Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process

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Abstract

The operations of *t*-norm and *t*-conorm introduced by Dombi was known as Dombi operations can have a lead of good flexibility with the general parameter. The Dombi operations have so far not yet been applied for bipolar fuzzy sets. Motivated from Dombi operations, we have proposed bipolar fuzzy Dombi weighted averaging operator, bipolar fuzzy Dombi order weighted averaging operator, bipolar fuzzy Dombi hybrid weighted averaging operator, bipolar fuzzy Dombi weighted geometric operator, bipolar fuzzy Dombi order weighted geometric operator and bipolar fuzzy Dombi hybrid weighted geometric operator as these operators have good advantage of flexibility in operational behavior. Many properties of these operators are investigated. Then, we developed a model for multiple attribute decision-making problem for bipolar fuzzy Dombi aggregation operators under the bipolar fuzzy environment. Finally, a practical example for selection of investment alternatives is given to demonstrate for the utility and application of the proposed work.

Keywords Bipolar fuzzy elements · Dombi operations · Arithmetic averaging operators · Geometric averaging operators · Multiple attribute decision making

1 Introduction

Nowadays, multi-attribute decision-making (MADM) problem is a potential research tool in modern decision science. The main aim of this method is to select the best alternative among the finite set of alternatives as claimed by decision makers under the preference values of the alternatives. It has been extensively applied with quantitative or qualitative attribute values and has a board application in operation research (Pourhassan and Raissi 2017), management science (Levy et al. 2016; Teixeira et al. 2018), economic (Ronaynea and Brown-Gordon 2017), market prediction and engineering technology (Abbasian et al. 2018; Viriyasitavat 2016), etc. As our modern society move forward with the decision-making process, so it always faces imprecise, vague and uncertain facts to take a decision in solving decision-making

problems. In order to solve imprecise and uncertain data, (Atanassov 2012) initiated the idea of intuitionistic fuzzy set (IFS) characterize by membership function and non-membership function, a powerful extension of fuzzy set (Zadeh 1965) whose part has only membership function.

The aggregated results for the execution of the criteria for alternatives, weighted and order weighted aggregation operators (Yager 1998; Yager and Kacprzyk 1997) takes a significant role during the combination of the information process. At that point, Xu (2007) built up a novel work on averaging operators like IFWA operator, IFOWA operator, and IFHA operator. Xu and Yager (2006) also built up some geometric aggregation operators, such as IFWG operator, IFOWG operator and IFHG operator and provided an example of the real application of IFHG operator to MADM issues. For more information on other operators and terminology, the readers are referred to Beliakov et al. (2007), Beliakov et al. (2011), Chen and Tan (1994), Chen and Chiou (2015), Garg (2016), He et al. (2015), Kumar and Garg (2018), Li (2011), Li and Ren (2015), Liu et al. (2016), Lourenzuttia and Krohling (2013), Wan and DF (2014), Wan and DF (2015), Wei (2008), Wei (2009), Wei (2010), Wei and Zhang (2018) and Zhang et al. (2018).

Although, intuitionistic fuzzy sets (IFS) and interval-valued intuitionistic fuzzy sets (IVIFS) successfully applied

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(He et al. 2015a; Ye 2009; Zhang et al. 2018a) to solve the uncertainty of the real world problems, but it has seen generally for the information analysis of an object that corresponding to each property there exists a counter property. In that view, Zhang (1994, 1998) originated another extension of fuzzy sets called bipolar fuzzy set (BFS) whose membership degree extended to $[-1, 1]$. The bipolar fuzzy sets was characterized by two-component pair, one is positive membership degree belongs to the interval $[0, 1]$ and other is negative membership degree belongs to the interval $[-1, 0]$. Then BFS treated as a new tool to depict uncertainty in decision science. Bipolar fuzzy sets have not only applied in bipolar logical reasoning and bipolar set theory (Han et al. 2015; Zhang and Zhang 2004) but also applied in other application areas such as computational psychiatry (Zhang et al. 2011), medicine science (Lu and Busemeyer 2014; Zhang et al. 2009), bipolar quantum logic-based computing (Zhang 2013; Zhang and Peace 2014), quantum cellular combinatorics (Zhang 2013), decision analysis and organizational modeling (Fink and Yolles 2015; Li 2016), physics and philosophy (Zhang 2016) and bipolar fuzzy graph and its applications (Samanta and Pal 2012, 2012a, 2014; Singh and Kumar 2014; Yang et al. 2013). Lately, Gul (2015) introduced the concept of bipolar fuzzy aggregation operators, defined bipolar fuzzy weighted averaging operator (BFWAA) and bipolar fuzzy weighted geometric operator (BFWGA) and then utilized these operators to develop a multiple-attribute group decision-making problems. Thereafter, some decision-making problems have been developed using in the environment of bipolar fuzzy numbers as for example, Wang et al. (2018) introduced the notion of Frank Choquet Bonferroni mean operators on the bipolar neutrosophic environment and then utilized it to multi-criteria decision-making problems. Wei et al. (2018b) studied recently an MADM problem based on bipolar fuzzy Hamacher aggregation operator. They have proposed bipolar fuzzy Hamacher weighted averaging (BFHWA) operator, bipolar fuzzy Hamacher ordered weighted averaging (BFHOWA) operator, bipolar fuzzy Hamacher hybrid weighted averaging (BFHHWA) operator, bipolar fuzzy Hamacher weighted geometric (BFHWGA) operator, bipolar fuzzy Hamacher ordered weighted geometric (BFHOWGA) operator and bipolar fuzzy Hamacher hybrid weighted geometric (BFHHWGA) operator and explained related properties of these operators. Also, Gao et al. (2018) combined the concept of Hamacher operations and prioritized aggregation operators, and then defined dual hesitant bipolar fuzzy hamacher prioritized weighted average (DHBFHPWA) operator and dual hesitant bipolar fuzzy hamacher prioritized weighted geometric (DHBFHPWG) operator and studied a MADM problem using these operators for the relevance and effectiveness of the proposed methodology. Wei et al. (2017) proposed hesitant bipolar fuzzy weighted averaging (HBFWA)

operator, hesitant bipolar fuzzy ordered weighted averaging (HBFOWA) operator, hesitant bipolar fuzzy hybrid weighted averaging (HBFHWA) operator, hesitant bipolar fuzzy weighted geometric (HBFWGA) operator, hesitant bipolar fuzzy ordered weighted geometric (HBFOWGA) operator and hesitant bipolar fuzzy hybrid weighted geometric (HBFHWGA) operator, and then developed a MADM problem for the evaluation of quality constructional engineering software selection. Xu and Wei (2017) provided dual hesitant bipolar fuzzy arithmetic aggregation operators and dual hesitant bipolar fuzzy geometric aggregation operators and then solved an MADM problem by using these proposed operators. Xu (1990) used the concept of linguistic variables to rank alternatives among the set of criteria, with emphasis placed on modeling the decision-making process. In (Garg and Nancy 2018), utilized linguistic prioritized aggregation operators to develop a multiple-attribute decision-making method under single-valued neutrosophic environment. Chen et al. (2018) represented a method combining social relation analysis with linguistic VIKOR to simultaneously evaluate and select a new project involving ambient intelligence products in accordance with the overall performance of a new product and the degree of influence of this product on the existing ambient intelligence products. Lu et al. (2017) introduced the notion of bipolar 2-tuple linguistic set, and using this concept defined bipolar 2-tuple linguistic hybrid average (B2TLHA) operator and bipolar 2-tuple linguistic hybrid geometric (B2TLHG) operator. Therefore, they utilized these operators to develop MADM problem for studying a real-world decision-making method. Wei et al. (2018a) studied risk evaluation of enterprise human capital investment problem in the environment of interval-valued bipolar 2-tuple linguistic numbers (IVB2TLNs). They investigated arithmetic and geometric aggregation operators with IVB2TLN information and analyze differently properties of these operators. In (Deschrijver et al. 2004; Deschrijver and Kerre 2002; Xia et al. 2012; Xu and Xia 2011), proposed some aggregation operators based on algebraic working laws for intuitionistic fuzzy numbers (IFNs), which is a special case of t-norm and t-conorm. Different generalizations of t-norms and t-conorms are available in literature, such as Archimedean t-norms and t-conorms, Hamacher t-norms and t-conorms, Algebraic t-norms and t-conorms, Einstein t-norms and t-conorms, Frank t-norms and t-conorms and Dombi t-norms and t-conorms. Liu (2014) utilized Hamacher aggregation operators in interval-valued intuitionistic fuzzy numbers (IVIFNs) and developed MAGDM methods. Zhang (2017) proposed Frank aggregation operators for IVIFNs and their applications to multiple attribute group decision-making. Zhao and Wei (2013) proposed Einstein hybrid aggregation operators for IFNs and apply it to multiple attribute decision-making method. Yu (2013) introduced Choquet aggregation operator based

on Einstein operational laws for IFNs. Dombi (1982) introduced a new operations known as Dombi t-norm and Dombi t-conorm, which has a good precedence of change with the operation of parameters. For this advantage, Liu et al. (2018) utilized Dombi operations on intuitionistic fuzzy sets and developed multiple attribute group decision making problem using Dombi Bonferroni mean operator in the environment of intuitionistic fuzzy information. Chen and Ye (2017) developed multiple attribute decision-making problem using Dombi operations in single-valued neutrosophic environment. He (2018) used Dombi operations in hesitant fuzzy environment and based on this theory typhoon disaster assessment investigated. Keeping in mind from the fact that the bipolar fuzzy set has an effective capacity to prove the questionable and uncertain data which emerges in real-world issues. Therefore, decision-making problems in different uncertain fuzzy aggregation environment under Dombi operations makes us enough motivation to develop our present paper. There is a significant interest to research aggregation operators based on Dombi operation and their application to MADM problems. Therefore, based on Dombi operation, how to aggregate these bipolar fuzzy numbers is a tremendous topic. Tosolve this issue, in this paper, we shall define some bipolar fuzzy Dombi aggregation operators on the basis of traditional arithmetic (Xu 2007; Yager 1998), geometric operations (Xu and Yager 2006; Xu and Da 2003) and Dombi operations (Chen and Ye 2017; Dombi 1982; He 2018; Liu et al. 2018; Wei and Wei 2018).

The remainder of this paper is organized as follows: In the next Section, we briefly introduced some essential definition of the BFEs and some operational principles for BFEs. In Sect. 3, we defined Dombi operations for the bipolar fuzzy set. In Sect. 4, we developed bipolar fuzzy Dombi weighted averaging (BFDWAA) operator, bipolar fuzzy Dombi order weighted averaging (BFDOWAA) operator, bipolar fuzzy Dombi hybrid weighted averaging (BFDHWAA) operator. In Sect. 5, we proposed bipolar fuzzy Dombi weighted geometric (BFDWGA) operator, bipolar fuzzy Dombi order weighted geometric (BFDOWGA) operator, bipolar fuzzy Dombi hybrid weighted geometric (BFDHWGA) operator. In the next Section, we utilized those operators to create bipolar fuzzy multi-attribute decision-making problems. An interpretative case is specified for selection emerging technology enterprise systems in Sect. 7. In Sect. 8, we analyzed the effect of a parameter on decision-making results. Finally, in Sect. 9, the conclusion and scope of future research are outlined and discussed.

2 Preliminaries

In this section, we recall some basic concepts related to bipolar fuzzy sets (BFSs) over the universe of discourse X.

2.1 Bipolar fuzzy sets

Definition 1 Wei et al. (2018b) A bipolar fuzzy sets (BFS) is defined over the universe of discourse X as

$$\widetilde{S} = \{\langle x, \hat{\mu}_S^+(x), \hat{v}_S^-(x)\rangle | x \in X\}, \tag{1}$$

where $\hat{\mu}_S^+(x) : X \to [0, 1]$ represent positive degree of membership to satisfy corresponding property of an element x to a BFS and $\hat{v}_S^-(x) : X \to [-1, 0]$ represent negative degree of membership to satisfy counter-property of an element x to a BFS, for every $x \in X$. The set $\langle(\mu_S^+, v_S^-)\rangle$ denotes bipolar fuzzy numbers (BFNs), i.e, bipolar fuzzy elements (BFEs).

