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Axially symmetric relativistic structures and the Riemann curvature tensor

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Abstract: Despite the existence of structure scalars in different geometries and modified theories, their importance of static and axially symmetric systems for *f*(*T*) gravity (where *T* is responsible for torsional effects) is still questionable. The novel approach to comprehending the role of structure scalars on static axial symmetric systems in the presence of *f*(*T*) gravity is performed in this manuscript. An extensive structure for static and axially symmetric geometry is presented. The line element with compatible anisotropic fluid (which serves as the origin for exterior Weyl spacetime) is contemplated. We initiate by exploring *f*(*T*) field equations with the support of non-diagonal tetrads for static axial symmetry. The structure scalars are determined in our scenario. We attain eight distinct sorts of scalars, which are trace and trace-free parts. The three distinct scalars $\mathbb{Y}_{TF_1}$, $\mathbb{Y}_{TF_2}$ and $\mathbb{Y}_{TF_3}$ with *f*(*T*) corrections are responsible for the complexity of the system; whereas, the inhomogeneity of the system is controlled by the three other scalars $\mathbb{X}_{TF_1}$, $\mathbb{X}_{TF_2}$ and $\mathbb{X}_{TF_3}$ with *f*(*T*) corrections. We assemble the hydrostatic equilibrium equations in terms of *f*(*T*) gravity and construct two conformal equations with the support of these equations. In the end, we computed a few analytical solutions in the frame of *f*(*T*) gravity. One of them is about the dense spheroid comprising isotropic pressure, and the other is about anisotropic fluid content dealing with inhomogeneity. In the first case, our findings indicate that joining with the Weyl exterior geometry is not possible. On the other hand, the solution associated with anisotropic fluids has smooth joining with the Weyl exterior geometry.

Keywords: Self-gravitating systems; Relativistic structure; Spacetime topology

1. Introduction

The role of structure scalars in comprehending the structure and evolution of celestial objects is the essence of our work. The idea is based on the concept of the orthogonal breakdown of the Riemann curvature tensor (RCrT). Herrera et al. [1] evaluated these structure scalars for the first time by utilizing the technique of orthogonal breakdown of RCrT in a scenario of general relativity (GR). The anisotropic fluid equipped with dissipation is taken for the symmetric sphere. The accomplished scalars are five in number, they are trace and trace-free constituents of the tensors $\mathbb{X}_{\omega\upsilon}$, $\mathbb{Y}_{\omega\upsilon}$ and $\mathbb{Z}_{\omega\upsilon}$. The impact of these scalars on the physical attributes of fluid distribution has the utmost importance. The contribution of each scalar in understanding the features of fluid components is studied by Herrera et al. [2] in the presence of electric charge. The scalar $\mathbb{Z}$ is associated with all sorts of dissipative fluxes. The energy density and inhomogeneities within them are analyzed with the aid of scalar $\mathbb{X}_T$ and $\mathbb{X}_{TF}$, respectively. The impact of density inhomogeneity and anisotropy on the Tolman mass is investigated by the factor $\mathbb{Y}_{TF}$. The relation with Tolman mass density is proportional to the factor $\mathbb{Y}_T$. Herrera [3] enlarged the idea and entitled the name of complexity factor (C_F) to the one of the scalar, i.e., $\mathbb{Y}_{TF}$. He assumed the static sphere as geometry endowed with anisotropic fluid content. The idea of diminishing C_F is established and the interior solutions for this case are evaluated in the scenario of GR. The scheme of C_F for non-static anisotropic sphere beholding dissipation is presented by Herrera et al. [4]. They marked the homologous constraint as the simplest way of evolution. The role of diminishing C_F together with cases of dissipation and non-

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dissipation are explained. In the end, the stability analysis for $\mathbb{Y}_{TF} = 0$ is performed. Ospino and Núñez [5] studied the Karmarkar constraint in the form of structure scalars attained through Herrera's strategy. The outcome expression is the algebraic relation associated with physical parameters. The approach to attain the static solution for spheres of embedding class I is implemented with the aid of the Karmarkar scalar constraint. The dynamic Karmarkar collapse in case of dissipation to found a new class of solutions is examined at the end.

GR is the most mathematically alluring, firmly established, and suitably composed theory. The precise geometrical explanation along with valid predictions provide sufficient ground to nominate it as a fruitful theory. Although it is a remarkable achievement, GR goes short in attempts to comprehend the most latest cosmological facts. More precisely, it is unable to forecast a late time rapid growth and does not adequately characterize the cosmic growth in the early universe. As a consequence, several modified theories of gravity (MTsG) have been recommended [6]. Out of many examples, one of the most appealing, determined, and in essence of geometrization of gravity is to universalize the affine connection. To implement the idea, one has to deal with a non-Riemannian geometric structure [7], where the main features are because of torsion, curvature, and the factor of non-metricity. Limiting the preceding non-Riemann geometric structure in a specific way, one attains distinct formalisms of gravity. As an illustration, the diminishing torsion and factor of non-metricity result in notable metric theories, in which GR is treated as a particular case. On the other side, the diminishing curvature and the factor of non-metricity turn out into standard teleparallel gravity (TG) [8]. The symmetric TG [9] is attained when the curvature and torsion are kept zero and leave the factor of non-metricity unchanged. In this manuscript, our focus is on the extension of TG.

The TG name refers to the standard three-parameter theory. While the TG equivalent to GR can be accepted as a gauge theory for the translation group which is grounded on Weitzenböck geometry. In TG, the force similar to Lorentz force is responsible for gravitational interactions, where the curvature has a diminishing nature and the torsion is performing the role of force [10]. In TG, the elemental entity is the non-trivial tetrad field $\mathfrak{h}^{a}{}_{\omega}$, where the elemental entity in GR is the metric tensor. Although both these theories have differences at fundamental points, these theories supplied the equivalent explanation of gravitational interactions [11]. This demonstrates that torsion and curvature may be merely two distinct methods of presenting the gravitational field, and it additionally endorses the idea that the energy-momentum (EM) tensor is an ingredient in both gravitational theories, serving as the cause of both torsion and curvature in TG and GR, respectively. It is significant to highlight that the tetrad framework offers numerous benefits, most notably its absence of dependency on the equivalence principle and the resulting validity in investigating quantum issues. Although in the normal framework of GR, the EM density for the gravitational field is continually portrayed as a pseudo tensor, TG appears to offer a more suitable setting to address the energy issue. The claimed $f(T)$ theories, for which T corresponds to a torsion scalar, are an extension of TG equivalent to GR. The characteristics of $f(T)$ theories are appropriately addressed in [12].

The $f(T)$ field equations correspond to second-order, and the $f(T)$ gravity incorporates intriguing cosmological solutions that offer several explanations for the universe's expanding phases. In this instance of $f(T)$ theory, Atazadeh and Mousavi [13] set up novel vacuum solutions and executed extra constraints on constituents of the spherically symmetric metric. They abolished metric components to solve the analytical design of the $f(T)$ theory. DeBenedictis and Ilijić [14] determined the vacuum solutions for $f(T)$ model in a covariant framework by employing spherical symmetry. They performed the perturbative examination for $f(T) = T + \frac{\alpha}{2}T^2$ model and found the perturbative gravitational impacts in this way. Bahamonde et al. [15] explored the perihelion shift and the photon sphere in the background of weak $f(T)$ gravity. They implied the perturbations up to an order of one to the two distinct geometric structures, i.e., Minkowski and Schwarzschild. The observations were analyzed and compared with previous results as found in the literature. Bhatti et al. [16, 17] investigated the notion of stability for various kinds of $f(T)$ theory in the context of non-static plane and cylindrical geometrical structure in detail. They identified the stable and unstable sections in these properties using the concept of perturbation. They understood the value of the adiabatic index in identifying unstable zones as a consequence. Nashed [18] found the $f(T)$ solutions for AdS static black holes in terms of charged and uncharged. They employed the power law model which has great significance in analysis as per observations. The solutions have a dependence on the modification terms along with an electric charge in the presence or absence of a cosmological constant. They are novel solutions as they do not reduce GR limitation.

Rotations about the reflection terms, symmetry axis, and meridional movements are all part of a line element that describes an axially symmetric arrangement. By accepting that the source is stationary and omitting the non-diagonal expressions, the structure is made less complicated. The Weyl metrics are employed in GR to address the field equations for a static, axially symmetric distribution. The Schwarzschild solution is thought by most to be the only static, asymptotically flat, and vacuum solution comprising a regular horizon. The physical constituents of RCrT contain singularity for every Weyl solution except Zipoy & Voorhees solution [19]. A rational query is: Which precise vacuum solution most accurately reflects the change from a spherical to a non-spherical static, axially symmetric distribution? The query does not provide a unique remedy. Spherical symmetry is essential in the investigation of self-gravitating bodies and consequently provides insight into how neutron stars, white dwarfs, and black holes originate. To obtain precise solutions, it is necessary to depart from spherical to non-spherical geometries in the case of powerful gravitational forces. It is generally accepted that distinct spacetimes can meet the field equations for various substantially significant EM tensors.

Nayak [20] explored the relationship between Einstein's equations and the inertial forces in the case of axially symmetric geometry admitting the feature of being stationary. He utilized the Geroch formulation to explain the Einstein equations in the form of inertial forces for vacuum cases. The formulation proposed by Hansen and Winicour is used to develop the relation in the case of a perfect fluid. He concluded that the field equations are misrepresented by the gradients of inertial forces. Dain [21] studied the geometric inequalities in the case of an axially symmetric black hole. These inequalities portrayed a significant role in the phenomena of gravitational collapse and the angular momentum (quasi-local) is properly defined in this way. He inspected the recent findings in this context and provided the main concepts of proof. In the end, he provided a complete list of pending issues. Vollmer [22] explored the space of the Killing vector for the class of Weyl spacetime (Which is a sub-class of static and axially symmetric spacetime) for the field of valence three. He exhibited the importance of the Killing vector and found that the Killing tensor field containing valence three is produced by Killing vector fields in the case of axially symmetric vacuum geometry. Hernandez-Pastora et al. [23] presented a general way to determine the interior solutions to Einstein field equations having axial symmetry. They deduced that the resulting solutions of all kinds meet the energy conditions and can be linked smoothly to exterior Weyl family solutions. They found that the solutions meet the incompressible fluids for delimiting the case of spherical symmetry. Ospino et al. [24] provided a way to determine all geodesics in the case of axially symmetric geometric structure. They found the Schwarzschild geodesic as a delimiting case of Kerr metric. The universal Killing tensor along with its related constant of motion is calculated in this scenario. They deduced that these results are impactful in comprehending the nature of celestial objects.

