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The radial scalar power potential and its application to quarkonium systems

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Abstract: The current study employs the Nikiforov-Uvarov method to solve the Schrödinger equation for quarkonium systems, utilizing the radial scalar power potential. The eigenvalues of energy and their corresponding wave functions are determined by including the spin–spin, spin–orbit, and tensor interactions in the radial scalar power potential. The mass spectra of charmonia, bottomonia, and bottom-charm in their S, P, D, and F states were determined. Our theoretical states for quarkonium systems align with experimental data across a range of spin levels, as evidenced by our comparison. The total percentage error of our work was computed, yielding a high level of accuracy. The cumulative percentage error for the meson masses of charmonia and bottomonia was determined to be 0.324% and 0.333%, respectively. The masses of the bottom-charm mesons had a total percentage error of 0.012%. Consequently, the present potential yields favorable outcomes for the quarkonium masses, surpassing previous theoretical studies and aligning well with experimental data.

1. Introduction

Generally, particles can be divided into groups based on their roles in matter and interactions. Particles like electrons, protons, and neutrons are considered fermions, while force carriers like photons are considered bosons. The distinction between fermions and bosons lies in their quantum mechanical properties, such as spin and helicity. Hadrons are composite particles made up of fundamental particles called quarks [1]. Theorists have attempted to explain various aspects of the quark-antiquark system, including mass spectra and properties related to decay modes [2–6]. Several researchers have employed different theoretical frameworks to study this phenomenon, including the lattice quantum chromodynamics approach [7–10], semirelativistic potential models [4], and nonrelativistic

potential models [11, 12]. These models all incorporated the Coulomb and linear potentials [13, 14]. The properties of these particles have been accurately characterized through the application of quantum chromodynamics (QCD) theory. This theory features color confinement, spontaneous symmetry breaking, and asymptotic freedom. Hadrons can be studied using the relativistic and non-relativistic quark models [15–19]. The non-relativistic approach effectively explains heavy meson spectroscopic properties like mass spectrum, decay rates, radius, etc. Joshi and Mitra [20] examined the spectroscopy of heavy mesons by employing the Schrödinger equation (SE) with a harmonic oscillator and an inverse square potential. Finding an analytical solution to the SE with the addition of spin-orbit coupling to the potential function is challenging, leading to limited attention in the literature [21–25]. In such scenarios, numerical solutions are often employed [26–29]. Decay characteristics can be determined by incorporating a spin component into the potential model [30]. For instance, Ali et al. [31] examined meson energy spectra using Numerov's method and compared their

findings to empirical elementary particle data. Luz et al. [32] solved the SE using the Cornell potential to examine the Wigner function and the associated Airy function of the charmonium meson. Gupta and Mehrotra [33] studied heavy quark systems in a non-relativistic framework using an energy-dependent global potential. They calculated root mean square radii (RMSR) for quantum mechanical states, derived mass spectra, and studied leptonic decay, noting that the energy-dependent potential component saturates mass spectra. Boroun and Abdolmalki [34] employed the SE solution with a global potential to find the radial expectation values for heavy and heavy-light mesons (HLMs), as well as the wave function at the origin. Furthermore, they calculated the average masses and RMSR of heavy mesons using a potential derived from superstring theory [35]. Several authors have investigated the mass spectra of heavy and light mesons with the Cornell and generalized Cornell potential [36–40]. Additionally, authors have solved the modified SE [41–45]. In this study, the radial scalar power potential (RSPP) is used, as it contains both the Coulomb and linear terms of the standard Cornell potential. The work is driven by two goals: (a) solving meson-bound states under spin-spin interaction analytically, and (b) determining mesonic system mass spectra using the bound state solution. To the best of our knowledge, this work has never been published.

2. Theory

For bound systems in the quarkonium system, a non-relativistic approach is suitable. The Schrödinger equation (SE) for a spherically symmetric potential is given by [46].

$$\frac{d^2R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \quad (1)$$

where l, μ, r and $\hbar$ denote the angular momentum quantum number, the reduced mass of the quarkonium particle, the distance between particles, and the reduced Planck constant, respectively.

Our potential of interest is of the form [47]

$$V(r) = ar + br^2 + d - \frac{g}{r} + \frac{k}{r^2} \quad (2)$$

where $a, b, d. g,$ and k represents the potential strength.

It is common practice to model quarkonium systems using the first and fourth terms, which represent linear confinement and Coulombic terms, respectively. The power term in the radial scalar potential provides greater flexibility in characterizing the confining force, allowing for the selection of a precise power value to better fit experimental data. Additionally, it has been demonstrated that a potential with more fitting parameters fits

experimental data better than a potential with fewer parameters [48, 49].

