i_k\big)_{i \in I}\Big)\Big\rangle\Big) \tag{6.2}$$

for all $(r^i_j)_{i \in I}, (r^i_k)_{i \in I} \in \prod_{i \in I} \mathcal{R}_i$. Since $\mathfrak{D}_i$ is the LDF-ideal of $\mathcal{R}_i$ for each $i \in I$, we obtain

$$\begin{aligned} \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big) &= \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j - r^i_k\Big)_{i \in I}\Big) \\ &= \bigwedge_{i \in I} \mathfrak{U}_{\mathfrak{D}_i}\Big(r^i_j - r^i_k\Big) \geq \bigwedge_{i \in I} \Big(\mathfrak{U}_{\mathfrak{D}_i}\Big(r^i_j\Big) \wedge \mathfrak{U}_{\mathfrak{D}_i}\big(r^i_k\big)\Big) \\ &= \Big(\bigwedge_{i \in I} \mathfrak{U}_{\mathfrak{D}_i}\Big(r^i_j\Big)\Big) \wedge \Big(\bigwedge_{i \in I} \mathfrak{U}_{\mathfrak{D}_i}\big(r^i_k\big)\Big) \\ &= \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I}\Big) \wedge \mathfrak{U}_{\widetilde{\Pi}\mathfrak{D}_i}\big((r^i_k)_{i \in I}\big), \end{aligned} \tag{6.3}$$

According to the similar discussion, the following inequalities are also true.

$$\begin{aligned} &\mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big) \\ &\leq \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I}\Big) \vee \mathfrak{S}_{\widetilde{\Pi}\mathfrak{D}_i}\big((r^i_k)_{i \in I}\big), \end{aligned} \tag{6.4}$$

$$\begin{aligned} &\alpha^{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I} - \big(r^i_k\big)_{i \in I}\Big) \\ &\geq \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\Big(\Big(r^i_j\Big)_{i \in I}\Big) \wedge \alpha^{\widetilde{\Pi}\mathfrak{D}_i}\big((r^i_k)_{i \in I}\big), \end{aligned} \tag{6.5}$$

$$\beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(r^i_j\right)_{i\in I}-\left(r^i_k\right)_{i\in I}\right) \le \beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(r^i_j\right)_{i\in I}\right)\vee\beta^{\widetilde{\Pi}\mathfrak{D}_i}\left(\left(r^i_k\right)_{i\in I}\right). \tag{6.6}$$

Thus, we obtain that $\prod_{i\in I}\mathfrak{D}_i((r^i_j)_{i\in I}-(r^i_k)_{i\in I})\ge\prod_{i\in I}\mathfrak{D}_i((r^i_j)_{i\in I})\wedge\prod_{i\in I}\mathfrak{D}_i((r^i_k)_{i\in I})$ for all $(g^i_j)_{i\in I},(g^i_k)_{i\in I}\in\prod_{i\in I}\mathcal{G}_i$. By using similar techniques, it is easily shown that $\prod_{i\in I}\mathfrak{D}_i((r^i_j)_{i\in I}\cdot(r^i_k)_{i\in I})\ge\prod_{i\in I}\mathfrak{D}_i((r^i_j)_{i\in I})\vee\prod_{i\in I}\mathfrak{D}_i((r^i_k)_{i\in I})$ for all $(g^i_j)_{i\in I},(g^i_k)_{i\in I}\in\prod_{i\in I}\mathcal{G}_i$. Hence $\prod_{i\in I}\mathfrak{D}_i$ is an LDF-ideal of the ring $\prod_{i\in I}\mathcal{G}_i$. □

Theorem 6.7 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two classical rings and $\Psi_{rng}:\mathcal{R}_1\to\mathcal{R}_2$ be a ring homomorphism. If $\mathfrak{D}_1$ is an LDF-ideal of $\mathcal{R}_1$, then the image $\Psi_{rng}(\mathfrak{D}_1)$ of $\mathfrak{D}_1$ is also an LDF-ideal of $\mathcal{R}_2$.*

Proof With the discussions similar to the proof of Theorem 3.8, it can be verified by considering Definition 6.2. □

Theorem 6.8 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two classical rings and $\Psi_{rng}:\mathcal{R}_1\to\mathcal{R}_2$ be a ring homomorphism. If $\mathfrak{D}_2$ is an LDF-ideal of $\mathcal{R}_2$, then the preimage $\Psi^{-1}_{rng}(\mathfrak{D}_2)$ of $\mathfrak{D}_2$ is also an LDF-ideal of $\mathcal{R}_1$.*

Proof With the discussions similar to the proof of Theorem 3.9, it can be verified by considering Definition 6.2. □

7 Linear Diophantine fuzzy subfield

This section is devoted to the linear Diophantine fuzzy subfield and its theoretical results.

Definition 7.1 Let $(\mathcal{F},+,\cdot)$ be a classical field and $\mathfrak{D}$ be an LDFS on $\mathcal{F}$. Then, $\mathfrak{D}$ is said to be a *linear Diophantine fuzzy subfield* (LDF-subfield) of $\mathcal{F}$ if and only if the following properties are satisfied:

(F1) $\mathfrak{D}(f_j+f_k)\ge\mathfrak{D}(f_j)\wedge\mathfrak{D}(f_k)\forall f_j,f_k\in\mathcal{F}$.
(F2) $\mathfrak{D}(-f_j)=\mathfrak{D}(f_j)\forall f_j\in\mathcal{F}$.
(F3) $\mathfrak{D}(f_j\cdot f_k)\ge\mathfrak{D}(f_j)\wedge\mathfrak{D}(f_k)\forall f_j,f_k\in\mathcal{F}$.
(F4) $\mathfrak{D}(f_j^{-1})=\mathfrak{D}(f_j)\forall f_j(\ne 0_{\mathcal{F}})\in\mathcal{F}$.

Note that the collection of all LDF-subfields of $\mathcal{R}$ is denoted by $LDFsF(\mathcal{F})$.

Example 7.2 Let us consider the field $\mathcal{F}=\mathbb{Q}(\sqrt{2})$, where $\mathbb{Q}$ denotes the set of rational numbers. We define an LDFS $\mathfrak{D}$ on $\mathcal{F}$ as follows:

$$\mathfrak{D}(f_j)=\begin{cases}(\langle 1,0\rangle,\langle 1,0\rangle), & \text{if} f_j\in\mathbb{Q}\\(\langle 0,1\rangle,\langle 0,1\rangle), & \text{otherwise}\end{cases}$$

Then, the LDFS $\mathfrak{D}$ is an LDF-subfield of $\mathcal{F}$.