Many authors have worked on the significance of axial symmetry in the framework of TG and extended TG. Nashed [25] attainted the definite charged solution using axial symmetry with structure function of form $\mathbb{G}(\xi) = 1 - \xi^2 - 2Am\xi^3 - A^2q^2\xi^4$ in frame of TG equivalent of GR. By utilizing coordinate transformation, a new kind of tetrad field is acquired, and the related metric results in Reissner–Nordström spacetime. Then he investigated the singularities of spacetime and obtained the compatible value of energy by taking the gravitational EM. Nashed [26] employed a universal tetrad field comprising sixteen indefinite functions to $f(T)$ theory and determined an analytical solution for vacuum case along with two distinct integrating constants. He deduced that the resulting field has axial symmetry and diminishing torsion scalar. He found the associated metric as a Kerr geometry and explained that in terms of two distinct Lorentz transformations acting at the local level. Bahamonde et al. [27] utilized one of the families of extended TG to determine solutions of axial spacetimes. The field equations of theory can be distributed into two components, one is symmetric and the other is antisymmetric. Specifically, they found the solutions related to the universality of the Taub-NUT metric along with the Kerr metric. Until now, no structure scalars for axial symmetry have been obtained in this theory employing the strategy of orthogonal division of the RCrT. The subject of the current investigation is to analyze the structure scalars for axial symmetry in $f(T)$ gravity.

A class of Fuchsian equations is studied by Beyer and LeFloch [28]. They are related to the evolution of compressible fluids in cosmological spacetime. They presented a numerical technique for the singular initial value problem employing the approach of lines. They used a variety of numerical approaches, including Runge–Kutta and pseudo-spectral, to approximate the Fuchsian kind of situation. Their main idea is a thorough examination of the numerical error, which originates from two different places. The key suggestion is to equalize the errors that occur at the discrete and continuous levels of approximation. They offered some tests that adequately validated the theoretical findings. In the end, compressible fluid flows propagating on a Kasner spacetime are considered using this technique. In the so-called subcritical domain, they provided numerical evidence of the nonlinear stability of such flows.

Cao et al. [29] studied the evolution of an isothermal compressible fluid on a future-expanding or future-contracting cosmic background. They handled the two nonlinear hyperbolic balancing laws that make up the Euler equations controlling such a flow in both one and two spatial dimensions. They created a finite volume approach that has second-order spatial accuracy and fourth-order temporal accuracy. This approach is adequate by design, preserving exactly a particular group of solutions, and it enables them to determine weak solutions incorporating shock waves. To validate their observations, detailed numerical experiments are conducted in both one and two spatial dimensions. Weinberger and Hernquist [30] described a novel technique in this investigation for modeling multi-phase gas states in cosmic hydrodynamic simulations. They used the moving-mesh finite volume approach to deal with the compressible two-fluid hydrodynamic equations. Next, they defined operator-split source expressions for the interchange of mass, energy, and momentum across the phases. Their application allows for the handling of both settled and unsettled multi-phase fluids since it can sustain volume fraction irregularities in pressure equilibrium to machine accuracy employing a stratified flow framework. They employed the source and sink components from an interstellar media two-phase design that was already in existence. Comparing it to its successful equation of state execution from the point where they illustrated the significance of this kind of method in simulations of galaxy development. They remarked on how it would be beneficial in the next, extensive simulations of galaxy development.

Our focus is to determine the structure scalars in the case of *f*(*T*) theory for axially symmetric anisotropic fluid configuration. For which the approach of splitting of RCrT is implemented. The solutions are identified without applying any kind of numerical technique. The role of incompressible fluid in this way has great significance. We studied their role and discussed its significance in the frame of *f*(*T*) gravity. All these authors made great contributions to resolving cosmological issues by using numerical simulations in the case of compressible fluids.

2. Axially symmetric source

The axially symmetric (stationary in nature) spacetimes feature the Weyl metrics as a subclass. By eliminating the rotational velocity, one can turn down these metrics from the Lewis–Papapetrou metric [19]. In GR, the Weyl exterior solutions to the field equations refer to all feasible static, axially symmetric geometric structures. They can be portrayed as sequential expansions of adequately stated relativistic multipole events [31]. As an outcome, a particular arrangement of these multipoles characterizes a unique Weyl metric. For this group of spacetimes, it would be intriguing to look at the structure scalars in the frame of the *f*(*T*) theory. It is represented in spherical coordinates (t, r, θ, ϕ) as a line element of the form

$$ds^2 = -I^2(r,\theta)dt^2 + K^2(r,\theta)(dr^2 + r^2 d\theta^2) + L^2(r,\theta)d\phi^2. \tag{1}$$

The EM tensor in our scenario has the representation of the form

$$T^{(m)}_{\omega\upsilon} = (\rho + p)u_\omega u_\upsilon + p g_{\omega\upsilon} + \Pi_{\omega\upsilon}, \tag{2}$$

where each quantity in Eq. (2) has its own significance in the EM tensor. The quantity p is assigned to the anisotropic pressure, $g_{\omega\upsilon}$ is a mathematical notation for metric tensor, the anisotropic tensor is denoted by $\Pi_{\omega\upsilon}$, u_ω is the four-velocity and density is designated by ρ. In a co-moving frame, $u_\omega = (-I, 0, 0, 0)$, $k = (0, K, 0, 0)$ and $l = (0, 0, Kr, 0)$. These variables fulfill the constraints

$$\begin{aligned}
\Pi_{\omega\upsilon} &= 2p_{xy}k_{(\omega}l_{\upsilon)} + (p_{yy} - p_{zz})(l_\omega l_\upsilon - \frac{h_{\omega\upsilon}}{3}) \\
&\quad + (p_{xx} - p_{zz})(k_\omega k_\upsilon - \frac{h_{\omega\upsilon}}{3}), \\
h_{\omega\upsilon} &= g_{\omega\upsilon} + u_\omega u_\upsilon, \\
p &= \frac{p_{xx} + p_{yy} + p_{zz}}{3},
\end{aligned}$$

which help design *f*(*T*) field equations along with structure scalars in our situation.

3. Formalism of *f*(*T*) field equations with non-diagonal tetrad

TG was an effort by Einstein to merge the idea of gravity and electromagnetism. Concerning GR, the concept of TG is initiated with a metric based on the manifold $\mathcal{M}$. Later on, the connection is selected in the way to define the concept of parallel transport. This connection must be the Lorentz connection and it is supposed to contain the vanishing curvature and non-diminishing torsion. To infer the idea, the tetrad field $\mathfrak{h}_a^{\ \omega}$ is chosen, which behaves as a set of orthonormal bases at every point of $\mathcal{M}$. The chosen connection must fulfill the concept of absolute parallelism, i.e., the constraint of vanishing of the covariant derivative of $\mathfrak{h}_a^{\ \omega}$. In this case, this possibility holds for the Weitzenböck connection. The theory of TG is the one that employs tetrads, which are the essential assembling elements that lead to the Weitzenböck connection. Let us initiate by revising the elemental concepts of TG [32]. The Greek indices, $\omega, \upsilon, ... = 0, 1, 2, 3$ are employed for spacetime manifolds; while, the Latin ones $a, b = 0, 1, 2, 3$ are connected to tangent space. The constituents of the tetrad field associated with an assigned metric with $g_{\omega\upsilon}$ can be computed by $g_{\omega\upsilon} = \zeta_{ab}\mathfrak{h}^a_{\ \omega}\mathfrak{h}^b_{\ \upsilon}$, where $\zeta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The Weitzenböck connection, which meets the requirement for absolute parallelism, serves as the TG connection

$$\nabla_\upsilon \mathfrak{h}^a{}_\omega = \partial_\upsilon \mathfrak{h}^a{}_\omega - \bar{\Gamma}^\pi{}_{\omega\upsilon}\mathfrak{h}^a{}_\pi,$$

that result into

$$\bar{\Gamma}^\pi{}_{\omega\upsilon} = \mathfrak{h}_a{}^\pi \partial_\upsilon \mathfrak{h}^a{}_\omega.$$

In addition, the torsion tensor is specified by

$$T^\pi{}_{\omega\upsilon} = \bar{\Gamma}^\pi{}_{\upsilon\omega} - \bar{\Gamma}^\pi{}_{\omega\upsilon} = \mathfrak{h}_a{}^\pi(\partial_\upsilon \mathfrak{h}^a{}_\omega - \partial_\omega \mathfrak{h}^a{}_\upsilon).$$

It has an antisymmetric structure in the two indices ω and υ. The action for $f(T)$ theory is specified as

$$S_{f(T)} = \int d^4x \left(\mathcal{L}_M + \frac{f(T)}{2\kappa^2}\right)|\mathfrak{h}|, \tag{3}$$