Wei et al. (2018b) provided some basic operations on BFEs given as follows:

Definition 2 Wei et al. (2018b) Let $\tilde{S} = (\langle \hat{\mu}_S^+(x), \hat{v}_S^-(x)\rangle)$ and $\tilde{T} = (\langle \hat{\mu}_T^+(x), \hat{v}_T^-(x)\rangle)$ be two BFEs over the universe X. The following operations between two BFEs are defined as:

(i) $\tilde{S} \subseteq \tilde{T}$, if $\hat{\mu}_S^+(x) \leq \hat{\mu}_T^+(x), \hat{v}_S^-(x) \geq \hat{v}_T^-(x)$ for all $x \in X$
(ii) $\tilde{S} \cup \tilde{T} = \{\langle x, \max\{\hat{\mu}_S^+(x), \hat{\mu}_T^+(x)\}, \min\{\hat{v}_S^-(x), \hat{v}_T^-\}\rangle | x \in X\}$
(iii) $\tilde{S} \cap \tilde{T} = \{\langle x, \min\{\hat{\mu}_S^+(x), \hat{\mu}_T^+(x)\}, \max\{\hat{v}_S^-(x), \hat{v}_T^-\}\rangle | x \in X\}$
(iv) $\overline{S} = \{\langle x, 1 - \hat{\mu}_S^+(x), |\hat{v}_S^-(x)| - 1\rangle | x \in X\}$ for all $x \in X$.

Definition 3 Wei et al. (2018b) Let $\tilde{S} = (\hat{\mu}_S, \hat{v}_S)$ be a bipolar fuzzy element (BFE), then score function $\hat{E}$ and accuracy function $\hat{L}$ for BFEs is calculated as follows:

$$\hat{E}(\tilde{S}) = \frac{1 + \hat{\mu}_S^+ + \hat{v}_S^-}{2}, \quad \hat{E}(\tilde{S}) \in [0, 1] \tag{2}$$

and accuracy function is evaluated as:

$$\hat{L}(\tilde{S}) = \frac{\hat{\mu}_S^+ - \hat{v}_S^-}{2}, \quad \hat{L}(\tilde{S}) \in [0, 1] \tag{3}$$

Based on score function $\hat{E}(\tilde{S})$ and accuracy function $\hat{L}(\tilde{S})$, defined order relation on two bipolar fuzzy elements $\tilde{S} = (\langle \hat{\mu}_S^+(x), \hat{v}_S^-(x)\rangle)$ and $\tilde{T} = (\langle \hat{\mu}_T^+(x), \hat{v}_T^-(x)\rangle)$ as follows:

Definition 4 Wei et al. (2018b) Let $\tilde{S}$ and $\tilde{T}$ be any two BFEs.

(i) If $\hat{E}(\tilde{S}) < \hat{E}(\tilde{T})$, then $\tilde{S} \prec \tilde{T}$
(ii) If $\hat{E}(\tilde{S}) > \hat{E}(\tilde{T})$, then $\tilde{S} \succ \tilde{T}$
(iii) If $\hat{E}(\tilde{S}) = \hat{E}(\tilde{T})$, then

    (1) If $\hat{L}(\tilde{S}) < \hat{L}(\tilde{T})$, then $\tilde{S} \prec \tilde{T}$.
    (2) If $\hat{L}(\tilde{S}) > \hat{L}(\tilde{T})$, then $\tilde{S} \succ \tilde{T}$.

(3) If $\hat{L}(\tilde{S}) = \hat{L}(\tilde{T})$, then $\tilde{S} \sim \tilde{T}$.

Wei et al. (2018b) provided some operations on bipolar fuzzy numbers that are as follows:

Definition 5 Wei et al. (2018b) Let $\tilde{S} = (\langle \hat{\mu}_S^+(x), \hat{v}_S^-(x) \rangle)$ and $\tilde{T} = (\langle \hat{\mu}_T^+(x), \hat{v}_T^-(x) \rangle)$ be two BFEs over the universe X, then following operations are defined as follows:

(i) $\tilde{S} \wedge \tilde{T} = \{\langle x, \min\{\hat{\mu}_S^+(x), \hat{\mu}_T^+(x)\}, \max\{\hat{v}_S^-(x), \hat{v}_T^-\} \rangle | x \in X\}$

(ii) $\tilde{S} \vee \tilde{T} = \{\langle x, \max\{\hat{\mu}_S^+(x), \hat{\mu}_T^+(x)\}, \min\{\hat{v}_S^-(x), \hat{v}_T^-\} \rangle | x \in X\}$

(iii) $\tilde{S} \oplus \tilde{T} = (\langle \hat{\mu}_S^+(x) + \hat{\mu}_T^+(x) - \hat{\mu}_S^+(x)\hat{\mu}_T^+(x), -|\hat{v}_S^-(x)||\hat{v}_T^-(x)| \rangle)$

(iv) $\tilde{S} \otimes \tilde{T} = (\langle \hat{\mu}_S^+(x)\hat{\mu}_T^+(x), \hat{v}_S^-(x) + \hat{v}_T^-(x) - \hat{v}_S^-(x)\hat{v}_T^-(x) \rangle)$

(v) $\lambda\tilde{S} = (1 - (1 - \hat{\mu}_S^+(x))^\lambda, -|\hat{v}_S(x)|^\lambda)$

(vi) $\tilde{S}^\lambda = (\hat{\mu}_S^\lambda(x), -1 + |1 + \hat{v}_S^-(x)|^\lambda)$.

Based on the Definition 5, Wei et al. (2018b) derived the following operations:

Definition 6 Wei et al. (2018b) Let $\tilde{S} = (\langle \hat{\mu}_S^+, \hat{v}_S^- \rangle)$ and $\tilde{T} = (\langle \hat{\mu}_T^+, \hat{v}_T^- \rangle)$ be two BFEs over the universe X and $\lambda, \lambda_1, \lambda_2 > 0$, then

(i) $\tilde{S} \oplus \tilde{T} = \tilde{T} \oplus \tilde{S}$

(ii) $\tilde{S} \otimes \tilde{T} = \tilde{T} \otimes \tilde{S}$

(iii) $\lambda(\tilde{S} \oplus \tilde{T}) = \lambda\tilde{S} \oplus \lambda\tilde{T}$

(iv) $(\tilde{S} \otimes \tilde{T})^\lambda = \tilde{S}^\lambda \otimes \tilde{T}^\lambda$

(v) $\lambda_1\tilde{S} \oplus \lambda_2\tilde{S} = (\lambda_1 + \lambda_2)\tilde{S}$

(vi) $\tilde{S}^{\lambda_1} \otimes \tilde{S}^{\lambda_2} = \tilde{S}^{(\lambda_1 + \lambda_2)}$

(vii) $(\tilde{S}^{\lambda_1})^{\lambda_2} = \tilde{S}^{\lambda_1\lambda_2}$.

3 Dombi operations on bipolar fuzzy numbers

3.1 Dombi operations

Dombi proposed operations Dombi product and Dombi sum which are special causes of t-norms and t-conorms given in the following definitions.

Definition 7 Dombi (1982) Let x and y be any two real numbers. Then, Dombi t-norms and Dombi t-conorms are defined in the following expression

$$Dom(x, y) = \frac{1}{1 + \left\{\left(\frac{1-x}{x}\right)^k + \left(\frac{1-y}{y}\right)^k\right\}^{1/k}} \quad (4)$$

$$Dom^*(x, y) = 1 - \frac{1}{1 + \left\{\left(\frac{x}{1-x}\right)^k + \left(\frac{y}{1-y}\right)^k\right\}^{1/k}} \quad (5)$$

where $k \geq 1$ and $(x, y) \in [0, 1] \times [0, 1]$.

Based on the Dombi t-norm and Dombi t-conorm, we defined Dombi operations on BFEs.

3.2 Dombi operations on bipolar fuzzy elements

In this section, we shall introduce the notion of Dombi operations on bipolar fuzzy sets and find some properties of these operations. Let $\tilde{B}_1$ and $\tilde{B}_2$ be two bipolar fuzzy sets and $\lambda > 0$, then Dombi product and Dombi sum of the two BFEs $\tilde{B}_1$ and $\tilde{B}_2$ are respectively denoted as $(\tilde{B}_1 \otimes \tilde{B}_2)$ and for right side and $(\tilde{B}_1 \oplus \tilde{B}_2)$ and defined by

(i) $\tilde{B}_1 \oplus \tilde{B}_2 = \left\langle 1 - \frac{1}{1 + \left\{\left(\frac{\hat{\mu}_1^+}{1-\hat{\mu}_1^+}\right)^k + \left(\frac{\hat{\mu}_2^+}{1-\hat{\mu}_2^+}\right)^k\right\}^{1/k}}, \right.$
$$\left. \frac{-1}{1 + \left\{\left(\frac{1+\hat{v}_1^-}{|\hat{v}_1^-|}\right)^k + \left(\frac{1+\hat{v}_2^-}{|\hat{v}_2^-|}\right)^k\right\}^{1/k}} \right\rangle$$

(ii) $\tilde{B}_1 \otimes \tilde{B}_2 = \left\langle \frac{1}{1 + \left\{\left(\frac{1-\hat{\mu}_1^+}{\hat{\mu}_1^+}\right)^k + \left(\frac{1-\hat{\mu}_2^+}{\hat{\mu}_2^+}\right)^k\right\}^{1/k}}, \right.$
$$\left. -1 + \frac{1}{1 + \left\{\left(\frac{|\hat{v}_1^-|}{1+\hat{v}_1^-}\right)^k + \left(\frac{|\hat{v}_2^-|}{1+\hat{v}_2^-}\right)^k\right\}^{1/k}} \right\rangle$$

(iii) $\lambda . \tilde{B}_1 = \left\langle 1 - \frac{1}{1 + \left\{\lambda\left(\frac{\hat{\mu}_1^+}{1-\hat{\mu}_1^+}\right)^k\right\}^{1/k}}, \right.$
$$\left. \frac{-1}{1 + \left\{\lambda\left(\frac{1+\hat{v}_1^-}{|\hat{v}_1^-|}\right)^k\right\}^{1/k}} \right\rangle$$

(iv) $(\tilde{B}_1)_1^\lambda = \left\langle \frac{1}{1 + \left\{\lambda\left(\frac{1-\hat{\mu}_1^+}{\hat{\mu}_1^+}\right)^k\right\}^{1/k}}, \right.$
$$\left. -1 + \frac{1}{1 + \left\{\lambda\left(\frac{|\hat{v}_1^-|}{1+\hat{v}_1^-}\right)^k\right\}^{1/k}} \right\rangle.$$

4 Bipolar fuzzy Dombi arithmetic aggregation operators

In this section, we propose Dombi arithmetic aggregation operators with bipolar fuzzy numbers such as bipolar fuzzy Dombi weighted averaging operator (BFDWA), bipolar fuzzy Dombi ordered weighted averaging operator (BFDOWA) and bipolar fuzzy Dombi hybrid weighted averaging operator (BFDHWA).

Definition 8 Let $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ be a collection of BFEs. Then bipolar fuzzy Dombi weighted averaging BFDWA operator is a function $\tilde{B}^n \to \tilde{B}$ such that

$$BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigoplus_{t=1}^{n}(\mho_t \tilde{B}_t) \qquad (6)$$

where $\mho = (\mho_1, \mho_2, \ldots, \mho_n)^T$ be the weight vector of B_t $(t = 1, 2, \ldots, n)$ with $\mho_j > 0$ and $\sum_{t=1}^{n} \mho_j = 1$.

We get the following theorem that follows the Dombi operations on BFEs.

