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LRS Bianchi type I with anisotropic bulk viscosity matter cosmological typical and quadratic deceleration parameter in $\mathbf{f(R, \sum, T)}$ gravity

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Abstract: The objective of this article is to explore the properties of a spatially homogeneous but anisotropic Bianchi type-I universe within the framework of $f(R, \sum, T)$ gravity. In this typical, the universe contains bulk viscosity matter, and R is the Ricci scalar, $\sum$ is the torsion scalar, T is the trace of the stress-energy momentum tensor, and η is an arbitrary parameter that defines the functional form of $f(R, \Sigma, T) = R + \Sigma + 2\eta T$. The field equations are solved utilizing the quadratic deceleration parameter, and a comprehensive analysis is conducted to examine and discuss the influence of torsion on the physical and kinematic characteristics of the typical in relation to the future evolution of the universe. Furthermore, we explore the weak energy conditions, dominant energy conditions, and strong energy conditions within our typical. Our results indicate that the universe is undergoing acceleration, and that this phenomenon is attributed to the existence of bulk viscosity matter.

Keywords: $f(R, \Sigma, T)$ Gravity; LRS Bianchi type-I space–time; Bulk viscosity; Quadratic deceleration parameter

Mathematics Subject Classification: 83D05; 83B05

1. Introduction

During the last century, numerous observational studies in cosmology, such as those conducted by Riess et al.[1], Eisenstein et al. [2], Astier et al. [3], Naess et al. [4] and Ade et al. [5]**,** provided accurate evidence of the acceleration of the universe. As a result, researchers began to utilize various deceleration parameters in their investigations of the universe's evolution during its late, current, and advanced stages. These deceleration parameters include the linearly varying deceleration parameter (LVDP) [6], the periodically varied deceleration parameter (PVDP) [7, 8], the periodic universe with the varying deceleration parameter of the second degree (PUVDP) [9], the varying polynomial deceleration parameter [10], to mention a few [11, 12]. The current research focuses on examining the evolution of the universe by employing modified versions of Einstein's field equations of general relativity, which can be derived through the Einstein-Hilbert action principle. In these modified theories, the matter Lagrangian is substituted with a customizable function, making them a compelling option for addressing the mysteries of the accelerated expansion of the universe and the dark energy problem. In 2007, a specific modified field theory was suggested, wherein the conventional Einstein-Hilbert action was altered by incorporating an arbitrary function $f(R)$, which is combined with the matter Lagrangian density l_{m}. This theory provides a viable gravitational alternative to dark energy and can explain both the early universe and the late-time cosmic acceleration of the universe. Numerous studies have been conducted to explore

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and refine this theory [13–26]. In 2011, a new modified theory of gravity called $f(R,T)$-gravity was proposed by Harko et al. [27]. The gravitational component of this theory is dependent on the Ricci scalar and a function of the trace of the stress energy–momentum tensor. The theory's applications include cosmology in scalar-tensor $f(R,T)$ gravity [28], gravitational wave echoes from compact stars in $f(R,T)$ gravity [29], and many other important lines of research, as documented in Refs,[30–32]. More recently, in 2023, Bakry and Ibraheem have proposed a new modified theory of gravity called $f(R,\sum,T)$-gravity [33]. The field equation for $f(R,\sum,T)$ gravity is derived by considering the metric-dependent Lagrangian density l_{m} and employing the Hilbert-Einstein variational principle. Similarly, the action for $f(R,\sum,T)$ gravity is described in accordance with the reference [33].

$$I=\frac{1}{16\pi}\int_{\Omega}\sqrt{-g}\Big(f(R,\sum,T)+l_{\mathrm{m}}\Big)d^4x, \tag{1}$$

where $g=\det|g_{i\alpha}|$, l_{m} is the matter Lagrangian density,

The equations governing $f(R,\sum,T)$-gravity can be derived by taking the variation of the action in Eq. (1) with respect to the metric tensor. A detailed explanation of this method can be found in Bakry and Ibraheem's work [33].

$$\begin{aligned}R_{\alpha\beta}\frac{\partial f}{\partial R}&+\sum_{\alpha\beta}\frac{\partial f}{\partial\Sigma}-\frac{1}{2}g_{\alpha\beta}f+(g_{\alpha\beta}\nabla^{\gamma}\nabla_{\gamma}\\&-\nabla_{\alpha}\nabla_{\beta})\left(\frac{\partial f}{\partial R}-\frac{\partial f}{\partial\Sigma}\right)\\&=8\pi T_{\alpha\beta}+(T_{\alpha\beta}+Pg_{\alpha\beta})\frac{\partial f}{\partial T}.\end{aligned} \tag{2}$$

In the context of this article, our assumption that $f(R,\sum,T)$-gravity is given by,

$$f(R,\sum,T)=R+\sum+2\eta T, \tag{3}$$

where η is an arbitrary parameter.

In this case, the field equations are given as

$$R_{\alpha\beta}+\sum_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}(R+\sum)=2(4\pi+\eta)T_{\alpha\beta}+\eta g_{\alpha\beta}(T+2P), \tag{4}$$

where,

$$R_{\alpha\gamma}=\left\{\begin{matrix}\beta\\\alpha\gamma\end{matrix}\right\}_{,\beta}-\left\{\begin{matrix}\beta\\\alpha\beta\end{matrix}\right\}_{,\gamma}+\left\{\begin{matrix}k\\\alpha\gamma\end{matrix}\right\}\left\{\begin{matrix}\beta\\k\beta\end{matrix}\right\}-\left\{\begin{matrix}k\\\alpha\beta\end{matrix}\right\}\left\{\begin{matrix}\beta\\k\gamma\end{matrix}\right\}, \tag{5}$$

$$R=g^{\alpha\gamma}R_{\alpha\gamma}, \tag{6}$$

and the torsion tensor and torsion scalar are,

$$\sum_{\alpha\gamma}=b\Big(\psi^{\beta}_{\alpha\gamma;\beta}-\psi^{\beta}_{\alpha\beta;\gamma}\Big)+b^2\Big(\psi^{k}_{\alpha\gamma}\psi^{\beta}_{k\beta}-\psi^{k}_{\alpha\beta}\psi^{\beta}_{k\gamma}\Big), \tag{7}$$