Here, $\mathcal{L}_M$ is representation for the matter Lagrangian and κ is symbolized for coupling constant. The mathematical expression $\mathfrak{h}^\omega_\upsilon$ is the dynamical entity of theory and it is defined by $|\mathfrak{h}| = det(\mathfrak{h}^\omega_\upsilon)$. The contortion tensor distinguishes the Levi-Civita and Weitzenböck connections from one another as

$$K^{\omega\upsilon}_\pi = -\frac{1}{2}\left(T^{\omega\upsilon}_\pi + T^{\upsilon\omega}_\pi - T_\pi{}^{\omega\upsilon}\right),$$

and it assists in defining the super-potential as

$$S_\pi{}^{\omega\upsilon} = \frac{\upsilon^\omega_\pi}{2}T^{\beta\upsilon}{}_\beta - \frac{\upsilon^\upsilon{}_\pi}{2}T^{\beta\omega}{}_\beta + \frac{1}{4}\left(T^{\omega\upsilon}_\pi + T^{\upsilon\omega}_\pi - T_\pi{}^{\omega\upsilon}\right).$$

The aforementioned values permit one to set out the $f(T)$ Lagrangian density, a quantity that's nothing more than the torsion scalar, as

$$T = S_\pi{}^{\omega\upsilon} T^\pi{}_{\omega\upsilon}.$$

The field equations are generated by modifying the action Eq. (3) in relation to the tetrad as

$$\begin{aligned}&\mathfrak{h}_a{}^\pi S_\pi{}^{\omega\upsilon}\partial_\omega T f_{TT} + \frac{f}{4}\mathfrak{h}_a{}^\upsilon + \frac{f_T}{\mathfrak{h}}\partial_\omega(\mathfrak{h}\mathfrak{h}_a{}^\pi S_\pi{}^{\omega\upsilon}) \\ &\quad + \mathfrak{h}_a{}^\pi T^\alpha{}_{\omega\pi} S_\alpha{}^{\upsilon\omega} f_T = \frac{\kappa^2}{2}\mathfrak{h}_a{}^\pi T_\pi^{\upsilon(m)}.\end{aligned} \tag{4}$$

Since the sole difference between the torsion scalar T and the Ricci scalar R is a total derivative, so in the covariant approach we found the interaction of the nature

$$\Upsilon_{\omega\upsilon} f_{TT} - \frac{T}{2}\left(f_T - \frac{f}{T}\right)g_{\omega\upsilon} + G_{\omega\upsilon} f_T = \kappa^2 T^{(m)}_{\omega\upsilon},$$

where $f_T \equiv \frac{\partial f}{\partial T}$, $f_{TT} \equiv \frac{\partial^2 f}{\partial T^2}$ and $\Upsilon_{\omega\upsilon} = S_{\omega\upsilon}{}^\pi \nabla_\pi T$. The geometric form of the Einstein tensor is indicated as $G_{\omega\upsilon}$. The theory's corrections terms are following to the reassembling of Eq. (4), which appears as

$$G_{\omega\upsilon} = \frac{\kappa^2}{f_T}\left(T^{(T)}_{\omega\upsilon} + T^{(m)}_{\omega\upsilon}\right), \tag{5}$$

and

$$T^{(T)}_{\omega\upsilon} = -\frac{1}{\kappa^2}\left\{\Upsilon_{\omega\upsilon} f_{TT} - \frac{1}{4}\left(\Upsilon f_{TT} - R f_T - T\right)g_{\omega\upsilon}\right\}. \tag{6}$$

The $f(T) = T$ constraint gives rise to TG equivalent to GR equations. The emergence of relativistic stars in the frame of $f(T)$ gravity is discussed by Böhmer et al. [33] by utilizing two sorts of tetrads, one is diagonal and the other is non-diagonal. They build up numerous groups of static solutions in this context. The idea of "good tetrad" was initially employed by Tamanini and Bohmer in [34], which relates to a tetrad that is free of added constraints on the functional version of $f(T)$, thereby making it conceivable to look into a very wide group of $f(T)$ cosmology. The decision to go with non-diagonal tetrad has been suggested by researchers, and numerous features of spherically symmetric geometry have been discussed (for an overview, we encourage the reader to study the existing literature, e.g., [35, 36]). The non-diagonal tetrad matrix of the subsequent form will be utilized in the present investigation to derive $f(T)$ field equations [33]

$$\mathfrak{h}^a{}_\omega = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & K\sin\theta\cos\phi & K\cos\theta\cos\phi & -K\sin\theta sin\phi \\ 0 & Kr\sin\theta\sin\phi & Kr\cos\theta\sin\phi & Kr\sin\theta\cos\phi \\ 0 & L\cos\theta & -L\sin\theta & 0 \end{pmatrix}. \tag{7}$$

The inverse of the aforementioned matrix is as follows

$$\mathfrak{h}_a{}^\omega = \begin{pmatrix} \frac{1}{I} & 0 & 0 & 0 \\ 0 & \frac{\sin\theta\cos\phi}{K} & \frac{\sin\theta\sin\phi}{Kr} & \frac{\cos\theta}{L} \\ 0 & \frac{\cos\theta\cos\phi}{K} & \frac{\cos\theta\sin\phi}{Kr} & \frac{-\sin\theta}{L} \\ 0 & \frac{-\sin\theta\sin\phi}{K} & \frac{\sin\theta\cos\phi}{Kr} & 0 \end{pmatrix}. \tag{8}$$

As an outcome, the $f(T)$ field equations listed in Eq. (5) have non-vanishing constituents if the values of tetrad from Eqs. (7 and 8) adjoining the values of torsion, contorsion, and super-potential tensors, we get

$$\begin{aligned}G_{00} = \frac{8\pi}{f_T}\Bigg[\rho - \frac{I^2}{8\pi}\Bigg\{\frac{Tf_T - f}{2} + \frac{f_{TT}}{2K^2}\left(\frac{I'}{2I} + \mathbb{W}_1\right)T' \\ + \frac{1}{r^2}\left(\frac{I_\theta}{2I} - \mathbb{W}_2\right)T_\theta\Bigg\}\Bigg],\end{aligned} \tag{9}$$

$$\begin{aligned}G_{11} = \frac{8\pi}{f_T}\Bigg[p_{xx} + \frac{K^2}{8\pi}\Bigg\{\frac{Tf_T - f}{2} \\ + \frac{f_{TT}}{2K^2r^2}\left(\frac{I_\theta}{I} + \mathbb{W}_3\right)T'\Bigg\}\Bigg],\end{aligned} \tag{10}$$

$$\begin{aligned}G_{22} = \frac{8\pi}{f_T}\Bigg[p_{yy} - \frac{K^2r^2}{8\pi}\Bigg\{\frac{Tf_T - f}{2} \\ - \frac{f_{TT}}{2K^2}\left(\mathbb{W}_4 - \frac{I'}{I}\right)T'\Bigg\}\Bigg],\end{aligned} \tag{11}$$

$$G_{33}=\frac{8\pi}{f_T}\left[p_{zz}+\frac{L^2}{8\pi}\left\{\frac{Tf_T-f}{2}-\frac{f_{TT}}{2K^2}\left(\mathbb{W}_5-\frac{I'}{I}\right)T'+\frac{1}{r^2}\left(\mathbb{W}_6-\frac{I_\theta}{I}\right)T_\theta\right\}\right], \tag{12}$$

$$G_{12}=\frac{8\pi}{f_T}\left[p_{xy}+\frac{f_{TT}}{16\pi}\left\{\mathbb{W}_7-\frac{I'}{I}\right\}T_\theta\right], \tag{13}$$

where $H=\sin\phi\cos\theta$, $J=\sin\theta\cos\phi$, $I'=\frac{\partial I}{\partial r}$, $I_\theta=\frac{\partial I}{\partial \theta}$, $T'=\frac{\partial T}{\partial r}$, $T_\theta=\frac{\partial T}{\partial \theta}$. The G_{aa} (where $a=0,1,2,3$) are enlisted in [37] and $\mathbb{W}_a$ (where $a=1,2,...,7$) are *f*(*T*) gravity components enlisted in Appendix A.

4. The conformal scalar and computation of structure scalars

The characteristics of gravitational curvature and how it influences spacetime have become more obvious because of the conformal scalars. They have a variety of uses, including characterizing the curvature of relativistic bodies, studying gravitational waves, and investigating the dynamical aspects of the cosmos. The phrases "electric" and "magnetic" are employed in the framework of the conformal scalars to characterize specific attributes of the gravitational tidal impacts and refrain from explicitly stating the availability of electromagnetic fields. The various constituents of the conformal tensor are called out as "electric" and "magnetic" using analog from electromagnetism. Likewise in the way the electromagnetic field tensor serves the curvature of the electromagnetic field induced by the occurrence of currents and charges, the conformal tensor portrays the curvature of spacetime induced by pure gravitational techniques. Since the conformal tensor is traceless and symmetric, it reflects the electromagnetic field tensor in multiple aspects. The constituents of the electric conformal tensor can be easily computed from

$$\mathbb{E}_{\varrho\sigma}=C_{\varrho\omega\sigma\upsilon}u^{\omega}u^{\upsilon},$$

where $C_{\varrho\omega\sigma\upsilon}$ is symbolized for the conformal tensor. By the built-in symmetries of spacetime, the magnetic constituent of the conformal tensor can likewise fade away in an axial symmetry. Since symmetry of rotation is implied by axial symmetry, the conformal tensor constituent may become zero as a result of this symmetry. The constituents of the conformal tensor, i.e., electric along with magnetic, describe the tidal influence of anisotropy in a gravitational field. So, the electric components with support of constraints of Eq. (2) turn out as

$$\mathbb{E}_{\omega\upsilon}=\psi_1(k_\omega l_\upsilon+l_\omega k_\upsilon)+\psi_2(k_\omega k_\upsilon-\frac{h_{\omega\upsilon}}{3})+\psi_3(l_\omega l_\upsilon-\frac{h_{\omega\upsilon}}{3}), \tag{14}$$

where ψ_1, ψ_2 and ψ_2 are three conformal scalars in this way and they their expressions are enlisted in Appendix B. They can provide support to express the electric constituents of the conformal tensors as

