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ORIGINAL RESEARCH

An entropy based solid transportation problem in uncertain environment

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Abstract The uncertain solid transportation problem considers material dispatching with uncertain elements like demands. Now, it plays an increasingly important role in logistics managements. Traditionally, the transportation cost is used as the optimization objective, while the dispersals of trips between origins and destinations are usually neglected. In order to minimize the transportation penalties and ensure uniform distribution of goods between origins and destinations, this paper employs entropy function of dispersals of trips between origins and destinations as a second objective function. Within the framework of uncertainty theory, the uncertain entropy based solid transportation model is transformed into its deterministic equivalent, which can be solved by general optimization methods. Finally, a numerical example is given for illustrating purpose.

Keywords Solid transportation problem · Entropy function · Uncertain variable · Expected value

1 Introduction

The classic transportation problem (TP) is designed for the transportation problem which can be described as transporting goods from several of its sources to several of its destinations. TP is essentially an optimizing problem which aim to minimize the transportation fee under the source and destination constraints. In many industrial problems, a homogeneous product is delivered from an origin to destination by different means of conveyance, such as trucks, cargo flights, etc. When the transportation problem considers different conveyances, then it is called solid transportation problem (STP). It was first introduced by Haley (1962). With the development of the logistics managements, STP holds in high regard today. Many models about STP have been established by lots of scholars in both fuzzy environment and uncertain environment.

In practice, it is often difficult to estimate the actual penalties (e.g., transportation cost, delivery time), demands, availabilities, the capacities of different conveyances. Depending upon different aspects, they fluctuate due to uncertainty in judgement, lack of evidence, insufficient information, etc. Therefore, we can not get the probability distributions of the variables. In this time, we can invite some experts to give the belief degrees that each event will occur. A natural idea is to treat belief degrees as probability distributions. However, Liu (2012) showed that it is inappropriate to model belief degrees through probability theory, it will lead to counterintuitive results. In order to model belief degrees, uncertainty theory was founded by Liu (2007) and refined by Liu (2009a). Uncertainty theory

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is a branch of mathematics based on normality, duality, subadditivity, and product axioms. Nowadays, uncertainty theory has been widely applied in many fields such as economics (Yang and Gao 2013, 2016, 2017), finance (Chen and Gao 2013; Liu 2013; Liu and Yao 2017; Xiao et al 2016; Zhang et al 2017), management (Chen et al 2017; Gao et al 2017; Yao and Liu 2017) and Engineering (Gao and Yao 2015; Yang and Ni 2017; Yao et al 2013; Yao and Gao 2015).

In literature, uncertain STP models have been discussed by some researchers within the framework of uncertainty theory. Cui and Sheng (2013) assumed that all the parameters of constraints are prescribed by uncertain variables, and introduced uncertain programming models for STP. By assuming that there was corresponding cost when the transportation activity took place, Zhang et al (2016) considered an extension of the traditional TP that is called uncertain fixed charge transportation problem. Das and Bera (2015) established a bi-objective uncertain solid transportation problem under the assumption that there were more than one penalty.

Entropy function acts as a measure of dispersals of trips between origins and destinations. It will be more practical to minimize the transportation penalties as well as to maximize entropy amount. This ensures uniform distribution of goods between origins and destinations. In this paper, we employ entropy function of dispersal of trips between origins and destinations as a second objective function, and introduce an entropy based uncertain solid transportation model in Sect. 2. Within the framework of uncertainty theory, this model is transformed into its deterministic equivalence that can be solved by general optimization methods. In Sect. 3, we give an example of coal transportation with three coal mines as sources, four cities as destinations, and two kinds of conveyance (trains and cargo ships). Then we use the theory developed in the former section to establish its model and derive its determinant. By using Matlab 2014a optimization tool, we solve the crisp model and get the transportation plans of train and cargo ship. In addition, we provide some useful concepts in uncertainty theory in the Appendix section.