$$\sum=g^{\alpha\gamma}\sum{}_{\alpha\gamma}, \tag{8}$$

The comma symbol (,) and the semicolon symbol (;) represent ordinary partial differentiation and covariant differentiation, respectively. The contortion tensor is defined as follows [34]:

$$\psi^{\gamma}_{\alpha\beta}=\Gamma^{\gamma}_{\alpha\beta}-\left\{\begin{matrix}\gamma\\\alpha\beta\end{matrix}\right\}, \tag{9}$$

where the affine connection $\Gamma^{\gamma}_{\alpha\beta}$ is defined as [34]

$$\Gamma^{\gamma}_{\alpha\beta}=\lambda^{\gamma}_{i}\lambda_{i\alpha,\beta}=-\lambda_{i\alpha}\lambda^{\gamma}_{i,\beta}, \tag{10}$$

where $\lambda_i{}^{\mu}$ is a tetrad vector defined in the conventional Absolute Parallelism –space (AP).

The covariant component of $\lambda_i{}^{\mu}$ is defined as,

$$\lambda_i{}^{\gamma}\lambda_{i\alpha}=\delta_{\alpha}{}^{\gamma},\lambda_i{}^{\alpha}\lambda_{j\alpha}=\delta_{ij}. \tag{11}$$

The torsion tensor is defined as [35],

$$\Lambda^{\gamma}_{\alpha\beta}=\Gamma^{\gamma}_{\alpha\beta}-\Gamma^{\gamma}_{\beta\alpha}=-\Lambda^{\gamma}_{\beta\alpha}, \tag{12}$$

The parameterized contortion tensor $\psi^{\gamma}{}_{\alpha\beta}$ is given by [33],

$$\psi_{\alpha\beta\gamma}=\frac{1}{2}(\Lambda_{\alpha\beta\gamma}+\Lambda_{\beta\gamma\alpha}+\Lambda_{\gamma\beta\alpha}). \tag{13}$$

By utilizing the foundational elements of the AP-space, it is possible to define a second-order symmetric tensor and a square line element as follows [36–38]:

$$g_{\alpha\beta}=\eta_{ij}\lambda_{i\alpha}\lambda_{j\beta}\ and\ ds^2=g_{\alpha\beta}dx^{\alpha}dx^{\beta}, \tag{14}$$

where η_{ij} = diag (+ 1, − 1, − 1, − 1).

The general linear connection is defined as [34, 39]

$$\nabla^{\gamma}_{\alpha\beta}=\left\{\begin{matrix}\gamma\\\alpha\beta\end{matrix}\right\}+b\psi^{\gamma}_{\alpha\beta}. \tag{15}$$

The interaction between torsion and space–time generated by the background field can be understood through the connection (15), where $b\neq 1$ is a dimensionless parameter. The COW experiment conducted by Wanas et al. provided insight into the physical interpretation of this parameter. The values of b are determined by the results of this experiment and are given in [39].

$$b=\frac{n}{2}\alpha\gamma, \tag{16}$$

where $n=0, 1, 2, 3, \ldots$, α and γ is the natural number, the fine structure constant, and the free parameter, respectively.

The curvature tensor is given by [34]

$$\overline{B}^{\varepsilon}_{\alpha\beta\gamma} = \nabla^{\varepsilon}_{\alpha\gamma,\beta} - \nabla^{\varepsilon}_{\alpha\beta,\gamma} + \nabla^{k}_{\alpha\gamma}\nabla^{\varepsilon}_{k\beta} - \nabla^{k}_{\alpha\beta}\nabla^{\varepsilon}_{k\gamma}, \tag{17}$$

This gives,

$$\overline{B}^{\varepsilon}_{\alpha\beta\gamma} = R^{\varepsilon}_{\alpha\beta\gamma}\{\} + \sum_{\alpha\beta\gamma}^{\varepsilon}, \tag{18}$$

$$\overline{B} = g^{\alpha\gamma}\overline{B}_{\alpha\gamma} = R\{\} + \sum \tag{19}$$

From Eqs.(3) and (19), we can rewrite $f(R, \sum, T)$ as follows

$$f(R, \sum, T) = f(\overline{B}, T) \tag{20}$$

where $\overline{B}$ represents the effect of the gravitational action with torsion.

When $b = 0(ie.\, \Sigma = 0)$, the field equations presented in (4) describe the behavior of $f(R, \Sigma, T)$-gravity, where $b<0$ corresponds to a strong gravitational field, while $b>1$ represents a strong anti-gravitational field, as noted in [33].

2. Some important mathematical accounts

In this section, we will define the tetrad field $\lambda_i^{\ \mu}$ for the spatially homogeneous LRS Bianchi type-I. In the curvature coordinates, the tetrad field in the coordinate $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ is given by,

$$\lambda_i^{\ \mu} = diag(1,\ 1/A,\ 1/B,\ 1/B) \tag{21}$$

where A and B are undetermined functions of only t.

Using (11) and (21), one gets

$$\lambda_{i\mu} = diag(1,\ A,\ B,\ B) \tag{22}$$

Substituting from (22) into (14), one obtains

$$g_{\alpha\beta} = diag\left(1,\ -A^2,\ -B^2,\ -B^2\right). \tag{23}$$

Substituting from (23) into (14), we get the line element

$$ds^2 = dt^2 - A^2\,dx^2 - B^2(dy^2 + dz^2). \tag{24}$$

The scale factor a is defined as,

$$a^3 = V = A\,B^2, \tag{25}$$

where V is a spatial volume.