Theorem 1 $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ be a collection of bipolar fuzzy elements, then aggregated value of them using the BFDWA operation is also a BFEs, and

$$BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigoplus_{t=1}^{n}(\mho_t \tilde{B}_t)$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{\hat{\mu}_t^+}{1 - \hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1 + \hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle \qquad (7)$$

where $\mho = (\mho_1, \mho_2, \ldots, \mho_n)$ be the weight vector of $\tilde{B}_t$ $(t = 1, 2, \ldots, n)$ such that $\mho_t > 0$, and $\sum_{t=1}^{n} \mho_t = 1$.

Thus Theorem 1 can be prove by the method of mathematical induction as follows:

Proof (i) When $n = 2$, based on Dombi operations on BFEs we obtain the following results

$$BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2) = \tilde{B}_1 \bigoplus \tilde{B}_2 = (\hat{\mu}_1^+, \hat{v}_1^-) \bigoplus (\hat{\mu}_2^+, \hat{v}_2^-) \quad (8)$$

and for right side of (8), we have

$$\left\langle 1 - \frac{1}{1 + \left\{ \mho_1 \left(\frac{\hat{\mu}_1^+}{1 - \hat{\mu}_1^+} \right)^k + \mho_2 \left(\frac{\hat{\mu}_2^+}{1 - \hat{\mu}_2^+} \right)^k \right\}^{1/k}}, \right.$$
$$\left. \frac{-1}{1 + \left\{ \mho_1 \left(\frac{1 + \hat{v}_1^-}{|\hat{v}_1^-|} \right)^k + \mho_2 \left(\frac{1 + \hat{v}_2^-}{|\hat{v}_2^-|} \right)^k \right\}^{1/k}} \right\rangle$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{2} \mho_t \left(\frac{\hat{\mu}_t^+}{1 - \hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{2} \mho_t \left(\frac{1 + \hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle. \qquad (9)$$

Thus, (7) holds is true for $n = 2$. (ii) Assume that (7) holds for $n = p$, where $p \in N(set of natural numbers)$, then based on Eq. (9), we have

$$BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{b}_p) = \bigoplus_{t=1}^{p}(\mho_t \tilde{B}_t)$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{p} \mho_t \left(\frac{\hat{\mu}_t^+}{1 - \hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{p} \mho_t \left(\frac{1 + \hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle. \qquad (10)$$

Now for $n = p + 1$, then $BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_p, \tilde{B}_{p+1})$
$$= \bigoplus_{t=1}^{p}(\mho_t B_t) \bigoplus(\mho_{p+1} B_{p+1})$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{p} \mho_t \left(\frac{\hat{\mu}_t^+}{1 - \hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{p} \mho_t \left(\frac{1 + \hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle$$
$$\bigoplus \left\langle 1 - \frac{1}{1 + \left\{ \mho_{p+1} \left(\frac{\hat{\mu}_{p+1}^+}{1 - \hat{\mu}_{p+1}^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \mho_{p+1} \left(\frac{1 + \hat{v}_{p+1}^-}{|\hat{v}_{p+1}^-|} \right)^k \right\}^{1/k}} \right\rangle$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{p+1} \mho_j \left(\frac{\hat{\mu}_t^+}{1 - \hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{p+1} \mho_t \left(\frac{1 + \hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle. \qquad (11)$$

Thus, (7) is true for $n = p + 1$.

Hence, we conclude that (7) is true for any $n \in N$. $\qquad \square$

Example 1 Suppose there are four BFEs $B_1 = (0.6, -0.3)$, $B_2 = (0.5, -0.4)$, $B_3 = (0.7, -0.2)$ and $B_4 = (0.2, -0.3)$, $\mho = (0.2, 0.1, 0.3, 0.4)$ is the weight vector for these BFEs B_t $(t = 1, 2, 3, 4)$. Then $BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{b}_4) = \bigoplus_{t=1}^{4}(\mho_t \tilde{B}_t)$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{4} \mho_t \left(\frac{\hat{\mu}_t^+}{1-\hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{4} \mho_t \left(\frac{1+\hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle$$

$$= \left\langle 1 - \frac{1}{1 + \left(0.2 \times (\frac{0.6}{1-0.6})^2 + 0.1 \times (\frac{0.5}{1-0.5})^2 + 0.3 \times (\frac{0.7}{1-0.7})^2 + 0.4 \times (\frac{0.2}{1-0.2})^2 \right)^2},\right.$$

$$\left. \frac{-1}{1 + \left(0.2 \times (\frac{1-0.3}{|-0.3|})^2 + 0.1 \times (\frac{1-0.4}{|-0.4|})^2 + 0.3 \times (\frac{1-0.2}{|-0.2|})^2 + 0.4 \times (\frac{1-0.3}{|-0.3|})^2 \right)^2} \right\rangle$$

$$= \langle 0.5978, -0.2578 \rangle$$

We prove easily the following properties by using the operator BFDWA.

Theorem 2 (Idempotency Property) *If $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ ($t = 1, 2, \ldots, n$) be a collection of BFEs and are all equal, i.e., $\tilde{B}_t = \tilde{B}$ for all t, then*

$$BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \tilde{B}. \tag{12}$$

Proof Since $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-) = \tilde{B}$ ($t = 1, 2, \ldots, n$). Then, we have by Eq. (10),

$$BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n)$$

$$= \bigoplus_{t=1}^{n} (\mho_t \tilde{B}_t) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{\hat{\mu}_t^+}{1-\hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \right.$$

$$\left. \frac{-1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1+\hat{v}_t^-}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \left(\frac{\hat{\mu}^+}{1-\hat{\mu}^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \left(\frac{1+\hat{v}^-}{|\hat{v}^-|} \right)^k \right\}^{1/k}} \right\rangle$$

$$= \left\langle 1 - \frac{1}{1 + \frac{\hat{\mu}^+}{1-\hat{\mu}^+}}, \frac{-1}{1 + \frac{1+\hat{v}^-}{|\hat{v}^-|}} \right\rangle = (\hat{\mu}^+, \hat{v}^-) = \tilde{B}.$$

Thus, $BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \tilde{B}$ holds. $\square$

Theorem 3 (Boundedness Property) *Let $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ ($t = 1, 2, \ldots, n$) be a collection of BFEs. Let $\tilde{B}^- = \min(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = (\hat{\mu'}^-, \hat{v'}^-)$ and $\tilde{B}^+ = \max(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = (\hat{\mu'}^+, \hat{v'}^+)$.*
Then, $\tilde{B}^- \leq BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq \tilde{B}^+$.

Proof Let $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ ($t = 1, 2, \ldots, n$) be a collection of BFEs. Let $\tilde{B}^- = \min(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = (\hat{\mu'}^-, \hat{v'}^-)$ and

$\tilde{B}^+ = \max(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = (\hat{\mu'}^+, \hat{v'}^+)$. We have $\hat{\mu'}^- = \min_t \{\hat{\mu}_t^+\}$, $\hat{v'}^- = \max_t \{\hat{v}_t^-\}$, $\hat{\mu'}^+ = \max_t \{\hat{\mu}_t^+\}$, and $\hat{v'}^+ = \min_t \{\hat{v}_t^-\}$. Then,

$$1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{\hat{\mu'}^-}{1-\hat{\mu'}^-} \right)^k \right\}^{1/k}} \leq 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{\hat{\mu}_t^+}{1-\hat{\mu}_t^+} \right)^k \right\}^{1/k}}$$

$$\leq 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{\hat{\mu'}^+}{1-\hat{\mu'}^+} \right)^k \right\}^{1/k}} \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1+\hat{v'}^-}{|\hat{v'}^-|} \right)^k \right\}^{1/k}}$$

$$\leq \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1+\hat{v}^-}{|\hat{v}^-|} \right)^k \right\}^{1/k}} \leq \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1+\hat{v'}^+}{|\hat{v'}^+|} \right)^k \right\}^{1/k}}.$$

This result conclude between

$$\tilde{B}^- \leq BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq \tilde{B}^+. \tag{13}$$

$\square$

Theorem 4 (Monotonicity Property) *Let $\tilde{B}_t$ ($t = 1, 2, \ldots, n$) and $\tilde{B}'_t$ ($t = 1, 2, \ldots, n$) be two sets of BFEs, if $\tilde{B}_t \leq \tilde{B}'_t$ for all t, then $BFDWA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq BFDWA_{\mho}(\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_n)$.*

Proof Proof follows from definition. $\square$

Now, we introduce bipolar fuzzy Dombi ordered weighted averaging operator BFDOWA.

Definition 9 Let $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ ($t = 1, 2, \ldots, n$) be a collection of BFEs. A bipolar fuzzy Dombi ordered weighted average (BFDOWA) operator of dimension n is a function $BFDOWA : \tilde{B}^n \to \tilde{B}$ with associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_t > 0$, and $\sum_{t=1}^{n} w_t = 1$.
Therefore,

$$BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigoplus_{t=1}^{n} (\mho_t \tilde{B}_{t(\sigma)}) \tag{14}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ are the permutation of $\sigma(t)$ $(t = 1, 2, \ldots, n)$, for which $\tilde{B}_{\sigma(t-1)} \geq \tilde{B}_{\sigma(t)}$ for all $t = 1, 2, \ldots, n$.

The following theorem is develop based on Dombi product operation on BFEs.

Theorem 5 *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *be a collection of BFEs. A bipolar fuzzy Dombi ordered weighted average (BFDOWA) operator of dimension n is a function* $BFDOWA : \tilde{B}^n \to \tilde{B}$ *with associated vector* $w = (w_1, w_2, \ldots, w_n)^T$ *such that* $w_t > 0$, *and* $\sum_{t=1}^{n} w_t = 1$. *Then,*

$$BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigoplus_{t=1}^{n}(w_t \tilde{B}_{t(\sigma)})$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} w_t \left(\frac{\hat{\mu}_{\sigma(t)}^+}{1 - \hat{\mu}_{\sigma(t)}^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{n} w_t \left(\frac{1 + \hat{v}_{\sigma(t)}^-}{|\hat{v}_{\sigma(t)}^-|} \right)^k \right\}^{1/k}} \right\rangle$$
(15)

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *are the permutation of* $\sigma(t)$ $(t = 1, 2, \ldots, n)$, *for which* $B_{\sigma(t-1)} \geq B_{\sigma(t)}$ *for all* $t = 1, 2, \ldots, n$.

Example 2 Let $\tilde{B}_1 = (0.6, -0.4)$, $\tilde{B}_2 = (0.3, -0.5)$, $\tilde{B}_3 = (0.5, -0.4)$ and $\tilde{B}_4 = (0.3, -0.6)$ be four BFEs and $w = (0.2, 0.1, 0.3, 0.4)^T$ is the weight vector of these BFEs. Then aggregated values of BFEs for $(k = 3)$ and by Definition 9, scores of $\tilde{B}_t$ $(t = 1, 2, 3, 4)$ can be evaluated as follows:

$\hat{E}(\tilde{B}_1) = \frac{1 + 0.6 - 0.4}{2} = 0.6$, $\hat{E}(\tilde{B}_2) = \frac{1 + 0.3 - 0.5}{2} = 0.4$, $\hat{E}(\tilde{B}_3) = \frac{1 + 0.5 - 0.4}{2} = 0.55$, $\hat{E}(\tilde{B}_4) = \frac{1 + 0.3 - 0.6}{2} = 0.35$. Since,

$\hat{E}(\tilde{B}_1) > \hat{E}(\tilde{B}_3) > \hat{E}(\tilde{B}_2) > \hat{E}(\tilde{B}_4)$,

then $\tilde{B}_{\sigma(1)} = \tilde{B}_1 = (0.6, -0.4)$, $\tilde{B}_{\sigma(2)} = \tilde{B}_3 = (0.5, -0.4)$, $\tilde{B}_{\sigma(3)} = \tilde{B}_2 = (0.3, -0.5)$ and $\tilde{B}_{\sigma(4)} = \tilde{B}_4 = (0.3, -0.6)$. Then, by definition of BFDOWA operator:

The following properties easily proved by BFDOWA operator.

