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ORIGINAL RESEARCH

Existence and uniqueness theorem for uncertain heat equation

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Abstract Uncertain heat equation is a type of uncertain partial differential equations, whose heat source is affected by uncertain interference. This paper proves an existence and uniqueness theorem of solutions for general uncertain heat equations under linear growth condition and Lipschitz condition. Moreover, for several special uncertain heat equations, the conditions of existence and uniqueness are derived.

Keywords Uncertain heat equation · Liu process · Existence and uniqueness theorem

1 Introduction

As a new mathematical system, uncertainty theory is based on normality, duality, subadditivity and product axioms to model human belief degrees. It was founded by Liu (2007) and perfected by Liu (2009). As an important concept in uncertainty theory, uncertain process is a sequence of uncertain variables indexed by totally ordered set to describe the dynamical behavior of uncertain phenomena. The origin of uncertain process was traced to the pioneering work of Liu (2008). As a counterpart of Wiener process, Liu process is designed by Liu (2009) to deal with white noise. It is a Lipschitz continuous uncertain process with stationary and independent increments which are normal uncertain variables. Later on, uncertain calculus was developed by Liu (2009) to handle the integration and differentiation of functions of uncertain processes. By now, uncertain process has been successfully extended in many directions, including uncertain renewal process (Liu 2008), uncertain renewal reward process (Liu 2010), uncertain alternating renewal process (Yao and Li 2012), and uncertain contour process (Yao 2015).

As a type of differential equations, uncertain differential equation driven by Liu process was first developed by Liu (2008). The existence and uniqueness theorem for an uncertain differential equation was proved by Chen and Liu (2010). Moreover, Yao and Chen (2013) proved that the solution of an uncertain differential equation can be represented by a spectrum of ordinary differential equations. For more detailed exposition of uncertain differential equation, the readers may consult Yao's recent book (Yao 2016). At present, uncertain differential equation has been widely applied in many fields such as uncertain finance (Liu 2009, 2013; Chen and Gao 2013; Liu et al. 2015), uncertain optimal control (Zhu 2010), and uncertain differential game (Yang and Gao 2013, 2016).

Uncertain partial differential equation driven by Liu process was first proposed by Yang and Yao (2016). They also studied uncertain heat equation whose heat source is affected by uncertain interference. And they obtained the solution and inverse uncertainty distribution of solution for a special linear uncertain heat equation. Based on this work, this paper will prove an existence and uniqueness theorem of solution for general uncertain heat equation under linear growth condition and Lispchitz condition. The rest of the paper is arranged as follows. Section 2 reviews some basic definitions and results of uncertainty theory. Section 3 introduces uncertain partial differential equation.

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Section 4 proves an existence and uniqueness theorem. At last, Section 5 gives a brief summary.

2 Preliminaries

In this section, we introduce some fundamental concepts and properties in uncertainty theory including uncertain variable, uncertain process and uncertain field.

Definition 1 (Liu (2007)) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}:\mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms,

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;
Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;
Axiom 3. For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

In order to provide the operational law, Liu (2009) defined the product uncertain measure on the product σ -algebre L, it is called product axiom.

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure M is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \ldots$, respectively.

Definition 2 (Liu (2007)) An uncertain variable is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, such that, for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

In order to describe uncertain variable in practice, uncertainty distribution $\Phi:\Re \to [0,1]$ of an uncertain variable ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \le x\}$. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \ \lim_{x\to+\infty} \Phi(x) = 1.$$

If ξ is an uncertain variable with regular uncertainty distribution $\Phi(x)$, then $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

The operational law of uncertain variables was proposed by Liu (2010) to calculate the inverse uncertainty distribution of strictly monotonous function as the following theorem.

Theorem 1 (Liu (2010)) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions$\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to$x_1, x_2, \ldots, x_m$ and strictly decreasing with$x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Definition 3 (Liu (2008)) Let T be a totally ordered set (e.g. time) and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t.

An uncertain process X_t is said to have independent increments if $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \ldots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$. An uncertain process X_t is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Definition 4 (Liu (2009)) An uncertain process C_t is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
(ii) C_t has stationary and independent increments;
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \; x \in \Re.$$

Theorem 2 (Liu (2015)) *Let C_t be a Liu process. Then for each time $t > 0$, the ratio C_t/t is a normal uncertain variable with expected value 0 and variance 1. That is,*

$$\frac{C_t}{t} \sim \mathcal{N}(0,1)$$

for any $t > 0$.

Definition 5 (Liu (2009)) Let X_t be an uncertain process and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be Liu integrable.