Calculating the Christoffel symbols using the metric (24), one gets

$$\begin{aligned} &\left\{\begin{matrix} 0 \\ 1\ 1 \end{matrix}\right\} = A\frac{dA}{dt}, \quad \left\{\begin{matrix} 0 \\ 2\ 2 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 3\ 3 \end{matrix}\right\} = B\frac{dB}{dt}, \\ &\left\{\begin{matrix} 1 \\ 0\ 1 \end{matrix}\right\} = \left\{\begin{matrix} 1 \\ 1\ 0 \end{matrix}\right\} = \frac{1}{A}\frac{dA}{dt}, \quad \left\{\begin{matrix} 2 \\ 0\ 2 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 2\ 0 \end{matrix}\right\} \\ &\qquad = \left\{\begin{matrix} 3 \\ 0\ 3 \end{matrix}\right\} = \left\{\begin{matrix} 3 \\ 3\ 0 \end{matrix}\right\} = \frac{1}{B}\frac{dB}{dt}. \end{aligned} \tag{26}$$

The non-vanishing components of the general connection $\Gamma^{\alpha}_{\beta\gamma}$ are given by

$$\Gamma^{1}_{10} = \frac{1}{A}\frac{dA}{dt}, \Gamma^{2}_{20} = \Gamma^{3}_{30} = \frac{1}{B}\frac{dB}{dt}, \tag{27}$$

Also, the contortion tensor is given by,

$$\begin{aligned} &\Psi^{0}_{11} = -A\frac{dA}{dt}, \quad \Psi^{0}_{22} = \Psi^{0}_{33} = -B\frac{dB}{dt}, \\ &\Psi^{1}_{01} = -\frac{1}{A}\frac{dA}{dt}, \quad \Psi^{2}_{02} = \Psi^{3}_{03} = -\frac{1}{B}\frac{dB}{dt}. \end{aligned} \tag{28}$$

The non-vanishing components of the torsion tensor $\Lambda^{\gamma}{}_{\alpha\beta}$ are,

$$\begin{aligned} &\Lambda^{1}{}_{10} = -\Lambda^{1}{}_{01} = \frac{1}{A}\frac{dA}{dt},\ \Lambda^{2}{}_{20} = -\Lambda^{2}{}_{02} = \Lambda^{3}{}_{30} = -\Lambda^{3}{}_{03} \\ &\quad = \frac{1}{B}\frac{dB}{dt}. \end{aligned} \tag{29}$$

It follows from Eqs. (12) and (13) that the contortion tensor has only one independent contraction. Specifically, we can express this contraction as follows:

$$\Lambda_{\alpha} = \Lambda^{\beta}{}_{\alpha\beta} = \psi^{\beta}{}_{\alpha\beta} = -\Lambda^{\beta}{}_{\beta\alpha} \tag{30}$$

where Λ_{α} is called a torsion vector.

The torsion vector is obtained as Λ_{α} is

$$\Lambda_0 = \Lambda_t = -\left(\frac{1}{A}\frac{dA}{dt} + \frac{1}{B}\frac{dB}{dt}\right) \tag{31}$$

3. Field equations and solution

Bulk viscous fluid is a type of fluid that exhibits both viscosity and volume elasticity. In the field of cosmology, it is often chosen as a model for the dark energy component of the universe. Unlike other models, such as the cosmological constant, bulk viscous fluid allows for the possibility of an evolving energy density. One of the main reasons for choosing bulk viscous fluid in cosmology is its ability to account for the accelerated expansion of the universe. The presence of viscosity in the fluid introduces dissipative effects, which can act as a driving force for the accelerated expansion. This is in contrast to other models where the acceleration is attributed to a constant energy density. The energy momentum tensor for bulk viscous fluid is considered in the form [40–42]

$$T_{\alpha\beta} = (\rho + \overline{P})u_\alpha u_\beta - g_{\alpha\beta}\overline{P}, \tag{32}$$

The bulk viscous pressure

$$\overline{P} = P - 3\zeta H, \tag{33}$$

where the isotropic pressure $P = \gamma\rho$, $0 < \gamma < 1$, ζ is a coefficient of bulk viscosity and $u^\alpha = \delta_0^\alpha$ is the four velocity vector satisfying $u^\alpha u_\alpha = 1$.

From (32), the trace of energy momentum tensor is given as

$$T = \rho - 3\overline{P}. \tag{34}$$

Substituting from (26), (28), (32) and (33) into the field Eqs. (4), one gets

$$\frac{(b-1)^2}{B^2}\left(\frac{dB}{dt}\right)^2 + \frac{2(b-1)^2}{AB}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right) = (8\pi + 3\eta)\rho - \eta\overline{P}, \tag{35}$$

$$\frac{2(b-1)}{B}\left(\frac{d^2B}{dt^2}\right) - \frac{(b-1)^2}{B^2}\left(\frac{dB}{dt}\right)^2 = (8\pi + 3\eta)\overline{P} - \eta\rho, \tag{36}$$

$$\frac{(b-1)}{A}\left(\frac{d^2A}{dt^2}\right) + \frac{(b-1)}{B}\left(\frac{d^2B}{dt^2}\right) - \frac{(b-1)^2\dot{A}\dot{B}}{AB}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right) = (8\pi + 3\eta)\overline{P} - \eta\rho. \tag{37}$$

Equations (35–37) consist of three equations involving four unknowns:A, B, ρ, and $\overline{P}$. Therefore, to solve these equations, an additional equation is required. In this regard, we employ the quadratic deceleration parameter, as discussed in the paper [9].

$$q(t) = (8n^2 - 1) - 12nt + 3t^2, \tag{38}$$

where $n > 0$ is an arbitrary constant. The relation between the deceleration parameter and the Hubble parameter is $q(t) = \frac{dH^{-1}}{dt} - 1$ [43],