Theorem 6 (Idempotency Property) *If* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *are all equal, i.e.* $\tilde{B}_t = \tilde{B}$ *for all t, then* $BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \tilde{B}$.

Theorem 7 (Boundedness Property) *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *be a collection of BFEs. Let* $B^- = \min_t \tilde{B}_t$, *and* $\tilde{B}^+ = \max_t \tilde{B}_t$. *Then,*

$$\tilde{B}^- \leq BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq \tilde{B}^+.$$
(16)

Theorem 8 (Monotonicity Property) *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *and* $\tilde{B}_t'$ $(t = 1, 2, \ldots, n)$ *be two sets of BFEs, if* $\tilde{B}_t \leq \tilde{B}_t'$ *for all t, then*

$$BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq BFDOWA_w(\tilde{B}_1', \tilde{B}_2', \ldots, \tilde{B}_n').$$
(17)

Theorem 9 (Commutativity Property) *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *and* $\tilde{B}_t'$ $(t = 1, 2, \ldots, n)$ *be two sets of BFEs, then* $BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = BFDOWA_w(\tilde{B}_1', \tilde{B}_2', \ldots, \tilde{B}_n')$ *where* $\tilde{B}_t'$ $(t = 1, 2, \ldots, n)$ *is any permutation of* $\tilde{B}_t$ $(t = 1, 2, \ldots, n)$.

Above Definitions 8 and 9, we see that BFDWA operator weights only the bipolar fuzzy values, on the other hand the BFDOWA operator weights only the ordered positions of the bipolar fuzzy values instead of weights of the bipolar fuzzy values themselves. Therefore, the weights used in the operators BFDWA and BFDOWA have in different aspects. But, they are considered only one of them. To avoid this disadvantage, we introduce bipolar fuzzy Dombi hybrid averaging (BFDHA) operator.

$$BFDOWA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_4) = \bigoplus_{t=1}^{4}(w_t \tilde{B}_{\sigma(t)})$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{4} w_t \left(\frac{\hat{\mu}_{\sigma(t)}^+}{1 - \hat{\mu}_{\sigma(t)}^+} \right)^3 \right\}^{1/3}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{4} w_t \left(\frac{1 + \hat{v}_{\sigma(t)}^-}{|\hat{v}_{\sigma(t)}^-|} \right)^3 \right\}^{1/3}} \right\rangle$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ 0.2 \times \left(\frac{0.6}{1 - 0.6} \right)^3 + 0.1 \times \left(\frac{0.5}{1 - 0.5} \right)^3 + 0.3 \times \left(\frac{0.3}{1 - 0.3} \right)^3 + 0.4 \times \left(\frac{0.3}{1 - 0.3} \right)^3 \right\}^{1/3}}, \right.$$

$$\left. \frac{-1}{1 + \left\{ 0.2 \times \left(\frac{1 - 0.4}{|-0.4|} \right)^3 + 0.1 \times \left(\frac{1 - 0.4}{|-0.4|} \right)^3 + 0.3 \times \left(\frac{1 - 0.5}{|-0.5|} \right)^3 + 0.4 \times \left(\frac{1 - 0.6}{|-0.6|} \right)^3 \right\}^{1/3}} \right\rangle = \langle (0.4845, -0.3381) \rangle.$$

Definition 10 A bipolar fuzzy Dombi hybrid average (BFDHA) operator of dimension n is a function $BFDHA : \tilde{B}^n \to \tilde{B}$, with associated weight vector $w = (w_1, w_2, \ldots, w_n)$ such that $w_t > 0$, and $\sum_{t=1}^{n} w_t = 1$. Therefore, BFDHWA operator can be evaluated as

$$BFDHWA_{w,\eth}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n)$$
$$= \langle w_1 \dot{\tilde{B}}_{\sigma(1)} \rangle + \langle w_2 \dot{\tilde{B}}_{\sigma(2)} \rangle + \ldots + \langle w_n \dot{\tilde{B}}_{\sigma(n)} \rangle$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} w_t \left(\frac{\hat{\dot{\mu}}^+_{\sigma(t)}}{1 - \hat{\dot{\mu}}^+_{\sigma(t)}} \right)^k \right\}^{1/k}}, \right.$$
$$\left. \frac{-1}{1 + \left\{ \sum_{t=1}^{n} w_t \left(\frac{1 + \hat{\dot{v}}^-_{\sigma(t)}}{|\dot{v}^-_{\sigma(t)}|} \right)^k \right\}^{1/k}} \right\rangle \qquad (18)$$

where $\dot{\tilde{B}}_{\sigma(t)}$ is the tth largest weighted bipolar fuzzy values $\dot{\tilde{B}}_t$ ($\dot{\tilde{B}}_t = n\eth_t B_t, t = 1, 2, \ldots, n$), and $\eth = (\eth_1, \eth_2, \ldots, \eth_n)^T$ be the weight vector of $\dot{\tilde{B}}_t$ with $\eth_t > 0$ and $\sum_{t=1}^{n} \eth_t = 1$, where n is the balancing coefficient. When $w = (1/n, 1/n, \ldots, 1/n)$, then BFDWA operator is a special case of BFDHA operator. Let $\eth = (1/n, 1/n, \ldots, 1/n)$, then BFDOWA is a special case of the operator BFDHA. Thus, BFDHA operator is a generalization of both the operators BFDWA and BFDOWA, which reflects the degrees of the given arguments and their ordered positions.

Example 3 There are four BFEs $\tilde{B}_1 = (0.5, -0.3)$, $\tilde{B}_2 = (0.6, -0.3)$, $\tilde{B}_3 = (0.7, -0.3)$ and $\tilde{B}_4 = (0.2, -0.4)$, and $\eth = (0.20, 0.30, 30, 0.20)^T$ weight vector of these four BFEs and $W = (0.2, 0.1, 0.3, 0.4)^T$ is the associated weight vector. Then, by Definition 10 aggregated of BFEs for ($k = 3$), by the way

$$\dot{\tilde{B}}_1 = \left\langle \left(1 - \frac{1}{1 + \left\{ 4 \times 0.20 \times \left(\frac{0.5}{1 - 0.5} \right)^3 \right\}^{1/3}}, \frac{-1}{1 + \left\{ 4 \times 0.20 \times \left(\frac{1 - 0.3}{|-0.3|} \right)^3 \right\}^{1/3}} \right) \right\rangle$$
$$= \langle 0.4814, -0.3158 \rangle$$

$$\dot{\tilde{B}}_2 = \left\langle \left(1 - \frac{1}{1 + \left\{ 4 \times 0.30 \times \left(\frac{0.6}{1 - 0.6} \right)^3 \right\}^{1/3}}, \frac{-1}{1 + \left\{ 4 \times 0.30 \times \left(\frac{1 - 0.3}{|-0.3|} \right)^3 \right\}^{1/3}} \right) \right\rangle$$
$$= \langle 0.6145, -0.2874 \rangle$$

$$\dot{\tilde{B}}_3 = \left\langle \left(1 - \frac{1}{1 + \left\{ 4 \times 0.30 \times \left(\frac{0.7}{1 - 0.7} \right)^3 \right\}^{1/3}}, \frac{-1}{1 + \left\{ 4 \times 0.30 \times \left(\frac{1 - 0.3}{|-0.3|} \right)^3 \right\}^{1/3}} \right) \right\rangle$$
$$= \langle 0.7126, -0.2874 \rangle$$

$$\dot{\tilde{B}}_4 = \left\langle \left(1 - \frac{1}{1 + \left\{ 4 \times 0.20 \times \left(\frac{0.2}{1 - 0.2} \right)^3 \right\}^{1/3}}, \frac{-1}{1 + \left\{ 4 \times 0.20 \times \left(\frac{1 - 0.4}{|-0.4|} \right)^3 \right\}^{1/3}} \right) \right\rangle$$
$$= \langle 0.1884, -0.4180 \rangle.$$

Scores of $\tilde{B}_t$ (t=1,2,3,4) calculated as follows:

$\hat{E}(\dot{B}_1) = \frac{1+0.4814-0.3158}{2} = 0.5828,$

$\hat{E}(\dot{B}_2) = \frac{1+0.6145-0.2874}{2} = 0.6636,$

$\hat{E}(\dot{B}_3) = \frac{1+0.7126-0.2874}{2} = 0.7126 ,$

$\hat{E}(\dot{B}_4) = \frac{1+0.1884-0.4180}{2} = 0.3852.$

Since,

$$\hat{E}(\dot{B}_3) > \hat{E}(\dot{B}_2) > \hat{E}(\dot{B}_1) > \hat{E}(\dot{B}_4).$$

Then, $\dot{B}_{\sigma(1)} = \dot{B}_3 = (0.7126, -0.2874),$

$\dot{B}_{\sigma(2)} = \dot{B}_2 = (0.6145, -0.2874),$

$\dot{B}_{\sigma(3)} = \dot{B}_1 = (0.4814, -0.3158)$ and

$\dot{B}_{\sigma(4)} = \dot{B}_4 = (0.1884, -0.4180)$. Therefore, aggregated values of BFEs ($k = 3$) by the definition of BFDHWA operator:

$BFDHWA_{\eth}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_4)$

$$= \bigoplus_{t=1}^{4} (\eth_t \dot{B}_{\sigma(t)}) = \left\langle 1 - \frac{1}{1+\left\{\sum_{t=1}^{4} \eth_t \left(\frac{\hat{\dot{\mu}}^+_{\sigma(t)}}{1-\hat{\dot{\mu}}^+_{\sigma(t)}}\right)^3\right\}^{1/3}}, \frac{-1}{1+\left\{\sum_{t=1}^{4} \eth_t \left(\frac{1+\hat{\dot{v}}^-_{\sigma(t)}}{|\hat{\dot{v}}^-_{\sigma(t)}|}\right)^3\right\}^{1/3}} \right\rangle$$

$$= \left\langle 1 - \frac{1}{1+\left\{0.2 \times \left(\frac{0.7126}{1-0.7126}\right)^3 + 0.1 \times \left(\frac{0.6145}{1-0.6145}\right)^3 + 0.3 \times \left(\frac{0.4814}{1-0.4814}\right)^3 + 0.4 \times \left(\frac{0.1884}{1-0.1884}\right)^3\right\}^{1/3}}, \right.$$

$$\left. \frac{-1}{1+\left\{0.2 \times \left(\frac{1-0.2874}{|-0.2874|}\right)^3 + 0.1 \times \left(\frac{1-0.2874}{|-0.2874|}\right)^3 + 0.3 \times \left(\frac{1-0.3158}{|-0.3158|}\right)^3 + 0.4 \times \left(\frac{1-0.4180}{|-0.4180|}\right)^3\right\}^{1/3}} \right\rangle$$

$$= \langle (0.6073, -0.3271) \rangle.$$

5 Bipolar fuzzy Dombi geometric aggregation operators

To this part, we shall propose Dombi geometric aggregation operators with bipolar fuzzy numbers such as bipolar fuzzy Dombi weighted geometric operator (BFDWGA), bipolar fuzzy Dombi ordered weighted geometric operator (BFDOWGA) and bipolar fuzzy Dombi hybrid weighted geometric operator (BFDHWGA).

Definition 11 Let $\tilde{B}_t = (\hat{\mu}^+_t, \hat{v}^-_t)$ $(t = 1, 2, \ldots, n)$ be a collection of BFEs. Then, BFDWGA operator is a function $\tilde{B}^n \to \tilde{B}$ such that

$$BFDWGA_{\eth}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigotimes_{t=1}^{n} (\tilde{B}_t)^{\eth_t} \qquad (19)$$

where $\eth = (\eth_1, \eth_2, \ldots, \eth_n)^T$ be the weight vector of $\tilde{B}_t$ $(t = 1, 2, \ldots, n)$ such that $\eth_t > 0$ and $\sum_{t=1}^{n} \eth_t = 1$.