Theorem 3 (Chen and Liu (2010)) *Let X_t be a Liu integrable uncertain process on [a, b]. Then for a sample path $C_t(\gamma)$ with a Lipschitz constant $Q(\gamma)$, we have*

$$\left|\int_a^b X_t(\gamma)\mathrm{d}C_t(\gamma)\right| \le Q(\gamma)\int_a^b \left|X_t(\gamma)\right|\mathrm{d}t.$$

Let $h(t, c)$ be a continuously differentiable function. Then $Z_t = h(t, C_t)$ has an uncertain differential

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

Uncertain field is a generalization of uncertain process when the index set T becomes a partially ordered set (e.g. time×space, or surface). A formal definition is given below.

Definition 6 (Liu (2014)) Let T be a partially ordered set (e.g. time×space) and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain field is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each t.

Definition 7 (Gao and Chen (2017)) An uncertain field X_t is said to have an uncertainty distribution $\Phi_t(x)$ if for each t, the uncertain variable X_t has an uncertainty distribution $\Phi_t(x)$.

An uncertainty distribution $\Phi_t(x)$ of uncertain field X_t is said to be regular if for any t, it is a continuous and strictly increasing function with respect to x such that $0 < \Phi_t(x) < 1$, and

$$\lim_{x\to-\infty} \Phi_t(x) = 0, \; \lim_{x\to+\infty} \Phi_t(x) = 1.$$

If X_t is an uncertain field with regular uncertainty distribution $\Phi_t(x)$, we call the inverse function $\Phi_t^{-1}(\alpha)$ as the inverse uncertainty distribution of ξ.

3 Uncertain heat equation

As a classic type of partial differential equation, heat equation describes the variation of temperature in a given region over time. However, heat source is often affected by the interference of noise in practice. In order to model noise, two processes are used, one is a Wiener process that is based on probability theory, another is a Liu process that is based on uncertainty theory. If we consider noise as Wiener process, then heat equation turns into stochastic heat equation. Nevertheless, Yang and Yao (2016) pointed that it is unreasonable to model the heat conduction process via stochastic heat equation. Therefore, Yang and Yao (2016) proposed an uncertain heat equation whose the noise of heat source is described by Liu process as follows,

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - a^2\dfrac{\partial^2 U_{t,x}}{\partial x^2} = f(t,x) + \sigma(t,x)\dot{C}_t \\ U_{0,x} = \varphi(x), \qquad t > 0, x \in \Re \end{cases} \tag{1}$$

where a^2 is the constant thermal diffusivity $(a > 0)$, $\dot{C}_t = \mathrm{d}C_t/\mathrm{d}t$ denotes the time white noise, C_t is a Liu process, $f(t, x)$ is a heat source, $\sigma(t, x)$ is the diffusion term of heat source, and $\varphi(x)$ is a given initial temperature at time $t = 0$. They proved that the solution of uncertain heat equation (1) is

$$\begin{aligned} U_{t,x} = &\int_{-\infty}^{+\infty} K(t, x-y)\varphi(y)\mathrm{d}y \\ &+ \int_0^t \int_{-\infty}^{+\infty} K(t-s, x-y)f(s,y)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_{-\infty}^{+\infty} K(t-s, x-y)\sigma(s,y)\mathrm{d}y\mathrm{d}C_s \end{aligned} \tag{2}$$

where

$$K(t,x)=\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right).$$

Example 1 The uncertain heat equation

$$\begin{cases}\dfrac{\partial U_{t,x}}{\partial t}-\dfrac{\partial^2 U_{t,x}}{\partial x^2}=\sin(x)\cdot\dot{C}_t\\ U_{0,x}=0, \quad t>0, x\in\mathfrak{R}\end{cases}$$

has a solution

$$\begin{aligned}U_{t,x}&=\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\sin y\mathrm{d}y\mathrm{d}C_s\\ &=e^{-t}\sin(x)\cdot\int_0^t e^s\mathrm{d}C_s\end{aligned}$$

with inverse uncertainty distribution

$$\Phi_{t,x}^{-1}(\alpha)=\frac{\sqrt{3}}{\pi}|\sin x|\left(1-e^{-t}\right)\ln\frac{\alpha}{1-\alpha}$$

which is shown in Fig.1.