Thus, Eq. (38) gives

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{t(2n-t)(4n-t)}. \tag{39}$$

From the above equation, we obtain

$$a(t) = \left(\frac{t(4n-t)}{(2n-t)^2}\right)^{1/8n^2}, \tag{40}$$

Using (25) and (40), the values of A; B and their derivatives are given by,

$$A(t) = \left(\frac{t(4n-t)}{(2n-t)^2}\right)^{1/4n^2}, \frac{dA}{dt} = \frac{2}{(2n-t)^3}\left(\frac{t(4n-t)}{(2n-t)^2}\right)^{\frac{1}{4n^2}-1}, \tag{41}$$

$$B(t) = \left(\frac{t(4n-t)}{(2n-t)^2}\right)^{1/16n^2}, \frac{dB}{dt} = \frac{1}{2(2n-t)^3}\left(\frac{t(4n-t)}{(2n-t)^2}\right)^{\frac{1}{16n^2}-1}, \tag{42}$$

$$\frac{d^2A}{dt^2} = \frac{8}{(2n-t)^4}\left(\frac{t(4n-t)}{(2n-t)^2}\right)^{\frac{1}{4n^2}-1}\left(\frac{(1-4n^2)}{(2n-t)^2}\left(\frac{t(4n-t)}{(2n-t)^2}\right)^{-1}\right), \tag{43}$$

$$\frac{d^2B}{dt^2} = \frac{1}{4(2n-t)^4}\left(\frac{t(4n-t)}{(2n-t)^2}\right)^{\frac{1}{16n^2}-1}\left(\frac{(1-16n^2)}{(2n-t)^2}\left(\frac{t(4n-t)}{(2n-t)^2}\right)^{-1}\right). \tag{44}$$

Therefore, the metric (24) takes the following form

$$ds^2 = dt^2 - \left(\frac{t(4n-t)}{(2n-t)^2}\right)^{1/2n^2}dx^2 - \left(\frac{t(4n-t)}{(2n-t)^2}\right)^{1/8n^2}(dy^2 + dz^2). \tag{45}$$

Solving the field Eqs. (35–37), we get

$$\rho = \frac{1}{(8\pi+3\eta)^2 - \eta^2}\left(\frac{2(4\pi+\eta)(b-1)^2}{B^2}\left(\frac{dB}{dt}\right)^2, + \frac{2(8\pi+3\eta)(b-1)^2}{AB}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right) + \frac{2\eta(b-1)}{B}\left(\frac{d^2B}{dt^2}\right)\right) \tag{46}$$

$$\bar{P} = \frac{1}{(8\pi+3\eta)^2 - \eta^2}\left(\frac{\eta(b-1)^2}{B}\left(\frac{1}{B}\left(\frac{dB}{dt}\right)\right)^2 + \frac{2}{A}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right), + \frac{(8\pi+3\eta)(b-1)}{B}\left(2\left(\frac{d^2B}{dt^2}\right)\right) + \frac{(b-1)}{B}\left(\frac{dB}{dt}\right)^2\right) \tag{47}$$

and the pressure is given by,

$$P = \gamma\rho = \frac{\gamma}{(8\pi+3\eta)^2 - \eta^2}\left(\frac{2(4\pi+\eta)(b-1)^2}{B^2}\left(\frac{dB}{dt}\right)^2 + \frac{2(8\pi+3\eta)(b-1)^2}{AB}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right) + \frac{2\eta(b-1)}{B}\left(\frac{d^2B}{dt^2}\right)\right). \tag{48}$$

Also, the bulk viscous coefficient is,

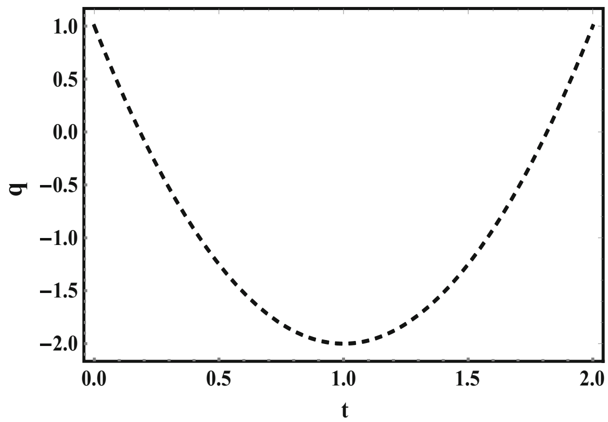

Fig. 1 q versus cosmic time

$$\zeta = (P - \overline{P})/3H. \tag{49}$$

Figure 1, indicates that the quadratic deceleration parameter q is a positive value at $t = 0$ (Big Bang) with the acceleration of the universe, and that it moved to a negative value until it reached $t = 1$ (Big Rip), then it reached the stage of a Big Crunch at $t = 2$. For more details, see Ref.[9]. From the observational results (Cunha 2009), the present value of the deceleration parameter is $q = -0.73$. Accordingly, we take $t_{\text{day}} = 0.35$ as a unit representing the now time(where one unit = 39.4 Gyr, $t_{\text{day}} = 13.8$ Gyr), with $q_{\text{day}} = -0.73$.Fig. 2 indicates that the scale factor $a = 0$ at $t = 0$ (Big Bang) with the acceleration of the universe, and that it became infinite at $t = 1$(Big Rip at $t_{\text{BR}} = 39.4$ Gyr); then it reached the stage of the Big Crunch at $t = 2$, and $a = 0$. For more details, see Ref.[9].

The results from using the quadratic deceleration parameter and its agreement with astronomical observations have been studied previously; see Ref.[9]. As we mentioned previously, the aim of this paper is to study the behavior of the universe in different fields. In $f(R, \sum, T)$-gravity, we can define three different fields: neutral gravity $(b = 0)$, strong gravity field $(b = -1)$, and strong anti-gravity field $(b = 2)$; please refer to Ref. [33]. This theory also includes other theories as a special case. For example, when $(b = \eta = 0)$, the theory refers to $f(R)$ gravity; when $(\eta = 0)$, it refers to a gravitational field with torsion, see ref.[44], and when $(b = 0)$, it gives $f(R, T)$,see Ref [27].