Bipolar fuzzy BFDWGA operator is evaluated as in the following theorem and can be prove by mathematical induction.

Theorem 10 $\tilde{B}_t = (\hat{\mu}^+_t, \hat{v}^-_t)$ $(t = 1, 2, \ldots, n)$ be a collection of bipolar fuzzy elements, then aggregated value of them using the BFDWGA operator is also a BFE, and

$$BFDWGA_{\eth}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigotimes_{t=1}^{n} (\tilde{B}_t)^{\eth_t}$$

$$= \left\langle \frac{1}{1+\left\{\sum_{t=1}^{n} \eth_t \left(\frac{1-\hat{\mu}^+_t}{\hat{\mu}^+_t}\right)^k\right\}^{1/k}}, -1 + \frac{1}{1+\left\{\sum_{t=1}^{n} \eth_t \left(\frac{|\hat{v}^-_t|}{1+\hat{v}^-_t}\right)^k\right\}^{1/k}} \right\rangle \qquad (20)$$

where $\eth = (\eth_1, \eth_2, \ldots, \eth_n)$ be the weight vector of $\tilde{B}_t$ $(t = 1, 2, \ldots, n)$ such that $\eth_t > 0$, and $\sum_{t=1}^{n} \eth_t = 1, \gamma > 0$.

Proof Proof of the theorem follows from Theorem 1. $\qquad \square$

Example 4 Suppose there are four BFEs $B_1 = (0.6, -0.3)$, $B_2 = (0.5, -0.4)$, $B_3 = (0.7, -0.2)$ and $B_4 = (0.2, -0.3)$, and $\eth = (0.2, 0.1, 0.3, 0.4)$ is the weight vector for these BFEs B_t $(t = 1, 2, 3, 4)$. Then $BFDWGA_{\eth}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_4)$ $= \bigoplus_{t=1}^{4} (\eth_t \tilde{B}_t)$

$$= \left\{ \frac{1}{1 + \left\{ \sum\limits_{t=1}^{4} \mho_t \left(\frac{1-\hat{\mu}_t^+}{\hat{\mu}_t^+} \right)^k \right\}^{1/k}}, -1 + \frac{1}{1 + \left\{ \sum\limits_{t=1}^{4} \mho_t \left(\frac{|\hat{v}_t^-|}{1+\hat{v}_t^-} \right)^k \right\}^{1/k}} \right\}.$$

$$= \left\langle \frac{1}{1 + \left(0.2 \times \left(\frac{1-0.6}{0.6} \right)^3 + 0.1 \times \left(\frac{1-0.5}{0.5} \right)^3 + 0.3 \times \left(\frac{1-0.7}{0.7} \right)^3 + 0.4 \times \left(\frac{1-0.2}{0.2} \right)^3 \right)^{1/3}}, \right.$$

$$\left. -1 + \frac{1}{1 + \left(0.2 \times \left(\frac{|-0.3|}{1-0.3} \right)^3 + 0.1 \times \left(\frac{|-0.4|}{1-0.4} \right)^3 + 0.3 \times \left(\frac{|-0.2|}{1-0.2} \right)^3 + 0.4 \times \left(\frac{|-0.3|}{1-0.3} \right)^3 \right)^{1/3}} \right\rangle$$

$$= \langle (0.2529, -0.3025) \rangle.$$

Following properties for BFDWGA operator can be prove easily.

Theorem 11 (Idempotency Property) *If* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *collection of BFEs and all are equal, i.e.,* $\tilde{B}_t = \tilde{B}$ *for all t, then* $BFDWGA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \tilde{B}$.

Theorem 12 (Boundedness Property) *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *be a collection of BFEs. Let* $\tilde{B}^- = \min_t \tilde{B}_t$, $\tilde{B}^+ = \max_t \tilde{B}_t$. *Then*

$$\tilde{B}^- \leq BFDWGA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq \tilde{B}^+. \tag{21}$$

Theorem 13 (Monotonicity Property) *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *and* $\tilde{B}_t'$ $(t = 1, 2, \ldots, n)$ *be two sets of BFEs, if* $\tilde{B}_t \leq \tilde{B}_t'$ *for all t, then* $BFDWGA_{\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq BFDWGA_{\mho}$ $(\tilde{B}_1', \tilde{B}_2', \ldots, \tilde{B}_n')$.

Now, we introduce bipolar fuzzy Dombi ordered weighted geometric BFDOWGA operator.

Definition 12 Let $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ be a collection of BFEs. A bipolar fuzzy Dombi ordered weighted geometric (BFDOWGA) operator of dimension n is a function $BFDOWGA : \tilde{B}^n \to \tilde{B}$ with associated vector

$w = (w_1, w_2, \ldots, w_n)^T$ such that $w_t > 0$, and $\sum_{t=1}^n w_t = 1$. Therefore,

$$BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigotimes_{t=1}^{n} (\tilde{B}_{t(\sigma)})^{w_t} \tag{22}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ are the permutation of $(t = 1, 2, \ldots, n)$, for which $\tilde{B}_{\sigma(t-1)} \geq \tilde{B}_{\sigma(t)}$ for all $t = 1, 2, \ldots, n$.

The following theorem is develop based on Dombi product operation on BFEs using BFDOWGA operator.

Theorem 14 *Let* $\tilde{B}_t = (\hat{\mu}_t^+, \hat{v}_t^-)$ $(t = 1, 2, \ldots, n)$ *be a collection of BFEs. A BFDOWGA operator of dimension n is a function* $BFDOWGA : \tilde{B}^n \to \tilde{B}$. *Furthermore,*

$$BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigotimes_{t=1}^{n} (\tilde{B}_t)^{w_t}$$

$$= \left\langle \frac{1}{1 + \left\{ \sum\limits_{t=1}^{n} w_t \left(\frac{1-\hat{\mu}_t^+}{\hat{\mu}_t^+} \right)^k \right\}^{1/k}}, -1 + \frac{1}{1 + \left\{ \sum\limits_{t=1}^{n} w_t \left(\frac{|\hat{v}_t^-|}{1+\hat{v}_t} \right)^k \right\}^{1/k}} \right\rangle \tag{23}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is the permutation of $\sigma(t)$ $(t = 1, 2, \ldots, n)$, for which $\tilde{B}_{\sigma(t-1)} \geq \tilde{B}_{\sigma(t)}$ for all $(t = 1, 2, \ldots, n)$, with associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_t > 0$, and $\sum_{t=1}^n w_t = 1$.

Example 5 Let $\tilde{B}_1 = (0.6, -0.4)$, $\tilde{B}_2 = (0.3, -0.5)$, $\tilde{B}_3 = (0.5, -0.4)$ and $\tilde{B}_4 = (0.3, -0.6)$ be four BFEs with $w = (0.2, 0.1, 0.3, 0.4)^T$ is the weight vector of these BFEs. Then aggregated BFEs is for $(k = 3)$ and by Definition 9, scores of $\tilde{B}_t$ $(t = 1, 2, 3, 4)$ can be evaluated as follows:

$\hat{E}(\tilde{B}_1) = \frac{1+0.6-0.4}{2} = 0.6$, $\hat{E}(\tilde{B}_2) = \frac{1+0.3-0.5}{2} = 0.4$, $\hat{E}(\tilde{B}_3) = \frac{1+0.5-0.4}{2} = 0.55$, $\hat{E}(\tilde{B}_4) = \frac{1+0.3-0.6}{2} = 0.35$.

Since,

$$\hat{E}(\tilde{B}_1) > \hat{E}(\tilde{B}_3) > \hat{E}(\tilde{B}_2) > \hat{E}(\tilde{B}_4),$$

then $\tilde{B}_{\sigma(1)} = \tilde{B}_1 = (0.6, -0.4)$, $\tilde{B}_{\sigma(2)} = \tilde{B}_3 = (0.5, -0.4)$, $\tilde{B}_{\sigma(3)} = \tilde{B}_2 = (0.3, -0.5)$ and $\tilde{B}_{\sigma(4)} = \tilde{B}_4 = (0.3, -0.6)$. Then, by definition of BFDOWGA operator:

$$BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_4) = \bigoplus_{t=1}^{4}(w_t\tilde{B}_{\sigma(t)})$$

$$= \left\langle \frac{1}{1 + \left\{ \sum\limits_{t=1}^{4} w_t \left(\frac{1-\hat{\mu}^+_{\sigma(t)}}{\hat{\mu}^+_{\sigma(t)}} \right)^3 \right\}^{1/3}}, -1 + \frac{1}{1 + \left\{ \sum\limits_{t=1}^{4} w_t \left(\frac{|\hat{v}^-_{\sigma(t)}|}{1-\hat{v}^-_{\sigma(t)}} \right)^3 \right\}^{1/3}} \right\rangle$$

$$= \left\langle \frac{1}{1 + \left\{ 0.2 \times \left(\frac{1-0.6}{0.6} \right)^3 + 0.1 \times \left(\frac{1-0.5}{0.5} \right)^3 + 0.3 \times \left(\frac{1-0.3}{0.3} \right)^3 + 0.4 \times \left(\frac{1-0.3}{0.3} \right)^3 \right\}^{1/3}}, \right.$$

$$\left. -1 + \frac{1}{1 + \left\{ 0.2 \times \left(\frac{|-0.4|}{1-0.4} \right)^3 + 0.1 \times \left(\frac{|-0.4|}{1-0.4} \right)^3 + 0.3 \times \left(\frac{|-0.5|}{1-0.5} \right)^3 + 0.4 \times \left(\frac{|-0.6|}{1-0.6} \right)^3 \right\}^{1/3}} \right\rangle$$

$$= \left\langle \left(0.3242, -0.5460 \right) \right\rangle.$$

The following properties can be proved easily by BFDOWGA operator.

Theorem 15 (Idempotency Property) If $\tilde{B}_t = (\hat{\mu}^+_t, \hat{v}^-_t)$ $(t = 1, 2, \ldots, n)$ are equal BFEs,
    i.e., $\tilde{B}_t = \tilde{B}$ for all t, then $BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \tilde{B}$.

Theorem 16 (Boundedness Property) Let $\tilde{B}_t = (\hat{\mu}^+_t, \hat{v}^-_t)$ $(t = 1, 2, \ldots, n)$ be a collection of BFEs. Let $\tilde{B}^- = \min_t \tilde{B}_t$, and $\tilde{B}^+ = \max_t \tilde{B}_t$. Then, $\tilde{B}^- \leq BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq \tilde{B}^+$.

Theorem 17 (Monotonicity Property)
    Let $\tilde{B}_t = (\hat{\mu}^+_t, \hat{v}^-_t)$ $(t = 1, 2, \ldots, n)$ and
    $\tilde{B}'_t$ $(t = 1, 2, \ldots, n)$ be two sets of BFEs, if $\tilde{B}_t \leq \tilde{B}'_t$ for all t, then
    $BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \leq BFDOWGA_w(\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_n)$.

Theorem 18 (Commutativity Property) Let $\tilde{B}_t = (\hat{\mu}^+_t, \hat{v}^-_t)$ $(t = 1, 2, \ldots, n)$ and $\tilde{B}'_t$ $(t = 1, 2, \ldots, n)$ be two sets of BFEs, then $BFDOWGA_w(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = BFDOWGA_w(\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_n)$ where $\tilde{B}'_t$ $(t = 1, 2, \ldots, n)$ is any permutation of $\tilde{B}_t$ $(t = 1, 2, \ldots, n)$.

Follows from Definition 11 and Definition 12, we see that BFDWGA operator weights only the bipolar fuzzy values, on the other hand, the BFDOWGA operator weights only the ordered position of the bipolar fuzzy values instead of weights of the bipolar fuzzy values themselves. Therefore, weights represent in both the operators BFDWGA and BFDOWGA are in different aspects. But, they are considered only one of them. To avoid this disadvantage, we introduce bipolar fuzzy Dombi hybrid geometric (BFDHWGA) operator.