Proposition 7.3 *Let $(\mathcal{F},+,\cdot)$ be a classical field and $\mathfrak{D}$ be an LDF-subfield of $\mathcal{F}$. $\mathfrak{D}(0_{\mathcal{F}})\ge\mathfrak{D}(f_j)$ for all $r_j\in\mathcal{F}$, where $0_{\mathcal{F}}$ is the unit element of $\mathcal{F}$ related to the binary operation $+$.*

Proof It can be shown similar to the proof of Proposition 5.3. □

Theorem 7.4 *Let $\mathcal{F}$ be a classical field and $\mathfrak{D}$ be an LDFS on $\mathcal{F}$. Then, $\mathfrak{D}$ is an LDF-subfield of $\mathcal{F}$ iff the following axioms are provided.*

(i) $\mathfrak{D}(f_j-f_k)\ge\mathfrak{D}(f_j)\wedge\mathfrak{D}(f_k)\forall f_j,f_k\in\mathcal{F}$.
(ii) $\mathfrak{D}(f_j\cdot f_k^{-1})\ge\mathfrak{D}(f_j)\wedge\mathfrak{D}(f_k)\forall f_j,f_k(\ne 0_{\mathcal{F}})\in\mathcal{F}$.

Proof It can be seen by discussing similar to the proof of Theorem 5.4. □

Theorem 7.5 *Let $\mathcal{R}$ be a classical field and $\mathfrak{D}_i$ $(i\in I)$ be the LDFN-subfields of $\mathcal{F}$. Then,*

(i) $\bigcap_{i\in I}\mathfrak{D}_i$ is an LDFN-subfield of $\mathcal{R}$.
(ii) $\bigcup_{i\in I}\mathfrak{D}_i$ is an LDFN-subfield of $\mathcal{R}$.

Proof Let's prove (ii), the other assertion can similarly be proved. Let $\mathfrak{D}_i$ $(i\in I)$ be the LDF-subfields of the classical field $\mathcal{F}$. We have to prove that

$$\bigcup_{i\in I}\mathfrak{D}_i(f_j-f_k)\ge\bigcup_{i\in I}\mathfrak{D}_i(f_j)\wedge\bigcup_{i\in I}\mathfrak{D}_i(f_k)\quad\forall f_j,f_k\in\mathcal{F}$$

and

$$\bigcup_{i\in I}\mathfrak{D}_i\left(f_j\cdot f_k{}^{-1}\right)\ge\bigcup_{i\in I}\mathfrak{D}_i(f_j)\wedge\bigcup_{i\in I}\mathfrak{D}_i(f_k)\quad\forall f_j,(0_{\mathcal{F}}\ne)f_k\in\mathcal{F}.$$

Since $\mathfrak{D}_i$ $(i\in I)$ are the LDF-subfields of $\mathcal{F}$, we obtain

$$
\begin{aligned}
\mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(f_j - f_k) &= \bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(f_j - f_k) \\
&\geq \bigvee_{i\in I} (\mathfrak{U}_{\mathfrak{D}_i}(f_j) \wedge \mathfrak{U}_{\mathfrak{D}_i}(f_k)) \\
&= \left(\bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(f_j)\right) \wedge \left(\bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(f_k)\right) \\
&= \mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(f_j) \wedge \mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(f_k),
\end{aligned} \tag{7.1}
$$

and

$$
\begin{aligned}
\mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}\left(f_j \cdot f_k^{-1}\right) &= \bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}\left(f_j \cdot f_k^{-1}\right) \\
&\geq \bigvee_{i\in I} (\mathfrak{U}_{\mathfrak{D}_i}(f_j) \wedge \mathfrak{U}_{\mathfrak{D}_i}(f_k)) \\
&= \left(\bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(f_j)\right) \wedge \left(\bigvee_{i\in I} \mathfrak{U}_{\mathfrak{D}_i}(f_k)\right) \\
&= \mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(f_j) \wedge \mathfrak{U}_{\tilde{\cup}\mathfrak{D}_i}(f_k).
\end{aligned} \tag{7.2}
$$

By the similar observation,

$$
\begin{aligned}
&\mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(f_j - f_k) \leq \mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(f_j) \vee \mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(f_k), \\
&\mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(f_j \cdot f_k^{-1}) \leq \mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(f_j) \vee \mathfrak{S}_{\tilde{\cup}\mathfrak{D}_i}(f_k),
\end{aligned} \tag{7.3}
$$

$$
\begin{aligned}
&\alpha^{\tilde{\cup}\mathfrak{D}_i}(f_j - f_k) \geq \alpha^{\tilde{\cup}\mathfrak{D}_i}(f_j) \wedge \alpha^{\tilde{\cup}\mathfrak{D}_i}(f_k), \\
&\alpha^{\tilde{\cup}\mathfrak{D}_i}(f_j \cdot f_k^{-1}) \geq \alpha^{\tilde{\cup}\mathfrak{D}_i}(f_j) \wedge \alpha^{\tilde{\cup}\mathfrak{D}_i}(f_k),
\end{aligned} \tag{7.4}
$$

$$
\begin{aligned}
&\beta^{\tilde{\cup}\mathfrak{D}_i}(f_j - f_k) \leq \beta^{\tilde{\cup}\mathfrak{D}_i}(f_j) \vee \beta^{\tilde{\cup}\mathfrak{D}_i}(f_k), \\
&\beta^{\tilde{\cup}\mathfrak{D}_i}(f_j \cdot f_k^{-1}) \leq \beta^{\tilde{\cup}\mathfrak{D}_i}(f_j) \vee \beta^{\tilde{\cup}\mathfrak{D}_i}(f_k)
\end{aligned} \tag{7.5}
$$

are satisfied. Therefore, the union of the LDF-subfields $\mathfrak{D}_i$ ($i \in I$) is also an LDF-subfield. □

Note 2 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fields. $\mathcal{F}_1 \times \mathcal{F}_2$ may not be a field. For instance, $\mathbb{Z}_p$ (p is a prime) is a field but $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a field.

Theorem 7.6 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two classical fields and $\Psi_{fld} : \mathcal{F}_1 \to \mathcal{F}_2$ be a field homomorphism. If $\mathfrak{D}_1$ is an LDF-subfield of $\mathcal{F}_1$, then the image $\Psi_{fld}(\mathfrak{D}_1)$ of $\mathfrak{F}_1$ is also an LDF-subfield of $\mathcal{R}_2$.*

Proof By the calculations similar to the proof of Theorem 3.8, it can be easily obtained by considering Definition 7.1. □

Theorem 7.7 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two classical fields and $\Psi_{fld} : \mathcal{F}_1 \to \mathcal{F}_2$ be a field homomorphism. If $\mathfrak{D}_2$ is an LDF-ideal of $\mathcal{F}_2$, then the preimage $\Psi_{fld}^{-1}(\mathfrak{D}_2)$ of $\mathfrak{D}_2$ is also an LDF-ideal of $\mathcal{F}_1$.*

Proof By the calculations similar to the proof of Theorem 3.9, it can be easily obtained by considering Definition 7.1. □

8 An LDF-based approach to coding theory

In this section, we introduce the concept of linear Diophantine fuzzy code and thus propose a different approach to coding theory. Also, we investigate relationships between the linear Diophantine fuzzy codes and LDF-algebraic structures (such as LDF-subring, LDF-ideal).