In the nonrelativistic quark model, the quark-antiquark potential ($V_{q\bar{q}}(r)$) is composed of two components: the spin-independent potential denoted as ($V_{Sl}(r)$) which contain a vector and scalar parts, and the spin-dependent potential [50] denoted as ($V_{SD}(r)$) as follows

$$V_{q\bar{q}}(r) = V_{Sl}(r) + V_{SD}(r) \quad (3)$$

Introducing spin-dependent terms to the potential used in studying quarkonium systems accounts for the effects of the intrinsic spin of the quarks and antiquarks. These spin-dependent interactions arise due to the strong force mediated by gluons, which can couple to the spin of the quarks. The spin-dependent terms in the potential contribute to the overall energy levels and dynamics of the quarkonium system and are crucial for understanding properties like spin splittings and hyperfine structure.

$$V_{SD}(r) = V_T(r) + V_{SS}(r) \quad (4)$$

where

$$V_{SS}(r) = \frac{2}{3m_q m_{\bar{q}}} \nabla^2 V_V(r)\left[\vec{S}_q.\vec{S}_{\bar{q}}\right] \quad (5)$$

$$V_{Sl}(r) = \frac{1}{2m_q m_{\bar{q}} r}\left[3\frac{dV_v(r)}{dr} - \frac{dV_s(r)}{dr}\right]\left[\vec{L}.\vec{S}\right] \quad (6)$$

$$V_T(r) = \frac{1}{12m_q m_{\bar{q}}}\left[\frac{1}{r}\frac{dV_v(r)}{dr} - \frac{d^2V_v(r)}{dr^2}\right]$$
$$\left[6\left(\vec{S}_q.\frac{\vec{r}}{|r|}\right)\left(\vec{S}_{\bar{q}}.\frac{\vec{r}}{|r|}\right) - 2\vec{S}_q.\vec{S}_{\bar{q}}\right] \quad (7)$$

The spin–orbit term $V_{Sl}(r)$ describes a relatively small correction to the potential energy that a quarkonium system experiences as a result of the interaction between the quark and antiquark's spins and their relative motion. The tensor term $V_T(r)$ explains how the orbital motion of the quark and the antiquark, together with their combined effects, affect the potential energy and its characterizes the intricate details of fine structure of states, whereas the spin–spin term $V_{SS}(r)$ is the total effect that the quark and antiquark spins have on one another in a quarkonium system and also, delineates the phenomenon of hyperfine splitting. $m_q m_{\bar{q}}$ is the quark and antiquark mass, L is a quantum operator that represents angular momentum, while S is an operator that represents spin and $V_v(r)$ is the vector part, which are crucial tools for characterizing their directional aspects of a quarkonium systems. The scalar potential $V_s(r)$ typically represents the interaction between the quark and antiquark mediated by the exchange of scalar particles, such as mesons or gluons. The scalar potential plays a crucial role in determining the energy levels and the overall dynamics of the quarkonium system. It governs the binding

Table 1 Calculated potential function parameters

Parameters	Charmonium	Bottomonium	Bottomcharm
Mass (GeV)	$m_{c\bar{c}} = 1.2090$	$m_{b\bar{b}} = 5.1473$	$m_b = 5.1473, m_c = 1.2090$
$a(GeV)$	5.7787	3.9997	0.6754
$b(GeV^2)$	0.1498	0.2584	0.2346
$d(GeV^2)$	0.1653	0.3345	0.3654
$g(GeV)$	3.5647	2.7375	0.6148
$k(GeV)$	3.6402	0.2911	0.4532

Table 2 Charmonia mass spectrum in GeV for the S and P-states

State	Present study	[59]	[60]	[5]	[61]	[62]	[3]	EXP [58]
1^1S_0	2.9835	2.981	2.984	2.989	2.979	2.982	3.088	2.984 (0.02%)
1^3S_1	3.0957	3.096	3.097	3.094	3.097	3.090	3.168	3.097 (0.04%)
2^1S_0	3.6389	3.635	3.637	3.602	3.623	3.630	3.669	3.639 (0.01%)
2^3S_1	3.6857	3.685	3.679	3.681	3.673	3.672	3.707	3.686 (0.014%)
3^1S_0	4.0548	3.989	4.004	4.058	3.991	4.043	4.067	–
3^3S_1	4.0380	4.039	4.030	4.129	4.022	4.072	4.094	4.039 (0.02%)
4^1S_0	4.3987	4.401	4.264	4.448	4.250	4.384	4.398	–
4^3S_1	4.4206	4.427	4.281	4.514	4.273	4.406	4.420	4.421 (0.01%)
5^1S_0	4.6719	4.811	4.459	4.799	4.446	–	–	–
5^3S_1	4.6892	4.837	4.472	4.863	4.463	–	–	–
6^1S_0	5.1226	5.155	–	5.124	4.595	–	–	–
6^3S_1	5.1480	5.167	–	5.185	4.608	–	–	–
1^3P_0	3.4138	3.413	3.415	3.428	3.433	3.424	3.448	3.415 (0.04%)
1^3P_1	3.5107	3.511	3.521	3.468	3.510	3.505	3.520	3.511 (0.01%)
1^1P_1	3.5235	3.525	3.526	3.470	3.556	3.516	3.536	3.525 (0.04%)
1^3P_2	3.5549	3.555	3.553	3.480	3.842	3.556	3.564	3.556 (0.03%)
2^3P_0	3.9184	3.870	3.848	3.897	3.901	3.852	3.870	3.918 (0.01%)
2^3P_1	3.9375	3.906	3.914	3.938	3.908	3.925	3.934	–
2^1P_1	3.9440	3.926	3.916	3.943	3.937	3.934	3.950	–
2^3P_2	3.9252	3.949	3.937	3.955	4.131	3.972	3.976	3.927 (0.05%)
3^3P_0	4.2896	4.301	4.146	4.296	4.178	4.202	4.214	–
3^3P_1	4.3485	4.319	4.192	4.338	4.184	4.271	4.275	–
3^1P_1	4.3546	4.337	4.193	4.344	4.208	4.279	4.291	–
3^3P_2	4.3684	4.354	4.211	4.358	–	4.317	4.316	–
4^3P_0	4.6739	4.698	–	4.653	–	–	–	–
4^3P_1	4.6867	4.728	–	4.696	–	–	–	–
4^1P_1	4.7039	4.744	–	4.704	–	–	–	–
4^3P_2	4.7179	4.763	–	4.718	–	–	–	–