Definition 13 A bipolar fuzzy Dombi hybrid geometric (BFDHGA) operator of dimension n is a function $BFDHWGA : \tilde{B}^n \to \tilde{B}$, with associated weight vector $w = (w_1, w_2, \ldots, w_n)$ such that $w_t > 0$, and $\sum_{t=1}^{n} w_t = 1$. Therefore, BFDHWGA operator can be evaluated as

$$BFDHWGA_{w,\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) = \bigotimes_{t=1}^{n}(\dot{\tilde{B}}_{\sigma(t)})^{w_t}$$

$$= \left\langle \frac{1}{1 + \left\{ \sum\limits_{t=1}^{n} w_t \left(\frac{1-\hat{\dot{\mu}}_{\sigma(t)}}{\hat{\dot{\mu}}_{\sigma(t)}} \right)^k \right\}^{1/k}}, -1 + \frac{1}{1 + \left\{ \sum\limits_{t=1}^{n} w_t \left(\frac{|\hat{\dot{v}}_{\sigma(t)}|}{1+\hat{\dot{v}}_{\sigma(t)}} \right)^k \right\}^{1/k}} \right\rangle \tag{24}$$

where $\dot{\tilde{B}}_{\sigma(t)}$ is the tth largest weighted bipolar fuzzy values $\dot{\tilde{B}}_t$ ($\dot{\tilde{B}}_t = n\mho_t B_t, t = 1, 2, \ldots, n)$, and $\mho = (\mho_1, \mho_2, \ldots, \mho_n)^T$ be the weight vector of $\dot{\tilde{B}}_t$ with $\mho_t > 0$ and $\sum_{t=1}^{n} \mho_t = 1$, where n is the balancing coefficient. When $w = (1/n, 1/n, \ldots, 1/n)$, then BFDWGA operator is a special case of BFDHGA operator. Let $\mho = (1/n, 1/n, \ldots, 1/n)$, then BFDOWGA is a special case of the operator BFDHGA. Thus, BFDHGA operator is a generalization of both the operators BFDWGA and BFDOWGA, which reflects the degrees of the given arguments and their ordered positions.

Example 6 Let $B_1 = (0.5, -0.3)$, $B_2 = (0.6, -0.3)$, $B_3 = (0.7, -0.3)$ and $B_4 = (0.2, -0.4)$ be four BFEs and $\mho = (0.20, 0.30, 0.30, 0.20)^T$ is the weight vector of these four BFEs and $W = (0.2, 0.1, 0.3, 0.4)^T$ is the associated weight vector. Then, by Definition 13 for aggregated of BFEs for $(k = 3)$, by the way

$$\dot{B}_1 = \left\langle \left(\frac{1}{1 + \left\{ 4 \times 0.20 \times (\frac{1-0.5}{0.5})^3 \right\}^{1/3}}, -1 + \frac{1}{1 + \left\{ 4 \times 0.20 \times (\frac{|-0.3|}{1-0.3})^3 \right\}^{1/3}} \right) \right\rangle$$

$$= \left\langle \left(0.5186, -0.2846 \right) \right\rangle$$

$$\dot{B}_2 = \left\langle \left(\frac{1}{1 + \left\{ 4 \times 0.30 \times (\frac{1-0.6}{0.6})^3 \right\}^{1/3}}, -1 + \frac{1}{1 + \left\{ 4 \times 0.30 \times (\frac{|-0.3|}{1-0.3})^3 \right\}^{1/3}} \right) \right\rangle$$

$$= \left\langle \left(0.5853, -0.3129 \right) \right\rangle$$

$$\dot{B}_3 = \left\langle \left(\frac{1}{1 + \left\{ 4 \times 0.30 \times (\frac{1-0.7}{0.7})^3 \right\}^{1/3}}, -1 + \frac{1}{1 + \left\{ 4 \times 0.30 \times (\frac{|-0.3|}{1-0.3})^3 \right\}^{1/3}} \right) \right\rangle$$

$$= \left\langle \left(0.6871, -0.3129 \right) \right\rangle$$

$$\dot{B}_4 = \left\langle \left(\frac{1}{1 + \left\{ 4 \times 0.20 \times (\frac{1-0.2}{0.2})^3 \right\}^{1/3}}, -1 + \frac{1}{1 + \left\{ 4 \times 0.20 \times (\frac{|-0.4|}{1-0.4})^3 \right\}^{1/3}} \right) \right\rangle$$

$$= \left\langle \left(0.2122, -0.3823 \right) \right\rangle.$$

Scores of B_t (t = 1,2,3,4) calculated as follows:

$$\hat{E}(\dot{B}_1) = \frac{1+0.5186-0.2846}{2} = 0.617,$$

$$\hat{E}(\dot{B}_2) = \frac{1+0.5853-0.3129}{2} = 0.6362,$$

$$\hat{E}(\dot{B}_3) = \frac{1+0.6871-0.3129}{2} = 0.6871,$$

$$\hat{E}(\dot{B}_4) = \frac{1+0.2122-0.3823}{2} = 0.4150.$$

Since,

$$\hat{E}(\dot{B}_3) > \hat{E}(\dot{B}_2) > \hat{E}(\dot{B}_1) > \hat{E}(\dot{B}_4).$$

Then, $\dot{B}_{\sigma(1)} = \dot{B}_3 = (0.6871, -0.3129)$,

$\dot{B}_{\sigma(2)} = \dot{B}_2 = (0.5853, -0.3129)$,

$\dot{B}_{\sigma(3)} = \dot{B}_1 = (0.5186, -0.2846)$ and

$\dot{B}_{\sigma(4)} = \dot{B}_4 = (0.2122, -0.3823)$. Therefore, aggregated values of BFEs ($k = 3$) by the definition of BFDHWGA operator:

6 Model for MADM using bipolar fuzzy information

In this section, we develop multiple attribute decision making (MADM) method using bipolar fuzzy aggregation operators in which attribute weights are real numbers and attribute values are bipolar fuzzy element. Here multiple-attribute decision-making problem used to develop usefulness of evaluation emerging technology commercialization with bipolar fuzzy Dombi information. Let $\tilde{Q} = \{\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_m\}$ be the discrete set of alternatives and $\tilde{G} = \{\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n\}$ be the set of attributes. Let $\mho = (\mho_1, \mho_2, \ldots, \mho_n)$ be the weight vector of the attribute $\tilde{G}_t$ ($t = 1, 2, \ldots, n$) are completely known such that $\mho_t > 0$ and $\sum_{t=1}^{n} \mho_t = 1$. Suppose $\tilde{M} = (\hat{\mu}_{ht}, \hat{v}_{ht})_{m \times n}$ is a bipolar fuzzy decision matrix, where $\hat{\mu}_{ht}$ is the degree of the positive membership for which alternative $\tilde{Q}_t$ satisfies the

$$BFDHWGA_{w,\mho}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_4) = \bigotimes_{t=1}^{4} (\dot{B}_{\sigma(t)})^{w_t}$$

$$= \left\langle \frac{1}{1 + \left\{ \sum_{t=1}^{4} w_t \left(\frac{1-\dot{\mu}_{\sigma(t)}}{\dot{\mu}_{\sigma(t)}} \right)^3 \right\}^{1/3}}, -1 = \frac{1}{1 + \left\{ \sum_{t=1}^{4} w_t \left(\frac{|\dot{v}_{\sigma(t)}|}{1+\dot{v}_{\sigma(t)}} \right)^3 \right\}^{1/3}} \right\rangle$$

$$= \left\langle \frac{1}{1 + \left\{ 0.2 \times (\frac{1-0.6871}{0.6871})^3 + 0.1 \times (\frac{1-0.5853}{0.5853})^3 + 0.3 \times (\frac{1-0.5186}{0.5186})^3 + 0.4 \times (\frac{1-0.2122}{0.2122})^3 \right\}^{1/3}}, \right.$$

$$\left. -1 + \frac{1}{1 + \left\{ 0.2 \times (\frac{|-0.3129|}{1-0.3129})^3 + 0.1 \times (\frac{|-0.3129|}{1-0.3129})^3 + 0.3 \times (\frac{|-0.2846|}{1-0.2846})^3 + 0.4 \times (\frac{|-0.3823|}{1-0.3823})^3 \right\}^{1/3}} \right\rangle$$

$$= \left\langle (0.3582, -0.3429) \right\rangle.$$

attribute $\tilde{G}_t$ given by the decision makers, and $\hat{v}_{ht}$ provided the degree that the alternative $\tilde{Q}_h$ does not satisfy the attribute $\tilde{G}_t$ given by the decision maker, where $\hat{\mu}_{ht} \subset [0, 1]$ and $\hat{v}_{ht} \subset [-1, 0]$ such that $-1 \leq \hat{\mu}_{ht} + \hat{v}_{ht} \leq 1$, $(h = 1, 2, \ldots, m)$ and $(t = 1, 2, \ldots, n)$.

We propose the following algorithm to solve MADM problem with bipolar fuzzy information using BFDWA and BFDWGA operators.

Algorithm

Input: To select best alternatives.

Output: Best alternative.

Step 1. We employ the decision information given in matrix M, and the operator BFDWAA

$$\tilde{\beta}_h = BFDWAA(\tilde{\beta}_{h1}, \tilde{\beta}_{h2}, \ldots, \tilde{\beta}_{hn}) = \bigoplus_{t=1}^{n}(\mho_t\tilde{\beta}_{ht})$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{\hat{\mu}_t^+}{1-\hat{\mu}_t^+} \right)^k \right\}^{1/k}}, \frac{-1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1+\hat{v}_t^+}{|\hat{v}_t^-|} \right)^k \right\}^{1/k}} \right\rangle \tag{25}$$

or $\tilde{\beta}_h = BFDWGA(\tilde{\beta}_{h1}, \tilde{\beta}_{h2}, \ldots, \tilde{\beta}_{hn}) = \bigoplus_{t=1}^{n}(\beta_{ht})^{\mho_t}$

$$= \left\langle \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{1-\hat{\mu}_t^+}{\hat{\mu}_t^+} \right)^k \right\}^{1/k}}, -1 + \frac{1}{1 + \left\{ \sum_{t=1}^{n} \mho_t \left(\frac{|\hat{v}_t^-|}{1+\hat{v}_t^-} \right)^k \right\}^{1/k}} \right\rangle \tag{26}$$

to obtained the overall preference values $\tilde{\beta}_h$ $(h = 1, 2, \ldots, m)$ of the alternative $\tilde{Q}_h$.

Step 2. Calculate the score function $\hat{E}(\tilde{\beta}_h)$ $(h = 1, 2, \ldots, m)$ based on overall bipolar fuzzy information $\tilde{\beta}_h$ $(h = 1, 2, \ldots, m)$ in ordered to rank all the alternative $\tilde{Q}_h$ $(h = 1, 2, \ldots, m)$ to select best choice $\tilde{Q}_h$. If there is no difference between the score functions $\hat{E}(\tilde{\beta}_h)$ and $\hat{E}(\tilde{\beta}_t)$, then we proceed to calculate accuracy degrees of $\hat{L}(\tilde{\beta}_h)$ and $\hat{L}(\tilde{\beta}_t)$ based on overall bipolar fuzzy information of $\tilde{\beta}_h$ and $\tilde{\beta}_t$, and rank the alternative $\tilde{Q}_h$ depending on the accuracy degrees of $\hat{L}(\tilde{\beta}_h)$ and $\hat{L}(\tilde{\beta}_t)$.

Step 3. Rank all the alternatives $\tilde{Q}_h$ $(h = 1, 2, \ldots, m)$ in order to choose the best one(s) in accordance with $\hat{E}(\tilde{\beta}_h)$ $(h = 1, 2, \ldots, m)$.

Step 4. End.

