Journal of Ambient Intelligence and Humanized Computing (2021) 12:1541–1558
https://doi.org/10.1007/s12652-020-02227-0

ORIGINAL RESEARCH

Interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets and their application in decision-making

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Received: 9 December 2019 / Accepted: 13 June 2020 / Published online: 27 June 2020

Abstract

Although some statistical tools, such as mean and median, used for modelling a problem containing parameters or alternatives with multiple intuitionistic fuzzy values because these values are obtained in a specific period, decrease uncertainty, they lead to data loss. However, interval-valued intuitionistic fuzzy values can overcome such a concern. For this reason, the present study proposes the concept of interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets (*d*-sets) and presents several of its basic properties. Moreover, by using *d*-sets, we suggest a new soft decision-making method and apply it to a problem concerning the eligibility of candidates for two vacant positions in an online job advertisement. Since it is the first method proposed in relation to this structure (*d*-sets), it is impossible to compare this method with another in this sense. To deal with this difficulty, we introduce four new concepts, i.e. mean reduction, mean bireduction, mean bireduction-reduction, and mean reduction-bireduction. By using these concepts, we apply four state-of-the-art soft decision-making methods to the problem. We then compare the ranking performances of the proposed method with those of the four methods. Besides, we apply five methods to a real problem concerning performance-based value assignment to some filters used in image denoising and compare the ranking performances of these methods. Finally, we discuss *d*-sets and the proposed method for further research.

Keywords Soft sets · Interval-valued intuitionistic fuzzy sets · *d*-sets · Soft decision-making

Mathematics Subject Classification 03F55 · 03E72

1 Introduction

The standard mathematical tools are incompetent at overcoming some problems containing uncertainties in many areas, such as engineering, physics, computer science, economics, social sciences, and medical sciences. To overcome this drawback, many new mathematical tools, such as fuzzy sets (Zadeh 1965), intuitionistic fuzzy sets (Atanassov 1986), interval-valued fuzzy sets (Gorzałczany 1987; Zadeh 1975), and soft sets (Molodtsov 1999), have been proposed. Moreover, some hybrid versions of these concepts, such as fuzzy soft sets (Maji et al. 2001), fuzzy parameterized soft sets (Çağman et al. 2011a), fuzzy parameterized fuzzy soft sets (*fpfs*-sets) (Çağman et al. 2010), fuzzy parameterized interval-valued fuzzy soft sets (Alkhazaleh et al. 2011), intuitionistic fuzzy parameterized soft sets (Deli and Çağman 2015), intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets (*ifpifs*-sets) (Karaaslan 2016), fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) (Sulukan et al. 2019), and interval-valued fuzzy parameterized intuitionistic fuzzy soft sets (*ivfpifs*-sets) (Kamacı 2019) have been introduced. So far, the studies have been conducted on these concepts in many fields, such as decision-making (Çağman and Enginoğlu 2010a, b, 2012; Çağman et al. 2011b; Enginoğlu 2012; Enginoğlu and Çağman n.d.; Enginoğlu and Memiş 2018; Enginoğlu et al. 2018; Hao et al. 2018; Joshi 2020; Maji et al. 2002; Razak and Mohamad 2011, 2013; Selvachandran et al. 2017), algebra (Çıtak and Çağman 2015, 2017; Sezgin 2016; Sezgin et al. 2019; Ullah et al. 2018), topology (Atmaca 2017; Enginoğlu et al. 2015; Riaz and Hashmi 2017; Şenel 2016; Thomas and John 2016;

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Zorlutuna and Atmaca 2016), analysis (Riaz et al. 2018; Şenel 2018), and the other (Garg and Arora 2020; Iqbal and Rizwan 2019; Maji et al. 2003; Niewiadomski 2013).

However, without losing data, the concepts mentioned above cannot directly model a problem in which values are assigned to a parameter or an alternative with multiple measurement results. Suppose that there are twenty speedometers which send ten signals in an hour in the Dardanelles Strait and every ten signals are stored as a measurement result. Furthermore, assume that a signal is accepted as a positive signal if the instant flow is sufficient to turn the impeller of a turbine or as a negative signal if insufficient. Let a_n^x and b_n^x denote the numbers of positive and negative signals transmitted by a speedometer x for nth measurement, respectively. Here, $a_n^x + b_n^x = 10$, for all unsigned integer number n. Then, for $(a_n^x) = (5, 3, 6, 8, 1)$ which shows the results of five measurements and the membership function defined by $\mu(x) := \frac{1}{10n} \sum_{i=1}^{n} a_i^x$, the membership degree of the speedometer x to the fuzzy set μ can be computed as 0.46. Therefore, the nonmembership degree of the speedometer x is computed as $\nu(x) = 1 - \mu(x) = 0.54$. However, considering multi-values as a single value refers to data loss. On the other hand, for

$$\alpha(x) := \left[\frac{\min_n a_n^x}{\max_n a_n^x + \max_n b_n^x}, \frac{\max_n a_n^x}{\max_n a_n^x + \max_n b_n^x} \right]$$

$$\beta(x) := \left[\frac{\min_n b_n^x}{\max_n a_n^x + \max_n b_n^x}, \frac{\max_n b_n^x}{\max_n a_n^x + \max_n b_n^x} \right]$$

$(a_n^x) = (5, 3, 6, 8, 1)$, and $(b_n^x) = (5, 7, 4, 2, 9)$, the membership/nonmembership degree of the speedometer x can be computed as ${}^{[0.06,0.47]}_{[0.12,0.53]}$ via the concept of interval-valued intuitionistic fuzzy sets (*ivif*-sets) provided in Atanassov (2020) and Atanassov and Gargov (1989) and which is being "data-friendly". As a result, the values 0.46 and 0.54 denote that the positive and negative signal numbers for one hundred signals of speedometer x are 46 and 54, respectively. On the other hand, the value [0.06, 0.47] signifies that the positive signal numbers for one hundred signals of speedometer x range from 6 to 47. Similarly, the value [0.12, 0.53] suggests that the negative signal numbers for one hundred signals of speedometer x occur between 12 and 53. Therefore, since an interval-valued intuitionistic fuzzy value contains more information than fuzzy values, it is more convenient to model such a problem.

Recently, many researchers have focused on both theoretical and applied studies concerning the concept of *ivif*-sets. For example, Sotirov et al. (2018) have propounded an approach combining intuitionistic fuzzy logic via the intercriteria analysis method. Thereafter, Atanassov et al. (2019) have described a new intercriteria analysis method based on interval-valued intuitionistic fuzzy assessment. Moreover, Kim et al. (2018) have proposed a method by using *ivif*-sets for the assessment and evaluation of all the classes in question. Besides, Luo and Liang (2018) have suggested a novel similarity measure based on *ivif*-sets. They then have applied it to pattern recognition and medical diagnosis. Furthermore, Liu and Jiang (2020) have defined a new distance measure of *ivif*-sets and applied it in consideration of the well known ideal house selection.

The concept of *ivif*-sets has been used to overcome uncertainties, especially in multi-criteria decision-making problems. For example, Mishra and Rani (2018) have extended the scope of this concept to attend to a method called as a weighted aggregated sum product assessment and applied it to a decision-making problem. Moreover, Priyadharsini and Balasubramaniam (2019) have suggested a multi-criteria decision-making method using accuracy function and applied it to an investment company's selection problem. In addition to the aforementioned studies, the concept of *ivif*-sets has also been used in a wide range of fields, including topology and algebra (Hemavathi et al. 2018; Park 2016, 2017; Senapati and Shum 2019).

However, more general forms are needed for mathematical modelling of some problems in the event that the parameters or alternatives (objects) have more serious uncertainties. Therefore, in this paper, we define the concept of the interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets (*d*-sets) by combining the concepts of interval-valued intuitionistic fuzzy parameterized soft sets (Deli and Karataş 2016) and interval-valued intuitionistic fuzzy soft sets (Jiang et al. 2010; Min 2008). Since this concept has a great modelling ability and provides new fields of study for researchers, it is worth conducting the study.

In Sect. 2 of this study, we present some of the basic definitions and properties required in the next sections of the paper. In Sect. 3, we define the concept of *d*-sets and investigate some of its basic properties. In Sect. 4, we suggest a new soft decision-making method by using *d*-sets. This method provides selecting optimal elements from the alternatives. In Sect. 5, we apply this method to a problem of the determination of eligible candidates for the positions. In Sect. 6, we define four new concepts, i.e. mean reduction, mean bireduction, mean bireduction-reduction, and mean reduction-bireduction. By using these concepts, we apply four soft decision-making methods constructed via *ifpifs*-sets, *ivfpifs*-sets, *fpifs*-sets, and *fpfs*-sets to the problem mentioned above. We then compare the ranking performance of the proposed method with those of the four methods. In Sect. 7, we apply five methods to a real problem concerning performance-based value assignment to some filters used in image denoising, so that we can order them in terms of performance. Moreover, we compare the ranking performances of these methods. Finally, we discuss *d*-sets and the proposed method for further research. This study is a part of the first author's PhD dissertation.

2 Preliminaries

In this section, we first provide several well-known definitions. Throughout this paper, let *Int*([0, 1]) be the set of all closed classical subintervals of [0, 1].

Definition 1 Let $\gamma_1, \gamma_2 \in Int([0,1])$. For $\gamma_1 := [\gamma_1^-, \gamma_1^+]$ and $\gamma_2 := [\gamma_2^-, \gamma_2^+]$, if $\gamma_1^- \leq \gamma_2^-$ and $\gamma_1^+ \leq \gamma_2^+$, then γ_1 is called a subinterval of γ_2 and is denoted by $\gamma_1 \tilde{\subseteq} \gamma_2$.

Definition 2 Let $\gamma_1, \gamma_2 \in Int([0,1])$. Then, $\gamma_1 \tilde{\leq} \gamma_2 \Leftrightarrow \gamma_1 \tilde{\subseteq} \gamma_2$.

Definition 3 Let $\gamma, \gamma_1, \gamma_2 \in Int([0,1])$, $c \in \mathbb{R}^+$, $\gamma := [\gamma^-, \gamma^+]$, $\gamma_1 := [\gamma_1^-, \gamma_1^+]$, and $\gamma_2 := [\gamma_2^-, \gamma_2^+]$. Then, $\gamma_1 + \gamma_2 := [\gamma_1^- + \gamma_2^-, \gamma_1^+ + \gamma_2^+]$, $\gamma_1 - \gamma_2 := [\gamma_1^- - \gamma_2^+, \gamma_1^+ - \gamma_2^-]$, $\gamma_1 \cdot \gamma_2 := [\gamma_1^- \cdot \gamma_2^-, \gamma_1^+ \cdot \gamma_2^+]$, and $c \cdot \gamma := [c \cdot \gamma^-, c \cdot \gamma^+]$.

Secondly, we present the concept of interval-valued intuitionistic fuzzy sets (Atanassov and Gargov 1989) and some of its basic properties (Atanassov 1994; Atanassov and Gargov 1989).

Definition 4 (Atanassov and Gargov 1989) Let E be a universal set and κ be a function from E to $Int([0,1]) \times Int([0,1])$. Then, the set $\left\{ {}^{\alpha(x)}_{\beta(x)}x : x \in E \right\}$ being the graphic of κ is called an interval-valued intuitionistic fuzzy set (*ivif*-set) over E.

Here, for all $x \in E$, $\alpha(x) := [\alpha^-(x), \alpha^+(x)]$ and $\beta(x) := [\beta^-(x), \beta^+(x)]$ such that $\alpha^+(x) + \beta^+(x) \leq 1$. Moreover, α and β are called membership function and nonmembership function in an *ivif*-set, respectively.

Note 1 Since $[\alpha(x), \alpha(x)] := \alpha(x)$, for all $x \in E$, we use ${}^{\alpha(x)}_{\beta(x)}x$ instead of ${}^{[\alpha(x),\alpha(x)]}_{[\beta(x),\beta(x)]}x$. Moreover, we do not display the elements ${}^{0}_{1}x$ in an *ivif*-set.

In the present paper, the set of all *ivif*-sets over E is denoted by $IVIF(E)$. In $IVIF(E)$, since the $graph(\kappa)$ and κ generate each other uniquely, the notations are interchangeable. Therefore, as long as it does not cause any confusion, we denote an *ivif*-set $graph(\kappa)$ by κ.

Example 1 Let $E = \{x_1, x_2, x_3, x_4\}$ be a universal set. Then,

$$\kappa = \left\{ {}^{[0.2,0.4]}_{[0.4,0.6]}x_1, {}^{[0,0.2]}_{[0.5,0.7]}x_2, {}^{[0.3,0.5]}_{[0.1,0.2]}x_4 \right\}$$

is an *ivif*-set over E.

In the present paper, for $\lambda, \varepsilon \in Int([0,1])$, let ${}^{\lambda}_{\varepsilon}E$ denote an *ivif*-set κ over E such that $\alpha(x) = \lambda$ and $\beta(x) = \varepsilon$, for all $x \in E$.

Definition 5 (Atanassov 1994) Let $\kappa \in IVIF(E)$. For all $x \in E$, if $\alpha(x) = 0$ and $\beta(x) = 1$, then κ is called empty *ivif*-set and is denoted by ${}^{0}_{1}E$ or 0_E.

Definition 6 (Atanassov 1994) Let $\kappa \in IVIF(E)$. For all $x \in E$, if $\alpha(x) = 1$ and $\beta(x) = 0$, then κ is called universal *ivif*-set and is denoted by ${}^{1}_{0}E$ or 1_E.

Definition 7 (Atanassov and Gargov 1989) Let $\kappa_1, \kappa_2 \in IVIF(E)$. For all $x \in E$, if $\alpha_1(x) \tilde{\leq} \alpha_2(x)$ and $\beta_2(x) \tilde{\leq} \beta_1(x)$, then κ_1 is called a subset of κ_2 and is denoted by $\kappa_1 \tilde{\subseteq} \kappa_2$.

Proposition 1 *Let* $\kappa, \kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $\kappa \tilde{\subseteq} 1_E$, $0_E \tilde{\subseteq} \kappa$, $\kappa \tilde{\subseteq} \kappa$, *and* $[\kappa_1 \tilde{\subseteq} \kappa_2 \wedge \kappa_2 \tilde{\subseteq} \kappa_3] \Rightarrow \kappa_1 \tilde{\subseteq} \kappa_3$.

Definition 8 Let $\kappa_1, \kappa_2 \in IVIF(E)$. Then, $\kappa_1 \tilde{\leq} \kappa_2 \Leftrightarrow \kappa_1 \tilde{\subseteq} \kappa_2$.

Definition 9 (Atanassov and Gargov 1989) Let $\kappa_1, \kappa_2 \in IVIF(E)$. If $\kappa_1 \tilde{\leq} \kappa_2$ and $\kappa_2 \tilde{\leq} \kappa_1$, then κ_1 and κ_2 are called equal *ivif*-sets and is denoted by $\kappa_1 = \kappa_2$.

Proposition 2 *Let* $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $[\kappa_1 = \kappa_2 \wedge \kappa_2 = \kappa_3] \Rightarrow \kappa_1 = \kappa_3$.

Definition 10 Let $\kappa_1, \kappa_2 \in IVIF(E)$. If $\kappa_1 \tilde{\subseteq} \kappa_2$ and $\kappa_1 \neq \kappa_2$, then κ_1 is called a proper subset of κ_2 and is denoted by $\kappa_1 \tilde{\subsetneq} \kappa_2$.

Definition 11 (Atanassov and Gargov 1989) Let $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. For all $x \in E$, if $\alpha_3(x) = \sup\{\alpha_1(x), \alpha_2(x)\}$ and $\beta_3(x) = \inf\{\beta_1(x), \beta_2(x)\}$, then κ_3 is called union of κ_1 and κ_2 and is denoted by $\kappa_1 \tilde{\cup} \kappa_2$.

Proposition 3 (Atanassov 1994) *Let* $\kappa, \kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $\kappa \tilde{\cup} \kappa = \kappa$, $\kappa_1 \tilde{\cup} \kappa_2 = \kappa_2 \tilde{\cup} \kappa_1$, *and* $(\kappa_1 \tilde{\cup} \kappa_2) \tilde{\cup} \kappa_3 = \kappa_1 \tilde{\cup} (\kappa_2 \tilde{\cup} \kappa_3)$.

Proposition 4 *Let* $\kappa, \kappa_1, \kappa_2 \in IVIF(E)$. *Then,* $\kappa \tilde{\cup} 0_E = \kappa$, $\kappa \tilde{\cup} 1_E = 1_E$, *and* $[\kappa_1 \tilde{\subseteq} \kappa_2 \Rightarrow \kappa_1 \tilde{\cup} \kappa_2 = \kappa_2]$.

Definition 12 (Atanassov and Gargov 1989) Let $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. For all $x \in E$, if $\alpha_3(x) = \inf\{\alpha_1(x), \alpha_2(x)\}$ and $\beta_3(x) = \sup\{\beta_1(x), \beta_2(x)\}$, then κ_3 is called intersection of κ_1 and κ_2 and is denoted by $\kappa_1 \tilde{\cap} \kappa_2$.