2 An entropy based solid transportation model

2.1 Problem description

As we know, the solid transportation problem discusses how to minimize the cost of transporting homogeneous products from their sources to their destinations through several kinds of conveyances.

We assume that there are m sources, and the amount of the goods of the ith source which can be transported to the destinations is a_i, while there are n destinations, and the least requirements of the jth destination is b_j. Moreover, l kinds of conveyances are available for us, the bearing capacity of which is c_k. The quantity transports from source i to destination j by transportation k is x_{ijk}, while the cost of unit product transported from source i to destination j by conveyance k is ξ_{ijk}. For convenience, we denote $\xi = (\xi_{ijk})$ and $x = (x_{ijk})$ as the cost matrix and decision matrix, respectively.

Under the analysis above, we may easily get the total transportation cost function:

$$c(x,\xi) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\xi_{ijk},$$

and we can get the STP model as follows:

$$\begin{cases} \min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\xi_{ijk} \\ s.t \\ \quad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le a_i, \quad i = 1,2,\dots,m \\ \quad \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge b_j, \quad j = 1,2,\dots,n \\ \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le c_k, \quad k = 1,2,\dots,l \\ \quad x_{ijk} \ge 0, \quad i = 1,2,\dots,m, \\ \quad j = 1,2,\dots,n, \quad k = 1,2,\dots,l. \end{cases} \tag{1}$$

In this model, the first constraint means that the amount of the product from source $i (i = 1,2,\dots,m)$ being transported to the destinations should not excess a_i . The second constraint means that the amount of the product being sent to the destination $j (j = 1,2,\dots,n)$ should not less than b_j. The third constraint means that the amount of the product being transported by conveyance $k (k = 1,2,\dots,l)$ should not beyond its bearing capacity c_k.

Except for the transportation cost, there may be other penalties for this transportation problem, such as delivery time, quantity of goods delivered etc. Let ξ^p_{ijk} denote the p-th penalty of unit product transported from source i to destination j by means of k-th conveyance, then we can get the following multi-objective solid transportation problem model.

$$\begin{cases} \min \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} \xi_{ijk}^{p}, \quad p = 1, 2, \ldots, P \\ s.t \\ \quad \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} \le a_i, \quad i = 1, 2, \ldots, m \\ \quad \sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} \ge b_j, \quad j = 1, 2, \ldots, n \\ \quad \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \le c_k, \quad k = 1, 2, \ldots, l \\ \quad x_{ijk} \ge 0, \quad i = 1, 2, \ldots, m, \\ \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, l. \end{cases} \tag{2}$$

2.2 Entropy function

In general STP, some goods are transported from origins to some specific destinations through some conveyances so that total transportation cost is minimum. Here we introduce an entropy function as an additional objective function. In this case, the transported products are meaningfully distributed to all destinations, but the cost is higher than that for the case without entropy function. In a real-life transportation model we should try to cover all cells, if possible. So we introduce the entropy function of the STP. Let T be the transported amount i.e., $T = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk}$. We consider a function $F(X)$ which represents the number of possible assignments for the state $X = (x_{ijk})$.

The (Shannon) entropy function of a variable X is defined as follows:

$$F(X) = \frac{T!}{\prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{k=1}^{l} x_{ijk}!},$$

then

$$\begin{aligned} \ln(F(X)) &= \ln(T!) - \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \ln(x_{ijk}!) \\ &= \ln(e^{-T} T^{T}) - \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \ln(e^{-x_{ijk}} x_{ijk}^{x_{ijk}}) \\ &\qquad \text{[By using Stirlings approximation formula.]} \\ &= T \cdot \ln(T) - \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} \ln x_{ijk}, \end{aligned}$$

and $\dfrac{\ln(F(X))}{T} = \ln(T) - \dfrac{1}{T} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} \ln x_{ijk}$, and this is known as entropy function.