Now let's briefly talk about the concepts of code, binary code, codeword. For further information, we refer to Hill (1986); Pless (1989).

A q-ary code $\mathfrak{C}$ is a given set of sequences of symbols where each symbol is selected from a set $\mathbb{F}_q$ (is often taken to be the set $\mathbb{Z}_q$) of q distinct elements. The set $\mathbb{F}_q$ is said to be the alphabet. If q is a prime power (i.e. $q = p^\alpha$ for some prime number p and positive integer α) then we take the alphabet $\mathbb{F}_q$ to be the finite field of order q. If $q = 2$ then the code is described as a binary code. That is, a binary code is a set of sequences of 0's and 1's.

$\mathbb{F}_q^n$ will denote the set of all ordered n-tuples $\mathfrak{a} = a_1a_2 \dots a_n$ where $a_i \in \mathbb{F}_q^n$. The elements of $\mathbb{F}_q^n$ are said to be vectors or words, and n is termed to be the length of a_i. Also, any element of $\mathfrak{C} \subseteq \mathbb{F}_q^n$ is said to be a codeword. Observe that the set $\mathbb{F}_q^n$ has q^n elements. For example, $\mathbb{F}_2^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$ is a set of all ordered 3-tuples of 0's and 1's, and it has $q^n = 2^3$ elements. Also, $\mathfrak{C} = \{000, 111\}$ is a binary code of $\mathbb{F}_2^3$, and $\mathfrak{c}_1 = 000$ and $\mathfrak{c}_2 = 111$ are two codewords in $\mathfrak{C}$.

A vector $(a_1, a_2, \dots, a_n)$ will usually be written simply as $a_1a_2 \dots a_n$. We consider the ambient space $\mathbb{F}_2^n$ of all n-tuples of 0's and 1's with addition $(+)$ and multiplication $(\cdot)$ of vectors componentwise mod 2. That is, if two vectors are $\mathfrak{a}_j = (a_1^j, a_2^j, \dots, a_n^j)$ and $\mathfrak{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$ then

$$
\mathfrak{a}_j + \mathfrak{a}_k = \left(a_1^j + a_1^k, a_2^j + a_2^k, \dots, a_n^j + a_n^k\right)\ (mod 2) \tag{8.1}
$$

and

$$
\mathfrak{a}_j \cdot \mathfrak{a}_k = \left(a_1^j \cdot a_1^k, a_2^j \cdot a_2^k, \dots, a_n^j \cdot a_n^k\right)\ (mod 2). \tag{8.2}
$$

Then, we say that $(\mathbb{F}_2^n, +, .)$ is a ring with identity, where $0_{\mathbb{F}_2^n} = \underbrace{(0, 0, \dots, 0)}_{n} = \underbrace{00 \dots 0}_{n}$ is the (additive) unit of $\mathbb{F}_2^n$ and $1_{\mathbb{F}_2^n} = \underbrace{(1, 1, \dots, 1)}_{n} = \underbrace{11 \dots 1}_{n}$ is the (multiplicative) identity of $\mathbb{F}_2^n$. For example, if two codewords of binary code $\mathfrak{C}$ of $\mathbb{F}_2^5$

are $\mathfrak{c}_1 = 10110$ and $\mathfrak{c}_2 = 11010$ then $\mathfrak{c}_1 + \mathfrak{c}_2 = 01100$ and $\mathfrak{c}_1 \cdot \mathfrak{c}_2 = 10010$.

Note 3 From now on, we assume that the alphabet $\mathbb{F}_q$ is $\mathbb{F}_2$, and so we use the binary code (i.e., 2-ary code).

Definition 8.1 Let $\mathfrak{a}_j = d_1^j d_2^j \dots d_n^j$ be a word (vector) of $\mathbb{F}_2^n$. The weight of word $\mathfrak{a}_j$ is the sum of its elements (entries). It is denoted by $w(\mathfrak{a}_j) = \sum_{i=1}^{n} d_i^j$. If the word of $\mathbb{F}_2^n$ is $1_{\mathbb{F}_2^n}$, then $w(1_{\mathbb{F}_2^n}) = n$ and this is called the maximum weight, denoted by w_{max}.

Note 4 For q-ary codes, the concept of weight of a vector is given as "it is number of its elements that non-zero entries" in literature (see (Özkan and Özkan 2002) for details). For example, if we consider the codeword $\mathfrak{c}_j = 1022001$ for $\mathbb{F}_3^7$ then we obtain $w(\mathfrak{c}_j) = 4$. Since 2-ary codes consist of 0 and 1, Definition 8.1 is the same as this concept.

Definition 8.2 Let $\mathfrak{a}_j = d_1^j d_2^j \dots d_n^j$ be a word (vector) of $\mathbb{F}_2^n$. If $\varpi_1^j, \varpi_2^j, \dots, \varpi_n^j$ are described to be the positions of entries (i.e., $d_1^j, d_2^j, \dots, d_n^j$ resp.) in $\mathfrak{a}_j$ then the value $d_1^j \varpi_1^j + d_2^j \varpi_2^j + \dots + d_n^j \varpi_n^j$ are called the relative weight of word $\mathfrak{a}_j$. It is denoted by $\varpi(\mathfrak{a}_j) = \sum_{i=1}^{n} d_i^j \varpi_i^j$. If the word of $\mathbb{F}_2^n$ is $\mathfrak{a}_j = 1_{\mathbb{F}_2^n}$, then $\varpi(\mathfrak{a}_j) = \frac{n(n+1)}{2}$ and this is said to be the maximum relative weight, denoted by ϖ_{max}.

The concept of relative weight can be generalized as follows.