energy of the quark and antiquark, influencing properties like the spectrum of bound states and their wave functions. $\vec{S}_q.\vec{S}_{\bar{q}}$ describes the interaction between the quark and antiquark spins in the bound state. Both the relative orientation and the magnitude of their spins affect the interaction.

Putting Eqs. (5), (6), (7) into Eq. (4) and then substituting in Eqs.(3) and (2) then finally into Eq. (1) using natural units gives

Table 3 Charmonia mass spectrum in GeV for the D and F-states

State	Present study	[59]	[61]	[5]	[60]	[62]	[3]	EXP [58]
1^3D_3	3.8089	3.813	3.808	3.755	3.799	3.806	3.809	–
1^1D_1	3.8058	3.807	3.805	3.765	3.796	3.799	3.803	–
1^3D_2	3.8076	3.795	3.807	3.772	3.798	3.800	3.804	–
1^3D_1	3.7735	3.783	3.792	3.775	3.787	3.785	3.789	3.774 (0.01%)
2^3D_2	3.8234	4.220	4.112	4.176	4.103	4.167	4.167	3.824 (0.02%)
2^1D_2	3.9998	4.190	4.108	4.182	4.099	4.158	4.158	–
2^3D_2	4.1889	4.105	4.109	4.188	4.100	4.158	4.159	–
2^3D_1	4.1895	4.574	4.095	4.188	4.089	4.142	4.143	–
3^3D_3	4.2494	3.549	4.340	4.549	4.331	–	–	–
3^1D_2	4.3533	4.544	4.336	4.553	4.326	–	–	–
3^3D_2	4.4575	4.507	4337	4.557	4.327	–	–	–
3^3D_1	4.5656	4.920	4.324	4.555	4.317	–	–	–
4^3D_3	4.8979	4.898	–	4.890	–	–	–	–
4^1D_2	4.8923	4.896	–	4.892	–	–	–	–
4^3D_2	4.8964	4.857	–	4.896	–	–	–	–
4^3D_1	4.8909	4.041	–	4.891	–	–	–	–
1^3F_2	4.0010	4.068	–	3.990	–	4.029	–	–
1^3F_3	4.0126	4.071	–	4.012	–	4.029	–	–
1^1F_3	4.0169	4.093	–	4.017	–	4.026	–	–
1^3F_4	4.0364	4.361	–	4.036	–	4.021	–	–
2^3F_2	4.3779	4.400	–	4.378	–	4.351	–	–
2^3F_3	4.3958	4.406	–	4.396	–	3.352	–	–
2^1F_3	4.4099	4.434	–	4.400	–	4.350	–	–
2^3F_4	4.4155	–	–	4.415	–	4.348	–	–
3^3F_2	4.7314	–	–	4.730	–	–	–	–
3^3F_3	4.7464	–	–	4.746	–	–	–	–
3^1F_3	4.7492	–	–	4.749	–	–	–	–
3^3F_4	4.7609	–	–	4.761	–	–	–	–

$$\frac{d^2R(r)}{dr^2} + \left[2\mu E - \frac{2\mu F_2}{r^2} - \frac{2\mu F_3}{r} - \frac{2\mu F_4}{r} \right.$$
$$\left. + 2\mu F_5 - 2\mu br - 2\mu ar^2 - \frac{l(l-1)}{r^2} \right] R(r) = 0 \tag{8}$$

where

$$F_2 = \frac{3kv(sl)}{2m_q m_{\overline{q}}} + \frac{kv(t)}{4m_q m_{\overline{q}}} \tag{9}$$

$$F_3 = -\frac{2bv(ss)}{3m_q m_{\overline{q}}} + \frac{bv(t)}{12m_q m_{\overline{q}}} \tag{10}$$

$$F_4 = \frac{4gv(ss)}{3m_q m_{\overline{q}}} + \frac{3gv(sl)}{2m_q m_{\overline{q}}} + \frac{gv(t)}{12m_q m_{\overline{q}}} \tag{11}$$

$$F_5 = \frac{dv(sl)}{m_q m_{\overline{q}}} \tag{12}$$

and [51, 52]

$$v(ss) = \frac{s(s+1)}{2} - \frac{3}{4}$$

$$v(sl) = \frac{1}{2}(j(j+1)) - l(l+1) - s(s+1)$$

$$v(t) = \begin{cases} \frac{-l}{6(2l+3)}, & j = l+1 \\ \frac{1}{6}, & j = 1 \\ \frac{-l-1}{6(2l-1)}, & j = l-1 \end{cases}$$