This entropy function (Shannon) can be expressed as

$$En(X) = -\sum_{X} f(r)$$

where

$$f(r) = \begin{cases} p(r) \ln(p(r)), & \text{if} \quad p(r) \ne 0 \\ 0, & \text{if} \quad p(r) = 0. \end{cases}$$

In a transportation problem, normalizing the trip number x_{ijk} by dividing the total number of trips T, a probability distribution, $p(r) = x_{ijk}/T$ is formulated.

Therefore,

$$\begin{aligned} En(X) &= -\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} (x_{ijk}/T) \ln(x_{ijk}/T) \\ &= \ln(T) - \frac{1}{T} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} \ln x_{ijk}. \end{aligned}$$

In a transportation problem, this entropy function acts as a measure of dispersal of trips among origins, destinations and conveyances. So the entropy function is another useful objective for the decision maker.

2.3 Uncertain solid transportation model with entropy function

The traditional model is studied under certain conditions, that is, the parameters in the model are all fixed quantities. But due to the complexity of the real world, we always meet uncertain phenomena when constructing mathematical models. For such conditions, we can not determine the demands by certain numbers. So we must add the uncertain variables to the model. Hence, in this paper, we assume that the unit cost, the capacity of conveyance, the capacity of each source and that of each destination are all uncertain variables and denoted by ξ_{ijk}, a_i , b_j, c_k, respectively. Under expected-value criterion, the model is described below,

$$
\begin{cases}
\min & \mathrm{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\xi_{ijk}^{p}\right], \quad p=1,2,\dots,P \\
\min & -\ln(T)+\frac{1}{T}\frac{1}{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\ln x_{ijk} \\
s.t & \\
& \mathcal{M}\left\{\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}-a_i\right\} \geq \alpha_i, \quad i=1,2,\dots,m \\
& \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk}-b_j\right\} \geq \beta_j, \quad j=1,2,\dots,n \\
& \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}-c_k\right\} \geq \gamma_k, \quad k=1,2,\dots,l \\
& x_{ijk}\geq 0, \quad i=1,2,\dots,m, \\
& j=1,2,\dots,n, \quad k=1,2,\dots,l.
\end{cases}
\tag{3}
$$

Theorem 6 *Let $\xi_{ijk}^{p}, a_i, b_j, c_k$ be independent uncertain variables with uncertain distribution $\Phi_{\xi_{ijk}^{p}}, \Phi_{a_i}, \Phi_{b_j}, \Phi_{c_k}$. Then Model (3) can be converted to*

$$
\begin{cases}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\mathrm{d}\alpha, \; p=1,\dots,P \\
\min & -\ln(T)+\frac{1}{T}\frac{1}{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\ln x_{ijk} \\
s.t. & \\
& \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}-\Phi_{a_i}^{-1}(1-\alpha_i)\leq 0, \; i=1,2,\dots,m \\
& \Phi_{b_j}^{-1}(\beta_j)-\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk}\leq 0, \; j=1,2,\dots,n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}-\Phi_{c_k}^{-1}(1-\gamma_k)\leq 0, \; k=1,2,\dots,l \\
& x_{ijk}\geq 0, \; i=1,2,\dots,m, \\
& j=1,2,\dots,n, \; k=1,2,\dots,l
\end{cases}
\tag{4}
$$

Proof Since ξ_{ijk} are independent uncertain variables with distributions $\Phi_{\xi_{ijk}}^{p}$. Through Theorems (A.1) and (A.2), we have

$$
\begin{aligned}
&\mathrm{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\xi_{ijk}\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\mathrm{E}[\xi_{ijk}^{p}] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\int_{0}^{1}\Phi_{\xi_{ijk}^{p}}^{-1}(\alpha)\mathrm{d}\alpha.
\end{aligned}
$$