Definition 8.3 Let $\mathfrak{a}_j = d_1^j d_2^j \dots d_n^j$ be a word (vector) of $\mathbb{F}_2^n$. If $\varpi_1^j, \varpi_2^j, \dots, \varpi_n^j$ are described to be the positions of entries (i.e., $d_1^j, d_2^j, \dots, d_n^j$ resp.) in $\mathfrak{a}_j$ then the value $(d_1^j)^t(\varpi_1^j)^t + (d_2^j)^t(\varpi_2^j)^t + \dots + (d_n^j)^t(\varpi_n^j)^t$ are said to be the t-rung relative weight of word (vector) $\mathfrak{a}_j$, where t is a positive integer. It is denoted by $\varpi^{(t)}(\mathfrak{a}_j) = \sum_{i=1}^{n} (d_i^j)^t(\varpi_i^j)^t = \sum_{i=1}^{n} d_i^j(\varpi_i^j)^t$. If the word of $\mathbb{F}_2^n$ is $\mathfrak{a}_j = 1_{\mathbb{F}_2^n}$, then $\varpi^{(t)}(\mathfrak{a}_j)$ is called the maximum t-rung relative weight, denoted by $\varpi_{max}^{(t)}$. For example, $\varpi_{max}^{(3)} = (\frac{n(n+1)}{2})^2$ for $t = 3$.

In the year 1990, Šešelja and Tepavčević (1990) study on construction of codes by P-fuzzy sets, in which $(P, \leq)$ is a partially ordered set. They described the following componentwise defined order on the set of codewords belonging to a binary code.

Definition 8.4 (Šešelja and Tepavčević 1990) Let $\mathfrak{c}_j = c_1^j c_2^j \dots c_n^j$ and $\mathfrak{c}_k = c_1^k c_2^k \dots c_n^k$ be two any codewords of the binary code $\mathfrak{C}$ of $\mathbb{F}_2^n$. If $c_i^j \geq c_i^k$ for all $i = 1, 2, \dots, n$, then two codewords $\mathfrak{c}_j$ and $\mathfrak{c}_k$ belonging to binary code $\mathfrak{C}$ can be ordered and this order is $\mathfrak{c}_j \leq \mathfrak{c}_k$. Here, $\leq$ is the ordinary ordering relation on the lattice $(\{0, 1\}, \leq) : 0 < 1$. It is obvious that $\mathfrak{c}_j = \mathfrak{c}_k$ iff $c_i^j = c_i^k$ for all $i = 1, 2, \dots, n$.

Proposition 8.5 *Let $\mathfrak{C}$ be a binary code of $\mathbb{F}_2^n$.*

(i) $\mathfrak{c}_j < 0_{\mathbb{F}_2^n}$ for all $\mathfrak{c}_j (\neq 0_{\mathbb{F}_2^n}) \in \mathfrak{C}$.
(ii) $1_{\mathbb{F}_2^n} < \mathfrak{c}_j$ for all $\mathfrak{c}_j (\neq 1_{\mathbb{F}_2^n}) \in \mathfrak{C}$.

Proof Let $\mathfrak{C}$ be a binary code of $\mathbb{F}_2^n$.

(i) If $\mathfrak{c}_j (\neq 0_{\mathbb{F}_2^n}) \in \mathfrak{C}$, we can say that $c_i^j = 1 > 0$ for at least one $i \in \{1, 2, \dots, n\}$. So, we have $\mathfrak{c}_j < 0_{\mathbb{F}_2^n}$ by Definition 8.4.
(ii) It can be proved similar to (i), and therefore omitted.

□

Example 8.6 We consider the binary code $\mathfrak{C} = \{000, 001, 011, 111\}$ of $\mathbb{F}_2^3$. Then, we have $111 < 011 < 001 < 000$.

Proposition 8.7 *Let $\mathfrak{c}_j$ and $\mathfrak{c}_k$ be two codewords of the binary code $\mathfrak{C}$ of $\mathbb{F}_2^n$. If $\mathfrak{c}_j \leq \mathfrak{c}_k$ then $\mathfrak{c}_j \cdot \mathfrak{c}_k = \mathfrak{c}_k$.*

Proof Assume that $\mathfrak{c}_j = (c_1^j, c_2^j, \dots, c_n^j)$ and $\mathfrak{c}_k = (c_1^k, c_2^k, \dots, c_n^k)$ is two codewords of the code $\mathfrak{C}$ of $\mathbb{F}_2^n$, and $\mathfrak{c}_j \leq \mathfrak{c}_k$. From Definition 8.4, we write $c_i^j \geq c_i^k$ for all $i = 1, 2, \dots, n$. Since $c_i^j, c_i^k \in \{0, 1\}$ and $c_i^j \geq c_i^k$, we calculate (by Eq. (8.2))

$$\begin{aligned} \mathfrak{c}_j \cdot \mathfrak{c}_k &= \left(c_1^j \cdot c_1^k, c_2^j \cdot c_2^k, \dots, c_n^j \cdot c_n^k\right) \\ &= \left(c_1^k, c_2^k, \dots, c_n^k\right) = \mathfrak{c}_k. \end{aligned} \tag{8.3}$$

□

Now, we introduce the concept of linear Diophantine fuzzy code, which offers a different perspective on coding theory.

Definition 8.8 Let $\mathfrak{C}$ be a code of $\mathbb{F}_2^n$ and $\mathfrak{D}$ be an LDFS on $\mathbb{F}_2^n$. If the following properties are satisfied:

(C1) $\mathfrak{D}(\mathfrak{a}_j \cdot \mathfrak{a}_k) \leq \mathfrak{D}(\mathfrak{a}_j) \wedge \mathfrak{D}(\mathfrak{a}_k) \forall \mathfrak{a}_j, \mathfrak{a}_k \in \mathbb{F}_2^n$.
(C2) $\mathfrak{D}(\mathfrak{a}_j) \neq \mathfrak{D}(\mathfrak{a}_k)$ if $\mathfrak{a}_j \neq \mathfrak{a}_k$.