To further simplify Eq. (8), we set $x = 1/r$ and substitute it into Eq. (8). With simplification, we obtain

$$\frac{d^2R}{dx^2} + \frac{2}{x}\frac{dR}{dx} + \frac{1}{x^2}\left[\xi + \frac{H_4}{x^2} + H_2 x^2 + \frac{H_3}{x} - \frac{A}{x^3} - \frac{D}{x^4} \right] R = 0 \tag{13}$$

where

Table 4 Bottomonia mass spectrum in GeV for the S and P-states

State	Present study	[59]	[60]	[2]	[65]	[66]	[62]	EXP [58]
1^1S_0	9.3976	9.398	9.390	9.414	9.389	9.393	9.392	9.398 (0.040%)
1^3S_1	9.4608	9.460	9.460	9.490	9.460	9.460	9.460	9.460 (0.010%)
2^1S_0	9.9989	9.990	9.990	9.987	9.987	9.987	9.991	9.999 (0.001%)
2^3S_1	10.0226	10.023	10.015	10.089	10.016	10.023	10.024	10.023 (0.004%)
3^1S_0	10.3307	10.329	10.326	–	10.330	10.345	10.323	–
3^3S_1	10.3548	10.355	10.343	10.327	10.351	10.364	10.346	10.355 (0.002%)
4^1S_0	10.5537	10.573	10.584	–	10.595	10.632	10.558	–
4^3S_1	10.5746	10.586	10.597	–	10.611	10.643	10.575	10.579 (0.042%)
5^1S_0	10.7408	10.851	10.800	–	10.817	–	10.741	–
5^3S_1	10.8756	10.869	10.811	–	10.831	–	10.755	10.876 (0.004%)
6^1S_0	10.8921	11.061	10.997	–	11.011	–	10.892	–
6^3S_1	10.9994	11.088	10.988	–	10.1988	–	10.904	11.019 (0.177%)
1^3P_0	9.8589	9.859	9.864	9.815	9.865	9.861	9.862	9.859 (0.001%)
1^3P_1	9.8937	9.892	9.903	9.842	9.897	9.891	9.888	9.893 (0.007%)
1^1P_1	9.8992	9.900	9.909	9.806	9.903	9.900	9.896	9.899 (0.002%)
1^3P_2	9.9133	9.912	9.921	9.906	9.918	9.912	9.908	9.912 (0.013%)
2^3P_0	10.2333	10.233	10.220	10.254	10.226	10.230	10.241	10.232 (0.013%)
2^3P_1	10.2600	10.255	10.249	10.120	10.251	10.255	10.256	10.260 (0.00%)
2^1P_1	10.2689	10.260	10.254	10.154	10.256	10.262	10.261	10.269 (0.001%)
2^3P_2	10.2685	10.268	10.490	–	10.269	10.271	10.268	10.268 (0.049%)
3^3P_0	10.5019	10.521	10.515	10.303	10.502	–	10.511	–
3^3P_1	10.5236	10.541	10.519	–	10.524	–	10.507	–
3^1P_1	10.5289	10.544	10.528	–	10.529	–	10.497	–
3^3P_2	10.5498	10.550	–	–	10.540	–	10.516	–
4^3P_0	10.7335	10.781	–	–	10.732	–	–	–
4^3P_1	10.7532	10.802	–	–	10.753	–	–	–
4^1P_1	10.7674	10.812	–	–	10.767	–	–	–
4^3P_2	10.9750	–	–	–	10.975	–	–	–

$$\xi = 2\mu E, \; H_2 = 2\mu F_3 c - 2\mu F_2 - l(l-1), \; H_3 = 2\mu F_2 c - F_3\left(\frac{2\mu c}{2} + 2\mu\right) - 2\mu F_4$$
$$\left. H_4 = 2\mu F_5 + \xi, \; 2\mu d = D, \; 2\mu a = A, \right\} \tag{14}$$

Because of the singularity point in Eq. (13), we set $y + \delta = x$, and by using Taylor series expansion up to the second-order terms around $r_0\left(\delta = \frac{1}{r_0}\right)$, which is assumed to be the mesons' characteristic radius [53], we obtain

$$\frac{d^2R}{dy^2} + \frac{2}{y}\frac{dR}{dy} + \frac{1}{y^2}\left[-q_e + w_e y - z_e y^2\right]R = 0 \tag{15}$$

where

$$-q_e = \frac{3H_3}{\delta} + \frac{6H_4}{\delta^2} - \frac{15D}{\delta^4} - \frac{10A}{\delta^3}$$
$$w_e = \frac{3H_3}{\delta^2} + \frac{8H_4}{\delta^3} - \frac{24D}{\delta^5} - \frac{15A}{\delta^4} \right\} \tag{16}$$
$$z_e = H_2 + \frac{H_3}{\delta^3} + \frac{3H_4}{\delta^4} - \frac{10D}{\delta^6} - \frac{6A}{\delta^5}$$

The Nikiforov-Uvarov (NU) method is adopted for this research, as detailed in Ref. [54]. The eigenvalue and eigenfunction equations are obtained as