According to Theorem (A.3), we have

$$
\begin{aligned}
&\mathcal{M}\left\{\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}-a_i\leq 0\right\}\geq\alpha_i \\
&\Leftrightarrow \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}-\Psi_{a_i}^{-1}(1-\alpha_i)\leq 0, \;\; i=1,2,\dots,m \\
&\mathcal{M}\left\{b_j-\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk}\leq 0\right\}\geq\beta_j \\
&\Leftrightarrow \Psi_{b_j}^{-1}(\beta_j)-\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk}\leq 0, \;\; j=1,2,\dots,n \\
&\mathcal{M}\left\{c_k-\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}\leq 0\right\}\geq\gamma_k \\
&\Leftrightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}-\Psi_{c_k}^{-1}(1-\gamma_k)\leq 0, \;\; k=1,2,\dots,l.
\end{aligned}
$$

It is obviously that Model (3) is equivalent to Model (4). The theorem is proved.

3 Numerical example

In order to show the application of these models, we present an example of coal transportation problem in this section. Suppose that there are three coal mines. They supply the coal for four cities, and two kinds of conveyances are available, i.e., train and cargo ship. Now, the

Table 1 The supply capacities a_i of three coal mines

a_1	a_2	a_3
$\mathcal{L}(100,200)$	$\mathcal{L}(200,300)$	$\mathcal{L}(150,200)$

Table 2 The demands b_j of four cities

b_1	b_2	b_3	b_4
$\mathcal{L}(100,120)$	$\mathcal{L}(100,120)$	$\mathcal{L}(100,120)$	$\mathcal{L}(100,120)$

Table 3 The capacities c_k of two conveyances

c_1	c_2
$\mathcal{L}(300,600)$	$\mathcal{L}(200,500)$

Table 4 The direct costs by trains (ξ_{ij1}) and cargo ships (ξ_{ij2})

ξ_{ij1}	Cities			
	1	2	3	4
Mines				
1	$\mathcal{L}(2,4)$	$\mathcal{L}(4,6)$	$\mathcal{L}(3,5)$	$\mathcal{L}(5,7)$
2	$\mathcal{L}(2,5)$	$\mathcal{L}(5,6)$	$\mathcal{L}(2,8)$	$\mathcal{L}(3,4)$
3	$\mathcal{L}(3,4)$	$\mathcal{L}(2,7)$	$\mathcal{L}(2,4)$	$\mathcal{L}(1,5)$
ξ_{ij2}	Cities			
	1	2	3	4
Mines				
1	$\mathcal{L}(4,5)$	$\mathcal{L}(3,5)$	$\mathcal{L}(4,6)$	$\mathcal{L}(2,4)$
2	$\mathcal{L}(5,7)$	$\mathcal{L}(3,4)$	$\mathcal{L}(5,7)$	$\mathcal{L}(2,5)$
3	$\mathcal{L}(6,7)$	$\mathcal{L}(2,5)$	$\mathcal{L}(6,8)$	$\mathcal{L}(3,6)$

Table 5 The transportation plans of the trains (x_{ij1}) and cargo ships (x_{ij2})

x_{ij1}	Cities			
	1	2	3	4
Mines				
1	32.31	12.43	27.73	0
2	28.08	8.21	18.92	22.62
3	27.72	17.02	36.91	26.85
x_{ij2}	Cities			
	1	2	3	4
Mines				
1	18.55	21.61	18.56	26.85
2	5.16	26.56	9.74	22.62
3	0.21	26.20	0.17	13.09

decision maker should make a transportation plan for the next month such that the transportation cost minimized as well as the entropy function maximized. In this example, the notations a_i, b_j and c_k are employed to denote the sources, the demands and the transportation capacities, respectively. Assume that all uncertain variables are linear uncertain variables, which are listed in Tables 1, 2 and 3, respectively.

In addition, we also assume that the transportation fees are linear uncertain variables, which are shown in Table 4.