Example 2 The uncertain heat equation

$$\begin{cases}\dfrac{\partial U_{t,x}}{\partial t}-\dfrac{\partial^2 U_{t,x}}{\partial x^2}=e^{-t}\cos x+\dot{C}_t\\ U_{0,x}=\cos x, \quad t>0, x\in\mathfrak{R}\end{cases}$$

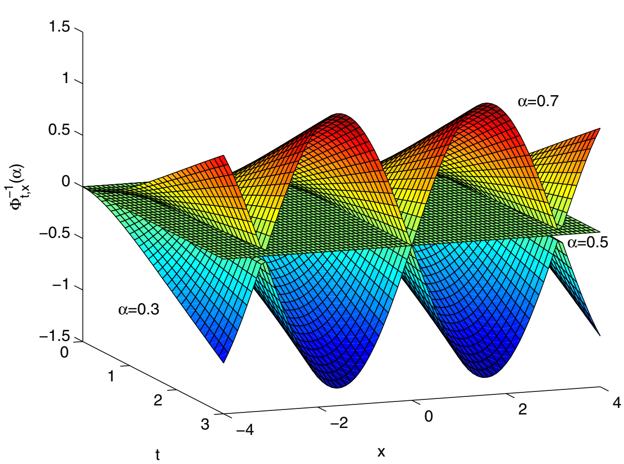

Fig. 1 Inverse uncertainty distribution of Example 1

has a solution

$$\begin{aligned}U_{t,x}&=\int_{-\infty}^{+\infty}K(t,x-y)\cos y\mathrm{d}y\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)e^{-s}\cos y\mathrm{d}y\mathrm{d}s\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\mathrm{d}y\mathrm{d}C_s\\ &=\int_{-\infty}^{+\infty}K(t,x-y)\cos y\mathrm{d}y\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)e^{-s}\cos y\mathrm{d}y\mathrm{d}s+C_t\\ &=e^{-t}\cos x+te^{-t}\cos x+C_t\\ &=(t+1)e^{-t}\cos x+C_t\end{aligned}$$

with inverse uncertainty distribution

$$\Phi_{t,x}^{-1}(\alpha)=(t+1)e^{-t}\cos x+\frac{\sqrt{3}t}{\pi}\ln\frac{\alpha}{1-\alpha}$$

which is shown in Fig. 2.

Example 3 The uncertain heat equation

$$\begin{cases}\dfrac{\partial U_{t,x}}{\partial t}-\dfrac{\partial^2 U_{t,x}}{\partial x^2}=\cos x+\dfrac{1}{t+1}\dot{C}_t\\ U_{0,x}=\sin x, \quad t>0, x\in\mathfrak{R}\end{cases}$$

has a solution

$$\begin{aligned}U_{t,x}&=\int_{-\infty}^{+\infty}K(t,x-y)\sin y\mathrm{d}y\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cos y\mathrm{d}y\mathrm{d}s\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\frac{1}{s+1}\mathrm{d}y\mathrm{d}C_s\\ &=e^{-t}\sin x+(1-e^{-t})\cos x+\int_0^t\frac{1}{s+1}\mathrm{d}C_s\end{aligned}$$

with inverse uncertainty distribution

$$\Phi_{t,x}^{-1}(\alpha)=e^{-t}(\sin x-\cos x)+\cos x+\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\ln(t+1)$$

which is shown in Fig. 3.

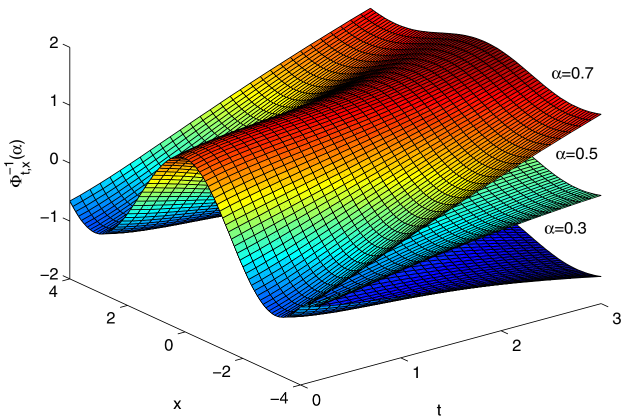

Fig. 2 Inverse Uncertainty Distribution of Example 2

4 Existence and uniqueness theorem

Let us consider the general uncertain heat equation as follows,

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - a^2 \dfrac{\partial^2 U_{t,x}}{\partial x^2} = f(t,x,U_{t,x}) + \sigma(t,x,U_{t,x})\dot{C}_t \\ U_{0,x} = \varphi(x), \qquad t > 0, x \in \Re \end{cases} \tag{3}$$

where $\dot{C}_t$ is a white noise in time, and $\varphi(x)$ is a bounded real-valued function.