$$\mathbb{E}_{11}=(\frac{2}{3}\psi_2-\frac{1}{3}\psi_3)K^2=\frac{1}{6}\left[\frac{2I''}{I}-\frac{K''}{K}-\frac{L''}{L}-\frac{3I'K'}{IK}-\frac{I'L'}{IL}+\left(\frac{K'}{K}\right)^2+\frac{3K'L'}{KL}+\frac{1}{r}\left(\frac{2L'}{L}-\frac{K'}{K}-\frac{I'}{I}\right)\right]+\frac{1}{6r^2}\left[\frac{-K_{\theta\theta}}{K}-\frac{I_{\theta\theta}}{I}+\frac{2L_{\theta\theta}}{L}+\frac{3I_\theta K_\theta}{IK}-\frac{I_\theta L_\theta}{IL}+\left(\frac{K_\theta}{K}\right)^2-\frac{3K_\theta L_\theta}{KL}\right], \tag{15}$$

$$\mathbb{E}_{22}=(\frac{2}{3}\psi_3-\frac{1}{3}\psi_2)K^2r^2=-\frac{r^2}{6}\left[\frac{I''}{I}+\frac{K''}{K}-\frac{2L''}{L}-\frac{3I'K'}{IK}+\frac{I'L'}{IL}-\left(\frac{K'}{K}\right)^2+\frac{3K'L'}{KL}+\frac{1}{r}\left(\frac{L'}{L}+\frac{K'}{K}-\frac{2I'}{I}\right)\right]-\frac{1}{6}\left[\frac{K_{\theta\theta}}{K}-\frac{2I_{\theta\theta}}{I}+\frac{L_{\theta\theta}}{L}+\frac{3I_\theta K_\theta}{IK}+\frac{I_\theta L_\theta}{IL}-\left(\frac{K_\theta}{K}\right)^2-\frac{3K_\theta L_\theta}{KL}\right], \tag{16}$$

$$\mathbb{E}_{33}=-(\frac{1}{3}\psi_2+\frac{1}{3}\psi_3)L^2=-\frac{L^2}{6K^2}\left[\frac{I''}{I}-\frac{2K''}{K}+\frac{L''}{L}-\frac{2I'L'}{IL}+2\left(\frac{K'}{K}\right)^2+\frac{1}{r}\left(\frac{L'}{L}-\frac{2K'}{K}+\frac{I'}{I}\right)\right]-\frac{L^2}{6K^2r^2}\left[\frac{-2K_{\theta\theta}}{K}+\frac{I_{\theta\theta}}{I}+\frac{L_{\theta\theta}}{L}-\frac{2I_\theta L_\theta}{IL}+2\left(\frac{K_\theta}{K}\right)^2\right], \tag{17}$$

$$\mathbb{E}_{12}=\psi_1K^2r=\frac{1}{2}\left[\frac{I'_\theta}{I}-\frac{L'_\theta}{L}+\frac{K_\theta L'}{KL}-\frac{I'K_\theta}{IK}-2\frac{K'I_\theta}{IK}+\frac{L_\theta K'}{LK}-\frac{1}{r}\left(\frac{I_\theta}{I}-\frac{L_\theta}{L}\right)\right]. \tag{18}$$

The constituents rely on each other and fulfill the subsequent relation

$$\mathbb{E}_{11}+\frac{1}{r^2}\mathbb{E}_{22}+\frac{K^2}{L^2}\mathbb{E}_{33}=0.$$

The mathematical expression of RCrT can be stated in terms of the conformal tensor and the Ricci scalar as below

$$R^{\varrho}_{\omega\upsilon\sigma}=C^{\varrho}_{\omega\upsilon\sigma}+\frac{1}{2}R^{\varrho}_{\upsilon}g_{\omega\sigma}-\frac{1}{2}R_{\omega\upsilon}\delta^{\varrho}_{\sigma}+\frac{1}{2}R_{\omega\sigma}\delta^{\varrho}_{\upsilon}-\frac{1}{2}R^{\varrho}_{\sigma}g_{\omega\upsilon}-\frac{1}{6}\Re(\delta^{\varrho}_{\upsilon}g_{\omega\sigma}-g_{\omega\upsilon}\delta^{\varrho}_{\sigma}). \tag{19}$$

The framework to review these structure scalars has been established on its own by Herrera et al. [1]. Each of these traces and traces-free constituents within the three tensors relates to a distinct relevance. Here, we established these scalars in the background of *f*(*T*) gravity. We employed the equation of the tensors $\mathbb{X}_{\omega\upsilon}$ and $\mathbb{Y}_{\omega\upsilon}$ of the type that is found in [38] to perform our evaluation contained within the background of *f*(*T*) gravity.

$$Y_{\omega\upsilon}=R_{\omega\varrho\upsilon\sigma}u^{\varrho}u^{\sigma},\quad X_{\omega\upsilon}={}^{\star}R^{\star}_{\omega\varrho\upsilon\sigma}u^{\varrho}u^{\sigma}=\frac{1}{2}\eta_{\omega\varrho}^{\ \ \epsilon\alpha}R^{\star}_{\epsilon\alpha\upsilon\sigma}u^{\varrho}u^{\sigma},$$

where $\star$ is the indication for the dual tensor, i.e., $R^{\star}_{\omega\upsilon\varrho\sigma}=\frac{1}{2}\eta_{\epsilon\alpha\varrho\sigma}R_{\omega\upsilon}^{\ \ \epsilon\alpha}$. By employing these tensors, illustrates the way the RCrT is possibly displayed. The Eq. (19) splits in a subsequent style considering the *f*(*T*) field equations

$$R^{\omega\sigma}_{\ \ \upsilon\varrho}=C^{\omega\sigma}_{\ \ \upsilon\varrho}+\frac{16\pi}{f_T}T^{(eff)[\omega}_{\ \ \ [\upsilon}\delta^{\sigma]}_{\varrho]}+\frac{8\pi}{f_T}T^{(eff)}\left(\frac{1}{3}\delta^{\omega}_{[\upsilon}\delta^{\sigma}_{\varrho]}-\delta^{[\omega}_{\ [\upsilon}\delta^{\sigma]}_{\varrho]}\right). \tag{20}$$

We attain the splitting of RCrT with support of Eq. (2) and Eq. (6) within Eq. (20) in this fashion

$$R^{\omega\sigma}_{\upsilon\varrho}=R^{\omega\sigma}_{(I)\upsilon\varrho}+R^{\omega\sigma}_{(II)\upsilon\varrho}+R^{\omega\sigma}_{(III)\upsilon\varrho},$$

here

$$R^{\omega\sigma}_{(I)\upsilon\varrho}=\frac{16\pi}{f_T}\left\{\rho u^{[\omega}u_{[\upsilon}\delta^{\sigma]}_{\varrho]}+Ph^{[\omega}_{[\upsilon}\delta^{\sigma]}_{\varrho]}\right\}+\frac{8\pi}{f_T}\left\{(-\rho+3P)+\frac{(Tf_T-f)}{8\pi}-\frac{f_{TT}}{8\pi}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_{\rho}T)\right\}\left(\frac{1}{3}\delta^{\omega}_{[\upsilon}\delta^{\sigma}_{\varrho]}-\delta^{[\omega}_{[\upsilon}\delta^{\sigma]}_{\varrho]}\right),$$

$$R^{\omega\sigma}_{(II)\upsilon\varrho}=\frac{16\pi}{f_T}\left\{\Pi^{[\omega}_{[\upsilon}\delta^{\sigma]}_{\varrho]}+\frac{1}{2\pi}\left(\frac{Tf_T-f}{2}\delta^{[\omega}_{[\upsilon}\delta^{\sigma]}_{\varrho]}-f_{TT}S_{[\upsilon}^{\ [\omega\rho}\delta^{\sigma]}_{\varrho]}\delta^{\rho}_{\rho}\nabla_{\rho}T\right)\right\},$$

$$R^{\omega\sigma}_{(III)\upsilon\varrho}=4u^{[\omega}u_{[\upsilon}E^{\sigma]}_{\varrho]}-\psi^{\omega\sigma}_{\alpha}\psi_{\upsilon\varrho\beta}E^{\alpha\beta}.$$

The orthogonal division of the RCrT, the idea to explore the structure of spacetime curvature and gravitational impacts in the backdrop of matter contributions, relies on the conformal scalars as one of its primary components. The conformal tensor and other concepts that take into consideration multiple features of the curvature and matter-energy composition belong to the concepts that are separated from the RCrT using the orthogonal division strategy. Researchers develop greater awareness of the delicate relationship between matter content and spacetime curvature along with how these two factors interact to influence the entire behavior of physical structures by including the conformal scalars into the orthogonal division of the RCrT. The second dual of the RCrT is referred to as $\mathbb{X}_{\omega\upsilon}$; while, the tensor $\mathbb{Y}_{\omega\upsilon}$ is known as the electric constituent. These two tensors can be fragmented down into the subsequent components combining the projection tensor along with the four-velocity of fluid

$$\mathbb{X}_{\omega\upsilon}=\mathbb{X}_{TF_1}(k_{\omega}l_{\upsilon}+k_{\upsilon}l_{\omega})+\mathbb{X}_{TF_2}(k_{\omega}k_{\upsilon}-\frac{h_{\omega\upsilon}}{3})+\mathbb{X}_{TF_3}(l_{\omega}l_{\upsilon}-\frac{h_{\omega\upsilon}}{3})+\frac{1}{3}\mathbb{X}_T h_{\omega\upsilon},$$

$$\mathbb{Y}_{\omega\upsilon}=\mathbb{Y}_{TF_1}(k_{\omega}l_{\upsilon}+k_{\upsilon}l_{\omega})+\mathbb{Y}_{TF_2}(k_{\omega}k_{\upsilon}-\frac{h_{\omega\upsilon}}{3})+\mathbb{Y}_{TF_3}(l_{\omega}l_{\upsilon}-\frac{h_{\omega\upsilon}}{3})+\frac{1}{3}\mathbb{Y}_T h_{\omega\upsilon}.$$