Figure 3 plots the relationship between the energy density ρ and t with three values of b. In the case $f(R, T)$-gravity $(b = 0)$ and the strong gravity field $(b = -1)$, the energy density is positive throughout the universe. But at the strong anti-gravity field $(b = 2)$, the energy density changes from a negative value at the Big Bang $(t = 0)$ to a positive value at the Big Rip $(t = 1)$, then it returns to a negative value again at the Big Crunch $(t = 2)$. Figure 4 depicts the behavior of the bulk viscous pressure $\overline{P}$ with three values of b. In the case $f(R, T)$-gravity $(b = 0)$ and the strong gravity field $(b = -1)$, the bulk viscous pressure density changes from a positive value at the Big Bang $(t = 0)$ to a negative value at the Big Rip $(t = 1)$, then it returns to a positive value again at the Big Crunch $(t = 2)$.At the strong anti-gravity field $(b = 2)$, the bulk viscous pressure is negative throughout the universe. Figures 3, 4, 5 and 6 allow a comparison of the theories under study with the evolution of the behavior of the universe. Figures 7, 8 and 9 also give us an idea of the extent to which the energy conditions are met for the typicals presented in the different fields.

Figure 5 represents the behaviour of the isotropic pressure P against cosmic time t. In the case of neutral gravity field $(b = 0)$ and the strong gravity field $(b = -1)$, the energy density is positive throughout the universe. But at the strong anti-gravity field $(b = 2)$, the energy density changes from a negative value at $(t = 0)$ to a positive value at $(t = 1)$, then it returns to a negative value again at $(t = 2)$. Figure 5 represents the behaviour of the isotropic pressure P against cosmic time t. Figure 6 plots the

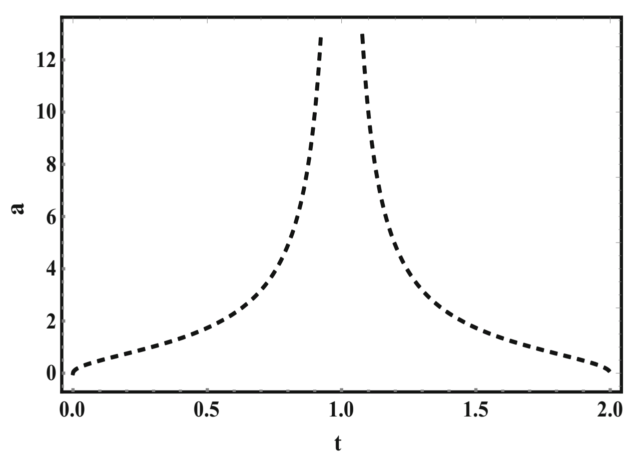

Fig. 2 a versus cosmic time

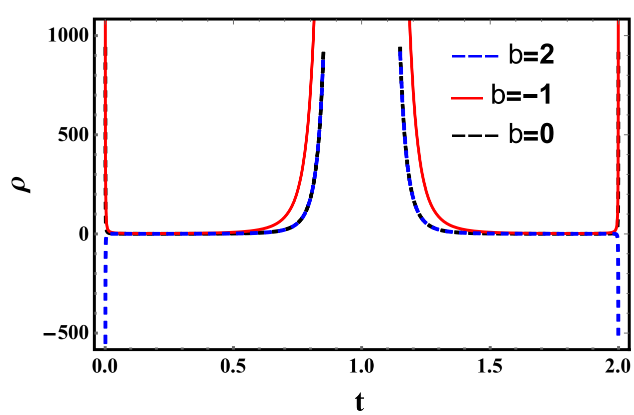

Fig. 3 ρ versus cosmic time with $\eta = 1$ and different values of b.

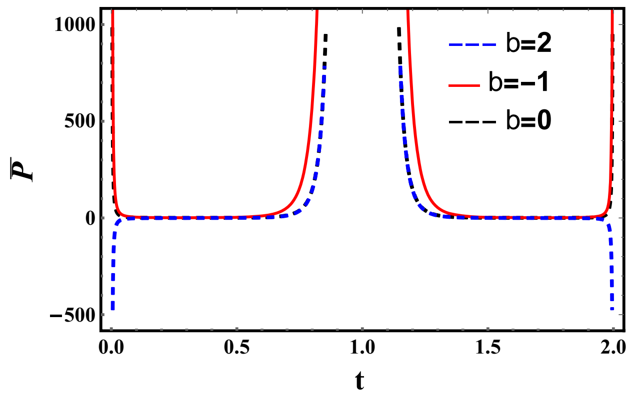

Fig. 4 $\overline{P}$ versus cosmic time with $\eta = 1$ and different values of b.

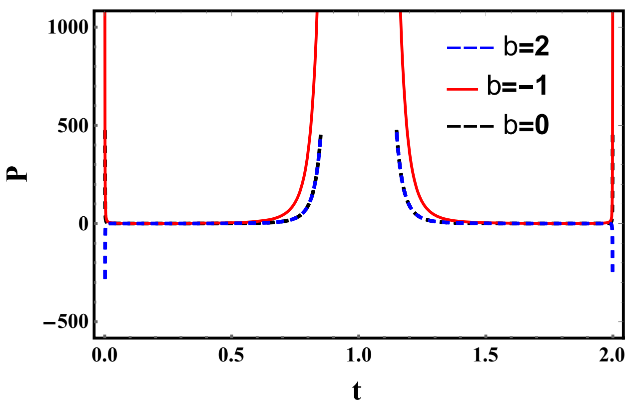

Fig. 5 P versus cosmic time with $\eta = 1; \gamma = 1/2$ and different values of b

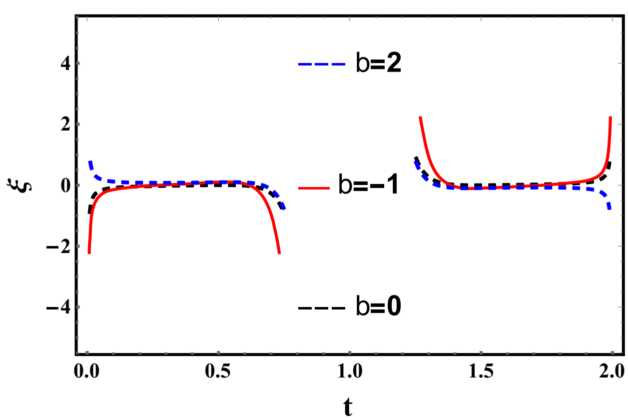

Fig. 6 ζ versus cosmic time with $\eta = 1; \gamma = 1/2$ and different values of b

dynamics of the bulk viscous coefficient ζ against cosmic time t. In the case $(b = 0)$ and $(b = -1)$**,** the bulk viscous coefficient changes from a negative value at $(t = 0)$ to a positive value at $(t = 1)$, and then it enters to a negative value again at $(t = 2)$. But at $(b = 2)$, the bulk viscous