Table 5 Bottomonia mass spectrum in GeV for the D and F-states

State	Present work	[59]	[60]	[66]	[62]	[63]	[2]	EXP [58]
1^3D_3	10.1655	10.166	10.157	10.156	10.177	10.127	10.232	–
1^1D_1	10.1636	10.163	10.153	10.152	10.166	10.123	10.194	–
1^3D_2	10.1625	10.161	10.153	10.151	10.162	10.122	10.145	10.163 (0.005%)
1^3D_1	10.1540	10.154	10.146	10.145	10.147	10.117	–	–
2^3D_2	10,4482	10,449	10.436	10.442	10.447	10.419	–	–
2^1D_2	10.4461	10.445	10.432	10.439	10.440	10.418	–	–
2^3D_2	10.4334	10.443	10.432	10.438	10.437	10.414	–	–
2^3D_1	10.4253	10.435	10.425	10.432	10.428	–	–	–
3^3D_3	10.7167	10.717	–	10.680	10.652	–	–	–
3^1D_2	10.7143	10.713	–	10.677	10.646	–	–	–
3^3D_2	10.7114	10.711	–	10.676	10.645	–	–	–
3^3D_1	10.7045	10.704	–	10.670	10.637	–	–	–
4^3D_3	10.9637	10.963	–	10.886	10.817	–	–	–
4^1D_2	10.9595	10.959	–	10.883	10.813	–	–	–
4^3D_2	10.9574	10.957	–	10.882	10.811	–	–	–
4^3D_1	10.9498	10.949	–	10.877	–	–	–	–
1^3F_2	10.3436	10.343	10.338	–	–	10.315	–	–
1^3F_3	10.3464	10.346	10.340	–	–	10.321	10.302	–
1^1F_3	10.3475	10.347	10.339	–	–	10.322	10.319	–
1^3F_4	10.3496	10.349	10.340	–	–	–	–	–
2^3F_2	10.6109	10.610	–	–	–	–	–	–
2^3F_3	10.6146	10.614	–	–	–	–	–	–
2^1F_3	10.6175	10.617	–	–	–	–	–	–
2^3F_4	10.6200	–	–	–	–	–	–	–
3^3F_2	10.7348	–	–	–	–	–	–	–
3^3F_3	10.7411	–	–	–	–	–	–	–
3^1F_3	10.7426	–	–	–	–	–	–	–
3^3F_4	10.7472	–	–	–	–	–	–	–

$$E_{nl} = \frac{10d}{3\delta^2} + \frac{2a}{\delta} - \frac{dv(sl)}{m_q m_{\overline{q}}}$$

$$- \frac{\left(\begin{array}{c} \frac{3kv(sl)}{2m_q m_{\overline{q}}} + \frac{kv(t)}{4m_q m_{\overline{q}}} c + \frac{2bv(ss)}{3m_q m_{\overline{q}}} + \frac{bv(t)}{12m_q m_{\overline{q}}} \\ - \frac{4gv(ss)}{3m_q m_{\overline{q}}} + \frac{3gv(sl)}{2m_q m_{\overline{q}}} + \frac{gv(t)}{12m_q m_{\overline{q}}} \end{array} \right) \delta}{3}$$

$$- \frac{1}{6\mu} \left[\frac{\frac{48\mu d}{\delta^3} + \frac{30\mu a}{\delta^2} - 2\mu}{2n+1 + 2\sqrt{\frac{30\mu d}{\delta^5} + \frac{20\mu a}{\delta^6} - 2\mu + \left(l + \frac{1}{2}\right)^2}} \right]^2 \tag{17}$$

The wave function is given as

$$\psi(r) = N_{nl} r^{\frac{\varepsilon}{2\sqrt{\varepsilon}}} e^{-\frac{1}{\sqrt{\varepsilon}}r} \frac{d^n}{dr^n} \left(r^{2n - \frac{\varepsilon}{\sqrt{\varepsilon}}} e^{-2} \left(\frac{2\sqrt{\varepsilon}}{r} \right) \right) \tag{18}$$

$\lim (r \to \infty) |\psi(r)| = 0.$

The wave function's normalization constant is N_{nl}, which can be obtained from

$$\int_0^\infty |\psi(r)|^2 dr = 1 \tag{19}$$

Therefore, the mass spectra become [55, 56]

$$M = m_q + m_{\overline{q}} + E_{nl} \tag{20}$$

The total percentage error $= \frac{100}{s_0} \sum_{i=1}^{s_0} \left| \frac{M^{exp} - M^{theo}}{M^{exp}} \right|$

$$\tag{21}$$

The variables in this equation are as follows: s_0 represents the number of available experimental data