We replace Eq. (3) by the following uncertain integral equation

$$\begin{aligned} U_{t,x} = & \int_{-\infty}^{+\infty} K(t,x-y)\varphi(y)\mathrm{d}y \\ & + \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y) f(s,y,U_{s,y})\mathrm{d}y\mathrm{d}s \\ & + \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)\sigma(s,y,U_{s,y})\mathrm{d}y\mathrm{d}C_s. \end{aligned} \tag{4}$$

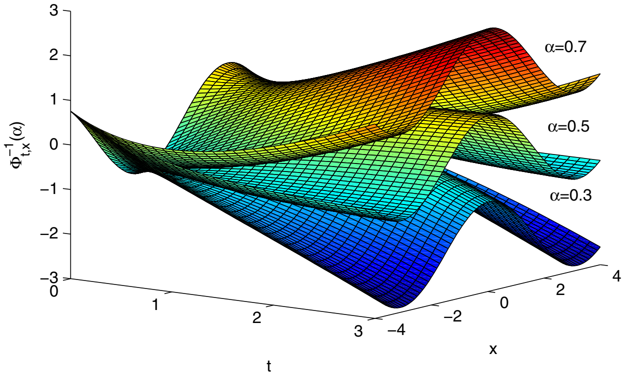

Fig. 3 Inverse uncertainty distribution of example 3

If the functions $f(t, x, u)$ and $\sigma(t,x,u)$ satisfy linear growth condition and Lipschitz condition, then we will prove that there exists a unique solution.

Theorem 4 *The uncertain heat equation* (3) *has a unique solution if the functions* $f(t, x, u)$ *and* $\sigma(t,x,u)$ *satisfy linear growth condition*

$$|f(t,x,u)| + |\sigma(t,x,u)| \le L(1+|u|), \quad \forall x \in \Re, t \ge 0 \tag{5}$$

and Lipschitz condition

$$\begin{aligned} & |f(t,x,u) - f(t,x,u')| + |\sigma(t,x,u) - \sigma(t,x,u')| \\ & \quad \le L|u-u'|, \quad \forall x \in \Re, t \ge 0 \end{aligned} \tag{6}$$

for some constant L, and the initial value $\varphi(x)$ *is a bounded real-valued function.*

Proof To prove the existence, a successive approximation method will be proposed to construct a solution of the uncertain heat equation (3). For each $\gamma \in \Gamma$, define $U_{t,x}^{(0)}(\gamma) = \varphi(x)$,

$$\begin{aligned} & U_{t,x}^{(n+1)}(\gamma) \\ & = \int_{-\infty}^{+\infty} K(t,x-y)\varphi(y)\mathrm{d}y \\ & \quad + \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y) f(s,y,U_{s,y}^{(n)}(\gamma))\mathrm{d}y\mathrm{d}s \\ & \quad + \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)\sigma(s,y,U_{s,y}^{(n)}(\gamma))\mathrm{d}y\mathrm{d}C_s(\gamma) \end{aligned}$$

and

$$D_{t,x}^{(n)}(\gamma) = \max_{0\le s\le t} \left| U_{s,x}^{(n+1)}(\gamma) - U_{s,x}^{(n)}(\gamma) \right|, \; n = 0,1,2,\cdots.$$

We claim that

$$\begin{aligned} D_{t,x}^{(n)}(\gamma) \le & \, 2N \frac{L^n(1+Q(\gamma))^n}{n!} t^n \\ & + (1+N) \frac{L^{n+1}(1+Q(\gamma))^{n+1}}{(n+1)!} t^{n+1}, \\ & n = 0,1,2,\ldots, t \in [0,T] \end{aligned}$$

where T is a constant, $|\varphi(x)| \le N$ and $Q(\gamma)$ is the Lipschitz constant of the sample path $C_t(\gamma)$ (see Theorem 3). Indeed for $n = 0$, we get