BINARY CODE	CHARACTER	BINARY CODE	CHARACTER	BINARY CODE	CHARACTER	BINARY CODE	CHARACTER	BINARY CODE	CHARACTER	BINARY CODE	CHARACTER	BINARY CODE	CHARACTER	BINARY CODE	CHARACTER
00000000	null	00100000	space	01000000	@	01100000	`	10000000	Ç	10100000	á	11000000	└	11100000	Ó
00000001	☺	00100001	!	01000001	A	01100001	a	10000001	ü	10100001	í	11000001	┴	11100001	ß
00000010	☻	00100010	"	01000010	B	01100010	b	10000010	è	10100010	ó	11000010	┬	11100010	Ô
00000011	♥	00100011	#	01000011	C	01100011	c	10000011	â	10100011	ú	11000011	├	11100011	Ò
00000100	♦	00100100	$	01000100	D	01100100	d	10000100	ä	10100100	ñ	11000100	─	11100100	õ
00000101	♣	00100101	%	01000101	E	01100101	e	10000101	à	10100101	Ñ	11000101	┼	11100101	Ô
00000110	♠	00100110	&	01000110	F	01100110	f	10000110	å	10100110	ª	11000110	ã	11100110	µ
00000111	•	00100111	'	01000111	G	01100111	g	10000111	ç	10100111	º	11000111	Ã	11100111	þ
00001000	◘	00101000	(	01001000	H	01101000	h	10001000	ê	10101000	¿	11001000	╚	11101000	Þ
00001001	○	00101001	)	01001001	I	01101001	i	10001001	ë	10101001	®	11001001	╔	11101001	Ú
00001010	◙	00101010	*	01001010	J	01101010	j	10001010	è	10101010	¬	11001010	╩	11101010	Û
00001011	♂	00101011	+	01001011	K	01101011	k	10001011	ï	10101011	½	11001011	╦	11101011	Ù
00001100	♀	00101100	,	01001100	L	01101100	l	10001100	î	10101100	¼	11001100	╠	11101100	ý
00001101	♪	00101101	-	01001101	M	01101101	m	10001101	ì	10101101	¡	11001101	═	11101101	Ý
00001110	♫	00101110	.	01001110	N	01101110	n	10001110	Ä	10101110	«	11001110	╬	11101110	¯
00001111	☼	00101111	/	01001111	O	01101111	o	10001111	Å	10101111	»	11001111	¤	11101111	´
00010000	►	00110000	0	01010000	P	01110000	p	10010000	È	10110000	░	11010000	ð	11110000	-
00010001	◄	00110001	1	01010001	Q	01110001	q	10010001	æ	10110001	▒	11010001	Ð	11110001	±
00010010	↕	00110010	2	01010010	R	01110010	r	10010010	Æ	10110010	▓	11010010	Ê	11110010	‗
00010011	‼	00110011	3	01010011	S	01110011	s	10010011	ô	10110011	│	11010011	Ë	11110011	¾
00010100	¶	00110100	4	01010100	T	01110100	t	10010100	ö	10110100	┤	11010100	È	11110100	¶
00010101	§	00110101	5	01010101	U	01110101	u	10010101	ò	10110101	Á	11010101	ı	11110101	§
00010110	▬	00110110	6	01010110	V	01110110	v	10010110	û	10110110	Â	11010110	Í	11110110	÷
00010111	↨	00110111	7	01010111	W	01110111	w	10010111	ù	10110111	À	11010111	Î	11110111	¸
00011000	↑	00111000	8	01011000	X	01111000	x	10011000	ÿ	10111000	©	11011000	Ï	11111000	°
00011001	↓	00111001	9	01011001	Y	01111001	y	10011001	Ö	10111001	╣	11011001	┘	11111001	¨
00011010	→	00111010	:	01011010	Z	01111010	z	10011010	Ü	10111010	║	11011010	┌	11111010	·
00011011	←	00111011	;	01011011	[	01111011	{	10011011	ø	10111011	╗	11011011	█	11111011	¹
00011100	∟	00111100	<	01011100	\	01111100	|	10011100	£	10111100	╝	11011100	▄	11111100	³
00011101	↔	00111101	=	01011101	]	01111101	}	10011101	Ø	10111101	¢	11011101	¦	11111101	²
00011110	▲	00111110	>	01011110	^	01111110	~	10011110	×	10111110	¥	11011110	Ì	11111110	■
00011111	▼	00111111	?	01011111	_	01111111	⌂	10011111	ƒ	10111111	┐	11011111	▀	11111111	nbsp

Fig. 1 Binary code-character for $\mathbb{F}_2^8$

then

$$\mathfrak{D}_{\mathfrak{C}} = \{(\mathfrak{c}_j, \langle \mathfrak{U}_{\mathfrak{D}}(\mathfrak{c}_j), \mathfrak{S}_{\mathfrak{D}}(\mathfrak{c}_j)\rangle, \langle \alpha^{\mathfrak{D}}(\mathfrak{c}_j), \beta^{\mathfrak{D}}(\mathfrak{c}_j)\rangle) : \mathfrak{c}_j \in \mathfrak{C}\}$$

is called a *linear Diophantine fuzzy code* (LDF-code) corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^n$.

Example 8.9 We consider $\mathbb{F}_2^8$ and Fig. 1 (Source: https://theasciicode.com.ar/).

Let us generate LDF-code corresponding to the code $\mathfrak{C}$ of 8-tuples encoding (converting information from a source into symbols) the text "LDFSs/2019". Then, we have the code $\mathfrak{C}$ as follows:

$$\mathfrak{C} = \begin{Bmatrix} \mathfrak{c}_1 \\ \mathfrak{c}_2 \\ \mathfrak{c}_3 \\ \mathfrak{c}_4 \\ \mathfrak{c}_5 \\ \mathfrak{c}_6 \\ \mathfrak{c}_7 \\ \mathfrak{c}_8 \\ \mathfrak{c}_9 \\ \mathfrak{c}_{10} \end{Bmatrix} = \begin{Bmatrix} 01001100 \\ 01000100 \\ 01000110 \\ 01010011 \\ 01110011 \\ 00101111 \\ 00110010 \\ 00110000 \\ 00110001 \\ 00111001 \end{Bmatrix}.$$

Also, we suppose that

$$\mathfrak{D} = \{(\mathfrak{a}_j, \langle \mathfrak{U}_{\mathfrak{D}}(\mathfrak{a}_j), \mathfrak{S}_{\mathfrak{D}}(\mathfrak{a}_j)\rangle, \langle \alpha^{\mathfrak{D}}(\mathfrak{a}_j), \beta^{\mathfrak{D}}(\mathfrak{a}_j)\rangle) : \mathfrak{a}_j \in \mathbb{F}_2^8\}$$

where

$$\mathfrak{U}_{\mathfrak{D}}(\mathfrak{a}_j) = \sqrt{\frac{\sum_{i=1}^{n} d_i^j (\varpi_i^j)^3}{\varpi_{max}^{(3)}}}, \quad \mathfrak{S}_{\mathfrak{D}}(\mathfrak{a}_j) = \sqrt{\frac{\sum_{i=1}^{n} (1-d_i^j)(\varpi_i^j)^3}{\varpi_{max}^{(3)}}},$$
$$\alpha^{\mathfrak{D}}(\mathfrak{a}_j) = \left(\frac{\sum_{i=1}^{n} d_i^j}{w^{max}}\right)^2, \quad \beta^{\mathfrak{D}}(\mathfrak{a}_j) = \left(\frac{\sum_{i=1}^{n} (1-d_i^j)}{w^{max}}\right)^2 \tag{8.4}$$

for all $\mathfrak{a}_j \in \mathbb{F}_2^8$.