However, subscripts like T and TF act as indicators to differentiate between trace and trace-free individual components. The structure scalars are capable of being expressed in the context of the matter profile of an axially symmetric object applying the *f*(*T*) field Equations (6) through Eq. (2) in addition to Eqs. (14 and 19) in this way

$$\mathbb{X}_T=8\pi\rho+\mathbb{W}_8,\ \ \mathbb{X}_{TF_2}=-\psi_2-4\pi(p_{xx}-p_{zz})+\mathbb{W}_{10}, \tag{21}$$

$$\mathbb{X}_{TF_1}=-\psi_1-4\pi p_{xy}+\mathbb{W}_9,\mathbb{X}_{TF_3}=-\psi_3-4\pi(p_{yy}-p_{zz})+\mathbb{W}_{11}, \tag{22}$$

where the values for $\mathbb{W}_8$, $\mathbb{W}_9$, $\mathbb{W}_{10}$ and $\mathbb{W}_{11}$ are enlisted in Appendix A. On similar way, the strategy is implemented for tensor $\mathbb{Y}_{\omega\upsilon}$ to attain

$$\mathbb{Y}_T=4\pi(\rho+3p)+\mathbb{W}_{12},\ \ \mathbb{Y}_{TF_2}=\psi_2-4\pi(p_{xx}-p_{zz})+\mathbb{W}_{14}, \tag{23}$$

$$\mathbb{Y}_{TF_1}=\psi_1-4\pi p_{xy}+\mathbb{W}_{13},\mathbb{Y}_{TF_3}=\psi_3-4\pi(p_{yy}-p_{zz})+\mathbb{W}_{15}, \tag{24}$$

where the values for $\mathbb{W}_{12}$, $\mathbb{W}_{13}$, $\mathbb{W}_{14}$ and $\mathbb{W}_{15}$ are enlisted in Appendix A. The scalars of *f*(*T*) featuring the extra curvature variables outlined prior resolve to those encountered in TG equivalent to GR. In a nutshell, employing the standard TG restriction, it is feasible to re-establish the structure formation and evolution of the universe as indicated in the TG equivalent to GR framework.

5. The hydrostatic equilibrium equation along with Weyl equations

The goal is to implement the essential equations in this part to illustrate an axially symmetric anisotropic fluid that has the characteristic of self-gravitation. Despite considering that specific examples of these equations (i.e., the field equations and the Bianchi identities) are not unrelated, we will nevertheless include them due to the situation at hand, it can be beneficial to use one subset compared to the other. These are a number of equations that are originated by the conservation formula

$T^{\omega\upsilon(eff)}_{;\upsilon} = 0.$

In GR, the covariant derivatives of the RCrT are associated mathematically with the Bianchi identities. They certainly have nothing to do with Hydro-Equal equations explicitly. However, for insight into the association between the composition of matter and spacetime curvature, Hydro-Eqili ought to be examined within the context of GR. In the state of Hydro-Eqili, the pulling force of gravity will be compensated by the pressure gradient when a fluid (such as a fluid star or a gas) is at rest or flowing at a constant speed. Examining how the distribution of mass and the curvature of spacetime correlate to the restoration of Hydro-Eqili. It is essential to keep in mind that the Hydro-Equili equation is unlikely to apply in a dynamic (non-static) instance, in such a scenario complicated equations incorporating non-gravitational forces along with time derivatives would be necessary. The way spacetime curvature and matter distribution are related in a static (equilibrium) situation is demonstrated by the Hydro-Eqili equation, which uses the covariant derivative of the usual as well as *f*(*T*) EM tensor and turns out as

$$p'_{xx} + \frac{I'}{I}(\rho + p_{xx}) + \frac{K'}{K}(p_{xx} - p_{yy}) + \frac{L'}{L}(p_{xx} - p_{zz}) + \frac{1}{r}\left[\left(\frac{I_\theta}{I} + 2\frac{K_\theta}{K} + \frac{L_\theta}{L}\right)p_{xy} + p_{xy,\theta} - (p_{yy} - p_{xx})\right] + \mathbb{Z}_1^{(D)} = 0, \tag{25}$$

$$p_{yy,\theta} + \frac{I_\theta}{I}(\rho + p_{yy}) - \frac{K_\theta}{K}(p_{xx} - p_{yy}) - \frac{L_\theta}{L}(p_{xx} - p_{zz}) + r\left[\left(\frac{I'}{I} + 2\frac{K'}{K} + \frac{L'}{L}\right)p_{xy} + p'_{xy}\right] + 2p_{xy} + \mathbb{Z}_2^{(D)} = 0, \tag{26}$$

here, $\mathbb{Z}_1^{(D)}$ and $\mathbb{Z}_1^{(D)}$ are the terms responsible for contribution of *f*(*T*) gravity and their values are assigned in Appendix B. The "Ellis–Bruni equations," are employed to analyze the evolution of anisotropic and inhomogeneous cosmos, and could be associated with the "Ellis equations." Bruni and Ellis [39] devised these equations. In a perturbed cosmological framework the growth of shear, density gradients, and vorticity is illustrated by a set of equations called the Ellis–Bruni equations. They emerged from the Einstein field equations and Hydro-Equal equations, and offer a theoretical basis for exploring how anisotropy and inhomogeneity impact the behavior of the cosmos at a vast scale. These mathematical models exert a unique significance for comprehending the dynamics of the cosmos at sizes where inhomogeneities at the local level can have an enormous effect [40, 41]. They aid in the analysis of the effects of small-scale density fluctuations on other observable, large-scale structures and the cosmic microwave background. They turn out as

$$\frac{\psi_{1,\theta}}{r} - \frac{1}{3}(\psi_3 - 2\psi_2)' + \frac{\psi_1}{r}\left(2\frac{K_\theta}{K} + \frac{L_\theta}{L}\right) + \psi_2\left(\frac{1}{r} + \frac{L'}{L} + \frac{K'}{K}\right) - \psi_3\left(\frac{1}{r} + \frac{K'}{K}\right) = \frac{4\pi}{3f_T}(3p + 2\rho)' + \frac{4\pi}{f_T}(\rho + p_{xx})\frac{I'}{I} + \frac{4\pi}{rf_T}p_{xy}\frac{I_\theta}{I} + \frac{4\pi}{3f_T}\left[(T_{11}^{(D)} + T_{22}^{(D)} + 2T_{00}^{(D)})' + 3(T_{11}^{(D)} + T_{00}^{(D)})\frac{I'}{I} + \frac{I_\theta}{Ir}T_{12}^{(D)}\right], \tag{27}$$

$$\frac{\psi_1'}{r} + \frac{1}{3r}(2\psi_3 - \psi_2)_\theta + \psi_1\left(\frac{2}{r} + \frac{L'}{L} + 2\frac{K'}{K}\right) - \frac{\psi_2 K_\theta}{rK} + \frac{\psi_3}{r}\left(\frac{K_\theta}{K} + \frac{L_\theta}{L}\right) = \frac{4\pi}{3rf_T}(2\rho + 3p)_\theta + \frac{4\pi}{f_T}(\rho + p_{yy})\frac{I_\theta}{rI} + \frac{4\pi}{3rf_T}\left[(T_{11}^{(D)} + T_{22}^{(D)} + 2T_{00}^{(D)})_{,\theta} + 3(T_{11}^{(D)} + T_{00}^{(D)})\frac{I_\theta}{I} + \frac{I'}{I}T_{12}^{(D)}\right] + \frac{4\pi}{f_T}p_{xy}\frac{I'}{I}, \tag{28}$$

here, $T_{aa}^{(D)}$ (where $a = 0, 1, 2, 3$) are corrections of theory stated in Appendix B. The implication of Eqs. (21, 22, 23 and 24) along with Eqs. (27 and 28) turn into

$$\frac{8\pi}{3f_T}\left[\rho' + T_{00}'^{(D)}\right] = \frac{1}{r}\left[\mathbb{Y}_{TF_{1,\theta}} + 8\pi p_{xy,\theta} - \mathbb{W}_{13,\theta} + (\mathbb{Y}_{TF_1} + 8\pi p_{xy} - \mathbb{W}_{13})(\ln K^2 L)_\theta\right] + \left[\frac{2}{3}(\mathbb{Y}'_{TF_2} + 8\pi(p_{xx} - p_{zz})' - \mathbb{W}'_{14}) + (\mathbb{Y}_{TF_2} + 8\pi(p_{xx} - p_{zz}) - \mathbb{W}_{14})(\ln KLr)'\right] - \left[\frac{1}{3}(\mathbb{Y}'_{TF_3} + 8\pi(p_{yy} - p_{zz})' - \mathbb{W}'_{15}) + 3(\mathbb{Y}_{TF_3} + 8\pi(p_{yy} - p_{zz}) - \mathbb{W}_{15})(\ln Br)'\right], \tag{29}$$

$$\frac{8\pi}{3f_T}\left[\rho_{,\theta}+T^{(D)}_{00,\theta}\right] = -\frac{1}{3r}\left[(\mathbb{Y}_{TF_{2,\theta}}+8\pi(p_{xx}-p_{zz})_{,\theta}-\mathbb{W}_{14,\theta}) + 3(\mathbb{Y}_{TF_2}+8\pi(p_{xx}-p_{zz})-\mathbb{W}_{14})(\ln K)_{\theta}\right] + \frac{1}{r}\left[\frac{2}{3}(\mathbb{Y}_{TF_{3,\theta}}+8\pi(p_{yy}-p_{zz})_{,\theta}-\mathbb{W}_{15,\theta})+(\mathbb{Y}_{TF_3} + 8\pi(p_{yy}-p_{zz})-\mathbb{W}_{15})(\ln KL)_{,\theta}\right] + \left[(\mathbb{Y}'_{TF_1}+8\pi p'_{xy}+\mathbb{W}'_{13})+(\mathbb{Y}_{TF_1} + 8\pi p_{xy}+\mathbb{W}_{13})(\ln K^2Lr^2)'\right], \tag{30}$$

where derivatives with respect to θ and r are indicated by subscript θ and prime, respectively.