$$
\begin{aligned}
D_{t,x}^{(0)}(\gamma) &= \max_{0\le s\le t}\left|U_{s,x}^{(1)} - U_{s,x}^{(0)}\right| = \max_{0\le s\le t}\left|\int_{-\infty}^{+\infty} K(s,x-y)\varphi(y)\mathrm{d}y - \varphi(x) + \int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)f(v,y,\varphi(y))\mathrm{d}y\mathrm{d}v\right.\\
&\left.+\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\sigma(v,y,\varphi(y))\mathrm{d}y\mathrm{d}C_v(\gamma)\right| \le \max_{0\le s\le t}\left|\int_{-\infty}^{+\infty} K(s,x-y)\varphi(y)\mathrm{d}y\right| + |\varphi(x)|\\
&+\max_{0\le s\le t}\left|\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)f(v,y,\varphi(y))\mathrm{d}y\mathrm{d}v\right| + \max_{0\le s\le t}\left|\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\sigma(v,y,\varphi(y))\mathrm{d}y\mathrm{d}C_v(\gamma)\right|\\
&\le N\max_{0\le s\le t}\int_{-\infty}^{+\infty} K(s,x-y)\mathrm{d}y + N + \max_{0\le s\le t}\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)|f(v,y,\varphi(y))|\mathrm{d}y\mathrm{d}v\\
&+Q(\gamma)\max_{0\le s\le t}\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)|\sigma(v,y,\varphi(y))|\mathrm{d}y\mathrm{d}v \le 2N + \int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)|f(v,y,\varphi(y))|\mathrm{d}y\mathrm{d}v\\
&+Q(\gamma)\int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)|\sigma(v,y,\varphi(y))|\mathrm{d}y\mathrm{d}v \le 2N + L\int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)(1+|\varphi(y)|)\mathrm{d}y\mathrm{d}v\\
&+LQ(\gamma)\int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)(1+|\varphi(y)|)\mathrm{d}y\mathrm{d}v = 2N + L(1+Q(\gamma))(1+N)\int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)\mathrm{d}y\mathrm{d}v\\
&= 2N + L(1+Q(\gamma))(1+N)t.
\end{aligned}
$$

This confirms the claim for $n=0$. Next we assume the claim is true for some $n-1$, then we obtain

$$
\begin{aligned}
D_{t,x}^{(n)}(\gamma) &= \max_{0\le s\le t}\left|U_{s,x}^{(n+1)}(\gamma) - U_{s,x}^{(n)}(\gamma)\right| = \max_{0\le s\le t}\left|\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\Big(f(v,y,U_{v,y}^{(n)}(\gamma)) - f(v,y,U_{v,y}^{(n-1)}(\gamma))\Big)\mathrm{d}y\mathrm{d}v\right.\\
&\left.+\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\Big(\sigma(v,y,U_{v,y}^{(n)}(\gamma)) - \sigma(v,y,U_{v,y}^{(n-1)}(\gamma))\Big)\mathrm{d}y\mathrm{d}C_v(\gamma)\right|\\
&\le \max_{0\le s\le t}\left|\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\Big(f(v,y,U_{v,y}^{(n)}(\gamma)) - f(v,y,U_{v,y}^{(n-1)}(\gamma))\Big)\mathrm{d}y\mathrm{d}v\right|\\
&+\max_{0\le s\le t}\left|\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\Big(\sigma(v,y,U_{v,y}^{(n)}(\gamma)) - \sigma(v,y,U_{v,y}^{(n-1)}(\gamma))\Big)\mathrm{d}y\mathrm{d}C_v(\gamma)\right|\\
&\le \max_{0\le s\le t}\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\left|f(v,y,U_{v,y}^{(n)}(\gamma)) - f(v,y,U_{v,y}^{(n-1)}(\gamma))\right|\mathrm{d}y\mathrm{d}v\\
&+Q(\gamma)\max_{0\le s\le t}\int_0^s\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\left|\sigma(v,y,U_{v,y}^{(n)}) - \sigma(v,y,U_{v,y}^{(n-1)})\right|\mathrm{d}y\mathrm{d}v\\
&\le \int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)\cdot\left|f(v,y,U_{v,y}^{(n)}(\gamma)) - f(v,y,U_{v,y}^{(n-1)}(\gamma))\right|\mathrm{d}y\mathrm{d}v\\
&+Q(\gamma)\int_0^t\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\left|\sigma(v,y,U_{v,y}^{(n)}(\gamma)) - \sigma(v,y,U_{v,y}^{(n-1)}(\gamma))\right|\mathrm{d}y\mathrm{d}v\\
&\le L\int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)\cdot\left|U_{v,y}^{(n)}(\gamma) - U_{v,y}^{(n-1)}(\gamma)\right|\mathrm{d}y\mathrm{d}v\\
&+LQ(\gamma)\int_0^t\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\left|U_{v,y}^{(n)}(\gamma) - U_{v,y}^{(n-1)}(\gamma)\right|\mathrm{d}y\mathrm{d}v\\
&\le L\int_0^t\int_{-\infty}^{+\infty} K(t-v,x-y)\cdot\left[2N\frac{L^{n-1}(1+Q(\gamma))^{n-1}}{(n-1)!}t^{n-1} + (1+N)\frac{L^n(1+Q(\gamma))^n}{n!}t^n\right]\mathrm{d}y\mathrm{d}v\\
&+LQ(\gamma)\int_0^t\int_{-\infty}^{+\infty} K(s-v,x-y)\cdot\left[2N\frac{L^{n-1}(1+Q(\gamma))^{n-1}}{(n-1)!}t^{n-1} + (1+N)\frac{L^n(1+Q(\gamma))^n}{n!}t^n\right]\mathrm{d}y\mathrm{d}v\\
&= L\int_0^t\left[2N\frac{L^{n-1}(1+Q(\gamma))^{n-1}}{(n-1)!}t^{n-1} + (1+N)\frac{L^n(1+Q(\gamma))^n}{n!}t^n\right]\mathrm{d}v\\
&+LQ(\gamma)\int_0^t\left[2N\frac{L^{n-1}(1+Q(\gamma))^{n-1}}{(n-1)!}t^{n-1} + (1+N)\frac{L^n(1+Q(\gamma))^n}{n!}t^n\right]\mathrm{d}v\\
&= L(1+Q(\gamma))\int_0^t\left[2N\frac{L^{n-1}(1+Q(\gamma))^{n-1}}{(n-1)!}t^{n-1} + (1+N)\frac{L^n(1+Q(\gamma))^n}{n!}t^n\right]\mathrm{d}v = 2N\frac{L^n(1+Q(\gamma))^n}{n!}t^n\\
&+(1+N)\frac{L^{n+1}(1+Q(\gamma))^{n+1}}{(n+1)!}t^{n+1}.
\end{aligned}
$$