From Definitions 8.1 and 8.3, we have $0 \le \mathfrak{U}_{\mathfrak{D}}(\mathfrak{a}_j), \mathfrak{S}_{\mathfrak{D}}(\mathfrak{a}_j), \alpha^{\mathfrak{D}}(\mathfrak{a}_j), \beta^{\mathfrak{D}}(\mathfrak{a}_j) \le 1$. Also, we calculate $\alpha^{\mathfrak{D}}(\mathfrak{a}_j) + \beta^{\mathfrak{D}}(\mathfrak{a}_j) = \left(\frac{\sum_{i=1}^n d_i^j}{w^{max}}\right)^2 + \left(\frac{\sum_{i=1}^n (1-d_i^j)}{w^{max}}\right)^2 \le \left(\frac{\sum_{i=1}^n d_i^j}{w^{max}} + \frac{\sum_{i=1}^n (1-d_i^j)}{w^{max}}\right)^2 = \left(\frac{\sum_{i=1}^n d_i^j + (1-d_i^j)}{w^{max}}\right)^2 = 1$. Thus, $\mathfrak{D}$ is an LDFS on $\mathbb{F}_2^8$. Since it satisfies the conditions (C1) and (C2) in Definition 8.8, we can say that

$$\mathfrak{D}_{\mathfrak{C}} = \begin{Bmatrix} (\mathfrak{c}_1, \langle \sqrt{\frac{349}{1296}}, \sqrt{\frac{947}{1296}} \rangle, \langle \frac{9}{64}, \frac{25}{64} \rangle), \\ (\mathfrak{c}_2, \langle \sqrt{\frac{224}{1296}}, \sqrt{\frac{1072}{1296}} \rangle, \langle \frac{1}{16}, \frac{9}{16} \rangle), \\ (\mathfrak{c}_3, \langle \sqrt{\frac{567}{1296}}, \sqrt{\frac{729}{1296}} \rangle, \langle \frac{9}{64}, \frac{25}{64} \rangle), \\ (\mathfrak{c}_4, \langle \sqrt{\frac{927}{1296}}, \sqrt{\frac{369}{1296}} \rangle, \langle \frac{1}{4}, \frac{1}{4} \rangle), \\ (\mathfrak{c}_5, \langle \sqrt{\frac{954}{1296}}, \sqrt{\frac{342}{1296}} \rangle, \langle \frac{25}{64}, \frac{9}{64} \rangle), \\ (\mathfrak{c}_6, \langle \sqrt{\frac{1223}{1296}}, \sqrt{\frac{73}{1296}} \rangle, \langle \frac{25}{64}, \frac{9}{64} \rangle)), \\ (\mathfrak{c}_7, \langle \sqrt{\frac{434}{1296}}, \sqrt{\frac{862}{1296}} \rangle, \langle \frac{9}{64}, \frac{25}{64} \rangle), \\ (\mathfrak{c}_8, \langle \sqrt{\frac{91}{1296}}, \sqrt{\frac{1205}{1296}} \rangle, \langle \frac{1}{16}, \frac{9}{16} \rangle), \\ (\mathfrak{c}_9, \langle \sqrt{\frac{603}{1296}}, \sqrt{\frac{693}{1296}} \rangle, \langle \frac{9}{64}, \frac{25}{64} \rangle), \\ (\mathfrak{c}_{10}, \langle \sqrt{\frac{728}{1296}}, \sqrt{\frac{568}{1296}} \rangle, \langle \frac{1}{4}, \frac{1}{4} \rangle) \end{Bmatrix}$$

$$= \begin{Bmatrix} (\mathfrak{c}_1, \langle 0.5188, 0.8548\rangle, \langle 0.1406, 0.3906\rangle), \\ (\mathfrak{c}_2, \langle 0.4156, 0.9094\rangle, \langle 0.0625, 0.5625\rangle), \\ (\mathfrak{c}_3, \langle 0.6614, 0.75\rangle, \langle 0.1406, 0.3906\rangle), \\ (\mathfrak{c}_4, \langle 0.8456, 0.5335\rangle, \langle 0.25, 0.25\rangle), \\ (\mathfrak{c}_5, \langle 0.8579, 0.5136\rangle, \langle 0.3906, 0.1406\rangle), \\ (\mathfrak{c}_6, \langle 0.9713, 0.2372\rangle, \langle 0.3906, 0.1406\rangle), \\ (\mathfrak{c}_7, \langle 0.5786, 0.8155\rangle, \langle 0.1406, 0.3906\rangle), \\ (\mathfrak{c}_8, \langle 0.2649, 0.9642\rangle, \langle 0.0625, 0.5625\rangle), \\ (\mathfrak{c}_9, \langle 0.682, 0.7312\rangle, \langle 0.1406, 0.3906\rangle), \\ (\mathfrak{c}_{10}, \langle 0.7494, 0.6619\rangle, \langle 0.25, 0.25\rangle) \end{Bmatrix}$$

is an LDF-code corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^8$. In addition, we give the matching in Table 1.

Table 1 The tabular form of LDF-code corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^8$

Character	Binary Code	LDF-code
L	01001100	$(\langle 0.5188, 0.8548\rangle, \langle 0.1406, 0.3906\rangle)$
D	01000100	$(\langle 0.4156, 0.9094\rangle, \langle 0.0625, 0.5625\rangle)$
F	01000110	$(\langle 0.6614, 0.75\rangle, \langle 0.1406, 0.3906\rangle)$
S	01010011	$(\langle 0.8456, 0.5335\rangle, \langle 0.25, 0.25\rangle)$
s	01110011	$(\langle 0.8579, 0.5136\rangle, \langle 0.3906, 0.1406\rangle)$
/	00101111	$(\langle 0.9713, 0.2372\rangle, \langle 0.3906, 0.1406\rangle)$
2	00110010	$(\langle 0.5786, 0.8155\rangle, \langle 0.1406, 0.3906\rangle)$
0	00110000	$(\langle 0.2649, 0.9642\rangle, \langle 0.0625, 0.5625\rangle)$
1	00110001	$(\langle 0.682, 0.7312\rangle, \langle 0.1406, 0.3906\rangle)$
9	00111001	$(\langle 0.7494, 0.6619\rangle, \langle 0.25, 0.25\rangle)$

Note 5 The the grades of reference parameters are important for the LDF-code. We assume that $\mathfrak{a}_j = 00000100$ and $\mathfrak{a}_k = 00111000$ (see: Fig. 1) then it is obtained that

$\mathfrak{U}_{\mathfrak{D}}(\mathfrak{a}_j) = \mathfrak{U}_{\mathfrak{D}}(\mathfrak{a}_k) = 0.4081$ and $\mathfrak{S}_{\mathfrak{D}}(\mathfrak{a}_j) = \mathfrak{S}_{\mathfrak{D}}(\mathfrak{a}_k) = 0.9128$ (by Eq. (8.4)). But we have $\mathfrak{D}(a_j) \neq \mathfrak{D}(a_k)$ by using the grades of reference parameters.