Proposition 5 (Atanassov 1994) *Let* $\kappa, \kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $\kappa \tilde{\cap} \kappa = \kappa$, $\kappa_1 \tilde{\cap} \kappa_2 = \kappa_2 \tilde{\cap} \kappa_1$, *and* $(\kappa_1 \tilde{\cap} \kappa_2) \tilde{\cap} \kappa_3 = \kappa_1 \tilde{\cap} (\kappa_2 \tilde{\cap} \kappa_3)$.

Proposition 6 *Let* $\kappa, \kappa_1, \kappa_2 \in IVIF(E)$. *Then,* $\kappa \tilde{\cap} 0_E = 0_E$, $\kappa \tilde{\cap} 1_E = \kappa$, *and* $[\kappa_1 \tilde{\subseteq} \kappa_2 \Rightarrow \kappa_1 \tilde{\cap} \kappa_2 = \kappa_1]$.

Proposition 7 (Atanassov 1994) *Let* $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $\kappa_1 \tilde{\cup} (\kappa_2 \tilde{\cap} \kappa_3) = (\kappa_1 \tilde{\cup} \kappa_2) \tilde{\cap} (\kappa_1 \tilde{\cup} \kappa_3)$ *and* $\kappa_1 \tilde{\cap} (\kappa_2 \tilde{\cup} \kappa_3) = (\kappa_1 \tilde{\cap} \kappa_2) \tilde{\cup} (\kappa_1 \tilde{\cap} \kappa_3)$.

Definition 13 Let $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. For all $x \in E$, if $\alpha_3(x) = \inf\{\alpha_1(x), \beta_2(x)\}$ and $\beta_3(x) = \sup\{\beta_1(x), \alpha_2(x)\}$, then κ_3 is called difference between κ_1 and κ_2 and is denoted by $\kappa_1 \tilde{\setminus} \kappa_2$.

Proposition 8 *Let* $\kappa \in IVIF(E)$. *Then,* $\kappa \tilde{\setminus} 0_E = \kappa$ *and* $\kappa \tilde{\setminus} 1_E = 0_E$.

Note 2 It must be noted that the difference is non-commutative and non-associative. For example, let $E = \{x\}$, $\kappa_1 = \left\{ {}^{[0.1,0.3]}_{[0.2,0.4]}x \right\}$, $\kappa_2 = \left\{ {}^{[0.4,0.5]}_{[0,0.1]}x \right\}$, and $\kappa_3 = \left\{ {}^{[0.5,0.7]}_{[0.1,0.2]}x \right\}$. Since $\kappa_1 \tilde{\setminus} \kappa_2 = \left\{ {}^{[0,0.1]}_{[0.4,0.5]}x \right\}$ and $\kappa_2 \tilde{\setminus} \kappa_1 = \left\{ {}^{[0.2,0.4]}_{[0.1,0.3]}x \right\}$, then $\kappa_1 \tilde{\setminus} \kappa_2 \neq \kappa_2 \tilde{\setminus} \kappa_1$. Similarly, since $\kappa_1 \tilde{\setminus} (\kappa_2 \tilde{\setminus} \kappa_3) = \left\{ {}^{[0.1,0.3]}_{[0.2,0.4]}x \right\}$ and $(\kappa_1 \tilde{\setminus} \kappa_2) \tilde{\setminus} \kappa_3 = \left\{ {}^{[0,0.1]}_{[0.5,0.7]}x \right\}$, then $\kappa_1 \tilde{\setminus} (\kappa_2 \tilde{\setminus} \kappa_3) \neq (\kappa_1 \tilde{\setminus} \kappa_2) \tilde{\setminus} \kappa_3$.

Definition 14 (Atanassov and Gargov 1989) Let $\kappa_1, \kappa_2 \in IVIF(E)$. For all $x \in E$, if $\alpha_2(x) = \beta_1(x)$ and $\beta_2(x) = \alpha_1(x)$, then κ_2 is called complement of κ_1 and is denoted by $\kappa_1^{\tilde{c}}$. It is clear that, $\kappa_1^{\tilde{c}} = 1_E \tilde{\setminus} \kappa_1$.

Proposition 9 *Let* $\kappa, \kappa_1, \kappa_2 \in IVIF(E)$. *Then,* $(\kappa^{\tilde{c}})^{\tilde{c}} = \kappa$, $0_E^{\tilde{c}} = 1_E$, $\kappa_1 \tilde{\setminus} \kappa_2 = \kappa_1 \tilde{\cap} \kappa_2^{\tilde{c}}$, *and* $[\kappa_1 \tilde{\subseteq} \kappa_2 \Rightarrow \kappa_2^{\tilde{c}} \tilde{\subseteq} \kappa_1^{\tilde{c}}]$.

Proposition 10 (Atanassov and Gargov 1989) *Let* $\kappa_1, \kappa_2 \in IVIF(E)$. *Then, De Morgan's laws are valid, i.e.* $(\kappa_1 \tilde{\cup} \kappa_2)^{\tilde{c}} = \kappa_1^{\tilde{c}} \tilde{\cap} \kappa_2^{\tilde{c}}$ *and* $(\kappa_1 \tilde{\cap} \kappa_2)^{\tilde{c}} = \kappa_1^{\tilde{c}} \tilde{\cup} \kappa_2^{\tilde{c}}$.

Definition 15 Let $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. For all $x \in E$, if

$$\alpha_3(x) = \sup\{\inf\{\alpha_1(x), \beta_2(x)\}, \inf\{\alpha_2(x), \beta_1(x)\}\}$$

and

$$\beta_3(x) = \inf\{\sup\{\beta_1(x), \alpha_2(x)\}, \sup\{\beta_2(x), \alpha_1(x)\}\}$$

then κ_3 is called symmetric difference between κ_1 and κ_2 and is denoted by $\kappa_1 \tilde{\triangle} \kappa_2$.

Proposition 11 *Let* $\kappa, \kappa_1, \kappa_2 \in IVIF(E)$. *Then,* $\kappa \tilde{\triangle} 0_E = \kappa$, $\kappa \tilde{\triangle} 1_E = \kappa^{\tilde{c}}$, *and* $\kappa_1 \tilde{\triangle} \kappa_2 = \kappa_2 \tilde{\triangle} \kappa_1$.

Note 3 It must be noted that the symmetric difference is non-associative. For example, let $E = \{x\}$, $\kappa_1 = \left\{ {}^{[0.2,0.3]}_{[0.4,0.7]}x \right\}$, $\kappa_2 = \left\{ {}^{[0.2,0.4]}_{[0.5,0.6]}x \right\}$, and $\kappa_3 = \left\{ {}^{[0.1,0.4]}_{[0,0.5]}x \right\}$. Since $\kappa_1 \tilde{\triangle} (\kappa_2 \tilde{\triangle} \kappa_3) = \left\{ {}^{[0.2,0.4]}_{[0.2,0.5]}x \right\}$ and $(\kappa_1 \tilde{\triangle} \kappa_2) \tilde{\triangle} \kappa_3 = \left\{ {}^{[0.1,0.4]}_{[0.2,0.5]}x \right\}$, then $\kappa_1 \tilde{\triangle} (\kappa_2 \tilde{\triangle} \kappa_3) \neq (\kappa_1 \tilde{\triangle} \kappa_2) \tilde{\triangle} \kappa_3$.

Definition 16 Let $\kappa_1, \kappa_2 \in IVIF(E)$. If $\kappa_1 \tilde{\cap} \kappa_2 = 0_E$, then κ_1 and κ_2 are called disjoint.

Definition 17 (Atanassov 1994) Let $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. For all $x \in E$, if $\alpha_3(x) = [\alpha_1^-(x) + \alpha_2^-(x) - \alpha_1^-(x)\alpha_2^-(x), \alpha_1^+(x) + \alpha_2^+(x) - \alpha_1^+(x)\alpha_2^+(x)]$ and $\beta_3(x) = [\beta_1^-(x)\beta_2^-(x), \beta_1^+(x)\beta_2^+(x)]$, then κ_3 is called sum of κ_1 and κ_2 and is denoted by $\kappa_1 \tilde{+} \kappa_2$.

Proposition 12 (Atanassov 1994) *Let* $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $\kappa_1 \tilde{+} \kappa_2 = \kappa_2 \tilde{+} \kappa_1$ *and* $(\kappa_1 \tilde{+} \kappa_2) \tilde{+} \kappa_3 = \kappa_1 \tilde{+} (\kappa_2 \tilde{+} \kappa_3)$.

Proposition 13 *Let* $\kappa \in IVIF(E)$. *Then,* $\kappa \tilde{+} 0_E = \kappa$ *and* $\kappa \tilde{+} 1_E = 1_E$.

Definition 18 (Atanassov 1994) Let $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. For all $x \in E$, if $\alpha_3(x) = [\alpha_1^-(x)\alpha_2^-(x), \alpha_1^+(x)\alpha_2^+(x)]$ and $\beta_3(x) = [\beta_1^-(x) + \beta_2^-(x) - \beta_1^-(x)\beta_2^-(x), \beta_1^+(x) + \beta_2^+(x) - \beta_1^+(x)\beta_2^+(x)]$, then κ_3 is called product of κ_1 and κ_2 and is denoted by $\kappa_1 \tilde{\cdot} \kappa_2$.

Proposition 14 (Atanassov 1994) *Let* $\kappa_1, \kappa_2, \kappa_3 \in IVIF(E)$. *Then,* $\kappa_1 \tilde{\cdot} \kappa_2 = \kappa_2 \tilde{\cdot} \kappa_1$ *and* $(\kappa_1 \tilde{\cdot} \kappa_2) \tilde{\cdot} \kappa_3 = \kappa_1 \tilde{\cdot} (\kappa_2 \tilde{\cdot} \kappa_3)$.

Proposition 15 *Let* $\kappa \in IVIF(E)$. *Then,* $\kappa \tilde{\cdot} 0_E = 0_E$ *and* $\kappa \tilde{\cdot} 1_E = \kappa$.

Proposition 16 (Atanassov 1994) *Let* $\kappa_1, \kappa_2 \in IVIF(E)$. *Then,* $(\kappa_1 \tilde{+} \kappa_2)^{\tilde{c}} = \kappa_1^{\tilde{c}} \tilde{\cdot} \kappa_2^{\tilde{c}}$ *and* $(\kappa_1 \tilde{\cdot} \kappa_2)^{\tilde{c}} = \kappa_1^{\tilde{c}} \tilde{+} \kappa_2^{\tilde{c}}$.

3 Interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets

In this section, we first define the concept of interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets and introduce some of its basic properties. The primary purpose of the present section is to make a theoretical contribution to the conceptualization of soft sets (Molodtsov 1999) and *ivif*-sets (Atanassov and Gargov 1989).

Definition 19 Let U be a universal set, E be a parameter set, $\kappa \in IVIF(E)$, and f be a function from κ to $IVIF(U)$. Then, the set $\left\{\left({}^{\alpha(x)}_{\beta(x)}x, f({}^{\alpha(x)}_{\beta(x)}x)\right) : x \in E\right\}$ being the graphic of f is called an interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft set (d-set) parameterized via E over U (or briefly over U).

Note 4 We do not display the elements $({}^{0}_{1}x, 0_U)$ in a d-set. Here, 0_U is the empty *ivif*-set over U.

In the present paper, the set of all d-sets over U is denoted by $D_E(U)$. In $D_E(U)$, since the *graph*(f) and f generate each other uniquely, the notations are interchangeable. Therefore, as long as it does not cause any confusion, we denote a d-set *graph*(f) by f.

Example 2 Let $E = \{x_1, x_2, x_3, x_4\}$ be a parameter set, $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universal set, $\kappa = \left\{{}^{[0.2,0.5]}_{[0.3,0.4]}x_1, {}^{[0.3,0.4]}_{[0.1,0.3]}x_2, {}^{1}_{0}x_4\right\}$, $f\left({}^{[0.2,0.5]}_{[0.3,0.4]}x_1\right) = \left\{{}^{[0.1,0.3]}_{[0.2,0.6]}u_2, {}^{[0.8,0.9]}_{[0,0.1]}u_3\right\}$, $f\left({}^{[0.3,0.4]}_{[0.1,0.3]}x_2\right) = 0_U$, $f({}^{0}_{1}x_3) = 0_U$, and $f({}^{1}_{0}x_4) = \left\{{}^{0.7}_{0.2}u_5\right\}$. Then, the d-set f over U is as follows:

$$f = \left\{\left({}^{[0.2,0.5]}_{[0.3,0.4]}x_1, \left\{{}^{[0.1,0.3]}_{[0.2,0.6]}u_2, {}^{[0.8,0.9]}_{[0,0.1]}u_3\right\}\right), \left({}^{[0.3,0.4]}_{[0.1,0.3]}x_2, 0_U\right), \left({}^{1}_{0}x_4, \left\{{}^{0.7}_{0.2}u_5\right\}\right)\right\}$$

Since ignoring a portion of the values of parameters or alternatives of a d-set may be a necessary or facilitating way to the solution in some decision-making problems, the definition of the restriction of a d-set, unlike the simple restriction, should be as follows. Thus, the restriction of a d-set again belongs to the $D_E(U)$. For more detail, see (Enginoğlu 2012; Enginoğlu and Çağman n.d.).

Definition 20 Let $f, f_1 \in D_E(U)$ and $A \subseteq E$. Then, Af_1-restriction of f, denoted by f_{Af_1}, is defined by

$$\alpha_{A\kappa_1}(x) := \begin{cases} \alpha(x), & x \in A \\ \alpha_1(x), & x \in E \backslash A \end{cases}$$

$$\beta_{A\kappa_1}(x) := \begin{cases} \beta(x), & x \in A \\ \beta_1(x), & x \in E \backslash A \end{cases}$$

and

$$f_{Af_1}\left({}^{\alpha_{A\kappa_1}(x)}_{\beta_{A\kappa_1}(x)}x\right) := \begin{cases} f({}^{\alpha(x)}_{\beta(x)}x), & x \in A \\ f_1({}^{\alpha_1(x)}_{\beta_1(x)}x), & x \in E \backslash A \end{cases}$$

Example 3 Let us consider the d-set f provided in Example 2, $A = \{x_1, x_3\}$, and $f_1 \in D_E(U)$ such that

$$f_1 = \left\{\left({}^{[0.1,0.6]}_{[0.2,0.3]}x_2, \left\{{}^{[0.7,0.8]}_{[0,0.1]}u_3, {}^{[0.5,0.7]}_{[0,0.1]}u_4\right\}\right), \left({}^{[0.1,0.3]}_{[0.2,0.4]}x_3, 1_U\right), \left({}^{1}_{0}x_4, \left\{{}^{[0.3,0.8]}_{[0.1,0.2]}u_1, {}^{[0.1,0.2]}_{[0,0.4]}u_2\right\}\right)\right\}$$

Then,

$$f_{Af_1} = \left\{\left({}^{[0.2,0.5]}_{[0.3,0.4]}x_1, \left\{{}^{[0.1,0.3]}_{[0.2,0.6]}u_2, {}^{[0.8,0.9]}_{[0,0.1]}u_3\right\}\right), \left({}^{[0.1,0.6]}_{[0.2,0.3]}x_2, \left\{{}^{[0.7,0.8]}_{[0,0.1]}u_3, {}^{[0.5,0.7]}_{[0,0.1]}u_4\right\}\right), \left({}^{1}_{0}x_4, \left\{{}^{[0.3,0.8]}_{[0.1,0.2]}u_1, {}^{[0.1,0.2]}_{[0,0.4]}u_2\right\}\right)\right\}$$

Definition 21 Let $f \in D_E(U)$. If $\kappa = 0_E$ and for all $x \in E$, $f({}^{0}_{1}x) = 0_U$, then f is called empty d-set and is denoted by $\tilde{0}$.

Definition 22 Let $f \in D_E(U)$. If $\kappa = 1_E$ and for all $x \in E$, $f({}^{1}_{0}x) = 1_U$, then f is called universal d-set and is denoted by $\tilde{1}$.

Definition 23 Let $f_1, f_2 \in D_E(U)$. If $\kappa_1 \tilde{\subseteq} \kappa_2$ and for all $x \in E$, $f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\subseteq} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_1 is called a subset of f_2 and is denoted by $f_1 \tilde{\subseteq} f_2$.