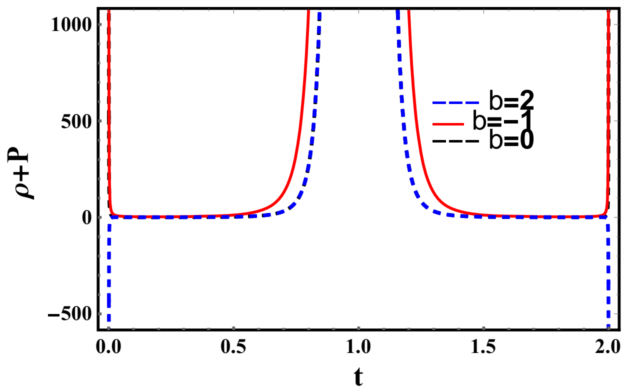

Fig. 7 $\rho + P$ versus cosmic time with $\eta = 1; \gamma = 1/2$ and different values of b

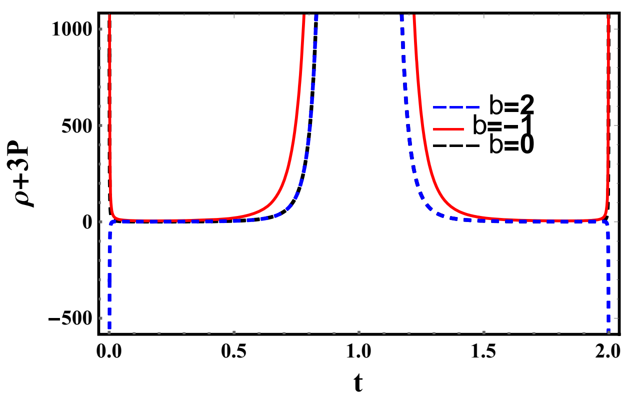

Fig. 8 $\rho + 3P$ versus cosmic time with $\eta = 1; \gamma = 1/2$ and different values of b

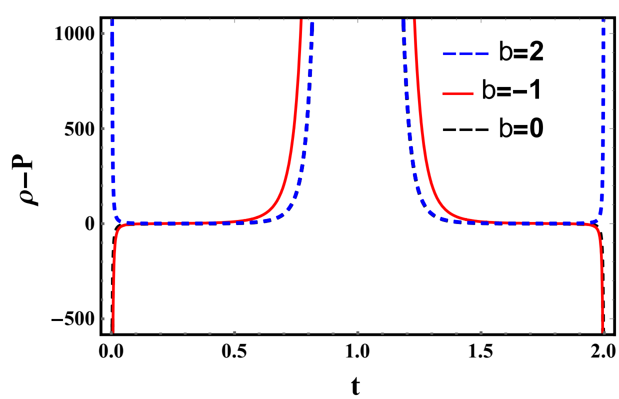

Fig. 9 $\rho - P$ versus cosmic time with $\eta = 1; \gamma = 1/2$ and different values of b

coefficient is positive at $t \in [0, 1[$ and it is negative at $t \in]0, 2]$.

4. The energy conditions

Energy conditions serve as boundary conditions that ensure the positivity of the energy density, as proposed by Ashton [45] and Hoehler [46]. However, these conditions do not necessarily reflect physical reality. The violation of strong energy conditions, which is evident in the observable effects of dark energy, is the most recent illustration of this fact. There are three key energy conditions that are particularly noteworthy:

(i) Weak energy condition

$$(WEC):\ \rho+P\geq 0,\ \rho\geq 0. \tag{50}$$

(ii) Null energy condition

$$(NEC):\ \rho+P\geq 0 \tag{51}$$

(iii) Strong energy condition

$$(SEC):\ \rho+3P\geq 0\,\rho+P\geq 0. \tag{52}$$

(iv) Dominant Energy Condition

$$(DEC):\ \rho\pm P\geq 0,\rho\geq 0 \tag{53}$$

Figures 7 and 8 depict the dynamics of $(\rho+P)$ and $(\rho+3P)$ against cosmic time t. In case $f(R,T)$-gravity $(b=0)$ and the strong gravity field $(b=-1)$,$(\rho+P)$ and $(\rho+3P)$ are positive throughout the universe. In the case of the strong anti-gravity field $(b=2)$, they are a negative value at the Big Bang $(t=0)$, come into a positive value at a Big Rip $(t=1)$, and then enter a negative value again at the Big Crunch $(t=2)$. Figure 9 shows the behavior of the dynamics of $(\rho-P)$ against cosmic time t. In case $(b=0)$ and $(b=-1)$, we see that $(\rho-P)$ is a negative value at a $(t=0)$, enters a positive value at $(t=1)$, and then comes into a negative value again at $(t=2)$. Also, in case $(b=2)$, it is positive throughout the universe.

The results may be obtained from Table 1 and 2 as follows.

The results presented in Table 1 are discussed previously when illustrating Figs.4, 5 and 6.