6. Solutions

In the situation of spherical symmetry, it has been demonstrated that the diminishing of the scalar related to the trace-free part of $\mathbb{X}_{\omega\upsilon}$ is the prerequisite and sufficient constraint for the disappearing of the spatial derivative of the energy density despite the lack of dissipation. This variable was given the suffix "inhomogeneity factor" (IF) for a variety of reasons. In a nutshell, the phrase "IF" corresponds to a set of geometrical and physical constituents which is a necessary and sufficient constraint for the homogeneity of the energy density (in the occurrence of dissipative flux) [42]. The IF can be determined by the two equations similar to Eqs. (29 and 30). On a similar side, the complexity of the system can be visualized by employing the limitations on the C_F. The analysis can be initiated by utilizing the easiest strategy for the matter content, i.e., the energy density possesses the constant value and pressure should be isotropic. By implying this constraint, we yield $\mathbb{X}_{TF_1}=\mathbb{X}_{TF_2}=\mathbb{X}_{TF_3}=0$. This is the indication that the energy density has a vanishing derivative with respect to spatial coordinate, i.e., (r,θ). Alternatively, put

$$\mathbb{X}_{TF_1}=\mathbb{X}_{TF_2}=\mathbb{X}_{TF_3}=0 \Longleftrightarrow \rho'=\rho_{\theta}=0. \tag{31}$$

Consequently, the demands of energy density in homogeneous form signifies $\mathbb{X}_{TF_1}=\mathbb{X}_{TF_2}=\mathbb{X}_{TF_3}=0$, which outcomes in

$$\begin{aligned}\mathbb{Y}_{TF_1}&=-8\pi p_{xy}+\mathbb{W}_9+\mathbb{W}_{13},\\ \mathbb{Y}_{TF_2}&=-8\pi(p_{xx}-p_{zz})+\mathbb{W}_{10}+\mathbb{W}_{14},\\ \mathbb{Y}_{TF_3}&=-8\pi(p_{yy}-p_{zz})+\mathbb{W}_{11}+\mathbb{W}_{15}.\end{aligned} \tag{32}$$

The scalars $\mathbb{Y}_{TF_a}$ $a=1,2,3$ will be recognized as the C_F based on the aforementioned factors and following the mathematical framework stated in the scenario of anisotropic sphere [3]. They dissolve in terms of incompressible fluids corresponding to isotropic pressure. They can additionally dissolve for inhomogeneous, anisotropic fluids if these two components combine to negate the three C_F. The analytical solutions corresponding to these ideas will be presented in the next section. Certain answers will be supplied in the next section. Keep in mind that we intend to describe how these models are set up. We have no concern about supplying solutions that reflect any particular physically significant compact objects.

6.1. Dense spheroid comprising isotropy

In the context of GR, an examination of incompressible fluids uncovers interesting features that differ from conventional fluid dynamics. Compared to its classical equivalent, an incompressible fluid is identified by its incapacity to vary its density as a consequence of pressure alterations. The behavior of incompressible fluids brings on another aspect in the framework of curved spacetime as defined by Einstein's equations. Though compressible fluids are the primary topic of a majority of relativistic fluid dynamics, the inquiry into incompressible fluids sheds light on the connection between fluid dynamics and spacetime curvature. In astrophysical circumstances where gravity forces have an immense effect on fluid behavior, this scenario has applicability. More specifically, incompressible fluid models aid in gaining knowledge of processes like fluid flows around enormous compact objects or accretion onto black holes. The behavior of incompressible fluids is going to be analyzed within the context of *f*(*T*) gravity. Many researchers found the analytical solutions for incompressible fluids in literature [43, 44].

The three C_F vanished for a fluid distribution depicted by the incompressible isotropic spheroid as discovered and examined in [37]. At this point, all we accomplish is a straightforward recurrence. Therefore it generates that solution is conformally flat from Eqs. (21, 31) and $p_{xx}=p_{yy}=p_{zz}=p$, $p_{xx}=0$, $\rho^{eff}=\rho_0$= constant. The idea of junction conditions has received a lot of interest in the inspection of relativistic phenomena. Their significance is revealed by examining the collapsing fluid and the physics of the thin shell. The comprehensive examination is performed for internal and external geometries of any compact object. They are split into two halves and matched smoothly by a hyper-surface Ξ. For our case, we assume that the boundary surface is specified by Ξ as below

$$r = r_1 = \text{constant}. \tag{33}$$

The conditions demand that the metric functions (i.e., I, K, L) must be continuous with regard to r and θ derivatives. The implication of the condition by using Eq. (10, 13) with support of Eq. (33) turn into

$$p \overset{\Xi}{=} \frac{TK^2}{16\pi}\left(f_T - \frac{f}{T}\right). \tag{34}$$

Implementing the aforementioned constraints to the Eqs. (25 and 26) yield into

$$p + \rho_0 = -\int_0^r \mathbb{Z}_1^{(D)}\mathrm{d}r + \frac{\zeta_1(\theta)}{I}, \tag{35}$$

$$p + \rho_0 = -\int_0^\theta \mathbb{Z}_2^{(D)}\mathrm{d}\theta + \frac{\zeta_2(r)}{I}, \tag{36}$$

where $\zeta_1(\theta)$ and $\zeta_2(r)$ are arbitrary functions attain in result of integration. As is the case, when the aforementioned conditions along with Eq. (34) are incorporated into Eqs. (35 and 36), we acquire these outcomes

$$I(r_1, \theta) = \frac{\vartheta}{p + \rho_0} = \text{constant} \quad \zeta_1 = \text{constant}. \tag{37}$$

Ultimately, it is feasible to restructure the line element for configuration of isotropic incompressible and conformally flat matter as states

$$\mathrm{d}s^2 = \frac{1}{(\mu r^2 + v + g r\cos\theta)^2} \left[-(\beta r^2 + \alpha + f r\cos\theta)^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right], \tag{38}$$

where the values of metric functions along with different parameters are as $I(r,\theta) = [I(r) + r\chi(\theta)]K(r,\theta)$, $\chi(\theta) = f\cos\theta$, $I(r) = \beta r^2 + \alpha$, $L(r,\theta) = K(r,\theta)r\sin\theta$, while $K(r,\theta) = \frac{1}{(\mu r^2 + v + g r\cos\theta)}$. Following is an estimation of its associated physical constituents

$$\frac{8\pi}{f_T}[\rho + T_{00}^{(D)}] = 12\mu v - 3g^2, \tag{39}$$

$$\frac{8\pi}{f_T}[p + T_{11}^{(D)}] = (3g^2 - 12\mu v) \times \left[1 - \frac{\beta r_1^2 + \alpha\mu r^2 + v + g r\cos\theta}{\mu r_1^2 + \beta v r^2 + \alpha + f r\cos\theta}\right], \tag{40}$$

where the Eq. (40) is attain after employing the Eqs. (35 and 37). In which the parameter ζ_1 and f must satisfies $\zeta_1 = \rho_0\frac{\beta r_1^2 + \alpha}{\mu r_1^2 + v}$ and $f = \frac{\beta r_1^2 + \alpha}{\mu r_1^2 + v} g$. The values of $f(T)$ corrections are assigned in the Appendix B. In the scenario of $f(T)$, the solutions are matchable to any kind of Weyl exterior only in the limit of the sphere and it behaves like a Schwarzschild solution (interior). The general idea is attained by soothing the concept of inhomogeneity and isotropy.

6.2. Anisotropic fluid content with inhomogeneity

Anisotropic fluids are fluid contents that demonstrate multiple pressures in multiple spatial directions. These fluids incorporate alterations in pressure based on the orientation upon which the fluid is evaluated. Contrary to isotropic fluids, the pressure is constant in all possible directions. This perspective is suitable for astrophysical instances, especially when modeling specific categories of compact objects like neutron stars. The structure-based characteristics and gravitational behavior of neutron stars are significantly influenced by the existence of anisotropic pressures within them. Anisotropic fluid contents have been included in gravitational theories using a variety of mathematical techniques. These are helpful for researchers in clarifying the characteristics and behavior of these peculiar occurrences. Herrera and Barreto [45] established the formulation in detail for the modeling of polytropic stars incorporating anisotropic pressure in the frame of GR. The two distinct sorts of polytropic equations are employed which turn in a similar form to the Lane–Emden equation. They utilized the Tolman mass to highlight the characteristics of the build-up model. Ovalle et al. [46] enlarged the isotropic solutions to the anisotropic in the case of the sphere by employing the strategy of gravitational decoupling. They produced a new class of anisotropic solutions from the single isotropic class of solutions. The idea of dynamical stability for the anisotropic composition in the form of $f(R, T, Q)$ gravity is analyzed by Yousaf et al. [47].