Proposition 17 *Let $f, f_1, f_2, f_3 \in D_E(U)$. Then,*

(i) $f \tilde{\subseteq} \tilde{1}$
(ii) $\tilde{0} \tilde{\subseteq} f$
(iii) $f \tilde{\subseteq} f$
(iv) $[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_3] \Rightarrow f_1 \tilde{\subseteq} f_3$

Remark 1 $f_1 \tilde{\subseteq} f_2$ does not imply that every element of f_1 is an element of f_2. For example, let $E = \{x_1, x_2, x_3\}$ be a parameter set, $U = \{u_1, u_2, u_3\}$ be a universal set,

$$f_1 = \left\{\left({}^{[0.1,0.7]}_{[0.2,0.3]}x_1, \left\{{}^{[0.3,0.4]}_{[0.2,0.5]}u_1, {}^{[0.3,0.4]}_{[0,0.1]}u_2, {}^{[0,0.2]}_{[0.5,0.6]}u_3\right\}\right), \left({}^{[0.2,0.3]}_{[0.4,0.5]}x_2, \left\{{}^{[0.1,0.3]}_{[0.4,0.5]}u_1, {}^{[0,0.1]}_{[0.3,0.9]}u_2, {}^{[0.3,0.4]}_{[0.4,0.6]}u_3\right\}\right), \left({}^{[0.5,0.8]}_{[0.1,0.2]}x_3, \left\{{}^{[0.2,0.3]}_{[0.2,0.6]}u_1, {}^{[0.3,0.4]}_{[0.4,0.6]}u_2, {}^{[0.2,0.3]}_{[0.4,0.7]}u_3\right\}\right)\right\},$$

and

$$f_2 = \left\{\left({}^{[0.2,0.8]}_{[0.1,0.2]}x_1, \left\{{}^{[0.4,0.5]}_{[0,0.3]}u_1, {}^{[0.5,0.8]}_{[0,0.1]}u_2, {}^{[0.2,0.4]}_{[0.3,0.5]}u_3\right\}\right), \left({}^{[0.3,0.4]}_{[0.2,0.5]}x_2, \left\{{}^{[0.2,0.4]}_{[0.2,0.5]}u_1, {}^{[0,0.1]}_{[0,0.9]}u_2, {}^{[0.4,0.5]}_{[0.3,0.4]}u_3\right\}\right), \left({}^{[0.7,0.8]}_{[0,0.1]}x_3, \left\{{}^{[0.3,0.4]}_{[0.1,0.5]}u_1, {}^{[0.6,0.8]}_{[0,0.1]}u_2, {}^{[0.4,0.5]}_{[0.1,0.3]}u_3\right\}\right)\right\}$$

Since $\kappa_1 \tilde{\subseteq} \kappa_2$ and $f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\subseteq} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, for all $x \in E$, then $f_1 \tilde{\subseteq} f_2$ .On the other hand, $f_1 \nsubseteq f_2$ because

$$\left({}^{[0.1,0.7]}_{[0.2,0.3]}x_1, \left\{{}^{[0.3,0.4]}_{[0.2,0.5]}u_1,\ {}^{[0.3,0.4]}_{[0,0.1]}u_2, {}^{[0,0.2]}_{[0.5,0.6]}u_3\right\}\right) \notin f_2$$

although it belong to f_1.

Definition 24 Let $f_1, f_2 \in D_E(U)$. If $\kappa_1 = \kappa_2$ and for all $x \in E$, $f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) = f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_1 and f_2 are called equal d-sets and is denoted by $f_1 = f_2$.

Proposition 18 *Let $f_1, f_2, f_3 \in D_E(U)$. Then,*

(i) $[f_1 = f_2 \wedge f_2 = f_3] \Rightarrow f_1 = f_3$
(ii) $[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_1] \Leftrightarrow f_1 = f_2$

Definition 25 Let $f_1, f_2 \in D_E(U)$. If $f_1 \tilde{\subseteq} f_2$ and $f_1 \neq f_2$, then f_1 is called a proper subset of f_2 and is denoted by $f_1 \tilde{\subsetneq} f_2$.

Definition 26 Let $f_1, f_2, f_3 \in D_E(U)$. If $\kappa_3 = \kappa_1 \tilde{\cup} \kappa_2$ and for all $x \in E$, $f_3({}^{\alpha_3(x)}_{\beta_3(x)}x) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cup} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_3 is called union of f_1 and f_2 and is denoted by $f_1 \tilde{\cup} f_2$.

Definition 27 Let $f_1, f_2, f_3 \in D_E(U)$. If $\kappa_3 = \kappa_1 \tilde{\cap} \kappa_2$ and for all $x \in E$, $f_3({}^{\alpha_3(x)}_{\beta_3(x)}x) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cap} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_3 is called intersection of f_1 and f_2 and is denoted by $f_1 \tilde{\cap} f_2$.

Proposition 19 *Let $f, f_1, f_2, f_3 \in D_E(U)$. Then,*

(i) $f \tilde{\cup} f = f$ *and* $f \tilde{\cap} f = f$
(ii) $f \tilde{\cup} \tilde{0} = f$ *and* $f \tilde{\cap} \tilde{0} = \tilde{0}$
(iii) $f \tilde{\cup} \tilde{1} = \tilde{1}$ *and* $f \tilde{\cap} \tilde{1} = f$
(iv) $f_1 \tilde{\cup} f_2 = f_2 \tilde{\cup} f_1$ *and* $f_1 \tilde{\cap} f_2 = f_2 \tilde{\cap} f_1$
(v) $(f_1 \tilde{\cup} f_2) \tilde{\cup} f_3 = f_1 \tilde{\cup} (f_2 \tilde{\cup} f_3)$
$(f_1 \tilde{\cap} f_2) \tilde{\cap} f_3 = f_1 \tilde{\cap} (f_2 \tilde{\cap} f_3)$
(vi) $f_1 \tilde{\cup} (f_2 \tilde{\cap} f_3) = (f_1 \tilde{\cup} f_2) \tilde{\cap} (f_1 \tilde{\cup} f_3)$
$f_1 \tilde{\cap} (f_2 \tilde{\cup} f_3) = (f_1 \tilde{\cap} f_2) \tilde{\cup} (f_1 \tilde{\cap} f_3)$
(vii) $[f_1 \tilde{\subseteq} f_2 \Rightarrow f_1 \tilde{\cup} f_2 = f_2]$
$[f_1 \tilde{\subseteq} f_2 \Rightarrow f_1 \tilde{\cap} f_2 = f_1]$

Definition 28 Let $f_1, f_2, f_3 \in D_E(U)$. If $\kappa_3 = \kappa_1 \tilde{\setminus} \kappa_2$ and for all $x \in E$, $f_3({}^{\alpha_3(x)}_{\beta_3(x)}x) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\setminus} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_3 is called difference between f_1 and f_2 and is denoted by $f_1 \tilde{\setminus} f_2$.

Proposition 20 *Let $f \in D_E(U)$. Then,*

(i) $f \tilde{\setminus} \tilde{0} = f$
(ii) $f \tilde{\setminus} \tilde{1} = \tilde{0}$

Note 5 It must be noted that the difference is non-commutative and non-associative.

Definition 29 Let $f_1, f_2 \in D_E(U)$. If $\kappa_2 = \kappa_1^{\tilde{c}}$ and for all $x \in E$, $f_2({}^{\beta_1(x)}_{\alpha_1(x)}x) = (f_1({}^{\alpha_1(x)}_{\beta_1(x)}x))^{\tilde{c}}$, then f_2 is called complement of f_1 and is denoted by $f_1^{\tilde{c}}$. That is, for all $x \in E$, $f_1^{\tilde{c}}({}^{\beta_1(x)}_{\alpha_1(x)}x) = (f_1({}^{\alpha_1(x)}_{\beta_1(x)}x))^{\tilde{c}}$. It is clear that, $f_1^{\tilde{c}} = \tilde{1} \tilde{\setminus} f_1$.

Proposition 21 *Let $f, f_1, f_2 \in D_E(U)$. Then,*

(i) $(f^{\tilde{c}})^{\tilde{c}} = f$
(ii) $\tilde{0}^{\tilde{c}} = \tilde{1}$
(iii) $f_1 \tilde{\setminus} f_2 = f_1 \tilde{\cap} f_2^{\tilde{c}}$
(iv) $f_1 \tilde{\subseteq} f_2 \Rightarrow f_2^{\tilde{c}} \tilde{\subseteq} f_1^{\tilde{c}}$

Proposition 22 *Let $f_1, f_2 \in D_E(U)$. Then, the following De Morgan's laws are valid.*

(i) $(f_1 \tilde{\cup} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cap} f_2^{\tilde{c}}$
(ii) $(f_1 \tilde{\cap} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cup} f_2^{\tilde{c}}$

Definition 30 Let $f_1, f_2, f_3 \in D_E(U)$. If $\kappa_3 = \kappa_1 \tilde{\triangle} \kappa_2$ and for all $x \in E$, $f_3({}^{\alpha_3(x)}_{\beta_3(x)}x) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\triangle} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_3 is called symmetric difference between f_1 and f_2 and is denoted by $f_1 \tilde{\triangle} f_2$.

Proposition 23 *Let $f, f_1, f_2 \in D_E(U)$. Then,*

(i) $f \tilde{\triangle} \tilde{0} = f$
(ii) $f \tilde{\triangle} \tilde{1} = f^{\tilde{c}}$
(iii) $f_1 \tilde{\triangle} f_2 = f_2 \tilde{\triangle} f_1$

Note 6 It must be noted that the symmetric difference operation is non-associative.

Definition 31 Let $f_1, f_2 \in D_E(U)$. If $f_1 \tilde{\cap} f_2 = \tilde{0}$, then f_1 and f_2 are called disjoint.

Example 4 Let $E = \{x_1, x_2, x_3\}$ be a parameter set, $U = \{u_1, u_2, u_3, u_4\}$ be a universal set,

$$\begin{aligned} f_1 = \Big\{ &\left({}^{[0.1,0.5]}_{[0.2,0.3]}x_1, \left\{{}^{1}_{0}u_2, {}^{[0,0.1]}_{[0.5,0.6]}u_3, {}^{[0,0.1]}_{[0.5,0.7]}u_4\right\}\right), \\ &\left({}^{1}_{0}x_2, \left\{{}^{[0.2,0.4]}_{[0.2,0.5]}u_1, {}^{[0,0.1]}_{[0,0.5]}u_2, {}^{[0.4,0.5]}_{[0.3,0.4]}u_3, {}^{[0.5,0.6]}_{[0.1,0.3]}u_4\right\}\right), \\ &\left({}^{[0.3,0.4]}_{[0,0.1]}x_3, \left\{{}^{[0.3,0.4]}_{[0.1,0.5]}u_1, {}^{[0,0.7]}_{[0.1,0.2]}u_3\right\}\right)\Big\}, \end{aligned}$$

and

$$\begin{aligned} f_2 = \Big\{ &\left({}^{[0.2,0.3]}_{[0,0.5]}x_2, \left\{{}^{[0.1,0.2]}_{[0.4,0.5]}u_1, {}^{[0.3,0.7]}_{[0.1,0.2]}u_2, {}^{[0.3,0.4]}_{[0.4,0.6]}u_3\right\}\right), \\ &\left({}^{[0.3,0.8]}_{[0,0.1]}x_3, \left\{{}^{1}_{0}u_1, {}^{[0.2,0.3]}_{[0,0.4]}u_3, {}^{[0.1,0.4]}_{[0.5,0.6]}u_4\right\}\right)\Big\} \end{aligned}$$

Then,

$$f_1\tilde{\cup}f_2 = \Big\{ \Big({}^{[0.1,0.5]}_{[0.2,0.3]}x_1, \Big\{ {}^{1}_{0}u_2, {}^{[0,0.1]}_{[0.5,0.6]}u_3, {}^{[0,0.1]}_{[0.5,0.7]}u_4 \Big\}\Big), \Big({}^{1}_{0}x_2, \Big\{ {}^{[0.2,0.4]}_{[0.2,0.5]}u_1, {}^{[0.3,0.7]}_{[0,0.2]}u_2, {}^{[0.4,0.5]}_{[0.3,0.4]}u_3, {}^{[0.5,0.6]}_{[0.1,0.3]}u_4 \Big\}\Big), \Big({}^{[0.3,0.8]}_{[0,0.1]}x_3, \Big\{ {}^{1}_{0}u_1, {}^{[0.2,0.7]}_{[0,0.2]}u_3, {}^{[0.1,0.4]}_{[0.5,0.6]}u_4 \Big\}\Big)\Big\},$$

$$f_1\tilde{\cap}f_2 = \Big\{ \Big({}^{[0.2,0.3]}_{[0,0.5]}x_2, \Big\{ {}^{[0.1,0.2]}_{[0.4,0.5]}u_1, {}^{[0,0.1]}_{[0.1,0.5]}u_2, {}^{[0.3,0.4]}_{[0.4,0.6]}u_3 \Big\}\Big), \Big({}^{[0.3,0.4]}_{[0,0.1]}x_3, \Big\{ {}^{[0.3,0.4]}_{[0.1,0.5]}u_1, {}^{[0,0.3]}_{[0.1,0.4]}u_3, \Big\}\Big)\Big\},$$

$$f_1\tilde{\veebar}f_2 = \Big\{ \Big({}^{[0.1,0.5]}_{[0.2,0.3]}x_1, \Big\{ {}^{1}_{0}u_2, {}^{[0,0.1]}_{[0.5,0.6]}u_3, {}^{[0,0.1]}_{[0.5,0.7]}u_4 \Big\}\Big), \Big({}^{[0,0.5]}_{[0.2,0.3]}x_2, \Big\{ {}^{[0.2,0.4]}_{[0.2,0.5]}u_1, {}^{[0,0.1]}_{[0.3,0.7]}u_2, {}^{[0.4,0.5]}_{[0.3,0.4]}u_3, {}^{[0.5,0.6]}_{[0.1,0.3]}u_4 \Big\}\Big), \Big({}^{[0,0.1]}_{[0.3,0.8]}x_3, \Big\{ {}^{[0,0.4]}_{[0.2,0.3]}u_3 \Big\}\Big)\Big\},$$

$$f_1^{\tilde{c}} = \Big\{ \Big({}^{[0.2,0.3]}_{[0.1,0.5]}x_1, \Big\{ {}^{1}_{0}u_1, {}^{[0.5,0.6]}_{[0,0.1]}u_3, {}^{[0.5,0.7]}_{[0,0.1]}u_4 \Big\}\Big), \Big({}^{0}_{1}x_2, \Big\{ {}^{[0.2,0.5]}_{[0.2,0.4]}u_1, {}^{[0,0.5]}_{[0,0.1]}u_2, {}^{[0.3,0.4]}_{[0.4,0.5]}u_3, {}^{[0.1,0.3]}_{[0.5,0.6]}u_4 \Big\}\Big), \Big({}^{[0,0.1]}_{[0.3,0.4]}x_3, \Big\{ {}^{[0.1,0.5]}_{[0.3,0.4]}u_1, {}^{1}_{0}u_2, {}^{[0.1,0.2]}_{[0,0.7]}u_3, {}^{1}_{0}u_4 \Big\}\Big)\Big\},$$

and

$$f_1\tilde{\triangle}f_2 = \Big\{ \Big({}^{[0.1,0.5]}_{[0.2,0.3]}x_1, \Big\{ {}^{1}_{0}u_2, {}^{[0,0.1]}_{[0.5,0.6]}u_3, {}^{[0,0.1]}_{[0.5,0.7]}u_4 \Big\}\Big), \Big({}^{[0,0.5]}_{[0.2,0.3]}x_2, \Big\{ {}^{[0.2,0.4]}_{[0.2,0.5]}u_1, {}^{[0,0.5]}_{[0.1,0.2]}u_2, {}^{[0.4,0.5]}_{[0.3,0.4]}u_3, {}^{[0.5,0.6]}_{[0.1,0.3]}u_4 \Big\}\Big), \Big({}^{[0,0.1]}_{[0.3,0.4]}x_3, \Big\{ {}^{[0.1,0.5]}_{[0.3,0.4]}u_1, {}^{[0.1,0.4]}_{[0,0.3]}u_3, {}^{[0.1,0.4]}_{[0.5,0.6]}u_4 \Big\}\Big)\Big\}$$

Definition 32 Let $f_1, f_2, f_3 \in D_E(U)$. If $\kappa_3 = \kappa_1 \tilde{+} \kappa_2$ and for all $x \in E$, $f_3({}^{\alpha_3(x)}_{\beta_3(x)}x) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{+} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_3 is called sum of f_1 and f_2 and is denoted by $f_1 \tilde{+} f_2$.

Definition 33 Let $f_1, f_2, f_3 \in D_E(U)$. If $\kappa_3 = \kappa_1 \tilde{\cdot} \kappa_2$ and for all $x \in E$, $f_3({}^{\alpha_3(x)}_{\beta_3(x)}x) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cdot} f_2({}^{\alpha_2(x)}_{\beta_2(x)}x)$, then f_3 is called product of f_1 and f_2 and is denoted by $f_1 \tilde{\cdot} f_2$.

Proposition 24 *Let $f, f_1, f_2, f_3 \in D_E(U)$. Then,*

(i) $f \tilde{+} \tilde{0} = f$ *and* $f \tilde{\cdot} \tilde{0} = \tilde{0}$
(ii) $f \tilde{+} \tilde{1} = \tilde{1}$ *and* $f \tilde{\cdot} \tilde{1} = f$
(iii) $f_1 \tilde{+} f_2 = f_2 \tilde{+} f_1$ *and* $f_1 \tilde{\cdot} f_2 = f_2 \tilde{\cdot} f_1$
(iv) $(f_1 \tilde{+} f_2) \tilde{+} f_3 = f_1 \tilde{+} (f_2 \tilde{+} f_3)$
$(f_1 \tilde{\cdot} f_2) \tilde{\cdot} f_3 = f_1 \tilde{\cdot} (f_2 \tilde{\cdot} f_3)$
(v) $(f_1 \tilde{+} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cdot} f_2^{\tilde{c}}$ *and* $(f_1 \tilde{\cdot} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{+} f_2^{\tilde{c}}$

Definition 34 Let $f_1 \in D_{E_1}(U)$, $f_2 \in D_{E_2}(U)$, and $f_3 \in D_{E_1 \times E_2}(U)$. For all $(x, y) \in E_1 \times E_2$, if $\alpha_3(x, y) = \inf\{\alpha_1(x), \alpha_2(y)\}$, $\beta_3(x, y) = \sup\{\beta_1(x), \beta_2(y)\}$, and $f_3({}^{\alpha_3(x,y)}_{\beta_3(x,y)}(x, y)) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cap} f_2({}^{\alpha_2(y)}_{\beta_2(y)}y)$, then f_3 is called *and*-product of f_1 and f_2 and is denoted by $f_1 \wedge f_2$.