Substitute these parameters into Model (4), we can get a multi-objective optimization model. We choose a weight λ for objective 1, and $1-\lambda$ for objective 2, then we can convert this model to a single objective model (Model 5). By using Matlab 2014a optimization tool, we can solve this model. The minimum cost is 853.6775, and the corresponding transportation plan is shown in Table 5.

$$\begin{cases} \min \lambda \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\int_0^1 \Phi^{-1}_{\xi^p_{ijk}}(\alpha)\mathrm{d}\alpha + (1-\lambda) \\ \quad \cdot(-\ln(T) + \frac{1}{T}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk}\ln x_{ijk}), \\ \quad p = 1,2,\ldots,P \\ s.t. \\ \quad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} - \Phi^{-1}_{a_i}(1-\alpha_i) \le 0,\ i = 1,2,3 \\ \quad \Phi^{-1}_{b_j}(\beta_j) - \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \le 0,\ j = 1,2,3,4 \\ \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} - \Phi^{-1}_{c_k}(1-\gamma_k) \le 0,\ k = 1,2 \\ \quad x_{ijk} \ge 0, i = 1,2,3, j = 1,2,3,4, k = 1,2. \end{cases} \tag{5}$$

4 Conclusion

In this paper, we mainly discussed the uncertain solid transportation model which involves entropy function as a new objective. As a result, a decision model under expected value criterion was proposed. Moreover, under the condition that the parameters were independent uncertain variables, we derived its crisp equivalent that could be solved by interior point method. Finally, a coal transportation problem is given as an example to illustrate the usefulness of the models developed in the work.

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Appendix: Uncertainty theory

Uncertain theory, founded by Liu (2007) and refined by Liu (2009a), is a branch of axiomatic mathematics for modeling human uncertainty. Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and each element Λ in $\mathcal{L}$ is called an event. Uncertain measure is defined as a function from $\mathcal{L}$ to [0,1].

Definition A.1 (Liu 2007) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies:

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;
Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;
Axiom 3. For any countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}; \tag{A.1}$$

Besides, in order to provide the operational law, Liu (2009a) defined the product uncertain measure on the product σ-algebra $\mathcal{L}$ as follows.

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2,\cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying:

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}. \tag{A.2}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2\cdots$, respectively. Based on the concept of uncertain measure, we can define an uncertain variable.

Definition A.2 (Liu 2007) An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.

In order to describe uncertain variable in practice, uncertainty distribution $\Phi{:}\Re \to [0, 1]$ of an uncertain variable is defined as

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \tag{A.3}$$

An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \lim_{x\to+\infty} \Phi(x) = 1. \tag{A.4}$$

If ξ is an uncertain variable with regular uncertainty distribution $\Phi(x)$, we call the inverse function $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of ξ.

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in \Re \tag{A.5}$$

denoted by N (e, σ) where e and σ are real numbers with $\sigma > 0$. The inverse uncertainty distribution of normal uncertain variable N (e, σ) is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{A.6}$$

Definition A.3 (Liu 2010) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{A.7}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Definition A.4 (Liu 2007) Let ξ be an uncertain variable. The expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\}\mathrm{d}x, \tag{A.8}$$

provided that at least one of the above two integrals is finite.

Theorem A.1 (Liu 2010) *Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the expected value exists, then*

$$\mathrm{E}[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \tag{A.9}$$

Theorem A.2 (Liu 2010) *Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$\mathrm{E}[a\xi + b\eta] = a\mathrm{E}[\xi] + b\mathrm{E}[\eta]. \tag{A.10}$$

Theorem A.3 (Liu 2009b) *Assume the constraint function $g(\boldsymbol{x}, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing with respect to ξ_{k+1}, $\xi_{k+2}, \cdots, \xi_n$.* If $\xi_1, \xi_2, \cdots, \xi_n$ *are independent uncertain variables with uncertain distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the chance constraint*

$$\mathcal{M}\left\{g(\boldsymbol{x}, \xi_1, \xi_2, \ldots, \xi_n) \leq 0\right\} \geq \alpha \tag{A.11}$$

holds if and only if

$$g\big(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\big) \leq 0. \tag{A.12}$$

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