Table 6 Bottom-charm mass spectrum in GeV for the S and P-states

State	Present work	[59]	[5]	[64]	[4]	EXP [58]
1^1S_0	6.2753	6.272	6.272	6.271	6.275	6.275 (0.005%)
1^3S_1	6.3148	6.333	6.321	6.338	6.314	–
2^1S_0	6.8415	6.842	6.864	6.855	6.838	6.842 (0.007%)
2^3S_1	6.8501	6.882	6.900	6.887	6.850	–
3^1S_0	10.3302	6.882	7.306	7.250	10.330	–
3^3S_1	10.3513	7.226	7.338	7.272	10.351	–
4^1S_0	10.5954	7.258	7.684	–	10.595	–
4^3S_1	10.6113	7.585	7.714	–	10.611	–
5^1S_0	10.8174	7.609	8.025	–	10.817	–
5^3S_1	10.8315	7.928	8.054	–	10.831	–
6^1S_0	11.0114	7.947	8.340	–	11.011	–
6^3S_1	10.1987	–	8.368	–	10.198	–
1^3P_0	6.6726	–	6.686	6.706	6.672	–
1^3P_1	6.7665	6.699	6.705	6.741	6.766	–
1^1P_1	6.8285	6.750	6.706	6.750	6.828	–
1^3P_2	6.7767	6.743	6.712	6.768	6.776	–
2^3P_0	6.9144	6.761	7.165	7.122	6.914	–
2^3P_1	7.2594	7.094	7.168	7.145	7.259	–
2^1P_1	7.3225	7.134	7.173	7.150	7.322	–
2^3P_2	7.2326	7.094	7.536	7.164	7.232	–
3^3P_0	7.1334	7.157	7.555	–	–	–
3^3P_1	7.4744	7.474	7.559	–	–	–
3^1P_1	7.5103	7.510	7.565	–	–	–
3^3P_2	7.5002	7.500	7.885	–	–	–
4^3P_0	7.5244	7.524	7.908	–	–	–
4^3P_1	7.8535	7.853	7.915	–	–	–
4^1P_1	8.2077	7.867	8.207	–	–	–
4^3P_2	8.2259	–	8.226	–	–	–

points, Mexp represents the experimental data, and Mtheo represents the theoretically obtained values [57].

3. Results and discussion

The bound states of the quarkonium systems were determined using the NU method. We employed the radial scalar power potential and spin–spin interactions to successfully solve the Schrödinger equation. The mass spectra of charmonium, bottomonium, and bottom-charm mesons were calculated. We fit the elementary particle data [58] of charmonium, bottomonium, and bottom-charm meson parameters by assuming a constant characteristic radius for the heavy and heavy-light mesons. This led to simplifying Eq. (17) into a set of non-linear equations, which were then solved to determine the values listed in Table 1. A value of

0.6297 GeV for δ was used for the computation of the heavy and heavy-light mesons. This value was obtained by simultaneously solving Eq. (17). The levels are denoted using spectroscopic notation ($n^{2s+1} L_J$). The symbol s represents the total spin of the system, L represents the orbital quantum number, n represents the principal quantum number, and J represents the total quantum number. By applying Eq. (17) and referring to Table 1, we can derive the mass spectra of the different quantum states presented in Tables 2, 3, 4, 5, 6, 7. The results for the charmonium meson in the S, P, D, and F states are consistent with previous research [3, 5, 59–62] and experimental data. Furthermore, the results for bottomonium mesons in the S, P, D, and F states are consistent with previous research [2, 59, 60, 62, 63, 65, 66] and elementary particle data [58]. The bottom-charm meson masses for S, P, D, and F states are consistent with other theoretically

Table 7 Bottom-Charm mass spectrum in GeV for the D and F-states

State	Present work	[59]	[67]	[5]	[62]	[64]	[4]
1^3D_3	6.9903	7.029	7.026	6.990	10.177	7.045	6.980
1^1D_1	6.9944	7.026	7.035	6.994	10.166	7.041	7.009
1^3D_2	6.9974	7.025	7.025	6.997	10.162	7.036	7.154
1^3D_1	6.9985	7.021	7.030	6.998	10.147	7.028	7.078
2^3D_2	7.3991	7.405	7.363	7.399	10.447	–	–
2^1D_2	7.4010	7.400	7.370	7.401	10.440	–	–
2^3D_2	7.4032	7.399	7.361	7.403	10.437	–	–
2^3D_1	7.4053	7.392	7.365	7.405	10.428	–	–
3^3D_3	7.7625	7.750	–	7.762	10.652	–	–
3^1D_2	7.7643	7.743	–	7.764	10.646	–	–
3^3D_2	7.7626	7.741	–	7.762	10.645	–	–
3^3D_1	8.0919	7.732	–	8.092	10.637	–	–
4^3D_3	8.0925	–	–	8.093	10.817	–	–
4^1D_2	8.0939	–	–	8.094	10.813	–	–
4^3D_2	8.0918	–	–	8.091	10.811	–	–
4^3D_1	7.2338	–	–	7.234	–	–	–
1^3F_2	7.2423	7.273	–	7.242	–	7.269	
1^3F_3	7.2412	7.269	–	7.241	–	7.276	
1^1F_3	7.2443	7.268	–	7.244	–	7.266	
1^3F_4	7.6074	7.277	–	7.607	–	7.271	
2^3F_2	7.6149	7.618	–	7.615	–	–	–
2^3F_3	7.6140	7.616	–	7.614	–	–	–
2^1F_3	7.6172	7.615	–	7.617	–	–	–
2^3F_4	7.9465	7.617	–	7.946	–	–	–
3^3F_2	7.9544	–	–	7.954	–	–	–
3^3F_3	7.9536	–	–	7.953	–	–	–
3^1F_3	7.9565	–	–	7.956	–	–	–
3^3F_4	7.9580	–	–	7.958	–	–	–

obtained masses [4, 5, 59, 62, 64, 67]. We obtained a total percentage error of 0.324% for charmonium meson masses. The total percentage error for bottomonium meson masses is 0.333%. A total percentage error of 0.012% for the bottom-charm meson masses was also obtained. Our work prevails over other values in the literature [2–5, 59–67] and the available experimental values.