Thus, the claim is proved. It follows from Weierstrass' criterion that, for each sample γ,

$$\begin{aligned}&\sum_{n=1}^{+\infty}\left[2N\frac{L^n(1+Q(\gamma))^n}{n!}t^n\right.\\&\quad\left.+(1+N)\frac{L^{n+1}(1+Q(\gamma))^{n+1}}{(n+1)!}t^{n+1}\right]\\&\leq\sum_{n=1}^{+\infty}\left[2N\frac{L^n(1+Q(\gamma))^n}{n!}T^n\right.\\&\quad\left.+(1+N)\frac{L^{n+1}(1+Q(\gamma))^{n+1}}{(n+1)!}T^{n+1}\right]\\&<+\infty.\end{aligned}$$

Then $U_{x,t}^{(n)}$ converges uniformly in $(t,x)\in([0,T]\times\mathfrak{R})$. We denote the limit by

$$U_{t,x}(\gamma)=\lim_{n\to\infty}U_{t,x}^{(n)}(\gamma),\quad \gamma\in\Gamma,(t,x)\in([0,T]\times\mathfrak{R}).$$

Then we have

$$\begin{aligned}U_{t,x}(\gamma)&=\int_{-\infty}^{+\infty}K(t,x-y)\varphi(y)\mathrm{d}y\\&\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)f(s,y,U_{s,y}(\gamma))\mathrm{d}y\mathrm{d}s\\&\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\sigma(s,y,U_{s,y}(\gamma))\mathrm{d}y\mathrm{d}C_s(\gamma).\end{aligned}$$

Therefore, the uncertain field $U_{t,x}$ is just the solution of the uncertain heat equation (3).

Next, we prove the uniqueness of the solution under the given conditions. Assume that $U_{t,x}$ and $U^*_{t,x}$ are two solutions of the uncertain heat equation (3) with a common initial value $\varphi(x)$. Then for almost every $\gamma\in\Gamma$, we have