7 Numerical Example

With the rapid development and wide application of information technology, the selection of emerging technology enterprise becomes more and more important. The aim of this paper is to evaluate the best emerging technology enterprise from the different companies of the emerging technology enterprise performances, that provide alternatives

of enterprise. Therefore, to this section, we shall present a numerical result to establish the potential assessment of technology commercialization depicted in Wei (2016) with bipolar fuzzy information in order to investigate our proposed method in this paper. There is a committee which selects five possible emerging technology enterprises $\tilde{Q}_h$ $(h = 1, 2, \ldots, 5)$. They choose four attributes to assess five possible emerging technology enterprises as follows:

G_1 : Technical Advancement
G_2 : Potential market and market risk
G_3 : Industrialization infrastructure, human resources and financial conditions
G_4 : The employment creation and the development of science and technology.

In order to avoid dominance against other, decision makers are required to exempt the four possible emerging technology enterprises $\tilde{Q}_h (h = 1, 2, \ldots, n)$ under the above attributes whose weight vector $\mho = (0.2, 0.1, 0.3, 0.4)^T$ presented by decision makers, where decision matrix $\tilde{M} = (\beta_{ht})_{5\times4}$ which is given in the following table, where β_{ht} are in the form of BFEs (Table 1).

In order to select the most desirable emerging technology enterprises $\tilde{Q}_h (h = 1, 2, \ldots, n)$, we utilize the BFDWA and BFDWGA operator to develop multi-attribute decision-making a theory with bipolar fuzzy information, which can be evaluated as follows:

- **Step 1.** For $k = 1$, using the BFDWA operator to calculate the overall preferences values $\tilde{\beta}_h$ of emerging technology enterprises $\tilde{A}_h (h = 1, 2, \ldots, 5)$
  $\tilde{\beta}_1 = (0.5862, -0.1980)$, $\quad \tilde{\beta}_2 = (0.5792, -0.2564,)$
  $\tilde{\beta}_3 = (0.3860, -0.2182,)$
  $\tilde{\beta}_4 = (0.5922, -0.1796)$, $\tilde{\beta}_5 = (0.4940, -0.2927)$
- **Step 2.** Calculate the values of the score functions $scor(\tilde{\beta}_h) (h = 1, 2, \ldots, 5)$ of the overall bipolar fuzzy elements $\tilde{\beta}_h (i = 1, 2, \ldots, 5)$ as follows:
  $\hat{E}(\tilde{\beta}_1) = 0.6941$, $\quad \hat{E}(\tilde{\beta}_2) = 0.6614$, $\quad \hat{E}(\tilde{\beta}_3) = 0.5839$,
  $\hat{E}(\tilde{\beta}_4) = 0.7063$, $\hat{E}(\tilde{\beta}_5) = 0.6007$
- **Step 3.** Rank all the emerging technology enterprise systems $\tilde{Q}_h (h = 1, 2, \ldots, 5)$ in accordance with the value of the score functions $\hat{E}(\tilde{\beta}_h) (h = 1, 2, \ldots, 5)$ of the overall bipolar fuzzy elements as $Q_4 > Q_1 > Q_2 > Q_5 > Q_3$.

Table 1 Bipolar fuzzy elements

	Q_1	Q_2	Q_3	Q_4	Q_5
G_1	$(0.6, -0.2)$	$(0.5, -0.5)$	$(0.6, -0.3)$	$(0.3, -0.5)$	$(0.4, -0.6)$
G_2	$(0.6, -0.4)$	$(0.3, -0.2)$	$(0.5, -0.4)$	$(0.4, -0.6)$	$(0.3, -0.4)$
G_3	$(0.7, -0.1)$	$(0.4, -0.3)$	$(0.3, -0.1)$	$(0.7, -0.3)$	$(0.4, -0.2)$
G_4	$(0.4, -0.5)$	$(0.7, -0.2)$	$(0.2, -0.6)$	$(0.6, -0.1)$	$(0.6, -0.3)$

- **Step 4.** Q_4 is selected as the most desirable emerging technology enterprise.

If **BFDWGA** operator is implemented instead, then the problem can be solved similarly as above.

- **Step 1.** For $k = 1$, using the BFDWGA operator to calculate the overall preferences values $\tilde{\beta}_h$ of emerging technology enterprises $\tilde{Q}_h$ $(h = 1, 2, \ldots, 5)$
  $\widetilde{\beta}_1 = (0.5035, -0.3548)$, $\widetilde{\beta}_2 = (0.4867, -0.3121)$, $\widetilde{\beta}_3 = (0.2830, 0.4400)$,
  $\widetilde{\beta}_4 = (0.4970, -0.3434)$, $\widetilde{\beta}_5 = (0.4444, -0.3801)$
- **Step 2.** Calculate the values of the score functions $scor(\tilde{\beta}_h)$ $(h = 1, 2, \ldots, 5)$ of the overall bipolar fuzzy elements $\tilde{\beta}_h$ $(h = 1, 2, \ldots, 5)$ as follows:
  $\hat{E}(\widetilde{\beta}_1) = 0.5744$, $\hat{E}(\widetilde{\beta}_2) = 0.5873$, $\hat{E}(\widetilde{\beta}_3) = 0.4215$, $scor(\widetilde{\beta}_4) = 0.5768$, $\hat{E}(\widetilde{\beta}_5) = 0.5322$
- **Step 3.** Rank all the emerging technology enterprise systems $\tilde{Q}_h$ $(h = 1, 2, \ldots, 5)$ in accordance with the value of the score functions $\hat{E}(\widetilde{\beta}_h)$ $(h = 1, 2, \ldots, 5)$ of the overall bipolar fuzzy elements as $Q_2 > Q_4 > Q_1 > Q_5 > Q_3$.
- **Step 4.** Return Q_2 is selected as the most desirable emerging technology enterprise system.

From the inspection, it is evident up to expectation though overall ranking values on the alternatives are distinctive through the use of two operators, but the ranking order concerning the alternatives are similar, the most desirable alternative is Q_4 for BFDWAA and Q_2 for BFDWGA operator.

In order to diagnose the effect of parameter $k \in [1, 10]$ on the ranking of the alternatives in the BFDWA and BFDWGA operators, which are shown in Tables 2 and 3.

8 Analysis on the effect of parameter k on decision making results

To describe the effect of the operational parameters k on multi-attribute decision-making results, we shall use different values of k to rank the alternatives. The results of the score function and ranking order of the alternatives Q_t $(t = 1, 2 \ldots, 5)$ in the range of $1 \leq k \leq 10$ based on BFDWA and BFDWGA operators are presented in Tables 2 and 3 correspondingly.

From Table 2, it is evident that when the value of k is changed for BFDWA operator, the rankings are separate, and the corresponding best alternatives are additionally non-identical. When, $1 \leq k \leq 2$, then we get

Table 2 Ranking order for various working parameters of the BFDWA operator

k	$\hat{E}(\beta_1)$	$\hat{E}(\beta_2)$	$\hat{E}(\beta_3)$	$\hat{E}(\beta_4)$	$\hat{E}(\beta_5)$	Ranking order
1	0.6941	0.6614	0.5839	0.7063	0.6007	$Q_4 > Q_1 > Q_2 > Q_5 > Q_3$
2	0.7267	0.6861	0.6390	0.7360	0.6231	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
3	0.7451	0.7021	0.6701	0.7518	0.6392	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
4	0.7565	0.7125	0.6885	0.7614	0.6508	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
5	0.7642	0.7195	0.7002	0.7679	0.6592	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
6	0.7697	0.7243	0.7083	0.7725	0.6654	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
7	0.7738	0.7280	0.7141	0.7761	0.6701	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
8	0.7770	0.7307	0.7186	0.7789	0.6738	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
9	0.7795	0.7329	0.7221	0.7812	0.6767	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$
10	0.7815	0.7346	0.7248	0.7830	0.6790	$Q_4 > Q_1 > Q_2 > Q_3 > Q_5$

Table 3 Ranking order for various working parameters of the BFDWGA operator

k	$\hat{E}(\beta_1)$	$\hat{E}(\beta_2)$	$\hat{E}(\beta_3)$	$\hat{E}(\beta_4)$	$\hat{E}(\beta_5)$	Ranking order
1	0.5744	0.5873	0.4215	0.5768	0.5322	$Q_2 > Q_4 > Q_1 > Q_5 > Q_3$
2	0.5426	0.5508	0.3803	0.5112	0.4951	$Q_2 > Q_1 > Q_4 > Q_5 > Q_3$
3	0.5189	0.5216	0.3583	0.4659	0.4662	$Q_2 > Q_1 > Q_5 > Q_4 > Q_3$
4	0.5039	0.5001	0.3453	0.4367	0.4457	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$
5	0.4943	0.4842	0.3368	0.4173	0.4310	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$
6	0.4871	0.4723	0.3309	0.4039	0.4199	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$
7	0.4820	0.4630	0.3266	0.3942	0.4113	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$
8	0.4780	0.4557	0.3233	0.3868	0.4044	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$
9	0.4749	0.4498	0.3207	0.3811	0.3988	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$
10	0.4725	0.4449	0.3186	0.3940	0.3945	$Q_1 > Q_2 > Q_5 > Q_4 > Q_3$

$Q_4 \succ Q_1 \succ Q_2 \succ Q_5 \succ Q_3$ and $Q_4 \succ Q_1 \succ Q_2 \succ Q_3 \succ Q_5$, thus best choice is Q_4. When $3 \le k \le 10$, then corresponding ranking is $Q_4 \succ Q_1 \succ Q_2 \succ Q_3 \succ Q_5$, then the best one is also Q_4. From Table 3, it is seen that when the value of k is changed for BFDWGA operator, the rankings are separate, and the corresponding best alternatives are additionally non-identical. When, $1 \le k \le 3$, then ranking orders are $Q_2 \succ Q_4 \succ Q_1 \succ Q_5 \succ Q_3$, $Q_2 \succ Q_1 \succ Q_4 \succ Q_5 \succ Q_3$, and $Q_2 \succ Q_1 \succ Q_5 \succ Q_4 \succ Q_3$ and best choice are Q_2. When $4 \le k \le 10$, then corresponding ranking is $Q_1 \succ Q_2 \succ Q_5 \succ Q_4 \succ Q_3$, the best one is Q_1.

To these MADM problems based on BFDWA and BFD-WGA operators, we see that the different values of working parameters, k can be changed corresponding ranking orders of the alternatives for BFDWGA operator, which is more responsive to k in this MADM process; while for various values of working parameters k could be changed raking forms corresponding to BFDWA operator, which is less responsive to k in this MADM process.

Analyze our introduced method with existing related methods (Chen and Ye 2017; He 2018; Liu et al. 2018), the MADM problem in this paper dealt with bipolar fuzzy environment, while existing methods (Chen and Ye 2017; He 2018; Liu et al. 2018) dealt with the single-valued neutrosophic environment, hesitant fuzzy environment or intuitionistic fuzzy environment but not in bipolar fuzzy sets.

Therefore, our proposed MADM method for BFDWA and BFDWGA operators investigate the improvement of its flexibility in real utilization. Thus, the advanced aggregation operators implement a new flexible measure for decision makers to control bipolar fuzzy MADM problems.

9 Conclusions

In this article, we have studied multi-attribute decision-making problem using bipolar fuzzy information. We have introduced arithmetic and geometric operations to develop some bipolar fuzzy Dombi aggregation operators from the motivation of Dombi operations as bipolar fuzzy Dombi weighted average (BFDWA) operator, bipolar fuzzy Dombi order weighted average (BFDOWA) operator, bipolar fuzzy Dombi hybrid weighted average (BFDHWA) operator, bipolar fuzzy Dombi weighted geometric (BFDWGA) operator, bipolar fuzzy Dombi order weighted geometric (BFDOWGA) operator and bipolar fuzzy Dombi hybrid weighted geometric (BFDHWGA) operator. The different feature of those recommend operators is deliberated. Then, we have used those operators to expand a few strategies to remedy multi-attribute decision-making issues. Ultimately, a realistic instance for emerging technology enterprise system selection is provided to develop a strategy and in accordance with the relevance and the effectiveness of the proposed methodology. In future, the application of our proposed model can be applied in decision-making theory, risk evaluation and other domains under ambiguous environments.