Definition 35 Let $f_1 \in D_{E_1}(U)$, $f_2 \in D_{E_2}(U)$, and $f_3 \in D_{E_1 \times E_2}(U)$. For all $(x, y) \in E_1 \times E_2$, if $\alpha_3(x, y) = \sup\{\alpha_1(x), \alpha_2(y)\}$, $\beta_3(x, y) = \inf\{\beta_1(x), \beta_2(y)\}$, and $f_3({}^{\alpha_3(x,y)}_{\beta_3(x,y)}(x, y)) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cup} f_2({}^{\alpha_2(y)}_{\beta_2(y)}y)$, then f_3 is called *or*-product of f_1 and f_2 and is denoted by $f_1 \vee f_2$.

Definition 36 Let $f_1 \in D_{E_1}(U)$, $f_2 \in D_{E_2}(U)$, and $f_3 \in D_{E_1 \times E_2}(U)$. For all $(x, y) \in E_1 \times E_2$, if $\alpha_3(x, y) = \inf\{\alpha_1(x), \beta_2(y)\}$, $\beta_3(x, y) = \sup\{\beta_1(x), \alpha_2(y)\}$, and $f_3({}^{\alpha_3(x,y)}_{\beta_3(x,y)}(x, y)) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cap} f_2^{\tilde{c}}({}^{\beta_2(y)}_{\alpha_2(y)}y)$, then f_3 is called *andnot*-product of f_1 and f_2 and is denoted by $f_1 \overline{\wedge} f_2$.

Definition 37 Let $f_1 \in D_{E_1}(U)$, $f_2 \in D_{E_2}(U)$, and $f_3 \in D_{E_1 \times E_2}(U)$. For all $(x, y) \in E_1 \times E_2$, if $\alpha_3(x, y) = \sup\{\alpha_1(x), \beta_2(y)\}$, $\beta_3(x, y) = \inf\{\beta_1(x), \alpha_2(y)\}$, and $f_3({}^{\alpha_3(x,y)}_{\beta_3(x,y)}(x, y)) = f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cup} f_2^{\tilde{c}}({}^{\beta_2(y)}_{\alpha_2(y)}y)$, then f_3 is called *ornot*-product of f_1 and f_2 and is denoted by $f_1 \underline{\vee} f_2$.

Proposition 25 *Let $f_1 \in D_{E_1}(U)$, $f_2 \in D_{E_2}(U)$, and $f_3 \in D_{E_3}(U)$. Then,*

(i) $(f_1 \vee f_2) \vee f_3 = f_1 \vee (f_2 \vee f_3)$
(ii) $(f_1 \wedge f_2) \wedge f_3 = f_1 \wedge (f_2 \wedge f_3)$

Proof Let $E_{123} = E_1 \times E_2 \times E_3$. Then, the proof of (i) is as follows:

$$\begin{aligned}
&(f_1 \vee f_2) \vee f_3 \\
&= \Big\{ \Big({}^{\sup\{\alpha_1(x),\alpha_2(y)\}}_{\inf\{\beta_1(x),\beta_2(y)\}}(x, y), f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cup} f_2({}^{\alpha_2(y)}_{\beta_2(y)}y)\Big) : (x, y) \in E_{12} \Big\} \\
&\quad \vee \Big\{ \Big({}^{\alpha_3(z)}_{\beta_3(z)}z, f_3({}^{\alpha_3(z)}_{\beta_3(z)}z)\Big) : z \in E_3 \Big\} \\
&= \Big\{ \Big({}^{\sup\{\sup\{\alpha_1(x),\alpha_2(y)\},\alpha_3(z)\}}_{\inf\{\inf\{\beta_1(x),\beta_2(y)\},\beta_3(z)\}}(x, y, z), \\
&\qquad \Big(f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \tilde{\cup} f_2({}^{\alpha_2(y)}_{\beta_2(y)}y)\Big) \tilde{\cup} f_3({}^{\alpha_3(z)}_{\beta_3(z)}z)\Big) : (x, y, z) \in E_{123} \Big\} \\
&= \Big\{ \Big({}^{\sup\{\alpha_1(x),\sup\{\alpha_2(y),\alpha_3(z)\}\}}_{\inf\{\beta_1(x),\inf\{\beta_2(y),\beta_3(z)\}\}}(x, y, z), f_1({}^{\alpha_1(x)}_{\beta_1(x)}x) \\
&\qquad \tilde{\cup} \Big(f_2({}^{\alpha_2(y)}_{\beta_2(y)}y) \tilde{\cup} f_3({}^{\alpha_3(z)}_{\beta_3(z)}z)\Big)\Big) : (x, y, z) \in E_{123} \Big\} \\
&= \Big\{ \Big({}^{\alpha_1(x)}_{\beta_1(x)}x, f_1({}^{\alpha_1(x)}_{\beta_1(x)}x)\Big) : x \in E_1 \Big\} \\
&\quad \vee \Big\{ \Big({}^{\sup\{\alpha_2(y),\alpha_3(z)\}}_{\inf\{\beta_2(y),\beta_3(z)\}}(y, z), f_2({}^{\alpha_2(y)}_{\beta_2(y)}y) \tilde{\cup} f_3({}^{\alpha_3(z)}_{\beta_3(z)}z)\Big) : (y, z) \in E_{23} \Big\} \\
&= f_1 \vee (f_2 \vee f_3)
\end{aligned}$$

□

Proposition 26 *Let $f_1 \in D_{E_1}(U)$ and $f_2 \in D_{E_2}(U)$. Then, the following De Morgan's laws are valid.*

(i) $(f_1 \vee f_2)^{\tilde{c}} = f_1^{\tilde{c}} \wedge f_2^{\tilde{c}}$
(ii) $(f_1 \wedge f_2)^{\tilde{c}} = f_1^{\tilde{c}} \vee f_2^{\tilde{c}}$

(iii) $(f_1 \underline{\vee} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \overline{\wedge} f_2^{\tilde{c}}$
(iv) $(f_1 \overline{\wedge} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \underline{\vee} f_2^{\tilde{c}}$

Note 7 It must be noted that the products mentioned above of *d*-sets are non-commutative and non-distributive. Moreover, *andnot*-product and *ornot*-product are non-associative.

4 The proposed soft decision-making method

In this section, we first define an aggregate *ivif*-set of a *d*-set.

Definition 38 (Huang et al. 2013) Let $\kappa \in IVIF(E)$ and $|E| = n$. Then, the average cardinality of κ, denoted by $|\kappa|_a$, is defined by

$$|\kappa|_a := \frac{1}{2} \sum_{i=1}^{n} \left(1 + \frac{\alpha^-(x_i) + \alpha^+(x_i)}{2} - \frac{\beta^-(x_i) + \beta^+(x_i)}{2} \right)$$

Definition 39 A function $\mathcal{A} : D_E(U) \to IVIF(U)$ defined by $\mathcal{A}(f) = f^*$ is called an aggregation operator over U and f^* is called aggregate *ivif*-set of *f*. Here, $f^* = \left\{ {}^{\omega^*(u)}_{\theta^*(u)} u : u \in U \right\}$ such that $\omega^*(u) = \frac{1}{|\kappa|_a} \sum_{x \in E} \alpha(x) \omega_x(u)$, and $\theta^*(u) = \frac{1}{|\kappa|_a} \sum_{x \in E} \beta(x) \theta_x(u)$, for $f = \left\{ \left({}^{\alpha(x)}_{\beta(x)} x, \left\{ {}^{\omega_x(u)}_{\theta_x(u)} u : u \in U \right\} \right) : x \in E \right\}$.

Secondly, we suggest a soft decision-making method that assigns a performance-based value to the alternatives via this aggregate *ivif*-set. Thus, we can choose the optimal elements among the alternatives.

Algorithm Steps of the Proposed Method

Step 1. Construct a *d*-set *f* over U
Step 2. Obtain the aggregate *ivif*-set f^* of *f*
Step 3. Obtain the values $s(u) = \omega^*(u) - \theta^*(u)$, for all $u \in U$
Step 4. Obtain the decision set $\{ {}^{d(u_k)} u_k | u_k \in U \}$ such that

$$d(u_k) = \left[\frac{s^-(u_k) + |\min_i s^-(u_i)|}{\max_i s^+(u_i) + |\min_i s^-(u_i)|}, \frac{s^+(u_k) + |\min_i s^-(u_i)|}{\max_i s^+(u_i) + |\min_i s^-(u_i)|} \right]$$

Step 5. Select the optimal elements among the alternatives via linear ordering relation provided in Xu and Yager (2006)

$$[\gamma_1^-, \gamma_1^+] \leq_{XY} [\gamma_2^-, \gamma_2^+] \Leftrightarrow \left[(\gamma_1^- + \gamma_1^+ < \gamma_2^- + \gamma_2^+) \vee (\gamma_1^- + \gamma_1^+ = \gamma_2^- + \gamma_2^+ \wedge \gamma_1^+ - \gamma_1^- \leq \gamma_2^+ - \gamma_2^-) \right]$$

Here, $s(u) := [s^-(u), s^+(u)]$, for all $u \in U$.

5 An illustrative example for the proposed method

In this section, we apply the proposed method to a problem concerning the eligibility of candidates for two vacant positions in a job advertisement. Assume that six candidates, denoted by $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, have applied for two vacant positions announced by a company. Then, the human resources department (HR) of the company has firstly determined a parameter set $E = \{x_1, x_2, x_3, x_4\}$ such that x_1 = "*knowledge of software*", x_2 = "*knowledge of foreign language*", x_3 = "*age*", and x_4 = "*experience*".

Secondly, HR has obtained the *ivif*-value of the parameters by the membership and the nonmembership functions defined by

$$\alpha(x) := \left[\frac{\min_n \mu_n^x}{\max_n \mu_n^x + \max_n \nu_n^x + \min\left\{ \min_{k \in I} \pi_k^x, \min_{t \in J} \pi_t^x \right\}}, \frac{\max_n \mu_n^x}{\max_n \mu_n^x + \max_n \nu_n^x + \min\left\{ \min_{k \in I} \pi_k^x, \min_{t \in J} \pi_t^x \right\}} \right]$$

and

$$\beta(x) := \left[\frac{\min_n \nu_n^x}{\max_n \mu_n^x + \max_n \nu_n^x + \min\left\{ \min_{k \in I} \pi_k^x, \min_{t \in J} \pi_t^x \right\}}, \frac{\max_n \nu_n^x}{\max_n \mu_n^x + \max_n \nu_n^x + \min\left\{ \min_{k \in I} \pi_k^x, \min_{t \in J} \pi_t^x \right\}} \right]$$

such that $I = \left\{ k : \max_n \mu_n^x = \mu_k^x \right\}$ and $J = \left\{ t : \max_n \nu_n^x = \nu_t^x \right\}$. Here, (μ_n^x), (ν_n^x), and (π_n^x) are ordered *s*-tuples which indicate the degrees of membership, nonmembership, and indeterminacy obtained according to the criteria determined by HR, for parameters.

For example, HR determines five software programs and, for the *n*th software programs, $\mu_n^{x_1}$, $\nu_n^{x_1}$, and $\pi_n^{x_1}$ denote the numbers of employees who have a valid certificate, who do not know how to use the software, and who hold the knowledge of the software but have no valid certificate, respectively. If $(\mu_n^{x_1}) = (18, 10, 15, 16, 12)$, $(\nu_n^{x_1}) = (1, 5, 3, 1, 7)$, and $(\pi_n^{x_1}) = (1, 5, 2, 3, 1)$, then the membership degree and the nonmembership degree of the parameter x_1 are $[0.38, 0.69]$ and $[0.04, 0.27]$, respectively. Similarly, the *ivif*-values of the other parameters can be constructed by HR. Thus, an *ivif*-set κ over E can be given as follows:

$$\kappa = \left\{ {}^{[0.38,0.69]}_{[0.04,0.27]}x_1, {}^{[0.53,0.6]}_{[0.34,0.37]}x_2, {}^{[0.05,0.25]}_{[0.32,0.45]}x_3, {}^{[0.4,0.52]}_{[0.26,0.38]}x_4 \right\}$$

Here, [0.38, 0.69] means that the positive effect of "*knowledge of software*" on success occurs between 38 and 69%. Moreover, [0.04, 0.27] means that the negative effect of "*knowledge of software*" on success ranges from 4 to 27%.

The application of the soft decision-making method proposed in Sect. 4 is as follows:

Step 1. The *d*-set *f* modelling the decision-making problem mentioned above is as follows:

$$f = \left\{ \left({}^{[0.38,0.69]}_{[0.04,0.27]}x_1, \left\{ {}^{[0.36,0.46]}_{[0.11,0.28]}u_1, {}^{[0,0.16]}_{[0.38,0.53]}u_2, {}^{[0.15,0.22]}_{[0,0.75]}u_3, {}^{[0.2,0.62]}_{[0.15,0.24]}u_4 \right\} \right), \left({}^{[0.53,0.6]}_{[0.34,0.37]}x_2, \left\{ {}^{[0.24,0.32]}_{[0.34,0.65]}u_1, {}^{[0.28,0.35]}_{[0,0.62]}u_2, {}^{[0.25,0.32]}_{[0.1,0.64]}u_3, {}^{[0.12,0.44]}_{[0.45,0.54]}u_4, {}^{1}_{0}u_5, {}^{[0,0.34]}_{[0,0.2]}u_6 \right\} \right), \left({}^{[0.05,0.25]}_{[0.32,0.45]}x_3, \left\{ {}^{[0.35,0.55]}_{[0.15,0.29]}u_1, {}^{[0.26,0.35]}_{[0.43,0.55]}u_3, {}^{[0.38,0.52]}_{[0.19,0.25]}u_4, {}^{[0.15,0.24]}_{[0.29,0.72]}u_6 \right\} \right), \left({}^{[0.4,0.52]}_{[0.26,0.38]}x_4, \left\{ {}^{0}_{[0.52,0.58]}u_1, {}^{1}_{0}u_2, {}^{[0.18,0.28]}_{[0.27,0.7]}u_3, {}^{0}_{[0.1,1]}u_4, {}^{[0.35,0.4]}_{[0,0.1]}u_5, {}^{[0.28,0.48]}_{[0.33,0.34]}u_6 \right\} \right) \right\}$$

The *ivif*-value of the candidates for each parameter has been obtained by the membership function $\omega_x(u)$ and the nonmembership function $\theta_x(u)$ defined by

$$\left[\frac{\min\limits_n (\mu^x_u)_n}{\max\limits_n (\mu^x_u)_n + \max\limits_n (\nu^x_u)_n + \min\left\{ \min\limits_{k\in I} (\pi^x_u)_k, \min\limits_{t\in J} (\pi^x_u)_t \right\}}, \frac{\max\limits_n (\mu^x_u)_n}{\max\limits_n (\mu^x_u)_n + \max\limits_n (\nu^x_u)_n + \min\left\{ \min\limits_{k\in I} (\pi^x_u)_k, \min\limits_{t\in J} (\pi^x_u)_t \right\}} \right]$$

and

$$\left[\frac{\min\limits_n (\nu^x_u)_n}{\max\limits_n (\mu^x_u)_n + \max\limits_n (\nu^x_u)_n + \min\left\{ \min\limits_{k\in I} (\pi^x_u)_k, \min\limits_{t\in J} (\pi^x_u)_t \right\}}, \frac{\max\limits_n (\nu^x_u)_n}{\max\limits_n (\mu^x_u)_n + \max\limits_n (\nu^x_u)_n + \min\left\{ \min\limits_{k\in I} (\pi^x_u)_k, \min\limits_{t\in J} (\pi^x_u)_t \right\}} \right]$$

respectively, such that $I = \left\{ k : \max\limits_n (\mu^x_u)_n = (\mu^x_u)_k \right\}$ and $J = \left\{ t : \max\limits_n (\nu^x_u)_n = (\mu^x_u)_t \right\}$. Here, $((\mu^x_u)_n)$, $((\nu^x_u)_n)$, and $((\pi^x_u)_n)$ are ordered s-tuples which indicate the degrees of membership, nonmembership, and indeterminacy according to the parameters of the candidates.