5. The behaviour of the function $f(R,\sum,T)$

The values of the Ricci scalar R, the torsion scalar $\sum$ and the trace of the matter source T are obtained as

$$R=-\left(\frac{2}{A}()()+\frac{4}{B}\left(\frac{d^2B}{dt^2}\right),\right.$$
$$\left.+\frac{4}{A\,B}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right)+\frac{1}{B^2}\left(\frac{dB}{dt}\right)^2\right) \tag{54}$$

$$\sum=\left(\frac{2b}{A}\left(\frac{d^2A}{dt^2}\right)+\frac{4\,b}{B}\left(\frac{d^2B}{dt^2}\right)-\frac{4b(b-2)}{AB}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right),\right.$$
$$\left.-\frac{b(b-2)}{B^2}\left(\frac{dB}{dt}\right)^2\right) \tag{55}$$

$$T=\frac{1}{(8\pi+3\eta)^2-\eta^2}$$
$$\left(\frac{2(4\pi+\eta)(b-1)^2}{B^2}\left(\frac{dB}{dt}\right)^2+\frac{2(8\pi+3\eta)(b-1)^2}{AB}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right),\right.$$
$$+\frac{2\eta(b-1)}{B}\left(\frac{d^2B}{dt^2}\right)-\frac{3\eta(b-1)^2}{B}\left(\frac{1}{B}\right)\left(\frac{dB}{dt}\right)^2+\frac{2}{A}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right)$$
$$\left.-\frac{3(8\pi+3\eta)(b-1)}{B}\left(2\left(\frac{d^2B}{dt^2}\right)\right)+\frac{(b-1)}{B}\left(\frac{dB}{dt}\right)^2\right) \tag{56}$$

Accordingly,

$$f(R,\prod,T)=R+\Sigma+2\eta\,T$$
$$=\left(\begin{array}{c}\frac{2\,(b-1)}{A}\left(\frac{d^2A}{dt^2}\right)-\frac{4\,(b-1)^2}{A\,B}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right)-\frac{(b-1)^2}{B^2}\left(\frac{d^2B}{dt^2}\right)+\\ \frac{2\eta}{(8\pi+3\eta)^2-\eta^2}\left(\begin{array}{c}\frac{2(4\pi+\eta)(b-1)^2}{B^2}\left(\frac{dB}{dt}\right)^2+\frac{2(8\pi+3\eta)(b-1)^2}{A\,B}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right)+\\ \frac{2\eta(b-1)}{B}\left(\frac{d^2B}{dt^2}\right)-\frac{3\,\eta\,(b-1)^2}{B}\left(\frac{1}{B}\left(\frac{dB}{dt}\right)\right)^2+\frac{2}{A}\left(\frac{dA}{dt}\right)\left(\frac{dB}{dt}\right)-\\ \frac{3\,(8\pi+3\eta)(b-1)}{B}\left(2\left(\frac{d^2B}{dt^2}\right)\right)+\frac{(b-1)}{B}\left(\frac{dB}{dt}\right)^2)\end{array}\right)\end{array}\right). \tag{57}$$

Table 1 Evolution behavior of the bulk viscous pressure, the isotropic pressure and the coefficient of bulk viscosity

Typical	$\overline{P}$	Interval	P	Interval	ζ	Interval
$b=0$	$+\to --\to +$	$t\in[0,1[$	$+$	$t\in[0,2]$	$-$	$t\in[0,1[$
		$t\in]1,2]$			$+$	$t\in]1,2]$
$b=-1$	$+\to -$	$t\in[0,1[$	$+$	$t\in[0,2]$	$-$	$t\in[0,1[$
	$-\to +$	$t\in]1,2]$			$+$	$t\in]1,2]$
$b=2$	$-$	$t\in[0,2]$	$-\to +$	$t\in[0,1[$	$+\to -$	$t\in[0,1[$
			$+\to -$	$t\in]1,2]$	$+\to -$	$t\in]1,2]$

Figure 10 shows the behaviour of the function $f(R,\Sigma,T)=R+\Sigma+2\eta T$. In case $(b=0)$, they are a positive value at $(t=0)$**,** come into a negative value at $(t=1)$, and then enter a positive value again at $(t=2)$. In the case of the strong gravity field $(b=-1)$, the function $f(R,\Sigma,T)$ is positive throughout the universe. In the case of the anti-gravity field $(b=2)$, the behaviour of the function $f(R,\Sigma,T)$ is negative throughout the universe (Fig. 11)

6. Non-singularity conditions of our cosmological typical

The modified Raychaudhuri equation, which was introduced by Wanas and Bakry in 2009, represents a generalized form of the original Raychaudhuri equation [47]. It is widely utilized in the study of the singularity problem in cosmology. The modified Raychaudhuri equation takes into account the proposed interaction between the quantum spin of a moving elementary particle and the torsion of the underlying gravitational field. The objective is to investigate the influence of torsion on the existence of an initial singularity in our proposed cosmological typicals. For this purpose, a modified version of the Raychaudhuri equation that incorporates the torsion term can be defined as follows [48].

$$\frac{d\theta}{d\tau}=2(\omega^2-\sigma^2)-\frac{1}{3}\theta^2-Z^\alpha Z^\beta\overline{B}_{\beta\alpha}+bZ^\alpha Z^\mu_{||\beta}\Lambda^\beta_{\alpha\mu},\tag{58}$$

where θ is the expansion scalar, ω is the rotation scalar, σ is the shear scalar, and $Z^\alpha=\frac{dx^\alpha}{d\tau}$ are the velocity components and the general parameterized absolute derivatives defined earlier as [34]

$$A_{\alpha||\beta}=A_{\alpha,\beta}-A_\gamma\nabla^\gamma_{\alpha\beta},\tag{59}$$

where $\nabla^\gamma_{\alpha\beta}$ is defined by Eq. (15). When b is equal to zero, Eq. (57) is simplified to the original Raychaudhuri equation.

Upon a lengthy but uncomplicated calculation, the modified Raychaudhuri equation provided in (58) is derived as follows:

$$\frac{d\theta}{d\tau}=(1-b)\left(\left(\frac{1}{A}\left(\frac{d^2A}{dt^2}\right)\right)+\frac{2}{B}\left(\frac{d^2B}{dt^2}\right)-\left(\frac{1}{A^2}\left(\frac{dA}{dt}\right)\right)^2+\frac{2}{B^2}\left(\frac{dB}{dt}\right)^2\right).\tag{60}$$

The existence of the initial singularity is primarily determined by the solution of Eq. (60), which depends on the sign of the term on the left-hand side $(\mathrm{d}\theta/\mathrm{d}\tau)$. This conclusion can also be inferred utilizing the standard conventions of the singularity theorems GR. In essence, the defining feature of singular typicals is their $\mathrm{d}\theta/\mathrm{d}\tau<0$.