Complaince with ethical standards

Conflict of interest There is no conflict of interest between the authors and the institute where the work has been carried out.

Ethical approval The article does not contain any studies with human participants or animals performed by any of the authors.

References

Abbasian NS, Salajegheh Gaspar AH, Brett PO (2018) Improving early OSV design robustness by applying 'Multivariate Big Data Analytics' on a ship's life cycle. J Ind Inform Integr 10:29–38

Atanassov KT (2012) On intuitionistic fuzzy sets theory. Studies in Fuzziness and Soft Computing 283: Springer-Verlag, Berlin Heidelberg

Beliakov G, Pradera A, Calvo T (2007) Aggregation Functions: a guide for practitioners. Springer, Heidelberg

Beliakov G, Bustince H, Goswami DP, Mukherjee UK, Pal NR (2011) On averaging operators for Atanassovs intuitionistic fuzzy sets. Inform Sci 181:1116–1124

Chen J, Ye J (2017) Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision-making. Symmetry 9(82):1–11

Chen SM, Tan JM (1994) Handling multicriteria fuzzy decision-making problems based on vague set theory. Fuzzy Sets Syst 67:163–172

Chen SM, Chiou CH (2015) Multiattribute decision making based on interval-valued intuitionistic fuzzy sets, PSO techniques, and evidential reasoning methodology. IEEE Trans Fuzzy Syst 23(6):1905–1916

Chen CT, Huang SF, Hung WZ (2018) Linguistic VIKOR method for project evaluation of ambient intelligence product. J Ambient Intell Hum Comput 1–11. https://doi.org/10.1007/s12652-018-0889-x

Deschrijver G, Cornelis C, Kerre EE (2004) On the representation of intuitionistic fuzzy *t*-norms and *t*-conorms. IEEE Trans Fuzzy Syst 12:45–61

Deschrijver G, Kerre EE (2002) A generalization of operators on intuitionistic fuzzy sets using triangular norms and conorms. Notes Intuit Fuzzy Sets 8:19–27

Dombi J (1982) A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. Fuzzy Sets Syst 8:149–163

Fink G, Yolles M (2015) Collective emotion regulation in an organization—a plural agency with cognition and affect. J Organ Change Manag 28(5):832–871

Gao H, Wei GW, Huang YH (2018) Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. IEEE Access 6(1):11508–11522

Garg H (2016) A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems. Appl Soft Comput 38:988–999

Garg H, Nancy (2018) Linguistic single.valued neutrosophic prioritized aggregation operators and their applications to multiple-attribute group decision making. J Ambient Intell Hum Comput. https://doi.org/10.1007/s12652-018-0723-5

Gul Z (2015) Some bipolar fuzzy aggregations operators and their applications in multicriteria group decision making, M.Phil Thesis

Han Y, Shi P, Chen S (2015) Bipolar-valued rough fuzzy set and its applications to decision information system. IEEE Trans Fuzzy Syst 23(6):2358–2370

He YD, He Z, Wang GD, Chen HY (2015) Hesitant fuzzy power bonferroni means and their application to multiple attribute decision making. IEEE Trans Fuzzy Syst 23(5):1655–1668

He YD, Chen HY, He Z, Zhou LG (2015a) Multi-attribute decision making based on neutral averaging operators for intuitionistic fuzzy information. Appl Soft Comput 27:64–76

He X (2018) Typhoon disaster assessment based on Dombi hesitant fuzzy information aggregation operators. Nat Hazards 90(3):1153–1175

Kumar K, Garg H (2018) TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment. Comput Appl Math 37(2):1319–1329

Levy R, Brodsky A, Luo J (2016) Decision guidance framework to support operations and analysis of a hybrid renewable energy system. J Manag Anal 3(4):285–304

Li DF (2011) Closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information. Appl Soft Comput 11:3402–3418

Li PP (2016) The global implications of the indigenous epistemological system from the east: how to apply Yin-Yang balancing to paradox management. Cross Cult Strateg Manag 23(1):42–47

Li DF, Ren HP (2015) Multi-attribute decision making method considering the amount and reliability of intuitionistic fuzzy information. J Intell Fuzzy Systems 28(4):1877–1883

Liu PD, Liu JL, Chen SM (2018) Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making. J Oper Res Soc 69(1):1–24

Liu PD (2014) Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. IEEE Trans Fuzzy Syst 22(1):83–97

Liu J, Chen HY, Xu Q, hou LG, Tao Z (2016) Generalized ordered modular averaging operator and its application to group decision making. Fuzzy Sets Syst 299:1–25

Lu M, Busemeyer JR (2014) Do traditional chinese theories of Yi Jing (Yin-Yang and Chinese Medicine) go beyond western concepts of mind and matter. Mind Matter 12(1):37–59

Lu M, Wei GW, Alsaadi FE, Tasawar H, Alsaedi A (2017) Bipolar 2-tuple linguistic aggregation operators in multiple attribute decision making. J Intell Fuzzy Systems 33(2):1197–1207

Lourenzuttia R, Krohling RA (2013) A study of TODIM in a intuitionistic fuzzy and random environment. Expert Syst Appl 40(16):6459–6468

Pourhassan MR, Raissi S (2017) An integrated simulation-based optimization technique for multi-objective dynamic facility layout problem. J Ind Inform Integr 8:49–58

Ronaynea D, Brown Gordon DA (2017) Multi-attribute decision by sampling: an account of the attraction, compromise and similarity effects. J Math Psychol 81:11–27

Samanta S, Pal M (2012) Bipolar fuzzy hypergraphs. Int J Fuzzy Logic Syst 2(1):17–28

Samanta S, Pal M (2012a) Irregular bipolar fuzzy graphs. Int J Appl Fuzzy Sets 2:91–102

Samanta S, Pal M (2014) Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs. J Fuzzy Math 22(2):253–262

Singh PK, Kumar CA (2014) Bipolar fuzzy graph representation of concept lattice. Infom Sci 288:437–488

Teixeira C, Lopes I, Figueiredo M (2018) Classification methodology for spare parts management combining maintenance and logistics perspectives. J Manag Anal 5(2):116–135

Viriyasitavat W (2016) Multi-criteria selection for services selection in service workflow. J Ind Infor Integr 1:20–25

Wan SP, Li DF (2014) Atanassovs intuitionistic fuzzy programming method for heterogeneous multiattribute group decision making with atanassovs intuitionistic fuzzy truth degrees. IEEE Trans Fuzzy Syst 22(2):300–312

Wan SP, Li DF (2015) Fuzzy mathematical programming approach to heterogeneous multiattribute decision-making with interval-valued intuitionistic fuzzy truth degrees. Inform Sci 325:484–503

Wang L, Zhang HY, Wang JQ (2018) Frank Choquet Bonferroni mean operators of bipolar neutrosophic sets and their application to multi-criteria decision-making problems. Int J Fuzzy Syst 20(1):13–28

Wei GW, Gao H, Wang J, Huang YH (2018a) Research on risk evaluation of enterprise human capital investment with interval-valued bipolar 2-tuple linguistic information. IEEE Access 6:35697–35712

Wei GW (2008) Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting. Knowl Based Syst 21:833–836

Wei GW (2009) Some geometric aggregation functions and their application to dynamic multiple attribute decision making in intuitionistic fuzzy setting. Int J Uncertain Fuzziness Knowl Based Syst 17:179–196

Wei GW (2010) Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Appl Soft Comput 10:423–431

Wei GW (2016) Picture fuzzy cross-entropy for multiple attribute decision making problems. J Bus Econ Manag 17(4):491–502

Wei GW, Alsaadi FE, Tasawar H, Alsaedi A (2018b) Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision Making. Int J Fuzzy Syst 20(1):1–12

Wei GW, Alsaadi FE, Tasawar H, Alsaedi A (2017) Hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. J Intell Fuzzy Systems 33(2):1119–1128

Wei GW, Wei Y (2018) Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making. J Intell Fuzzy Syst. https://doi.org/10.3233/JIFS-171741

Wei GW, Zhang Z (2018) Some single-valued neutrosophic Bonferroni power aggregation operators in multiple attribute decision making. J Ambient Intell Hum Comput:1–20. https://doi.org/10.1007/s12652-018-0738-y

Xia M, Xu Z, Zhu B (2012) Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. Knowl Based Syst 31:78–88

Xu L (1990) Linguistic approach to the multi-criteria ranking problem. Int J Syst Sci 21:1773–1782

Xu XR, Wei GW (2017) Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. Int J Knowl Based Intell Eng Syst 21(3):155–164

Xu Z, Xia M (2011) Induced generalized intuitionistic fuzzy operators. Knowl Based Syst 24(2):197–209

Xu ZS, Yager RR (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst 35:417–433

Xu ZS (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15:1179–1187

Xu ZS, Da QL (2003) An overview of operators for aggregating information. Int J Intell Syst 18(9):953–969

Yager RR (1998) On ordered weighted avergaing aggregation operators in multi-criteria decision making. IEEE Trans Syst Man Cybern 18(1):183–190

Yager RR, Kacprzyk J (1997) The ordered weighted averaging operators: theory and applications. Kluwer, Boston

Yang LH, Li SG, Yang WH, Lu Y (2013) Notes on Bipolar fuzzy graphs. Inform Sci 242:113–121

Ye J (2009) Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. Expert Syst Appl 36(2):899–6902

Yu D (2013) Intuitionistic fuzzy Choquet aggregation operator based on Einstein operation laws. Sci Iran 20(6):2109–2122

Zadeh LA (1965) Fuzzy sets. Inform Control 8:338–353

Zhang WR (1994) Bipolar fuzzy sets and relations, a computational frame work for cognitive modelling and multiagent decision analysis. In: Proceedings of IEEE conference, pp 305–309

Zhang WR (1998) Bipolar fuzzy sets In: IEEE Proceedings of FUZZY, pp 835–840

Zhang WR, Zhang L (2004) Bipolar logic and bipolar fuzzy logic. Inform Sci 165(34):265–287

Zhang WR, Pandurangi KA, Peace KE, Zhang Y, Zhao Z (2011) Mental squares a generic bipolar support vector machine for psychiatric disorder classification diagnostic analysis and neurobiological data mining. Int J Data Min Bioinf 5(5):532–572

Zhang WR, Zhang HJ, Shi Y, Chen SS (2009) Bipolar linear algebra and Yin Yang N element cellular networks for equilibrium based biosystem simulation and regulation. J Biol Syst 17(4):547–576

Zhang WR (2013) Bipolar quantum logic gates and quantum cellular combinatorics a logical extension to quantum entanglement. J Quant Inf Sci 3(2):93–105

Zhang WR, Peace KE (2014) Causality is logically definable-toward an equilibrium based computing paradigm of quantum agent and quantum intelligence. J Quant Inf Sci 4:227–268

Zhang WR (2016) G-CPT symmetry of quantum emergence and submergence an information conservational multi agent cellular automata unification of CPT symmetry and CP violation for equilibrium-based many world causal analysis of quantum coherence and decoherence. J Quant Inf Sci 6(2):62–97

Zhang XY, Wang JQ, Hu JH (2018) On novel operational laws and aggregation operators of picture 2-tuple linguistic information for MCDM problems. Int J Fuzzy Syst 20(3):958–969

Zhang Z (2017) Interval-valued intuitionistic fuzzy Frank aggregation operators and their applications to multiple attribute group decision making. Neural Comp Appl 28(6):1471–1501

Zhang C, Wang C, Zhang Z, Tian D (2018a) A novel technique for multiple attribute group decision making in interval-valued hesitant fuzzy environments with incomplete weight information. J Ambient Intell Hum Comput :1-17. https://doi.org/10.1007/s12652-018-0912-2

Zhao XF, Wei GW (2013) Some intuitionistic fuzzy einstein hybrid aggregation operators and their application to multiple attribute decision making. Knowl Based Syst 37:472–479