$$\begin{aligned}|U_{t,x}(\gamma)-U^*_{t,x}(\gamma)|&=\left|\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cdot\left[f(s,y,U_{s,y}(\gamma))-f(s,y,U^*_{s,y}(\gamma))\right]\mathrm{d}y\mathrm{d}s\right.\\&\quad\left.+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cdot\left[\sigma(s,y,U_{s,y}(\gamma))-\sigma(s,y,U^*_{s,y}(\gamma))\right]\mathrm{d}y\mathrm{d}C_s(\gamma)\right|\\&\leq\left|\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cdot\left[f(s,y,U_{s,y}(\gamma))-f(s,y,U^*_{s,y}(\gamma))\right]\mathrm{d}y\mathrm{d}s\right|+\left|\right.\\&\quad\left.\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cdot\left[\sigma(s,y,U_{s,y}(\gamma))-\sigma(s,y,U^*_{s,y}(\gamma))\right]\mathrm{d}y\mathrm{d}C_s(\gamma)\right|\\&\leq\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cdot\left|f(s,y,U_{s,y}(\gamma))-f(s,y,U^*_{s,y}(\gamma))\right|\mathrm{d}y\mathrm{d}s\\&\quad+Q(\gamma)\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\cdot\left|\sigma(s,y,U_{s,y}(\gamma))-\sigma(s,y,U^*_{s,y}(\gamma))\right|\mathrm{d}y\mathrm{d}s\\&\leq L\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\left|U_{s,y}(\gamma)-U^*_{s,y}(\gamma)\right|\mathrm{d}y\mathrm{d}s+\cdot LQ(\gamma)\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\left|\right.\\&\quad\left.U_{s,y}(\gamma)-U^*_{s,y}(\gamma)\right|\mathrm{d}y\mathrm{d}s=L(1+Q(\gamma))\cdot\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\left|U_{s,y}(\gamma)-U^*_{s,y}(\gamma)\right|\mathrm{d}y\mathrm{d}s.\end{aligned}$$

By the Grönwall's inequality, we get

$$\begin{aligned}&|U_{t,x}(\gamma)-U^*_{t,x}(\gamma)|\\&\quad\leq 0\cdot\exp\left(L(1+Q(\gamma))\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\mathrm{d}y\mathrm{d}s\right)\\&\quad=0.\end{aligned}$$

That means $U_{t,x}(\gamma)=U^*_{t,x}(\gamma)$ almost surely. The uniqueness of the solution is verified. The theorem is thus proved. □

Corollary 1 *If $f(t,x)$, $\sigma(t,x)$ and $\varphi(x)$ are both bounded functions, then the uncertain heat equation*

$$\begin{cases}\dfrac{\partial U_{t,x}}{\partial t}-a^2\dfrac{\partial^2U_{t,x}}{\partial x^2}=f(t,x)+\sigma(t,x)\dot{C}_t\\U_{0,x}=\varphi(x),\qquad t>0,x\in\mathfrak{R}\end{cases}\tag{7}$$

has a unique solution

$$\begin{aligned}U_{t,x}&=\int_{-\infty}^{+\infty}K(t,x-y)\varphi(y)\mathrm{d}y\\&\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s\\&\quad+\int_0^t\int_{-\infty}^{+\infty}K(t-s,x-y)\sigma(s,y)\mathrm{d}y\mathrm{d}C_s\end{aligned}\tag{8}$$

where

$$K(t,x)=\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right).$$

Proof From the work of Yang and Yao (2016), the Eq. (8) is a solution of uncertain heat equation (7). Then we just prove the uniqueness. Since $f(t,x)$ and $\sigma(t,x)$ are bounded functions, there exists a positive number L such that

$|f(t,x)| + |\sigma(t,x)| \leq L.$

That is, they are satisfy linear growth condition. And, it is obvious that $f(t, x)$ and $\sigma(t,x)$ satisfy Lipschitz condition. From Theorem 4, the corollary is thus proved. □

Example 4 Consider the uncertain heat equation in Example 1,

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - \dfrac{\partial^2 U_{t,x}}{\partial x^2} = \sin(x) \cdot \dot{C}_t \\ U_{0,x} = 0, \quad t > 0, x \in \mathfrak{R}. \end{cases}$$

Note that $f(t,x) = 0$, $\sigma(t,x) = \sin x$ and $\varphi(x) = 0$ are both bounded functions. By Corollary 1, the uncertain field

$$U_{t,x} = e^{-t} \sin x \cdot \int_0^t e^s \mathrm{d}C_s$$

is a unique solution of the above uncertain heat equation.

Example 5 Consider the uncertain heat equation in Example 2,

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - \dfrac{\partial^2 U_{t,x}}{\partial x^2} = e^{-t} \cos x + \dot{C}_t \\ U_{0,x} = \cos x, \quad t > 0, x \in \mathfrak{R}. \end{cases}$$

Note that $f(t,x) = e^{-t} \cos x$, $\sigma(t,x) = 1$ and $\varphi(x) = \cos x$ are both bounded functions. By Corollary 1, the uncertain field

$$U_{t,x} = (t+1)e^{-t} \cos x + C_t$$

is a unique solution of the above uncertain heat equation.