Proposition 8.10 *Let $\mathfrak{D}$ be an LDFS on $\mathbb{F}_2^n$ and $\mathfrak{D}_{\mathfrak{C}}$ be an LDF-code corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^n$. If $\mathfrak{c}_j \leq \mathfrak{c}_k$ for $\mathfrak{c}_j, \mathfrak{c}_k \in \mathfrak{C}$, then $\mathfrak{D}(\mathfrak{c}_j) \geq \mathfrak{D}(\mathfrak{c}_k)$.*

Proof Assume that $\mathfrak{D}_{\mathfrak{C}}$ is an LDF-code corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^n$ and $\mathfrak{c}_j \leq \mathfrak{c}_k$. Then, we have $\mathfrak{c}_j \cdot \mathfrak{c}_k = \mathfrak{c}_k$ from Proposition 8.7. By the property (C1) in Definition 8.8, we write

$$\begin{aligned}\mathfrak{D}(\mathfrak{c}_j \cdot \mathfrak{c}_k) \leq & \mathfrak{D}(\mathfrak{c}_j) \wedge \mathfrak{D}(\mathfrak{c}_k) \Rightarrow \mathfrak{D}(\mathfrak{c}_k) \\ \leq & \mathfrak{D}(\mathfrak{c}_j) \wedge \mathfrak{D}(\mathfrak{c}_k) \Rightarrow \mathfrak{D}(\mathfrak{c}_k) \leq \mathfrak{D}(\mathfrak{c}_j)\end{aligned} \tag{8.5}$$

for all $\mathfrak{c}_j, \mathfrak{c}_k \in \mathfrak{C}$. Thus, the proof is completed. □

Proposition 8.11 *Let $\mathfrak{C}$ be a code of $\mathbb{F}_2^n$ and $\mathfrak{D}$ be an LDFS on $\mathbb{F}_2^n$. Assume that $\mathfrak{D}_{\mathfrak{C}}$ is an LDF-code corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^n$. Then,*

(i) $\mathfrak{D}(0_{\mathbb{F}_2^n}) < \mathfrak{D}(\mathfrak{c}_j)$ for all $\mathfrak{c}_j(\neq 0_{\mathbb{F}_2^n}) \in \mathfrak{C}$.
(ii) $\mathfrak{D}(\mathfrak{c}_j) < \mathfrak{D}(1_{\mathbb{F}_2^n})$ for all $\mathfrak{c}_j(\neq 1_{\mathbb{F}_2^n}) \in \mathfrak{C}$.

Proof Let $\mathfrak{D}_{\mathfrak{C}}$ is an LDF-code corresponding to the code $\mathfrak{C}$ of $\mathbb{F}_2^n$.

(i) By the property (C1) in Definition 8.8, we can write $\mathfrak{D}(\mathfrak{c}_j \cdot 0_{\mathbb{F}_2^n}) \leq \mathfrak{D}(\mathfrak{c}_j) \wedge \mathfrak{D}(0_{\mathbb{F}_2^n}) \Rightarrow \mathfrak{D}(0_{\mathbb{F}_2^n}) \leq \mathfrak{D}(\mathfrak{c}_j)$ for all $\mathfrak{c}_j \in \mathfrak{C}$. Considering the property (C2) in Definition 8.8, we have $\mathfrak{D}(0_{\mathbb{F}_2^n}) < \mathfrak{D}(\mathfrak{c}_j)$ for all $\mathfrak{c}_j(\neq 0_{\mathbb{F}_2^n}) \in \mathfrak{C}$.
(ii) It can be proved similarly to the proof of (i).

□

Theorem 8.12 *Let $\mathfrak{D}$ be an LDFS on $\mathbb{F}_2^n$. If $\mathfrak{D}$ be an LDF-subring/LDF-ideal of $\mathbb{F}_2^n$, then there is no LDF-code corresponding to any binary code $\mathfrak{C}$ of $\mathbb{F}_2^n$. (or contrapositive)*

Proof The proof is clear from Definitions 5.1, 6.2, 8.8 and Propositions 5.3, 6.3, 8.11. □

Example 8.13 We consider $\mathbb{F}_2^2 = \{00, 01, 10, 11\}$. Then, we have Cayley tables in Example 6.4 (for $0 = 00, p = 01, q = 10, r = 11$). It obvious that $(\mathbb{F}_2^2, +, .)$ is a ring with identity. If $\mathfrak{D}$ is an LDF-subring/LDF-ideal of $\mathbb{F}_2^2$ then we have $\mathfrak{D}(p+q) \geq \mathfrak{D}(p) \wedge \mathfrak{D}(q) \Rightarrow \mathfrak{D}(r) \geq \mathfrak{D}(p) \wedge \mathfrak{D}(q)$, and $\mathfrak{D}(q) \geq \mathfrak{D}(p) \wedge \mathfrak{D}(r)$ and $\mathfrak{D}(p) \geq \mathfrak{D}(q) \wedge \mathfrak{D}(r)$. These imply $\mathfrak{D}(p) = \mathfrak{D}(q) = \mathfrak{D}(r)$. By considering Proposition 5.3, we have

$$\mathfrak{D}(0) \geq \mathfrak{D}(p) = \mathfrak{D}(q) = \mathfrak{D}(r). \tag{8.6}$$

By considering Definition 8.8 (C2) and Proposition 8.11, we say that if $\mathfrak{D}$ is an LDF-subring/LDF-ideal of $\mathbb{F}_2^2$ then there is no LDF-code corresponding to any binary code $\mathfrak{C}$ of $\mathbb{F}_2^2$.

Discussion Can the properties (R1, R2, R3) in Definition 5.1 or the properties (I1, I2) in Definition 6.2 be taken instead of properties (C1, C2) in Definition 8.8? Suppose that $\mathfrak{D}$ on $\mathbb{F}_2^n$ has the properties (R1),(R2) and (R3) in definition of LDF-subring. It is necessary in order to generate LDF-code that each codeword in $\mathfrak{C}$ matches a distinct LDF-number in $\mathfrak{D}$. So, we assume that if $\mathfrak{a}_j \neq \mathfrak{a}_k$ implies $\mathfrak{D}(\mathfrak{a}_j) \neq \mathfrak{D}(\mathfrak{a}_k)$. We will continue by considering this assumption.