In this section, the solutions for the case of anisotropic matter and density inhomogeneity fulfilling the zero C_F constraint will be explained. The values for metric functions in this case are

$$I(r,\theta) = \frac{f_1 r\sin\theta}{g_1 r^2 + g_2}, \quad K(r,\theta) = \frac{1}{g_1 r^2 + g_2}, \quad L(r,\theta) = \frac{g_1 r^2 - g_2}{g_1 r^2 + g_2} \times G\left(\frac{r\cos\theta}{g_1 r^2 - g_2}\right). \tag{41}$$

For the functions stated in Eq. (41), the $f(T)$ field equations in the form of physical variables furnish into

$$\frac{8\pi}{f_T}[\rho+T_{00}^{(D)}]=12g_1g_2-\frac{(g_1r^2+g_2)^2}{(g_1r^2-g_2)^2}\left(1+\frac{4g_1g_2r^2\cos^2\theta}{(g_1r^2-g_2)^2}\right)\frac{G_{zz}}{G},$$
$$\frac{8\pi}{f_T}[p+T_{11}^{(D)}]=-12g_1g_2+\frac{(g_1r^2+g_2)^2}{3(g_1r^2-g_2)^2}\left(1+\frac{4g_1g_2r^2\cos^2\theta}{(g_1r^2-g_2)^2}\right)\frac{G_{zz}}{G},$$
$$\frac{8\pi}{f_T}[(p_{xx}-p_{zz})+T_{11}^{(D)}-T_{33}^{(D)}]=\frac{G_{zz}}{4G}\frac{(g_1r^2+g_2)^2}{(g_1r^2-g_2)^2}\sin^2\theta,$$
$$\frac{8\pi}{f_T}[(p_{yy}-p_{zz})+T_{22}^{(D)}-T_{33}^{(D)}]=\frac{G_{zz}}{4G}\frac{(g_1r^2+g_2)^4}{(g_1r^2-g_2)^4}\cos^2\theta,$$
$$\frac{8\pi}{f_T}[p_{xy}+T_{12}^{(D)}]=-\frac{G_{zz}}{2G}\frac{r(g_1r^2+g_2)^3}{(g_1r^2-g_2)^3}\sin2\theta,$$

where $G(z)=\left(\frac{r\cos\theta}{g_1r^2-g_2}\right)$. The correction terms due to *f*(*T*) gravity are assigned in Appendix B. They are responsible for raising the IF and anisotropy factor within the relativistic structures.

A modified theory constructed on the torsion invariant *T* is called *f*(*T*) gravity; whereas, a theory of gravity constructed on the Gauss–Bonnet invariant *G* and the Ricci scalar *R* is called *f*(*G*) gravity. Furthermore, *f*(*G*) gravity has been used to study the formation of relativistic compact objects; while, *f*(*T*) gravity has been suggested as an alternate hypothesis to account for the rapid expansion of the universe. We have effectively established in our work that *f*(*T*) gravity is a useful tool for the analysis of static axially symmetric systems. We exhibited an incredible mathematical depth in our thorough investigation of structure scalars and in determining particular factors to system complexity. The study gains great significance from the construction of hydrostatic fluid equations and conformal equations as well as the finding of the analytical solutions.

A thorough understanding of *f*(*T*) gravity and its implications in astrophysics and cosmology will require more investigation and study. Based on the proposed approach, the potential limitations on *f*(*T*) gravity comprise the following:

- Findings from the research could be affected by the hypotheses or model parameters that were used. To gain more knowledge of the consistency of the suggested strategy, a sensitivity analysis might be used to determine the important parameters and their influence on the outcomes.
- Even though analytical solutions are found, the empirical usefulness of research can be increased by comparing the findings with observational data or experimental outcomes. A more thorough examination would address any inconsistencies or restrictions in matching theoretical expectations with empirical observations.
- The work emphasizes static axially symmetric systems, and it could be helpful if the discussion is made about how the results can be applied to a wider range of astrophysical or cosmological settings. Examining the constraints observed when applying the technique to various system configurations or sizes may provide helpful insights into the adaptability of the methodology.

In addition to enhancing the scientific accuracy of the research, addressing these potential constraints in the manuscript would promote a deeper understanding of the limits and real-world applications of the suggested technique in *f*(*T*) gravity.

7. Conclusion

This investigation intends to establish the extended TG version of an axially symmetric anisotropic model and determine the structure scalars that correspond to a particular model. Due to second-order field equations, *f*(*T*) gravity has a considerable benefit compared with *f*(*R*) gravity, because *f*(*R*) fourth-order equations can produce pathologies. In addition to receiving attention and enabling the reconstruction of many cosmological evolution's, this property has contributed to a fast-growing interest in the research community. To launch our examination, an overview of an axially symmetric geometry correlated with a source and a handful of equations that involve key parameters are supplied. We stated the core structure of *f*(*T*) gravity. The *f*(*T*) field equations were then developed using a non-diagonal tetrad in the context of an anisotropic distribution. The RCrT was divided in an orthogonal manner in this frame. We generated a list of structure scalars employing the aforementioned technique in this gravity. For a higher level of investigation, the Hydro-Eqili equation is developed with the aid of conservation law. To boost the degree of significance of structure scalars, a few differential equations are examined. To end up with the findings in this theory, we presented a couple of models.

By contracting with the fluid four vectors, the RCrT is broken down into its various tensor parts. Since these fluid vectors are set to fulfill being orthogonal to each other, the disintegration of RCrT into its trace and trace-free components is recognized as orthogonal splitting. We examined the implications of additional curvature relative to *f*(*T*) gravity on the scalar components developed by the orthogonal splitting since the RCrT serves to examine curvature. These scalars, formerly referred to as structure scalars, are vitally important for interpreting the development and design of the cosmos. We explored into these scalars for axial symmetry under the dark effects of *f*(*T*) gravity. In connection to our query, we evaluated the complete number of such scalars and came up with eight scalar factors that are significant and in comparison with spherical symmetric instance [48] where only five scalar variables are involved. In the current study, we assessed the significant impact of scalar variables. The Ellis perspective likewise serves to investigate a couple of differential equations to determine the factor underlying the inhomogeneity of the system. We observed that one of the aforementioned components is the trace-free element that makes up the second dual of the RCrT to the class of these components.

- In the case of axially symmetric structure, we attained two main sets of structure scalars in the frame of *f*(*T*) gravity. One of them is associated with the complexity of the system, for which the three scalars $\mathbb{Y}_{TF_a}$ (where $a = 1, 2, 3$) are acquired after splitting RCrT. The outcomes of C_F are similar as in the case of GR [3]. Meanwhile, the appearance of *f*(*T*) corrections terms slows down the variations process as they exhibit a repulsive nature. These scalars have a significant contribution to analyzing the homogeneity along stability of any relativistic system.
- The other set of scalars $\mathbb{X}_{TF_a}$ (where $a = 1, 2, 3$) corresponds the IF of the system. They are an amalgam of geometric and physical parameters. They indicate that the energy density will be homogenous when IF is vanishing.
- From two classes of the solutions, the model with isotropy and incompressible matter is matchable to the Weyl exterior only in the limit of the sphere. While the model is matchable with any kind of Weyl exterior if the matter is anisotropic and homogenous.
- All our findings easily turned into GR [49] by imposing the limit $f(T) \to T$.

The investigation of incompressible fluids proposes unique characteristics that deviate from conventional fluid dynamics into the structure of *f*(*T*) gravity. In contrast to its classical corresponding, an incompressible fluid preserves its density fixed under any sort of pressure. The TG equivalent of GR, an alternative theory to GR, provides the basis for *f*(*T*) gravity. The foundation of the TG equivalent of GR is the teleparallel sort of Einstein's theory, in which torsion assumes the function of curvature. In this way, analyzing incompressible fluids supplies insights into the interplay between spacetime torsion and fluid contents. Numerous astrophysical consequences occur due to different scenarios, specifically those involving strong gravitational fields and compact objects. Although gravitational theories have received the majority of attention in *f*(*T*) gravity [50, 51], the addition of incompressible fluid models contributes an additional level of complexity to our information on both cosmological geometry and fluid behavior. Considering these laboratories serve as essentials for the analysis of the *f*(*T*) theories, it will be appealing to discover additional examples of these analytical solutions. We intend to write on this topic in our upcoming projects.

When the density of a substance shows a recurring variation or pattern over time and space, it is referred to as periodic density inhomogeneity. The choice to investigate the comparatively unknown region of inhomogeneous exact solutions of field equations led to the investigation of inhomogeneous cosmologies. By determining and examining precise solutions for the field equations, the framework will be equipped to tackle any challenge presented by an inhomogeneous cosmos. Inhomogeneous solutions are particularly useful for addressing issues like the anisotropy in cosmic microwave background radiation caused by density inhomogeneities in the universe, singularity issues, void evolutions, potential effects of universe expansion on planetary orbits, and others, as stated by Krasinski [52]. The density inhomogeneities in our case are associated with the matter distribution.

Relating to periodic density inhomogeneity might require examining how the suggested design handles scenarios in which the density of matter inside the system exhibits a periodic pattern in light of our work on static axially symmetric systems and *f*(*T*) gravity. The characteristics of the scalars in our system are influenced by the periodic density fluctuations. The suggested framework offers insights into how the general design of the system and characteristics are impacted by these periodic inhomogeneities as can be seen in Sect. 4. In our study, we evaluated the analytical solutions. Therefore, we might need further methodology to describe the solutions in terms of periodic density inhomogeneities. The only approach to represent the behavior graphically and comprehend the energy density inhomogeneity pattern is to apply the numerical simulation approach. The periodic density inhomogeneity might be predicted or observable in cosmological or astrophysical scenarios. The results of both approaches may be connected, so indicating the usefulness of our framework in various scenarios.