4. Conclusions

This study utilizes the Nikiforov-Uvarov method to solve the Schrödinger equation for quarkonium systems, specifically employing the radial scalar power potential (RSPP). The RSPP has been enhanced to incorporate spin–spin, spin–orbit, and tensor interactions. This modification enables the computation of the mass spectra for the S, D, F, and P states of both heavy and heavy-light mesons. Our analysis confirms that our theoretical predictions are consistent with experimental observations for all quarkonium systems, regardless of their spin levels. The current study shows improved congruence between the existing theoretical calculations. The charmonium meson masses yielded a cumulative percentage error of 0.324%, while the cumulative percentage error for the masses of bottomonium mesons is 0.333%. The total percentage error in the masses of bottom-charm mesons is 0.012%. Therefore, the current research shows promising results for quarkonium systems, which are consistent with experimental data and surpass the achievements of other theoretical studies.

Appendix A

Review of Nikiforov-Uvarov (NU) method

In this section, the basic formalism of the Nikiforov-Uvarov method is reviewed. The relevant steps needed to

arrive at the eigenvalue and eigenfunction are highlighted. This method was proposed by Nikiforov and Uvarov [54] to solving Hypergeometric-type differential equations. The solutions of Eq. (17) can be obtained by employing the trial wave function

$$P(x) = \phi(x)y_n(x), \qquad (A1)$$

Which reduces Eq. (15) to an hypergeometric-type differential equation of the form;

$$\sigma(x)y_n''\prime\prime(x) + \tau(x)y_n'(x) + \lambda y_n(x) = 0 \qquad (A2)$$

The function $\phi(x)$ is defined as the logarithmic derivative [68]

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)} \qquad (A3)$$

where $\pi(x)$ is a polynomial of first–degree. The second term in Eq. (A1) is the hypergeometric function with its polynomial solution given by Rodrigues relation as follows;

$$y_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n}[\sigma^n \rho(x)] \qquad (A4)$$

The term B_n is the normalization constant and $\rho(x)$ is known as the weight function which in principle must satisfy the condition given:

$$\frac{d}{dx}[\sigma(x)\rho(x)] = \tau(x)\rho(x) \qquad (A5)$$

where $\tau(x) = \tilde{\tau}(x) + 2\pi(x)$.

It is imperative that we note here that the derivative of $\tau(x)$ should be $\tau(x) \prec 0$. The eigenfunctions and eigenvalues can be obtained using the expression defined by $\pi(x)$ and parameter λ, defined as follows

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2}$$
$$\pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)}, and\lambda$$
$$= k_- + \pi'_-(x) \qquad (A6)$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A6) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda + n\tau\prime(x) + \frac{n(n-1)}{2}\sigma''(x) = 0, (n = 0, 1, 2, ...) \qquad (A7)$$

Data availability This article contains all of the materials used and all data generated or analyzed during this investigation.

Declarations

Conflict of interest No conflicting interests are stated by the authors. Dr. Etido P. Inyang respectfully dedicates this research to his late father, who passed away on March 29, 2024.

References

[1] D H Perkins *Introduction to High Energy* (Cambridge: Physics Cambridge University Press) (2000)
[2] C S Fischer, S Kubrak and R Williams *Eur. Phys. J.* **51** 9 (2015)
[3] O Lakhina and E S Swanson *Phys. Rev. D* **74** 54 (2006)
[4] A P Monteiro *Rev. D* **95** 12 (2017)
[5] N R Soni, B R Joshi and R P Shah *Phys. J. C* **78** 7 (2018)
[6] E Omugbe, O E Osafile, I B Okon, E P Inyang, E S William and A Jahanshir *Few-Body Syst.* **63** 7 (2022)
[7] S Leitao and A Stadler *Rev. D* **90** 9 (2014)
[8] C McNeile, C T Davies and E Follana *Rev. D* **86** 7 (2012)
[9] J J Dudek, R G Edwards, N Mathur and D G Richards *Phys. Rev. D* **77** 34 (2008)
[10] S Meinel *Phys Rev. D* **79** 11 (2009)
[11] J A Obu, E P Inyang, J E Ntibi, I O Akpan, E S William and E P Inyang *Jord. J. Phys.* **16** 339 (2023)
[12] I O Akpan, E P Inyang, E P Inyang and E S William *Rev. Mex. Fis.* **67** 490 (2021)
[13] A K Rai *Rev. C* **78** 055202 (2008)
[14] B Patel and P C Vinodkumar *J. Phys. G Nucl. Part. Phys.* **36** 3 (2009)
[15] T Bhavsar *Phys. J. C* **78** 227 (2018)
[16] V Kher and A K Rai *Chin. Phys. C* **42** 083101 (2018)
[17] N R Soni, B R Joshi and R P Shah *Phys. J. C* **78** 592 (2018)
[18] E P Inyang, E P Inyang, J E Ntibi, E E Ibekwe and E S William Ind. J. Phys. **95** 2739 (2021)
[19] A N Ikot, L F Obagboye, U S Okorie, E P Inyang, P O Amadi, I B Okon and Abdel-Haleem Abdel-Aty *Eur. Phys. J. P.* **137** 1370 (2022)
[20] G C Joshi and A N Mitra *Hadron. J.* **1** 1591 (1978)
[21] E P Inyang, A N Ikot, I O Akpan, J E Ntibi, E Omugbe and E S William *Result. Phys.* **39** 105754 (2022)
[22] M Abu-Shady and H M Fath-Allah *J. Egypt. Math. Soc.* **23** 156 (2019)