For every $\mathfrak{a}_j(\neq 1_{\mathbb{F}_2^n}) \in \mathbb{F}_2^n$ there is an $\mathfrak{a}_k \in \mathbb{F}_2^n$ such that $\mathfrak{a}_j + \mathfrak{a}_k = 1_{\mathbb{F}_2^n}$, e.g., $\mathfrak{a}_k = 01010101 \in \mathbb{F}_2^8$ for $\mathfrak{a}_j = 10101010 \in \mathbb{F}_2^8$. From the property (R1) in Definition 5.1, we have

$$\mathfrak{D}(\mathfrak{a}_j + \mathfrak{a}_k) = \mathfrak{D}(1_{\mathbb{F}_2^n}) \geq \mathfrak{D}(\mathfrak{a}_j) \wedge \mathfrak{D}(\mathfrak{a}_k), \tag{8.7}$$

$$\mathfrak{D}(\mathfrak{a}_j + 1_{\mathbb{F}_2^n}) = \mathfrak{D}(\mathfrak{a}_k) \geq \mathfrak{D}(\mathfrak{a}_j) \wedge \mathfrak{D}(1_{\mathbb{F}_2^n}) \tag{8.8}$$

and

$$\mathfrak{D}(1_{\mathbb{F}_2^n} + \mathfrak{a}_k) = \mathfrak{D}(\mathfrak{a}_j) \geq \mathfrak{D}(1_{\mathbb{F}_2^n}) \wedge \mathfrak{D}(\mathfrak{a}_k). \tag{8.9}$$

Since $\mathfrak{a}_j \neq \mathfrak{a}_k$, this implies $\mathfrak{D}(\mathfrak{a}_j) \neq \mathfrak{D}(\mathfrak{a}_k)$. If $\mathfrak{D}(\mathfrak{a}_j) < \mathfrak{D}(\mathfrak{a}_k)$ then $\mathfrak{D}(\mathfrak{a}_j) \geq \mathfrak{D}(1_{\mathbb{F}_2^n})$ by Eq. (8.9) and $\mathfrak{D}(1_{\mathbb{F}_2^n}) \geq \mathfrak{D}(\mathfrak{a}_j)$ by Eq. (8.7). So, we obtain $\mathfrak{D}(\mathfrak{a}_j) = \mathfrak{D}(1_{\mathbb{F}_2^n})$. If $\mathfrak{D}(\mathfrak{a}_j) > \mathfrak{D}(\mathfrak{a}_k)$ then $\mathfrak{D}(\mathfrak{a}_k) \geq \mathfrak{D}(1_{\mathbb{F}_2^n})$ by Eq. (8.8) and $\mathfrak{D}(1_{\mathbb{F}_2^n}) \geq \mathfrak{D}(\mathfrak{a}_k)$ by Eq. (8.7). Hence, we obtain $\mathfrak{D}(\mathfrak{a}_j) = \mathfrak{D}(1_{\mathbb{F}_2^n})$. Therefore, if $\mathfrak{a}_j + \mathfrak{a}_k = 1_{\mathbb{F}_2^n}$ then $\mathfrak{D}(\mathfrak{a}_j) = \mathfrak{D}(1_{\mathbb{F}_2^n})$ or $\mathfrak{D}(\mathfrak{a}_k) = \mathfrak{D}(1_{\mathbb{F}_2^n})$. Consequently, we can say that $\mathfrak{D}(\mathfrak{a}_{j'}) = \mathfrak{D}(1_{\mathbb{F}_2^n})$ for some (at least $(2^{n-1} - 1)$) $\mathfrak{a}_{j'} \neq 1_{\mathbb{F}_2^n}$. This means that some codewords in $\mathfrak{C}$ match the same LDF-number. However, this is an undesirable result (contradiction).

Comparison In the year 2002, Özkan and Özkan published a seminal paper, which aims to introduce the notion of fuzzy code and thus propose a different approach to coding theory. In (Özkan and Özkan 2002), the authors endeavored to convert binary codes to fuzzy numbers by using not-so fuzzy ideal, and thus made major contribution to the literature. However, there are some flaws for the conditions (see: Definition 2.2) in the definition of fuzzy code in Özkan and Özkan (2002). The authors described the function J as $J : C \to [0, 1]$ where C is a binary code (in Definition 2.2 (Özkan and Özkan 2002)). For $x = 1000101 \in C$ and $y = 1100010 \in C$ in Example 2.2 (Özkan and Özkan 2002), it is obtained that $x + y = 0100111 \notin C$ and

$x \cdot y = 1000000 \notin C$. Hence, $J(x+y)$ and $J(x \cdot y)$ are not defined. Assume that

$$C = \begin{Bmatrix} 0000000 \\ 0110000 \\ 0010000 \\ 0100000 \end{Bmatrix}$$

(i.e., $x+y \in C$ and $x \cdot y \in C$ for all $x, y \in C$) then it is calculated as $J(0110000 + 0010000) \ngeq \min\{J(0110000), J(0010000)\}$. Therefore, we can say that the function J does not generate a fuzzy code for each code $\mathfrak{C}$ of $\mathbb{F}_2^7$, however can generate fuzzy codes for some specially specified codes. Especially, we take the code

$$C = \begin{Bmatrix} 0000000 \\ 0111100 \\ 0011000 \\ 0100100 \end{Bmatrix}$$

then J is a fuzzy code according to Definition 2.2 in Özkan and Özkan (2002), but $J(0011000) = J(0100100) = \frac{7}{28}$. Since $0011000 \neq 0100100$, this is a problematic situation. To eradicate these restrictions, we describe the novel concept of linear Diophantine fuzzy code (by using not-so LDF-subring/LDF-ideal). Thus, we argue that linear Diophantine fuzzy code can be generated for each code $\mathfrak{C}$ of $\mathbb{F}_2^n$ if the LDFS $\mathfrak{D}$ on $\mathbb{F}_2^n$ satisfies the conditions (C1) and (C2) in Definition 8.8. All of these are details demonstrating the advantage of the LDF-code proposed in this section.

9 Conclusion

The studies of generalized types of fuzzy sets in the algebraic structures like group, ring, field are interesting research topics. In this paper, we investigated the algebraic properties of linear Diophantine fuzzy sets in the structures of groups, rings and fields. Some related concepts, e.g., the linear Diophantine fuzzy subgroup, linear Diophantine fuzzy normal subgroup, linear Diophantine fuzzy subring, linear Diophantine fuzzy ideal, linear Diophantine fuzzy subfield were proposed. In addition, we made a theoretical study on their fundamental characteristic features analogous to those of ordinary groups, rings and fields. Also, we proposed the linear Diophantine fuzzy code corresponding to the binary code, which can be used for data compression, data storage, data transmission and cryptography.

We hope that this new notion will bring a new opportunity in the research and development of theory of linear Diophantine fuzzy set, which is a generalized form of fuzzy set. To extend this study, further research can be done by examining the properties of linear Diophantine fuzzy sets in other algebraic structures such as modules and lattices. In the near future, we will endeavor to describe these potential concepts for linear Diophantine fuzzy sets.

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