Example 6 Consider the uncertain heat equation in Example 3,

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - \dfrac{\partial^2 U_{t,x}}{\partial x^2} = \cos x + \dfrac{1}{t+1} \dot{C}_t \\ U_{0,x} = \sin x, \quad t > 0, x \in \mathfrak{R}. \end{cases}$$

Note that $f(t,x) = \cos x$, $\sigma(t,x) = 1/(t+1)$ and $\varphi(x) = \sin x$ are both bounded functions. By Corollary 1, the uncertain field

$$U_{t,x} = e^{-t} \sin x + (1 - e^{-t}) \cos x + \int_0^t \frac{1}{s+1} \mathrm{d}C_s$$

is a unique solution of the above uncertain heat equation.

Corollary 2 *If $f(t, x)$, $\sigma(t,x)$ and$\varphi(x)$ are both bounded functions, then each of the following three uncertain heat equations*

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - a^2 \dfrac{\partial^2 U_{t,x}}{\partial x^2} = f(t,x)U_{t,x} + \sigma(t,x)U_{t,x}\dot{C}_t \\ U_{0,x} = \varphi(x), \quad t > 0, x \in \mathfrak{R}, \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - a^2 \dfrac{\partial^2 U_{t,x}}{\partial x^2} = f(t,x)U_{t,x} + \sigma(t,x)\dot{C}_t \\ U_{0,x} = \varphi(x), \quad t > 0, x \in \mathfrak{R}, \end{cases} \tag{10}$$

and

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} - a^2 \dfrac{\partial^2 U_{t,x}}{\partial x^2} = f(t,x) + \sigma(t,x)U_{t,x}\dot{C}_t \\ U_{0,x} = \varphi(x), \quad t > 0, x \in \mathfrak{R} \end{cases} \tag{11}$$

both have a unique solution.

Proof Firstly, we prove the uncertain heat equation (9) has a unique solution. Since $f(t, x)$ and $\sigma(t,x)$ are bounded functions, there exists a positive number L such that

$$|f(t,x)u| + |\sigma(t,x)u| \leq L|u|$$

and

$$|f(t,x)u - f(t,x)v| + |\sigma(t,x)u - \sigma(t,x)v| \leq L|u - v|.$$

That is, they are satisfy linear growth condition and Lipschitz condition. From Theorem 4, the uncertain heat equation (9) has a unique solution. Similarly, we can prove the uncertain heat equations (10) and (11) both have a unique solution. Thus the corollary is proved. □

If the constant thermal diffusivity a^2 is close to 0 (thermal insulation material), then the uncertain heat equation (3) turns into

$$\begin{cases} \dfrac{\partial U_{t,x}}{\partial t} = f(t,x,U_{t,x}) + \sigma(t,x,U_{t,x})\dot{C}_t \\ U_{0,x} = \varphi(x), \quad t > 0, x \in \mathfrak{R} \end{cases} \tag{12}$$

which is an uncertain ordinary differential equation. Its solution can be expressed by an uncertain integral equation

$$U_{t,x} = U_{0,x} + \int_0^t f(t,x,U_{s,x})\mathrm{d}s + \int_0^t \sigma(t,x,U_{s,x})\mathrm{d}C_s \tag{13}$$

that can not be obtained from Eq. (2), because Eqs. (3) and (12) are different, the former is an uncertain partial differential equation and the latter is an uncertain ordinary differential equation.

Corollary 3 *The* Eq. (12) *has a unique solution if the functionsf*(t, x, u) and $\sigma(t,x,u)$ *satisfy linear growth condition*

$$|f(t,x,u)| + |\sigma(t,x,u)| \leq L(1+|u|), \forall x \in \Re, t \geq 0 \tag{14}$$

and Lipschitz condition

$$|f(t,x,u) - f(t,x,v)| + |\sigma(t,x,u) - \sigma(t,x,v)| \\ \leq L|u-v|, \quad \forall x \in \Re, t \geq 0 \tag{15}$$

for some constant L, and the initial value $\varphi(x)$ *is a bounded real-valued function.*

Proof The process of proof is similar to Theorem 4, it just use Eq. (13) to construct a solution of the uncertain ordinary differential equation (12). □

5 Conclusion

Uncertain heat equation is an important kind of partial differential equation in uncertain environments. For a class of linear uncertain heat equations, the analytic solution and the inverse uncertainty distribution of solution are already investigated (see Yang and Yao 2016). However, it is difficult to find analytic solutions of general uncertain heat equations. The contribution of this paper was first to prove an existence and uniqueness theorem under linear growth condition and Lipschitz condition.

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