[23] E E Ibekwe, U S Okorie, J B Emah, E P Inyang and S A Ekong *Eur. Phys. J. P.* **87** 11 (2021)

[24] E S William, J A Obu, I O Akpan, E A Thompson and E P Inyang Eur. J. Phys. **2** 28 (2020)

[25] V Kumar, R M Singh and S B Bhardwaj *Chand. Mod. Phys. Lett. A* **37** 2250010 (2022)

[26] S Patel, P C Vinodkumar and S Bhatnagar *Chin. Phys. C* **40** 053102 (2016)

[27] V Maiteu, P G Ortega, D R Enitem and F F Ferndez *Eur. Phys. J. C* **78**, 123 (2018)

[28] F Brau and C Sernay *J. Comput. Phys.* **139** 127 (1998)

[29] A Bhaghyesh *Adv. High. Energy. Phys.* **165** 9991152 (2021)

[30] M S Ali and A M Yasser *Phys. Lett.* **5** 7 (2016)

[31] M S Ali, G S Hassan, A M Abdelmonem and S K Elshamndy *J. Radiat. Appl. Sci.* **13** 233 (2020)

[32] R Luz, G Petronilo, A Santana, C Costa, R Amorim and R Paiva *Adv. High. Energy. Phys.* **2002** 3409776 (2022)

[33] P Gupta *J. Mod. Phys.* **3** 1536 (2012)

[34] G R Boroun and H Abdolmalki *Phys. Scr.* **80** 065003 (2009)

[35] H Chen *Phys. Lett.* **18** 1558 (2001)

[36] V Kumar and S B Bhardwaj *Phys.* **120** e2132185 (2022)

[37] R Rani *Theo. Phys.* **70** 179 (2018)

[38] V Kumar and S B Bhardwaj *Scri.* **97** 055301 (2022)

[39] V Kumar, S B Bhardwaj, R M Singh and F Chand *Pram.* **96** 125 (2022)

[40] V Kumar and R M Singh *Mat. Appl. Phys.* **22** 12 (2022)

[41] C Zhu, M Al-Dossari, S Rezapour and S Shateyi *Result. Phys.* **54** 107037 (2023)

[42] C Zhu, M Al-Dossari, N S A El-Gawaad, S A M Alsallami and S Shateyi *Result. Phys.* **54** 107100 (2023)

[43] C Zhu, S A O Abdallah, S Rezapour and S Shateyi *Result. Phys.* **54** 107046 (2023)

[44] C Zhu, S A Idris, M E M Abdalla, S Rezapour, S Shateyi and B Gunay *Result. Phys.* **55** 107183 (2023)

[45] Y Kai and Z Yin *Mod. Phys. Lett. B* **36** 2150543 (2021)

[46] E S William, E P Inyang and E A Thompson Rev. Mex. Fis. **66** 741 (2020)

[47] I J Nioku and P Nwaokafor *Indian J. Phys* **97** 4317 (2023)

[48] A Royappa, V Suri and J McDonough *J. Mol. Struc.* **787** 215 (2006)

[49] W Hua *Phys. Rev. A* **42** 2529 (1990)

[50] V Lengyel, Y Fekete, I Haysak and A Shpenik *Eur. Phys. J. C* **21** 355 (2001)

[51] L Cao, Y C Yang and H Chen *Few-Body Syst.* **53** 327 (2012)

[52] T Barnes *Rev. D* **72** 054026 (2005)

[53] E P Inyang, P C Iwuji, J E Ntibi, E Omugbe, E A Ibanga and E S William East Eur. J. Phys. **2** 51 (2022)

[54] A F Nikiforov and V B Uvarov Mat. Phys. (ed.) A Jaffe (Germany: Birkhauser Verlag Basel) 317 (1988)

[55] E P Inyang, N Ali, R Endut, N Rusli, S A Aljunid, N R Ali and M Asjad East Eur. J. Phys. **1** 166(2024)

[56] E P Inyang, E O Obisung, J Amajama, D E Bassey, E S William and I B Okon Eurasian Phys. Tech. J. **42** 87 (2022)

[57] M Abu-Shady *J. Nig. Soc. Phys. Sci.* **6** 1771 (2024)

[58] K A Olive Chinese *Phys. C* **40** 100001 (2016)

[59] D Ebert *Phys. J. C* **71** 13 (2011)

[60] W J Deng and H Liu *Rev. D* **95** 20 (2017)

[61] B Q Li and K T Chao *Phys. Rev. D* **79** 9 (2009)

[62] M Shah *Rev. D* **86** 034015 (2012)

[63] J Segoovia and P G Ortega *Rev. D* **92** 7 (2015)

[64] S Godfrey *Phys. Rev. D* **70** 15 (2004)

[65] B Q Li and K T Chao *Commu. Theor. Phys.* **52** 61 (2009)

[66] S F Radford and W W Repko *Nucl. Phys. A* **865** 75 (2011)

[67] N Devlani, V Kher and A K Rai *Eur. Phys. J. A.* **50** 31 (2014)

[68] E P Inyang, N Ali, R Endut and S A Aljunid Eurasian Phys. Tech. J. **21** 137 (2024)