For example, ten questions are asked to the candidates regarding each software program and they are asked to answer these questions using a three-level Likert scale, i.e. positive, negative, and indeterminant. Here, $(\mu^{x_1}_{u_5})_n$, $(\nu^{x_1}_{u_5})_n$, and $(\pi^{x_1}_{u_5})_n$ denote the number of positive, negative, and indeterminant answers according to the x_1 parameter of the candidate u_5, respectively. If $((\mu^{x_1}_{u_5})_n) = (0, 0, 0, 0, 0)$, $((\nu^{x_1}_{u_5})_n) = (10, 10, 10, 10, 10)$, and $((\pi^{x_1}_{u_5})_n) = (0, 0, 0, 0, 0)$, then the membership degree and the nonmembership degree of the candidate u_5 according to the parameter x_1 are $[0, 0] = 0$ and $[1, 1] = 1$, respectively. Similarly, the *ivif*-values of the other candidates can be constructed.

Step 2. f^* is as follows:

$$\left\{ {}^{[0.1253,0.2878]}_{[0.1349,0.2968]}u_1, {}^{[0.2440,0.3739]}_{[0.1491,0.3660]}u_2, {}^{[0.1221,0.2567]}_{[0.1076,0.4239]}u_3, {}^{[0.0706,0.3657]}_{[0.1094,0.3369]}u_4, {}^{[0.2981,0.3595]}_{[0.1602,0.3373]}u_5, {}^{[0.0532,0.2285]}_{[0.0973,0.3547]}u_6 \right\}$$

where

$$\omega^*(u_1) = \frac{[0.38, 0.69] \cdot [0.36, 0.46] + [0.53, 0.6] \cdot [0.24, 0.32]}{2.2475} + \frac{[0.05, 0.25] \cdot [0.35, 0.55] + [0.4, 0.52] \cdot [0, 0]}{2.2475} = [0.1253, 0.2878]$$

and

$$\theta^*(u_1) = \frac{[0.04, 0.27] \cdot [0.11, 0.28] + [0.34, 0.37] \cdot [0.34, 0.65]}{2.2475} + \frac{[0.32, 0.45] \cdot [0.15, 0.29] + [0.26, 0.38] \cdot [0.52, 0.58]}{2.2475} = [0.1349, 0.2968]$$

Step 3. For all $u \in U$, the values $s(u)$ are as follows:

$$s(u_1) = [-0.1715, 0.1529], s(u_2) = [-0.1220, 0.2248], s(u_3) = [-0.3018, 0.1491], s(u_4) = [-0.2663, 0.2563], s(u_5) = [-0.0392, 0.1993], s(u_6) = [-0.3015, 0.1313]$$

Step 4. The decision set is as follows:

$$\left\{ {}^{[0.2334,0.8148]}u_1, {}^{[0.3223,0.9436]}u_2, {}^{[0,0.8079]}u_3, {}^{[0.0636,1]}u_4, {}^{[0.4706,0.8980]}u_5, {}^{[0.0005,0.7760]}u_6 \right\}$$

where $d(u_1)$ is calculated as follows:

$$d(u_1) = \left[\frac{-0.1715 + |-0.3018|}{0.2563 + |-0.3018|}, \frac{0.1529 + |-0.3018|}{0.2563 + |-0.3018|} \right]$$

Step 5. According to the linear ordering relation ($\preceq_{XY}$), the ranking order $u_6 \prec u_3 \prec u_1 \prec u_4 \prec u_2 \prec u_5$ is valid.

The results show that u_5 and u_2 are more eligible for the vacant positions than the others. Thus, the candidates u_5 and u_2 are selected for the positions announced by the company.

6 Comparison results

In this section, we first provide the definitions of fuzzy sets (Zadeh 1965), intuitionistic fuzzy sets (Atanassov 1986), fuzzy parameterized fuzzy soft sets (Çağman et al. 2010), fuzzy parameterized intuitionistic fuzzy soft sets (Sulukan et al. 2019), and intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets (Karaaslan 2016) by taking into account the notations used throughout this paper.

Definition 40 (Zadeh 1965) Let E be a universal set and μ be a function from E to [0, 1]. Then, the set $\left\{{}^{\mu(x)}x : x \in E\right\}$ being the graphic of μ is called a fuzzy set (*f*-set) over E. Besides, the set of all *f*-sets over E is denoted by $F(E)$.

Definition 41 (Atanassov 1986) Let E be a universal set and η be a function from E to $[0,1]\times[0,1]$. Then, the set $\left\{{}^{\mu(x)}_{\nu(x)}x : x \in E\right\}$ being the graphic of η is called an intuitionistic fuzzy set (*if*-set) over E.

Here, for all $x \in E$, $0 \leq \mu(x) + \nu(x) \leq 1$. Moreover, μ and ν are called the membership function and the nonmembership function in an *if*-set, respectively, and $\pi(x) = 1 - \mu(x) - \nu(x)$ is called the degree of indeterminacy of the element $x \in E$. Further, the set of all *if*-sets over E is denoted by $IF(E)$.

Moreover, each fuzzy set can be written as $\left\{{}^{\mu(x)}_{1-\mu(x)}x : x \in E\right\}$.

Definition 42 (Karaaslan 2016) Let U be a universal set, E be a parameter set, $\eta \in IF(E)$, and g be a function from η to $IF(U)$. Then, the set $\left\{\left({}^{\mu(x)}_{\nu(x)}x, g({}^{\mu(x)}_{\nu(x)}x)\right) : x \in E\right\}$ being the graphic of g is called an intuitionistic fuzzy parameterized intuitionistic fuzzy soft set (*ifpifs*-set) parameterized via E over U (or briefly over U). Besides, the set of all *ifpifs*-sets over U is denoted by $IFPIFS_E(U)$.

Definition 43 (Çağman et al. 2010) Let U be a universal set, E be a parameter set, $\mu \in F(E)$, and h be a function from μ to $F(U)$. Then, the set $\left\{\left({}^{\mu(x)}x, h({}^{\mu(x)}x)\right) : x \in E\right\}$ being the graphic of h is called an fuzzy parameterized fuzzy soft set (*fpfs*-set) parameterized via E over U (or briefly over U). Besides, the set of all *fpfs*-sets over U is denoted by $FPFS_E(U)$.

Definition 44 (Sulukan et al. 2019) Let U be a universal set, E be a parameter set, $\mu \in F(E)$, and p be a function from μ to $IF(U)$. Then, the set $\left\{\left({}^{\mu(x)}x, p({}^{\mu(x)}x)\right) : x \in E\right\}$ being the graphic of p is called an fuzzy parameterized intuitionistic fuzzy soft set (*fpifs*-set) parameterized via E over U (or briefly over U). Besides, the set of all *fpifs*-sets over U is denoted by $FPIFS_E(U)$.

Since the proposed method in Sect. 4 is the first method proposed in relation to this structure (*d*-sets), it is impossible to compare this method with another in this sense. However, if the uncertainties in the modelled problem are decreased, it is possible to compare the method with the others in a substructure, such as *ifpifs*-sets, *fpifs*-sets, and *fpfs*-sets. For this reason, secondly, we define four new concepts, i.e. mean reduction, mean bireduction, mean bireduction-reduction, and mean reduction-bireduction.

Definition 45 Let $f \in D_E(U)$, that is $f := \left\{\left({}^{\alpha(x)}_{\beta(x)}x, \left\{{}^{\omega(u)}_{\theta(u)}u : u \in U\right\}\right) : x \in E\right\}$. Then, the *ifpifs*-set

$$\left\{\left({}^{\frac{\alpha^-(x)+\alpha^+(x)}{2}}_{\frac{\beta^-(x)+\beta^+(x)}{2}}x, \left\{{}^{\frac{\omega^-(u)+\omega^+(u)}{2}}_{\frac{\theta^-(u)+\theta^+(u)}{2}}u : u \in U\right\}\right) : x \in E\right\}$$

is called mean reduction of *f* and is denoted by f_{mr}.

Definition 46 Let $f \in D_E(U)$, that is $f := \left\{\left({}^{\alpha(x)}_{\beta(x)}x, \left\{{}^{\omega(u)}_{\theta(u)}u : u \in U\right\}\right) : x \in E\right\}$. Then, the *fpfs*-set

$$\left\{\left({}^{\frac{\alpha^-(x)+\alpha^+(x)-\beta^-(x)-\beta^+(x)+2}{4}}x, \left\{{}^{\frac{\omega^-(u)+\omega^+(u)-\theta^-(u)-\theta^+(u)+2}{4}}u : u \in U\right\}\right) : x \in E\right\}$$

is called mean bireduction of *f* and is denoted by f_{mb}.

Definition 47 Let $f \in D_E(U)$, that is $f := \left\{\left({}^{\alpha(x)}_{\beta(x)}x, \left\{{}^{\omega(u)}_{\theta(u)}u : u \in U\right\}\right) : x \in E\right\}$. Then, the *fpifs*-set

$$\left\{\left({}^{\frac{\alpha^-(x)+\alpha^+(x)-\beta^-(x)-\beta^+(x)+2}{4}}x, \left\{{}^{\frac{\omega^-(u)+\omega^+(u)}{2}}_{\frac{\theta^-(u)+\theta^+(u)}{2}}u : u \in U\right\}\right) : x \in E\right\}$$

is called mean bireduction-reduction of *f* and is denoted by f_{mbr}.

Definition 48 Let $f \in D_E(U)$, that is $f := \left\{\left({}^{\alpha(x)}_{\beta(x)}x, \left\{{}^{\omega(u)}_{\theta(u)}u : u \in U\right\}\right) : x \in E\right\}$. Then, the *ifpfs*-set

$$\left\{\left({}^{\frac{\alpha^-(x)+\alpha^+(x)}{2}}_{\frac{\beta^-(x)+\beta^+(x)}{2}}x,\ \left\{{}^{\frac{\omega^-(u)+\omega^+(u)-\theta^-(u)-\theta^+(u)+2}{4}}u : u \in U\right\}\right) : x \in E\right\}$$

is called mean reduction-bireduction of f and is denoted by f_{mrb}.

Example 5 f_{mr}, f_{mbr}, and f_{mb} for the d-set f provided in Section 5 are as follows:

$$\begin{aligned} f_{mr} = \Big\{ &\left({}^{0.53}_{0.16}x_1, \left\{{}^{0.41}_{0.2}u_1, {}^{0.08}_{0.46}u_2, {}^{0.19}_{0.38}u_3, {}^{0.41}_{0.2}u_4\right\}\right), \\ &\left({}^{0.57}_{0.36}x_2, \left\{{}^{0.28}_{0.5}u_1, {}^{0.32}_{0.31}u_2, {}^{0.29}_{0.37}u_3, {}^{0.28}_{0.5}u_4, {}^{1}_{0}u_5, {}^{0.17}_{0.1}u_6\right\}\right), \\ &\left({}^{0.15}_{0.39}x_3, \left\{{}^{0.45}_{0.22}u_1, {}^{0.31}_{0.49}u_3, {}^{0.45}_{0.22}u_4, {}^{0.2}_{0.51}u_6\right\}\right), \\ &\left({}^{0.46}_{0.32}x_4, \left\{{}^{0}_{0.55}u_1, {}^{1}_{0}u_2, {}^{0.23}_{0.49}u_3, {}^{0}_{0.55}u_4, {}^{0.38}_{0.05}u_5, {}^{0.38}_{0.34}u_6\right\}\right)\Big\} \\ f_{mbr} = \Big\{ &\left({}^{0.69}x_1, \left\{{}^{0.41}_{0.2}u_1, {}^{0.08}_{0.46}u_2, {}^{0.19}_{0.38}u_3, {}^{0.41}_{0.2}u_4\right\}\right), \\ &\left({}^{0.61}x_2, \left\{{}^{0.28}_{0.5}u_1, {}^{0.32}_{0.31}u_2, {}^{0.29}_{0.37}u_3, {}^{0.28}_{0.5}u_4, {}^{1}_{0}u_5, {}^{0.17}_{0.1}u_6\right\}\right), \\ &\left({}^{0.38}x_3, \left\{{}^{0.45}_{0.22}u_1, {}^{0.31}_{0.49}u_3, {}^{0.45}_{0.22}u_4, {}^{0.2}_{0.51}u_6\right\}\right), \\ &\left({}^{0.57}x_4, \left\{{}^{0}_{0.55}u_1, {}^{1}_{0}u_2, {}^{0.23}_{0.49}u_3, {}^{0}_{0.55}u_4, {}^{0.38}_{0.05}u_5, {}^{0.38}_{0.34}u_6\right\}\right)\Big\} \end{aligned}$$

and

$$\begin{aligned} f_{mb} = \Big\{ &\left({}^{0.69}x_1, \left\{{}^{0.61}u_1, {}^{0.31}u_2, {}^{0.41}u_3, {}^{0.61}u_4\right\}\right), \\ &\left({}^{0.61}x_2, \left\{{}^{0.39}u_1, {}^{0.5}u_2, {}^{0.46}u_3, {}^{0.39}u_4, {}^{1}u_5, {}^{0.54}u_6\right\}\right), \\ &\left({}^{0.38}x_3, \left\{{}^{0.62}u_1, {}^{0.41}u_3, {}^{0.62}u_4, {}^{0.35}u_6\right\}\right), \\ &\left({}^{0.57}x_4, \left\{{}^{0.22}u_1, {}^{1}u_2, {}^{0.37}u_3, {}^{0.22}u_4, {}^{0.66}u_5, {}^{0.52}u_6\right\}\right)\Big\} \end{aligned}$$

Thirdly, we present the soft decision-making methods in Çağman et al. (2010), Karaaslan (2016), Kamacı (2019), and Sulukan et al. (2019) by considering the notations used throughout this study. Moreover, to sort all the alternatives instead of selecting only one optimum alternative, we rearrange the last step of the method provided in Karaaslan (2016), faithfully to the original. Furthermore, just as the concepts of intuitionistic fuzzy sets and interval-valued fuzzy sets are equivalent (Atanassov and Gargov 1989), so are the concepts of *ivfpifs* (Kamacı 2019) and *ifpifs* (Karaaslan 2016). Therefore, we express the algorithm of the method, provided in Kamacı (2019), by using *ifpifs* instead of *ivfpifs*.

Algorithm Steps of Method 1 (Karaaslan 2016)

Step 1. Construct an *ifpifs*-set

$$g = \left\{\left({}^{\mu(x)}_{\nu(x)}x, \left\{{}^{\rho_x(u)}_{\sigma_x(u)}u : u \in U\right\}\right) : x \in E\right\}$$

over U

Step 2. Obtain the *if*-set $g^* = \left\{{}^{\rho^*(u)}_{\sigma^*(u)}u : u \in U\right\}$ such that $\rho^*(u) = \frac{1}{|E|}\sum_{x\in E}\mu(x)\rho_x(u)$ and $\sigma^*(u) = \frac{1}{|E|}\sum_{x\in E}\nu(x)\sigma_x(u)$. Here, $|E|$ is cardinality of E.

Step 3. For all $u \in U$, obtain the values

$$\xi(u) = \frac{\rho^*(u)}{\rho^*(u) + \sigma^*(u)}$$

Step 4. Obtain the decision set $\left\{{}^{d(u_k)}u_k | u_k \in U\right\}$ such that $d(u_k) = \frac{\xi(u_k)}{\max_i \xi(u_i)}$

Algorithm Steps of Method 2 (Kamacı 2019)

Step 1. Construct an *ifpifs*-set

$$g = \left\{\left({}^{\mu(x)}_{\nu(x)}x, \left\{{}^{\rho_x(u)}_{\sigma_x(u)}u : u \in U\right\}\right) : x \in E\right\}$$

over U

Step 2. Obtain the *ivif*-set $\kappa_1 = \left\{{}^{\alpha_1(u)}_{\beta_1(u)}u : u \in E\right\}$ such that

$$\alpha_1(u) = \left[1 - \prod_{x\in E}(1 - \mu(x)\rho_x(u)), 1 - \prod_{x\in E}(1 - (1 - \nu(x))\rho_x(u))\right]$$

and

$$\beta_1(u) = \left[\prod_{x\in E}\mu(x)\sigma_x(u), \prod_{x\in E}(1 - \nu(x))\sigma_x(u)\right]$$

Step 3. Obtain the *ivif*-set $\kappa_2 = \left\{{}^{\alpha_2(u)}_{\beta_2(u)}u : u \in U\right\}$ such that

$$\alpha_2(u) = \left[\prod_{x\in E}\mu(x)\rho_x(u), \prod_{x\in E}(1 - \nu(x))\rho_x(u)\right]$$

and

$$\beta_2(u) = \left[1 - \prod_{x\in E}(1 - \mu(x)\sigma_x(u)), 1 - \prod_{x\in E}(1 - (1 - \nu(x))\sigma_x(u))\right]$$

Step 4. Obtain the decision set κ_3 such that

$$\kappa_3 := \kappa_1 \tilde{+} \kappa_2$$

Step 5. Select the optimal elements among the alternatives via the ordering relation (Tan 2011; Xu 2007)

$${}^{\alpha}_{\beta} \preceq {}^{\tilde{\alpha}}_{\tilde{\beta}} \Leftrightarrow \left[\left(s_1({}^{\alpha}_{\beta}) < s_1({}^{\tilde{\alpha}}_{\tilde{\beta}})\right) \vee \left(s_1({}^{\alpha}_{\beta}) = s_1({}^{\tilde{\alpha}}_{\tilde{\beta}}) \wedge s_2({}^{\alpha}_{\beta}) \leq s_2({}^{\tilde{\alpha}}_{\tilde{\beta}})\right)\right]$$

such that

$$s_1\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right) = \frac{\alpha^- - \beta^- + \alpha^+ - \beta^+}{2}$$

and

$$s_2\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right) = \frac{\alpha^- + \beta^- + \alpha^+ + \beta^+}{2}$$

Here, ${}^{\alpha}_{\beta} := {}^{[\alpha^-,\alpha^+]}_{[\beta^-,\beta^+]}$ and ${}^{\tilde{\alpha}}_{\tilde{\beta}} := {}^{[\tilde{\alpha}^-,\tilde{\alpha}^+]}_{[\tilde{\beta}^-,\tilde{\beta}^+]}$ are *ivif*-values.