Appendix A

The values for $\mathbb{W}_a$ where $a = 1, 2, 3, ..., 15$ are stated as below

$$\mathbb{W}_1 = H(\theta,\phi)J(\theta,\phi)(\frac{K'}{K}-1) + H^2(\theta,\phi)(\frac{K'}{K}+\frac{1}{r}+1) + \frac{J(\theta,\phi)\sin\theta}{Kr}(L-L') + \frac{K'}{K} \times H(\theta,\phi)\sin\theta - \frac{J(\theta,\phi)\sin\theta}{L}\times(K'r+K),$$

$$\mathbb{W}_2 = H(\theta,\phi)J(\theta,\phi)cot\phi(\frac{K'}{K}-1) - H(\theta,\phi)J(\theta,\phi)cot\theta(\frac{K'r}{K}+1-r) - \frac{J^2(\theta,\phi)tan\phi}{Kcos\phi}(L-L') - \frac{KH^2(\theta,\phi)}{L\sin\phi} - \frac{KrJ(\theta,\phi)\cos\theta}{L},$$

$$\mathbb{W}_3 = \frac{H(\theta,\phi)J(\theta,\phi)cot\phi}{2}\times(\frac{K'}{K}-1) + \frac{H^2(\theta,\phi)cot\phi}{2}\times(\frac{K'}{K}+1-r) + \frac{J^2(\theta,\phi)tan\phi}{2\cos\phi} (L'-L) + \frac{H^2(\theta,\phi)K}{Lr^2\sin\phi} + \frac{J(\theta,\phi)rK\cos\theta}{L},$$

$$\mathbb{W}_4 = \frac{H(\theta,\phi)J(\theta,\phi)}{r}(\frac{K'}{K}-1) + H^2(\theta,\phi)(\frac{K'}{K}+\frac{1}{r}-1) + \frac{J(\theta,\phi)\sin\theta}{Kr}(L-L') - \frac{K'}{L}\times H(\theta,\phi)\sin\theta + \frac{J(\theta,\phi)\sin\theta}{L}\times(K'r+K),$$

$$\mathbb{W}_5 = \frac{K'}{2L}H(\theta,\phi)\sin\theta - \frac{J(\theta,\phi)\sin\theta}{2L}(K'r+K) - \frac{H(\theta,\phi)J(\theta,\phi)}{r}(\frac{K'}{K}-1) - H^2(\theta,\phi) (\frac{K'}{K}+\frac{1}{r}-1) - \frac{J(\theta,\phi)\sin\theta}{Kr}\times(L-L'),$$

$$\mathbb{W}_6 = (\frac{K}{L}+\frac{Kr}{L})\times H(\theta,\phi)\cos\theta + (cot\phi(\frac{K'}{K}-1) + cot\theta(\frac{K'r}{K}+1-r))H(\theta,\phi)J(\theta,\phi) + \frac{H(\theta,\phi)tan\theta\sin\theta}{K}\times(L'-L),$$

$$\mathbb{W}_7 = \frac{H(\theta,\phi)J(\theta,\phi)}{r}(\frac{K'}{K}-1) + H^2(\theta,\phi)(\frac{K'}{K}+\frac{1}{r}-1) + \frac{J(\theta,\phi)\sin\theta}{Kr}(L-L') - \frac{K'}{L}\times H(\theta,\phi)\sin\theta + \frac{J(\theta,\phi)\sin\theta}{L}\times(K'r+k),$$

$$\mathbb{W}_8 = 2\left\{(Tf_T-f) - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right\},$$

$$\mathbb{W}_9 = \left[\frac{Tf_T-f}{2}g_{\omega\upsilon} - f_{TT}S_{\upsilon}^{\ \omega\rho}g_{\omega\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right],$$

$$\mathbb{W}_{10} = \frac{1}{3}\left[\frac{Tf_T-f}{2} - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right],$$

$$\mathbb{W}_{11} = -\frac{2}{3}\left[(Tf_T-f) - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right],$$

$$\mathbb{W}_{12} = \left\{(Tf_T-f) - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T + 2f_{TT}S^{\rho}\nabla_\rho T + 2f_{TT}u^{\sigma}S_{\sigma}^{\ \rho}\nabla_\rho T\right\} + \left\{\frac{Tf_T-f}{2} - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right\},$$

$$\mathbb{W}_{13} = \frac{1}{2}\left\{\frac{Tf_T-f}{2}g_{\omega\upsilon} - f_{TT}S_{\upsilon}^{\ \omega\rho}g_{\omega\omega}\delta^{\rho}_{\rho}\nabla_\rho T + f_{TT}u_{\upsilon}S_{\omega}^{\ \rho}\nabla_\rho T + f_{TT}u_{\omega}S_{\upsilon}^{\ \rho}\nabla_\rho T + f_{TT}g_{\omega\upsilon}u^{\sigma}S_{\upsilon}^{\ \rho}\nabla_\rho T\right\},$$

$$\mathbb{W}_{14} = \frac{1}{3}\left\{\frac{Tf_T-f}{2} - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right\}h_{\omega\upsilon},$$

$$\mathbb{W}_{15} = \left[\frac{1}{2}\left\{(Tf_T-f) - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T + 2f_{TT}S^{\rho}\nabla_\rho T + 2f_{TT}u^{\sigma}S_{\sigma}^{\rho}\nabla_\rho T\right\} + \left(\frac{Tf_T-f}{2} - f_{TT}S_{\upsilon}^{\ \omega\rho}\delta^{\upsilon}_{\omega}\delta^{\rho}_{\rho}\nabla_\rho T\right)\right]h_{\omega\upsilon}.$$

Appendix B

Here, the values for corrections $\mathbb{Z}_1^{(D)}$, $\mathbb{Z}_2^{(D)}$ are given below

$$\mathbb{Z}_1^{(D)} = \frac{I'}{I^3K^2}T_{00}^{(T)} + \left\{\left(\frac{I'}{I}+\frac{1}{r}+\frac{L'}{L}\right)\frac{1}{K^4} + \left(\frac{2}{K^2}+1\right)\frac{K'}{K^3}\right\}T_{11}^{(T)} - \left(K'-\frac{1}{Kr}\right)\frac{T_{22}^{(T)}}{K^3r^2} - \frac{L'}{L^3K^2}T_{33}^{(T)} + T_{,1}^{11(T)} + +T_{,2}^{12(T)},$$

$$\mathbb{Z}_2^{(D)} = \frac{I_\theta}{I^3K^2r^2}T_{00}^{(T)} - \frac{K_\theta}{K^5r^2}T_{11}^{(T)}\left\{\left(\frac{I'}{I}+\frac{3K'}{K}\right)\frac{1}{K^4r^2} + \left(\frac{2}{r}+\frac{K'}{K}\right)\frac{1}{K^4r^2}\right\}T_{12}^{(T)} + \left\{\left(\frac{I_\theta}{I}+\frac{3K_\theta}{K}\right)\frac{1}{K^4r^4} + \frac{L_\theta}{K^4Lr^4}\right\}T_{22}^{(T)} - \frac{L_\theta}{K^2r^2L^3}T_{33}^{(T)} + T_{,2}^{22(T)} + +T_{,1}^{21(T)}.$$

The values for conformal scalars ψ_a where $a=1,2,3$ are as

$$\psi_1 = \frac{1}{2K^2}\left[\frac{1}{r}\left\{\frac{I'_\theta}{I} - \frac{L'_\theta}{L} - \frac{K_\theta I'}{KI} + \frac{L'K_\theta}{LK} - \frac{K'I_\theta}{KI} + \frac{L_\theta K'}{LK}\right\} + \frac{1}{r^2}\left(\frac{L_\theta}{L}-\frac{I_\theta}{I}\right)\right],$$

$$\psi_2 = -\frac{1}{2K^2}\left[-\frac{I''}{I}+\frac{K''}{K}+\frac{I'K'}{IK}+\frac{I'L'}{IL}-\left(\frac{K'}{K}\right)^2-\frac{K'L'}{KL} + \frac{1}{r}\left(\frac{K'}{K}-\frac{L'}{L}\right)\right] - \frac{1}{2K^2r^2}\left[\frac{K_{\theta\theta}}{K}-\frac{L_{\theta\theta}}{L}-\frac{I_\theta K_\theta}{IK}+\frac{I_\theta L_\theta}{IL}-\left(\frac{K_\theta}{K}\right)^2+\frac{K_\theta L_\theta}{KL}\right],$$

$$\psi_3 = -\frac{1}{2K^2}\left[\frac{K''}{K}-\frac{L''}{L}-\frac{I'K'}{IK}+\frac{I'L'}{IL}-\left(\frac{K'}{K}\right)^2+\frac{K'L'}{KL} + \frac{1}{r}\left(\frac{K'}{K}-\frac{I'}{I}\right)\right] - \frac{1}{2K^2r^2}\left[\frac{K_{\theta\theta}}{K}-\frac{I_{\theta\theta}}{I}+\frac{I_\theta K_\theta}{IK}+\frac{I_\theta L_\theta}{IL}-\left(\frac{K_\theta}{K}\right)^2-\frac{K_\theta L_\theta}{KL}\right].$$

The values for $f(T)$ correction terms $T_{aa}^{(D)}$ where $a=1,2,3$ are given below

$$T_{00}^{(D)} = -\frac{I^2}{8\pi}\left\{\frac{Tf_T - f}{2} + \frac{f_{TT}}{2K^2}\left(\frac{I'}{2I} + \mathbb{W}_1\right)T' + \frac{1}{r^2}\left(\frac{I_\theta}{2I} - \mathbb{W}_2\right)T_\theta\right\},$$

$$T_{11}^{(D)} = \frac{K^2}{8\pi}\left\{\frac{Tf_T - f}{2} + \frac{f_{TT}}{2K^2r^2}\left(\frac{I_\theta}{I} + \mathbb{W}_3\right)T'\right\},$$

$$T_{22}^{(D)} = -\frac{K^2r^2}{8\pi}\left\{\frac{Tf_T - f}{2} - \frac{f_{TT}}{2K^2}\left(\mathbb{W}_4 - \frac{I'}{I}\right)T'\right\},$$

$$T_{33}^{(D)} = \frac{L^2}{8\pi}\left\{\frac{Tf_T - f}{2} - \frac{f_{TT}}{2K^2}\left(\mathbb{W}_5 - \frac{I'}{I}\right)T' + \frac{1}{r^2}\left(\mathbb{W}_6 - \frac{I_\theta}{I}\right)T_\theta\right\},$$

$$T_{12}^{(D)} = \frac{f_{TT}}{16\pi}\left\{\mathbb{W}_7 - \frac{I'}{I}\right\}T_\theta.$$

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