Algorithm Steps of Method 3 (Sulukan et al. 2019)

Step 1. Construct an *fpifs*-set over U

$$p = \left\{ \left({}^{\mu(x)}x, \left\{ {}^{\rho_x(u)}_{\sigma_x(u)}u : u \in U \right\} \right) : x \in E \right\}$$

Step 2. Obtain the values

$$\omega(u) = \frac{1}{|E|} \sum_{x \in E} \mu(x)(\rho_x(u) - \sigma_x(u))$$

for all $u \in U$. Here, $|E|$ is cardinality of E.

Step 3. Obtain the decision set $\left\{ {}^{d(u_k)}u_k | u_k \in U \right\}$ such that $d(u_k) = \frac{\omega(u_k) + |\min_i \omega(u_i)|}{\max_i \omega(u_i) + |\min_i \omega(u_i)|}$

Algorithm Steps of Method 4 (Çağman et al. 2010)

Step 1. Construct an *fpfs*-set over U

$$h = \left\{ \left({}^{\mu(x)}x, \left\{ {}^{\nu_x(u)}u : u \in U \right\} \right) : x \in E \right\}$$

Step 2. Obtain the *f*-set $h^* = \left\{ {}^{\nu^*(u)}u : u \in U \right\}$ such that $\nu^*(u) = \frac{1}{|E|} \sum_{x \in E} \mu(x)\nu_x(u)$. Here, $|E|$ is cardinality of E.

Step 3. Obtain the decision set $\left\{ {}^{d(u_k)}u_k | u_k \in U \right\}$ such that $d(u_k) = \frac{\nu^*(u_k)}{\max_i \nu^*(u_i)}$

Fourthly, we apply the proposed method and Method 1, 2, 3, and 4 to f, f_{mr}, f_{mr}, f_{mbr}, and f_{mb} provided in Example 5, respectively. The decision sets and the ranking orders of the methods within their own structures are provided in Tables 1 and 2, respectively. The proposed method, Method 1, and Method 2 decide that the candidates u_5 and u_2 are eligible for the vacant positions. Thus, the candidates u_5 and u_2 are selected for the positions announced by the company. On the other hand, while Method 3 and 4 suggest the candidate u_2 for one of the two positions, it fails to decide between u_1 and u_4 for the other position. Moreover, these five methods propose that the candidates u_6 and u_3 are ineligible for the vacant positions. Furthermore, although the performances of the candidates u_1 and u_4 in the application of Method 1, 2, 3, and 4 are the same, the proposed method is capable of sorting them. Therefore, the proposed method has been successfully applied to the problem involving further uncertainties.

Table 2 The ranking orders of the five methods within their own structures

Methods	Structures	Ranking orders
Proposed method	*d*-sets	$u_6 \prec u_3 \prec u_1 \prec u_4 \prec u_2 \prec u_5$
Method 1	*ifpifs*-sets	$u_6 \prec u_3 \prec u_1 = u_4 \prec u_2 \prec u_5$
Method 2	*ifpifs*-sets	$u_6 \prec u_3 \prec u_1 = u_4 \prec u_2 \prec u_5$
Method 3	*fpifs*-sets	$u_6 \prec u_3 \prec u_5 \prec u_1 = u_4 \prec u_2$
Method 4	*fpfs*-sets	$u_6 \prec u_3 \prec u_5 \prec u_1 = u_4 \prec u_2$

7 An application of the proposed method to a performance-based value assignment problem

In this section, we apply the proposed method and four state-of-the-art methods mentioned in the previous section to the performance-based value assignment problem for seven known filters used in image denoising, namely based on pixel density filter (BPDF) (Erkan and Gökrem 2018), modified decision based unsymmetrical trimmed median filter (MDBUTMF) (Esakkirajan et al. 2011), decision based algorithm (DBA) (Pattnaik et al. 2012), noise adaptive fuzzy switching median filter (NAFSMF) (Toh and Isa 2010),

Table 1 The decision sets of the proposed method and Method 1, 2, 3, and 4

Methods	Decision sets
Proposed method	$\left\{ {}^{[0.2334,0.8148]}u_1, {}^{[0.3223,0.9436]}u_2, {}^{[0,0.8079]}u_3, {}^{[0.0636,1]}u_4, {}^{[0.4706,0.8980]}u_5, {}^{[0.0005,0.7760]}u_6 \right\}$
Method 1	$\left\{ {}^{0.8518}u_1, {}^{0.9565}u_2, {}^{0.7667}u_3, {}^{0.8518}u_4, {}^{1}u_5, {}^{0.6593}u_6 \right\}$
Method 2	$\left\{ {}^{[0.3866,0.6096]}_{[0.0001,0.0019]}u_1, {}^{[0.5772,0.7626]}_{0}u_2, {}^{[0.3600,0.5322]}_{[0.0004,0.0057]}u_3, {}^{[0.3866,0.6096]}_{[0.0001,0.0019]}u_4, {}^{[0.6452,0.7330]}_{0}u_5, {}^{[0.2771,0.4197]}_{[0.0002,0.0036]}u_6 \right\}$
Method 3	$\left\{ {}^{0.7792}u_1, {}^{1}u_2, {}^{0.5114}u_3, {}^{0.7792}u_4, {}^{0.6957}u_5, {}^{0}u_6 \right\}$
Method 4	$\left\{ {}^{0.9365}u_1, {}^{1}u_2, {}^{0.8543}u_3, {}^{0.9365}u_4, {}^{0.9057}u_5, {}^{0.6969}u_6 \right\}$

different applied median filter (DAMF) (Erkan et al. 2018), a new adaptive weighted mean filter (AWMF) (Tang et al. 2016), and adaptive Riesz mean filter (ARmF) (Enginoğlu et al. 2019). Hereinafter, let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be an alternative set such that u_1 = "BPDF", u_2 = "MDBUTMF", u_3 = "DBA", u_4 = "NAFSMF", u_5 = "DAMF", u_6 = "AWMF", and u_7 = "ARmF". Moreover, let $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ be a parameter set determined by a decision-maker such that x_1 = "*noise density* 10%", x_2 = "*noise density* 20%", x_3 = "*noise density* 30%", x_4 = "*noise density* 40%", x_5 = "*noise density* 50%", x_6 = "*noise density* 60%", x_7 = "*noise density* 70%", x_8 = "*noise density* 80%", and x_9 = "*noise density* 90%".

We consider four traditional images, i.e. "Cameraman", "Lena", "Jet Plane", and "Baboon", for the convenience of the experts who are asked to produce a suitable ranking order of the aforementioned filters. Thus, we can compare the results of the methods with the experts' judgements. To this end, we present the results of the filters in Enginoğlu et al. (2019) by Structural Similarity (SSIM) (Wang et al. 2004) for the images at noise densities ranging from 10 to 90%, in Table 3. Further, let bold numbers in a table point out the best scores therein. Let $((\mu_u^x)_n)$ be ordered s-tuples such that $(\mu_u^x)_n$ corresponds the SSIM results by nth image for filter u and noise density x. Moreover, the first, second, third, and fourth image are the Cameraman, Lena, Jet Plane, and Baboon, respectively.

Secondly, we construct the d-set f via the membership function $\omega_x(u)$ and the nonmembership function $\theta_x(u)$ defined by

Table 3 The SSIM results of the filters for the Cameraman, Lena, Jet Plane, and Baboon images

Filters	10%	20%	30%	40%	50%	60%	70%	80%	90%
Cameraman									
BPDF	0.9910	0.9783	0.9588	0.9306	0.8934	0.8406	0.7700	0.6665	0.4990
MDBUTMF	0.9897	0.9278	0.7945	0.7964	0.8844	0.9158	0.8962	0.8056	0.4451
DBA	0.9938	0.9847	0.9710	0.9520	0.9222	0.8843	0.8283	0.7584	0.6645
NAFSMF	0.9798	0.9636	0.9484	0.9329	0.9164	0.8954	0.8696	0.8335	0.7288
DAMF	0.9960	0.9906	0.9833	0.9749	0.9638	0.9492	0.9293	0.8973	0.8294
AWMF	0.9872	0.9839	0.9798	0.9748	0.9667	0.9541	0.9345	0.9015	0.8346
ARmF	**0.9969**	**0.9933**	**0.9885**	**0.9824**	**0.9735**	**0.9600**	**0.9395**	**0.9059**	**0.8376**
Lena									
BPDF	0.9834	0.9647	0.9400	0.9085	0.8649	0.8075	0.7213	0.5441	0.2861
MDBUTMF	0.9845	0.9341	0.8302	0.8205	0.8734	0.8840	0.8515	0.7515	0.3774
DBA	0.9867	0.9705	0.9499	0.9219	0.8862	0.8389	0.7748	0.6909	0.5701
NAFSMF	0.9831	0.9657	0.9473	0.9274	0.9046	0.8791	0.8471	0.8009	0.6900
DAMF	0.9897	0.9783	0.9645	0.9478	0.9285	0.9055	0.8751	0.8340	0.7611
AWMF	0.9811	0.9727	0.9622	0.9484	0.9319	0.9108	0.8805	0.8387	0.7675
ARmF	**0.9906**	**0.9806**	**0.9690**	**0.9545**	**0.9374**	**0.9155**	**0.8846**	**0.8416**	**0.7693**
Jet Plane									
BPDF	0.9883	0.9729	0.9496	0.9205	0.8773	0.8139	0.7216	0.5440	0.1615
MDBUTMF	0.9868	0.9112	0.7611	0.7644	0.8558	0.8956	0.8717	0.7609	0.3449
DBA	0.9875	0.9753	0.9581	0.9338	0.8997	0.8564	0.7952	0.7121	0.5915
NAFSMF	0.9843	0.9677	0.9497	0.9305	0.9109	0.8866	0.8562	0.8159	0.7067
DAMF	0.9937	0.9856	0.9756	0.9633	0.9493	0.9293	0.9050	0.8669	0.7943
AWMF	0.9846	0.9785	0.9717	0.9630	0.9517	0.9342	0.9097	0.8715	0.7995
ARmF	**0.9945**	**0.9882**	**0.9808**	**0.9710**	**0.9590**	**0.9406**	**0.9152**	**0.8758**	**0.8022**
Baboon									
BPDF	0.9794	0.9497	0.9090	0.8538	0.7778	0.6814	0.5584	0.3808	0.1039
MDBUTMF	0.9700	0.9212	0.8490	0.8140	0.8057	0.7720	0.7128	0.6051	0.3074
DBA	0.9827	0.9593	0.9265	0.8796	0.8152	0.7341	0.6309	0.4980	0.3521
NAFSMF	0.9610	0.9192	0.8752	0.8275	0.7736	0.7170	0.6500	0.5660	0.4406
DAMF	0.9883	0.9738	0.9559	0.9342	0.9058	0.8716	0.8213	0.7424	0.5946
AWMF	0.9717	0.9599	0.9475	0.9328	0.9110	0.8805	0.8311	0.7508	0.6019
ARmF	**0.9914**	**0.9811**	**0.9682**	**0.9517**	**0.9277**	**0.8944**	**0.8427**	**0.7594**	**0.6065**

Bold values indicate the best performance

$$\left[\frac{\min\limits_{n}(\mu_u^x)_n}{\max\limits_{n}(\mu_u^x)_n+\max\limits_{n}\{1-(\mu_u^x)_n\}},\frac{\max\limits_{n}(\mu_u^x)_n}{\max\limits_{n}(\mu_u^x)_n+\max\limits_{n}\{1-(\mu_u^x)_n\}}\right]$$

and

$$\left[\frac{\min\limits_{n}\{1-(\mu_u^x)_n\}}{\max\limits_{n}(\mu_u^x)_n+\max\limits_{n}\{1-(\mu_u^x)_n\}},\frac{\max\limits_{n}\{1-(\mu_u^x)_n\}}{\max\limits_{n}(\mu_u^x)_n+\max\limits_{n}\{1-(\mu_u^x)_n\}}\right]$$

For example, $((\mu_{u_3}^{x_1})_n)$ denote the SSIM results of four traditional images by DBA at noise density 10%, namely $((\mu_{u_3}^{x_1})_n)=(0.9938,0.9867,0.9875,0.9827)$. Since

$$\begin{aligned}\omega_{x_1}(u_3)=&\left[\frac{0.9827}{0.9938+0.0173},\frac{0.9938}{0.9938+0.0173}\right]\\=&[0.9719,0.9829]\end{aligned}$$

and

$$\begin{aligned}\theta_{x_1}(u_3)=&\left[\frac{0.0062}{0.9938+0.0173},\frac{0.0173}{0.9938+0.0173}\right]\\=&[0.0061,0.0171]\end{aligned}$$

then the membership degree and the nonmembership degree of the filter u_3 according to the parameter x_1 are [0.9719, 0.9829] and [0.0061, 0.0171], respectively. Similarly, the *ivif*-values of the other filters can be constructed. Suppose that the noise removal performances of the filters are more significant in high noise density, in which noisy pixels outnumber uncorrupted pixels, then performance-based success would be more important in the presence of high noise densities than of other densities. For example, let

$$\kappa=\left\{{}^{[0,0.01]}_{[0.9,0.95]}x_1,{}^{[0,0.05]}_{[0.85,0.9]}x_2,{}^{[0,0.1]}_{[0.8,0.85]}x_3,{}^{[0.05,0.35]}_{[0.25,0.5]}x_4,{}^{[0.2,0.45]}_{[0.2,0.45]}x_5,{}^{[0.25,0.5]}_{[0.05,0.35]}x_6,{}^{[0.8,0.85]}_{[0,0.1]}x_7,{}^{[0.85,0.9]}_{[0,0.05]}x_8,{}^{[0.9,0.95]}_{[0,0.01]}x_9\right\}$$

Therefore, the *d*-set *f*, the *ifpifs*-set f_{mr}, the *fpifs*-set f_{mbr}, and the *fpfs*-set f_{mb}, respectively are as follows:

$$\Big\{\Big({}^{[0,0.01]}_{[0.9,0.95]}x_1,\Big\{{}^{[0.9682,0.9796]}_{[0.0089,0.0204]}u_1,{}^{[0.9513,0.9706]}_{[0.0101,0.0294]}u_2,{}^{[0.9719,0.9829]}_{[0.0061,0.0171]}u_3,{}^{[0.9391,0.9619]}_{[0.0153,0.0381]}u_4,{}^{[0.9807,0.9884]}_{[0.0040,0.0116]}u_5,{}^{[0.9569,0.9721]}_{[0.0126,0.0279]}u_6,{}^{[0.9844,0.9907]}_{[0.0031,0.0093]}u_7\Big\}\Big),$$
$$\Big({}^{[0,0.05]}_{[0.85,0.9]}x_2,\Big\{{}^{[0.9233,0.9511]}_{[0.0211,0.0489]}u_1,{}^{[0.8908,0.9132]}_{[0.0644,0.0868]}u_2,{}^{[0.9355,0.9603]}_{[0.0149,0.0397]}u_3,{}^{[0.8767,0.9229]}_{[0.0308,0.0771]}u_4,{}^{[0.9577,0.9742]}_{[0.0092,0.0258]}u_5,{}^{[0.9374,0.9608]}_{[0.0157,0.0392]}u_6,{}^{[0.9683,0.9808]}_{[0.0066,0.0192]}u_7\Big\}\Big),$$
$$\Big({}^{[0,0.1]}_{[0.8,0.85]}x_3,\Big\{{}^{[0.8659,0.9133]}_{[0.0392,0.0867]}u_1,{}^{[0.6996,0.7804]}_{[0.1388,0.2196]}u_2,{}^{[0.8870,0.9296]}_{[0.0278,0.0704]}u_3,{}^{[0.8145,0.8839]}_{[0.0468,0.1161]}u_4,{}^{[0.9304,0.9571]}_{[0.0163,0.0429]}u_5,{}^{[0.9179,0.9491]}_{[0.0196,0.0509]}u_6,{}^{[0.9489,0.9688]}_{[0.0113,0.0312]}u_7\Big\}\Big),$$
$$\Big({}^{[0.05,0.35]}_{[0.25,0.5]}x_4,\Big\{{}^{[0.7929,0.8642]}_{[0.0645,0.1358]}u_1,{}^{[0.7238,0.7769]}_{[0.1700,0.2231]}u_2,{}^{[0.8202,0.8877]}_{[0.0448,0.1123]}u_3,{}^{[0.7486,0.8439]}_{[0.0607,0.1561]}u_4,{}^{[0.8977,0.9368]}_{[0.0241,0.0632]}u_5,{}^{[0.8952,0.9355]}_{[0.0242,0.0645]}u_6,{}^{[0.9234,0.9531]}_{[0.0171,0.0469]}u_7\Big\}\Big),$$
$$\Big({}^{[0.2,0.45]}_{[0.2,0.45]}x_5,\Big\{{}^{[0.6972,0.8008]}_{[0.0956,0.1992]}u_1,{}^{[0.7469,0.8199]}_{[0.1072,0.1801]}u_2,{}^{[0.7364,0.8331]}_{[0.0703,0.1669]}u_3,{}^{[0.6769,0.8019]}_{[0.0732,0.1981]}u_4,{}^{[0.8561,0.9110]}_{[0.0342,0.0890]}u_5,{}^{[0.8629,0.9157]}_{[0.0315,0.0843]}u_6,{}^{[0.8871,0.9309]}_{[0.0253,0.0691]}u_7\Big\}\Big),$$
$$\Big({}^{[0.25,0.5]}_{[0.05,0.35]}x_6,\Big\{{}^{[0.5878,0.7252]}_{[0.1375,0.2748]}u_1,{}^{[0.6749,0.8007]}_{[0.0736,0.1993]}u_2,{}^{[0.6382,0.7688]}_{[0.1006,0.2312]}u_3,{}^{[0.6085,0.7598]}_{[0.0888,0.2402]}u_4,{}^{[0.8088,0.8808]}_{[0.0471,0.1192]}u_5,{}^{[0.8201,0.8887]}_{[0.0428,0.1113]}u_6,{}^{[0.8393,0.9009]}_{[0.0375,0.0991]}u_7\Big\}\Big),$$
$$\Big({}^{[0.8,0.85]}_{[0,0.1]}x_7,\Big\{{}^{[0.4609,0.6355]}_{[0.1898,0.3645]}u_1,{}^{[0.6023,0.7573]}_{[0.0877,0.2427]}u_2,{}^{[0.5269,0.6917]}_{[0.1434,0.3083]}u_3,{}^{[0.5330,0.7130]}_{[0.1069,0.2870]}u_4,{}^{[0.7412,0.8387]}_{[0.0638,0.1613]}u_5,{}^{[0.7532,0.8469]}_{[0.0594,0.1531]}u_6,{}^{[0.7683,0.8566]}_{[0.0552,0.1434]}u_7\Big\}\Big),$$
$$\Big({}^{[0.85,0.9]}_{[0,0.05]}x_8,\Big\{{}^{[0.2962,0.5184]}_{[0.2594,0.4816]}u_1,{}^{[0.5040,0.6711]}_{[0.1619,0.3289]}u_2,{}^{[0.3951,0.6017]}_{[0.1917,0.3983]}u_3,{}^{[0.4465,0.6576]}_{[0.1314,0.3424]}u_4,{}^{[0.6428,0.7770]}_{[0.0889,0.2230]}u_5,{}^{[0.6525,0.7834]}_{[0.0856,0.2166]}u_6,{}^{[0.6624,0.7901]}_{[0.0821,0.2099]}u_7\Big\}\Big),$$
$$\Big({}^{[0.9,0.95]}_{[0,0.01]}x_9,\Big\{{}^{[0.0745,0.3577]}_{[0.3591,0.6423]}u_1,{}^{[0.2702,0.3912]}_{[0.4877,0.6088]}u_2,{}^{[0.2683,0.5063]}_{[0.2556,0.4937]}u_3,{}^{[0.3420,0.5658]}_{[0.2105,0.4342]}u_4,{}^{[0.4815,0.6717]}_{[0.1382,0.3283]}u_5,{}^{[0.4883,0.6771]}_{[0.1342,0.3229]}u_6,{}^{[0.4926,0.6804]}_{[0.1319,0.3196]}u_7\Big\}\Big)\Big\}$$

$$\Big\{\Big({}^{0.005}_{0.925}x_1, \Big\{{}^{0.9739}_{0.0146}u_1, {}^{0.9609}_{0.0198}u_2, {}^{0.9774}_{0.0116}u_3, {}^{0.9505}_{0.0267}u_4, {}^{0.9846}_{0.0078}u_5, {}^{0.9645}_{0.0202}u_6, {}^{0.9875}_{0.0062}u_7\Big\}\Big),$$
$$\Big({}^{0.025}_{0.875}x_2, \Big\{{}^{0.9372}_{0.0350}u_1, {}^{0.9020}_{0.0756}u_2, {}^{0.9479}_{0.0273}u_3, {}^{0.8998}_{0.0539}u_4, {}^{0.9660}_{0.0175}u_5, {}^{0.9491}_{0.0274}u_6, {}^{0.9746}_{0.0129}u_7\Big\}\Big),$$
$$\Big({}^{0.05}_{0.825}x_3, \Big\{{}^{0.8896}_{0.0630}u_1, {}^{0.7400}_{0.1792}u_2, {}^{0.9083}_{0.0491}u_3, {}^{0.8492}_{0.0815}u_4, {}^{0.9437}_{0.0296}u_5, {}^{0.9335}_{0.0352}u_6, {}^{0.9589}_{0.0212}u_7\Big\}\Big),$$
$$\Big({}^{0.2}_{0.375}x_4, \Big\{{}^{0.8286}_{0.1001}u_1, {}^{0.7504}_{0.1965}u_2, {}^{0.8540}_{0.0785}u_3, {}^{0.7963}_{0.1084}u_4, {}^{0.9172}_{0.0437}u_5, {}^{0.9154}_{0.0443}u_6, {}^{0.9382}_{0.0320}u_7\Big\}\Big),$$
$$\Big({}^{0.325}_{0.325}x_5, \Big\{{}^{0.7490}_{0.1474}u_1, {}^{0.7834}_{0.1436}u_2, {}^{0.7847}_{0.1186}u_3, {}^{0.7394}_{0.1356}u_4, {}^{0.8836}_{0.0616}u_5, {}^{0.8893}_{0.0579}u_6, {}^{0.9090}_{0.0472}u_7\Big\}\Big),$$
$$\Big({}^{0.375}_{0.2}x_6, \Big\{{}^{0.6565}_{0.2062}u_1, {}^{0.7378}_{0.1365}u_2, {}^{0.7035}_{0.1659}u_3, {}^{0.6841}_{0.1645}u_4, {}^{0.8448}_{0.0831}u_5, {}^{0.8544}_{0.0770}u_6, {}^{0.8701}_{0.0683}u_7\Big\}\Big),$$
$$\Big({}^{0.825}_{0.05}x_7, \Big\{{}^{0.5482}_{0.2772}u_1, {}^{0.6798}_{0.1652}u_2, {}^{0.6093}_{0.2258}u_3, {}^{0.6230}_{0.1969}u_4, {}^{0.7900}_{0.1125}u_5, {}^{0.8001}_{0.1062}u_6, {}^{0.8125}_{0.0993}u_7\Big\}\Big),$$
$$\Big({}^{0.875}_{0.025}x_8, \Big\{{}^{0.4073}_{0.3705}u_1, {}^{0.5875}_{0.2454}u_2, {}^{0.4984}_{0.2950}u_3, {}^{0.5521}_{0.2369}u_4, {}^{0.7099}_{0.1560}u_5, {}^{0.7180}_{0.1511}u_6, {}^{0.7263}_{0.1460}u_7\Big\}\Big),$$
$$\Big({}^{0.925}_{0.005}x_9, \Big\{{}^{0.2161}_{0.5007}u_1, {}^{0.3307}_{0.5483}u_2, {}^{0.3873}_{0.3747}u_3, {}^{0.4539}_{0.3224}u_4, {}^{0.5766}_{0.2332}u_5, {}^{0.5827}_{0.2286}u_6, {}^{0.5865}_{0.2258}u_7\Big\}\Big)\Big\}$$

$$\Big\{\Big({}^{0.04}x_1, \Big\{{}^{0.9739}_{0.0146}u_1, {}^{0.9609}_{0.0198}u_2, {}^{0.9774}_{0.0116}u_3, {}^{0.9505}_{0.0267}u_4, {}^{0.9846}_{0.0078}u_5, {}^{0.9645}_{0.0202}u_6, {}^{0.9875}_{0.0062}u_7\Big\}\Big),$$
$$\Big({}^{0.075}x_2, \Big\{{}^{0.9372}_{0.0350}u_1, {}^{0.9020}_{0.0756}u_2, {}^{0.9479}_{0.0273}u_3, {}^{0.8998}_{0.0539}u_4, {}^{0.9660}_{0.0175}u_5, {}^{0.9491}_{0.0274}u_6, {}^{0.9746}_{0.0129}u_7\Big\}\Big),$$
$$\Big({}^{0.1125}x_3, \Big\{{}^{0.8896}_{0.0630}u_1, {}^{0.7400}_{0.1792}u_2, {}^{0.9083}_{0.0491}u_3, {}^{0.8492}_{0.0815}u_4, {}^{0.9437}_{0.0296}u_5, {}^{0.9335}_{0.0352}u_6, {}^{0.9589}_{0.0212}u_7\Big\}\Big),$$
$$\Big({}^{0.4125}x_4, \Big\{{}^{0.8286}_{0.1001}u_1, {}^{0.7504}_{0.1965}u_2, {}^{0.8540}_{0.0785}u_3, {}^{0.7963}_{0.1084}u_4, {}^{0.9172}_{0.0437}u_5, {}^{0.9154}_{0.0443}u_6, {}^{0.9382}_{0.0320}u_7\Big\}\Big),$$
$$\Big({}^{0.5}x_5, \Big\{{}^{0.7490}_{0.1474}u_1, {}^{0.7834}_{0.1436}u_2, {}^{0.7847}_{0.1186}u_3, {}^{0.7394}_{0.1356}u_4, {}^{0.8836}_{0.0616}u_5, {}^{0.8893}_{0.0579}u_6, {}^{0.9090}_{0.0472}u_7\Big\}\Big),$$
$$\Big({}^{0.5875}x_6, \Big\{{}^{0.6565}_{0.2062}u_1, {}^{0.7378}_{0.1365}u_2, {}^{0.7035}_{0.1659}u_3, {}^{0.6841}_{0.1645}u_4, {}^{0.8448}_{0.0831}u_5, {}^{0.8544}_{0.0770}u_6, {}^{0.8701}_{0.0683}u_7\Big\}\Big),$$
$$\Big({}^{0.8875}x_7, \Big\{{}^{0.5482}_{0.2772}u_1, {}^{0.6798}_{0.1652}u_2, {}^{0.6093}_{0.2258}u_3, {}^{0.6230}_{0.1969}u_4, {}^{0.7900}_{0.1125}u_5, {}^{0.8001}_{0.1062}u_6, {}^{0.8125}_{0.0993}u_7\Big\}\Big),$$
$$\Big({}^{0.925}x_8, \Big\{{}^{0.4073}_{0.3705}u_1, {}^{0.5875}_{0.2454}u_2, {}^{0.4984}_{0.2950}u_3, {}^{0.5521}_{0.2369}u_4, {}^{0.7099}_{0.1560}u_5, {}^{0.7180}_{0.1511}u_6, {}^{0.7263}_{0.1460}u_7\Big\}\Big),$$
$$\Big({}^{0.96}x_9, \Big\{{}^{0.2161}_{0.5007}u_1, {}^{0.3307}_{0.5483}u_2, {}^{0.3873}_{0.3747}u_3, {}^{0.4539}_{0.3224}u_4, {}^{0.5766}_{0.2332}u_5, {}^{0.5827}_{0.2286}u_6, {}^{0.5865}_{0.2258}u_7\Big\}\Big)\Big\}$$

$$\Big\{\Big({}^{0.04}x_1, \Big\{{}^{0.9796}u_1, {}^{0.9706}u_2, {}^{0.9829}u_3, {}^{0.9619}u_4, {}^{0.9884}u_5, {}^{0.9721}u_6, {}^{0.9907}u_7\Big\}\Big),$$
$$\Big({}^{0.075}x_2, \Big\{{}^{0.9511}u_1, {}^{0.9132}u_2, {}^{0.9603}u_3, {}^{0.9229}u_4, {}^{0.9742}u_5, {}^{0.9608}u_6, {}^{0.9808}u_7\Big\}\Big),$$
$$\Big({}^{0.1125}x_3, \Big\{{}^{0.9133}u_1, {}^{0.7804}u_2, {}^{0.9296}u_3, {}^{0.8839}u_4, {}^{0.9571}u_5, {}^{0.9491}u_6, {}^{0.9688}u_7\Big\}\Big),$$
$$\Big({}^{0.4125}x_4, \Big\{{}^{0.8642}u_1, {}^{0.7769}u_2, {}^{0.8877}u_3, {}^{0.8439}u_4, {}^{0.9368}u_5, {}^{0.9355}u_6, {}^{0.9531}u_7\Big\}\Big),$$
$$\Big({}^{0.5}x_5, \Big\{{}^{0.8008}u_1, {}^{0.8199}u_2, {}^{0.8331}u_3, {}^{0.8019}u_4, {}^{0.9110}u_5, {}^{0.9157}u_6, {}^{0.9309}u_7\Big\}\Big),$$
$$\Big({}^{0.5875}x_6, \Big\{{}^{0.7252}u_1, {}^{0.8007}u_2, {}^{0.7688}u_3, {}^{0.7598}u_4, {}^{0.8808}u_5, {}^{0.8887}u_6, {}^{0.9009}u_7\Big\}\Big),$$
$$\Big({}^{08875}x_7, \Big\{{}^{0.6355}u_1, {}^{0.7573}u_2, {}^{0.6917}u_3, {}^{0.7130}u_4, {}^{0.8387}u_5, {}^{0.8469}u_6, {}^{0.8566}u_7\Big\}\Big),$$
$$\Big({}^{0.925}x_8, \Big\{{}^{0.5184}u_1, {}^{0.6711}u_2, {}^{0.6017}u_3, {}^{0.6576}u_4, {}^{0.7770}u_5, {}^{0.7834}u_6, {}^{0.7901}u_7\Big\}\Big),$$
$$\Big({}^{0.96}x_9, \Big\{{}^{0.3577}u_1, {}^{0.3912}u_2, {}^{0.5063}u_3, {}^{0.5658}u_4, {}^{0.6717}u_5, {}^{0.6771}u_6, {}^{0.6804}u_7\Big\}\Big)\Big\}$$

Thirdly, we apply the proposed method and Method 1, 2, 3, and 4 to f, f_{mr}, f_{mr}, f_{mbr}, and f_{mb} provided in this section, respectively. In Tables 4 and 5, we present the decision sets and the ranking orders of the filters for the five methods within their own structures, respectively. Based on the values in Table 3, the proposed method, Method 3, and Method 4 produce the same ranking order as proposed by the experts, including the authors herein. In other words, the proposed method has been able to produce a valid ranking of the seven filters in view of the four images, which suggests that the method is also applicable to a larger number of images. On the other hand, Method 1 and 2 are generally observed to yield a ranking different from the one created by the experts although they say, "ARmF outperforms the other filters". The above discussion shows that the proposed method can be successfully applied to performance based-value assignment problems so that alternatives can be ordered in terms of performance.

Table 4 The decision sets of filters for the proposed method and Method 1, 2, 3, and 4

Methods	Decision sets
Proposed method	$\left\{{}^{[0.2799,0.7496]}\text{BPDF}, {}^{[0.3680,0.7824]}\text{MDBUTMF}, {}^{[0.3834,0.8351]}\text{DBA}, {}^{[0.3749,0.8460]}\text{NAFSMF}, {}^{[0.5872,0.9817]}\text{DAMF}, {}^{[0.5870,0.9834]}\text{AWMF}, {}^{[0.6145,1]}\text{ARmF}\right\}$
Method 1	$\left\{{}^{0.9021}\text{BPDF}, {}^{0.8689}\text{MDBUTMF}, {}^{0.9413}\text{DBA}, {}^{0.9142}\text{NAFSMF}, {}^{0.9901}\text{DAMF}, {}^{0.9827}\text{AWMF}, {}^{1}\text{ARmF}\right\}$
Method 2	$\left\{{}^{[0.8754,0.9823]}_{0}\text{BPDF}, {}^{[0.9364,0.9925]}_{0}\text{MDBUTMF}, {}^{[0.9241,0.9912]}_{0}\text{DBA}, {}^{[0.9355,0.9917]}_{0}\text{NAFSMF}, {}^{[0.9773,0.9988]}_{0}\text{DAMF}, {}^{[0.9787,0.9989]}_{0}\text{AWMF}, {}^{[0.9804,0.9991]}_{0}\text{ARmF}\right\}$
Method 3	$\left\{{}^{0.5233}\text{BPDF}, {}^{0.6613}\text{MDBUTMF}, {}^{0.6822}\text{DBA}, {}^{0.7215}\text{NAFSMF}, {}^{0.9672}\text{DAMF}, {}^{0.9782}\text{AWMF}, {}^{1}\text{ARmF}\right\}$
Method 4	$\left\{{}^{0.7413}\text{BPDF}, {}^{0.8162}\text{MDBUTMF}, {}^{0.8276}\text{DBA}, {}^{0.8488}\text{NAFSMF}, {}^{0.9822}\text{DAMF}, {}^{0.9882}\text{AWMF}, {}^{1}\text{ARmF}\right\}$

Table 5 The ranking orders of the filters for the five methods within their own structures

Methods	Structures	Ranking orders
Proposed method	*d*-sets	BPDF ≺ MDBUTMF ≺ DBA ≺ NAFSMF ≺ DAMF ≺ AWMF ≺ **ARmF**
Method 1	*ifpifs*-sets	MDBUTMF ≺ BPDF ≺ NAFSMF ≺ DBA ≺ AWMF ≺ DAMF ≺ **ARmF**
Method 2	*ifpifs*-sets	BPDF ≺ DBA ≺ NAFSMF ≺ MDBUTMF ≺ DAMF ≺ AWMF ≺ **ARmF**
Method 3	*fpifs*-sets	BPDF ≺ MDBUTMF ≺ DBA ≺ NAFSMF ≺ DAMF ≺ AWMF ≺ **ARmF**
Method 4	*fpfs*-sets	BPDF ≺ MDBUTMF ≺ DBA ≺ NAFSMF ≺ DAMF ≺ AWMF ≺ **ARmF**

8 Conclusion

In this paper, we defined the concept of *d*-sets. We then suggested a new soft decision-making method via the aggregation operator and gave an application of this method to a problem of the determination of eligible candidates in the recruitment process of a company. Moreover, we provided an real application of this method to evaluate the performances of seven filters used in image denoising. To compare this method with another method, we defined four new concepts, i.e. mean reduction, mean bireduction, mean bireduction-reduction, and mean reduction-bireduction. By using these concepts, we applied the proposed method and the four state-of-the-art soft decision-making methods to the aforesaid problems. The results showed that the proposed method was successfully applied to the problems involving further uncertainties.

In the future, effective soft decision-making methods based on group decision-making can be developed by using *and*/*or*/*andnot*/*ornot*-products of *d*-sets. Thus, it will be possible to compare such soft decision-making methods constructed by the same structure with the method proposed in this paper. By doing so, the decision-making performances of the methods can be evaluated in a more consistent and down-to-earth manner. Besides, to obtain *ivif*-values of alternatives or parameters with multiple measurement results, the different membership/nonmembership functions can be defined and compared with the results provided in this study. Moreover, it is necessary and worthwhile to conduct theoretical and applied studies in various fields, such as algebra and topology, and on varied topics, e.g., similarity and distance measurement, by making use of the *d*-sets. Furthermore, to overcome decision-making problems containing a large number of data and multiple measurement results, defining the matrix representations of *d*-sets have an enormous significance. Therefore, we have been recently studying the concept of *d*-matrices that we believe will use and improve *d*-sets' skills in modelling.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Alkhazaleh S, Salleh AR, Hassan N (2011) Fuzzy parameterized interval-valued fuzzy soft set. Appl Math Sci 5(67):3335–3346

Atanassov K, Marinov P, Atanassova V (2019) Intercriteria analysis with interval-valued intuitionistic fuzzy evaluations. In: Int conf flexible query answering syst, pp 329–338. https://doi.org/10.1007/978-3-030-27629-4_30

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96. https://doi.org/10.1016/S0165-0114(86)80034-3

Atanassov KT (1994) Operators over interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 64(2):159–174. https://doi.org/10.1016/0165-0114(94)90331-X

Atanassov KT (2020) Interval-valued intuitionistic fuzzy sets. Studies in fuzziness and soft computing. Springer

Atanassov KT, Gargov G (1989) Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31(3):343–349. https://doi.org/10.1016/0165-0114(89)90205-4

Atmaca S (2017) Relationship between fuzzy soft topological spaces and (X, τ_e) parameter spaces. Cumhuriyet Sci J 38(4):77–85. https://doi.org/10.17776/csj.340541

Çağman N, Enginoğlu S (2010a) Soft matrix theory and its decision making. Comput Math Appl 59(10):3308–3314. https://doi.org/10.1016/j.camwa.2010.03.015

Çağman N, Enginoğlu S (2010b) Soft set theory and *uni-int* decision making. Eur J Oper Res 207(2):848–855. https://doi.org/10.1016/j.ejor.2010.05.004

Çağman N, Enginoğlu S (2012) Fuzzy soft matrix theory and its application in decision making. Iran J Fuzzy Syst 9(1):109–119. http://ijfs.usb.ac.ir/article_229.html

Çağman N, Çıtak F, Enginoğlu S (2010) Fuzzy parameterized fuzzy soft set theory and its applications. Turk J Fuzzy Syst 1(1):21–35

Çağman N, Çıtak F, Enginoğlu S (2011a) FP-soft set theory and its applications. Ann Fuzzy Math Inf 2(2):219–226. http://www.afmi.or.kr/papers/2011/Vol-02_No-02/AFMI-2-2(219-226)-J-110329R1.pdf

Çağman N, Enginoğlu S, Çıtak F (2011b) Fuzzy soft set theory and its applications. Iran J Fuzzy Syst 8(3):137–147. http://ijfs.usb.ac.ir/article_292.html

Çıtak F, Çağman N (2015) Soft int-rings and its algebraic applications. J Intell Fuzzy Syst 28(3):1225–1233. https://doi.org/10.3233/IFS-141406

Çıtak F, Çağman N (2017) Soft k-int-ideals of semirings and its algebraic structures. Ann Fuzzy Math Inf 13(4):531–538. https://doi.org/10.30948/afmi.2017.13.4.531

Deli I, Çağman N (2015) Intuitionistic fuzzy parameterized soft set theory and its decision making. Appl Soft Comput 28:109–113. https://doi.org/10.1016/j.asoc.2014.11.053

Deli I, Karataş S (2016) Interval valued intuitionistic fuzzy parameterized soft set theory and its decision making. J Intell Fuzzy Syst 30(4):2073–2082. https://doi.org/10.3233/IFS-151920

Enginoğlu S (2012) Soft matrices. PhD dissertation, Gaziosmanpaşa University, Tokat. https://tez.yok.gov.tr/UlusalTezMerkezi

Enginoğlu S, Çağman N (n.d.) Fuzzy parameterized fuzzy soft matrices and their application in decision-making. TWMS J Appl Eng Math, In Press

Enginoğlu S, Memiş S (2018) A configuration of some soft decision-making algorithms via fpfs-matrices. Cumhuriyet Sci J 39(4):871–881. https://doi.org/10.17776/csj.409915

Enginoğlu S, Çağman N, Karataş S, Aydın T (2015) On soft topology. El-Cezerî J Sci Eng 2(3):23–38. https://doi.org/10.31202/ecjse.67135

Enginoğlu S, Memiş S, Arslan B (2018) Comment (2) on soft set theory and uni-int decision-making [European Journal of Operational Research, (2010) 207, 848–855]. J New Theory (25):84–102. https://dergipark.org.tr/download/article-file/594503

Enginoğlu S, Erkan U, Memiş S (2019) Pixel similarity-based adaptive Riesz mean filter for salt-and-pepper noise removal. Multimed Tools Appl 78:35401–35418. https://doi.org/10.1007/s11042-019-08110-1

Erkan U, Gökrem L (2018) A new method based on pixel density in salt and pepper noise removal. Turk J Electr Eng Comput Sci 26(1):162–171. https://doi.org/10.3906/elk-1705-256

Erkan U, Gökrem L, Enginoğlu S (2018) Different applied median filter in salt and pepper noise. Comput Electr Eng 70:789–798. https://doi.org/10.1016/j.compeleceng.2018.01.019

Esakkirajan S, Veerakumar T, Subramanyam AN, PremChand CH (2011) Removal of high density salt and pepper noise through modified decision based unsymmetric trimmed median filter. IEEE Signal Process Lett 18(5):287–290. https://doi.org/10.1109/LSP.2011.2122333

Garg H, Arora R (2020) Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set. J Ambient Intell Humaniz Comput 11(1):375–410. https://doi.org/10.1007/s12652-019-01238-w

Gorzałczany MB (1987) A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst 21(1):1–17. https://doi.org/10.1016/0165-0114(87)90148-5

Hao F, Pei Z, Park DS, Phonexay V, Seo HS (2018) Mobile cloud services recommendation: a soft set-based approach. J Ambient Intell Humaniz Comput 9(4):1235–1243. https://doi.org/10.1007/s12652-017-0572-7

Hemavathi P, Muralikrishna P, Palanivel K (2018) On interval valued intuitionistic fuzzy β-subalgebras. Afr Mat 29(1–2):249–262. https://doi.org/10.1007/s13370-017-0539-z

Huang B, Zhuang YL, Li HX (2013) Information granulation and uncertainty measures in interval-valued intuitionistic fuzzy information systems. Eur J Oper Res 231(1):162–170. https://doi.org/10.1016/j.ejor.2013.05.006

Iqbal MN, Rizwan U (2019) Some applications of intuitionistic fuzzy sets using new similarity measure. J Ambient Intell Humaniz Comput. https://doi.org/10.1007/s12652-019-01516-7

Jiang Y, Tang Y, Chen Q, Liu H, Tang J (2010) Interval-valued intuitionistic fuzzy soft sets and their properties. Comput Math Appl 60(3):906–918. https://doi.org/10.1016/j.camwa.2010.05.036

Joshi R (2020) A new multi-criteria decision-making method based on intuitionistic fuzzy information and its application to fault detection in a machine. J Ambient Intell Humaniz Comput 11(2):739–753. https://doi.org/10.1007/s12652-019-01322-1

Kamacı H (2019) Interval-valued fuzzy parameterized intuitionistic fuzzy soft sets and their applications. Cumhuriyet Sci J 40(2):317–331. https://doi.org/10.17776/csj.524802

Karaaslan F (2016) Intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets with applications in decision making. Ann Fuzzy Math Inf 11(4):607–619. http://www.afmi.or.kr/papers/2016/Vol-11_No-04/PDF/AFMI-11-4(607-619)-H-150813-1R1.pdf

Kim T, Sotirova E, Shannon A, Atanassova V, Atanassov K, Jang LC (2018) Interval valued intuitionistic fuzzy evaluations for analysis of a student's knowledge in university e-learning courses. Int J Fuzzy Logic Intell Syst 18(3):190–195. https://doi.org/10.5391/IJFIS.2018.18.3.190

Liu Y, Jiang W (2020) A new distance measure of interval-valued intuitionistic fuzzy sets and its application in decision making. Soft Comput 24(9):6987–7003. https://doi.org/10.1007/s00500-019-04332-5

Luo M, Liang J (2018) A novel similarity measure for interval-valued intuitionistic fuzzy sets and its applications. Symmetry 10(10):1–13. https://doi.org/10.3390/sym10100441

Maji PK, Biswas R, Roy AR (2001) Fuzzy soft sets. J Fuzzy Math 9(3):589–602

Maji PK, Roy AR, Biswas R (2002) An application of soft sets in a decision making problem. Comput Math Appl 44(8–9):1077–1083. https://doi.org/10.1016/S0898-1221(02)00216-X

Maji PK, Biswas R, Roy AR (2003) Soft set theory. Comput Math Appl 45(4–5):555–562. https://doi.org/10.1016/S0898-1221(03)00016-6

Min WK (2008) Interval-valued intuitionistic fuzzy soft sets. J Korean Inst Intell Syst 18(3):316–322. https://doi.org/10.5391/JKIIS.2008.18.3.316

Mishra AR, Rani P (2018) Interval-valued intuitionistic fuzzy WASPAS method: application in reservoir flood control management policy. Group Decis Negotiat 27(6):1047–1078. https://doi.org/10.1007/s10726-018-9593-7

Molodtsov D (1999) Soft set theory-first results. Comput Math Appl 37(4–5):19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

Niewiadomski A (2013) Cylindric extensions of interval-valued fuzzy sets in data linguistic summaries. J Ambient Intell Humaniz Comput 4(3):369–376. https://doi.org/10.1007/s12652-011-0098-3
Park CK (2016) Interval-valued intuitionistic gradation of openness. Korean J Math 24(1):27–40. https://doi.org/10.11568/kjm.2016.24.1.27
Park CK (2017) ($[r, s]$, $[t, u]$)-interval-valued intuitionistic fuzzy generalized precontinuous mappings. Korean J Math 25(1):1–18 https://doi.org/10.11568/kjm.2017.25.1.1
Pattnaik A, Agarwal S, Chand S (2012) A new and efficient method for removal of high density salt and pepper noise through cascade decision based filtering algorithm. Proc Technol 6:108–117. https://doi.org/10.1016/j.protcy.2012.10.014
Priyadharsini J, Balasubramaniam P (2019) Multi-criteria decision making method based on interval-valued intuitionistic fuzzy sets. J Anal 27(1):259–276. https://doi.org/10.1007/s41478-018-0122-5
Razak SA, Mohamad D (2011) A soft set based group decision making method with criteria weight. World Acad Sci Eng Technol 5(10):1641–1646. https://doi.org/10.5281/zenodo.1062538
Razak SA, Mohamad D (2013) A decision making method using fuzzy soft sets. Malays J Fundam Appl Sci 9(2):99–104. https://doi.org/10.11113/mjfas.v9n2.91
Riaz M, Hashmi MR (2017) Fuzzy parameterized fuzzy soft topology with applications. Ann Fuzzy Math Inf 13(5):593–613. https://doi.org/10.30948/afmi.2017.13.5.593
Riaz M, Hashmi MR, Farooq A (2018) Fuzzy parameterized fuzzy soft metric spaces. J Math Anal 9(2):25–36. http://www.ilirias.com/jma/repository/docs/JMA9-2-3.pdf
Selvachandran G, John SJ, Salleh AR (2017) Decision making based on the aggregation operator and the intuitionistic fuzzy reduction method of intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets. J Telecommun Electron Comput Eng 9(1-3):123–127. http://journal.utem.edu.my/index.php/jtec/article/view/1756
Senapati T, Shum KP (2019) Atanassov's interval-valued intuitionistic fuzzy set theory applied in KU-subalgebras. Discrete Math Algorithms Appl 11(2):17. https://doi.org/10.1142/S179383091950023X
Şenel G (2016) A new approach to Hausdorff space theory via the soft sets. Math Probl Eng 2016:6. https://doi.org/10.1155/2016/2196743**(Article ID 2196743)**
Şenel G (2018) Analyzing the locus of soft spheres: illustrative cases and drawings. Eur J Pure Appl Math 11(4):946–957. https://doi.org/10.29020/nybg.ejpam.v11i4.3321
Sezgin A (2016) A new approach to semigroup theory I: Soft union semigroups, ideals and bi-ideals. Algebra Lett 2016(3):1–46, http://scik.org/index.php/abl/article/view/2989
Sezgin A, Çağman N, Çıtak F (2019) α-inclusions applied to group theory via soft set and logic. Commun Fac Sci Univ Ank Ser A1 Math Stat 68(1):334–352. https://doi.org/10.31801/cfsuasmas.420457
Sotirov S, Sotirova E, Atanassova V, Atanassov K, Castillo O, Melin P, Petkov T, Surchev S (2018) A hybrid approach for modular neural network design using intercriteria analysis and intuitionistic fuzzy logic. Complexity 2018:11. https://doi.org/10.1155/2018/3927951 **(Article ID 3927951)**
Sulukan E, Çağman N, Aydın T (2019) Fuzzy parameterized intuitionistic fuzzy soft sets and their application to a performance-based value assignment problem. J New Theory (29):79–88. https://dergipark.org.tr/tr/download/article-file/906764
Tan C (2011) A multi-criteria interval-valued intuitionistic fuzzy group decision making with choquet integral-based topsis. Expert Syst Appl 38(4):3023–3033. https://doi.org/10.1016/j.eswa.2010.08.092
Tang Z, Yang Z, Liu K, Pei Z (2016) A new adaptive weighted mean filter for removing high density impulse noise. In: Eighth international conference on digital image processing (ICDIP 2016), international society for optics and photonics, vol 10033, pp 1003353/1–5. https://doi.org/10.1117/12.2243838
Thomas J, John SJ (2016) A note on soft topology. J New Results Sci 5(11):24–29. https://dergipark.org.tr/tr/pub/jnrs/issue/27287/287227
Toh KKV, Isa NAM (2010) Noise adaptive fuzzy switching median filter for salt-and-pepper noise reduction. IEEE Signal Process Lett 17(3):281–284. https://doi.org/10.1109/LSP.2009.2038769
Ullah A, Karaaslan F, Ahmad I (2018) Soft uni-Abel-Grassmann's groups. Eur J Pure Appl Math 11(2):517–536. https://doi.org/10.29020/nybg.ejpam.v11i2.3228
Wang Z, Bovik AC, Sheikh HR, Simoncelli EP (2004) Image quality assessment: from error visibility to structural similarity. IEEE Trans Image Process 13(4):600–612. https://doi.org/10.1109/TIP.2003.819861
Xu Z, Yager RR (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst 35(4):417–433. https://doi.org/10.1080/03081070600574353
Xu ZS (2007) Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. Control Decis 22(2):215–219
Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning-I. Inf Sci 8(3):199–249. https://doi.org/10.1016/0020-0255(75)90036-5
Zorlutuna I, Atmaca S (2016) Fuzzy parametrized fuzzy soft topology. New Trends Math Sci 4(1):142–152. https://doi.org/10.20852/ntmsci.2016115658