7. Conclusions

We have reconnoitered the LRS Bianchi Type I with an anisotropic bulk viscosity matter cosmological typical in the presence of bulk viscosity in the scope of $f(R,\sum,T)$ gravity. Resorting to the choice of $f(R,\sum,T)$, we have presented three cosmological typicals. The exact solutions of the modified Einstein's field equations $f(R,\sum,T)$ have been obtained under the choice of the quadratic deceleration Parameter. The comments of the three typicals are as follows: The typical $(b=0)$ in the scope of $f(R,\sum,T)$ gravity leads to $f(R,T)$ gravity. The typical $(b=-1)$ in the framework of $f(R,\sum,T)$ represents the strong gravitational field with torsion, while at $(b=2)$ it represents the strong anti-gravitational field with torsion. The three typicals presented here are accelerating, and the expanding universe typical follows the quadratic deceleration Parameter. Energy density is a positive value in the cases $(b=0,-1)$, and also $\rho\to\infty$ when $t=0,1,2$.. In case $(b=2)$, energy density increases when $t\in[0,1]$, and it decreases at $t\in]0,2]$.The coefficient of bulk viscosity ζ is a negative value at $t\in[0,1]$, while it is a positive value at $t\in]0,2]$ in both cases $(b=0,-1)$. The bulk viscous pressure $\overline{P}$ decreases when $t\in[0,1]$, and it increases at $t\in]0,2]$. Energy conditions *WEC*, *SEC*, and *NEC* are satisfied for the typicals ($b=0$ and $b=-1$), while DEC

Table 2 The energy conditions for our typical

typical	$\rho \geq 0$	Interval	$\rho + P \geq 0$	Interval	$\rho - P \geq 0$	Interval	$\rho + 3P \geq 0$	Interval
$b = 0$	$+$	$t \in [0, 2]$	$+$	$t \in [0, 2]$	$- \rightarrow +$ $+ \rightarrow -$	$t \in [0, 1[$ $t \in]1, 2]$	$+$	$t \in [0, 2]$
$b = -1$	$+$	$t \in [0, 2]$	$+$	$t \in [0, 2]$	$- \rightarrow +$ $+ \rightarrow -$	$t \in [0, 1[$ $t \in]1, 2]$	$+$	$t \in [0, 2]$
$b = 2$	$- \rightarrow ++ \rightarrow -$	$t \in [0, 1[t \in]1, 2]$	$- \rightarrow +$ $+ \rightarrow -$	$t \in [0, 1[t \in]1, 2]$	$+$	$t \in [0, 2]$	$- \rightarrow +$ $+ \rightarrow -$	$t \in [0, 1[$ $t \in]1, 2]$

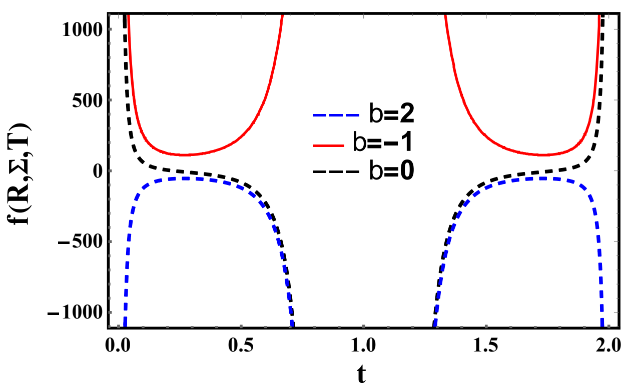

Fig. 10 Behaviour of $f(R, \sum, T)$ versus cosmic time t with $\eta = 1$; and different values of b

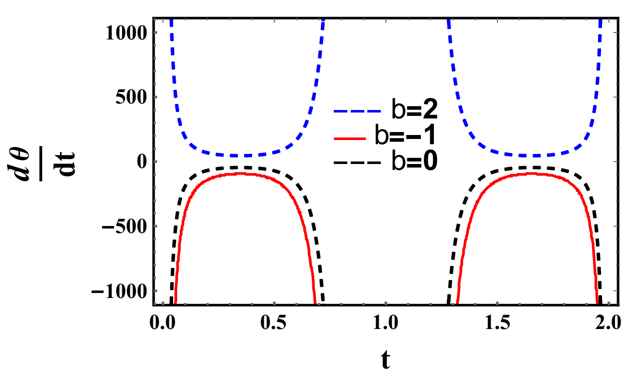

Fig. 11 Behaviour of $\mathrm{d}\theta/\mathrm{d}t$ versus cosmic time t with $\eta = 1$; and different values of b

are violated for these typical. Also, we have observed that all the energy conditions are violated for the typical $b = 2$. The function $f(R, \sum, T) = R + \sum + 2\eta T$ is a positive value in the cases $(b = -1)$, and it is a negative value in the cases $(b = 2)$, while it decreases when $t \in [0, 1]$, and it increases at $t \in]0, 2]$. The $f(R, \sum, T)$-gravity is a modified theory of gravity that incorporates both the Ricci scalar (R) and the trace of the energy–momentum tensor and torsion scalar in the gravitational action. This theory offers several distinctive features compared to other gravitational theories:

Extension of General Relativity: The $f(R, \sum, T)$-gravity extends the framework of General Relativity by considering additional terms in the gravitational action. By including the trace of the energy–momentum tensor and torsion tensor**,** it accounts for the effects of matter and torsion on the geometry of space–time. Dynamic Nature: Unlike many other gravitational theories, the $f(R, \sum, T)$-gravity allows for a dynamic gravitational constant. The gravitational coupling constant can vary with the energy density of matter, leading to time-dependent gravitational effects. This provides a more flexible and evolving description of gravity. Energy–Momentum Conservation: The $f(R, \sum, T)$-gravity ensures energy–momentum conservation by incorporating the trace of the energy–momentum tensor. This feature is particularly important in cosmological scenarios, where the conservation of energy and momentum plays a crucial role. Explanation of Dark Energy: The $f(R, \sum, T)$-gravity has been proposed as a possible explanation for the accelerated expansion of the universe, attributed to dark energy. By modifying the gravitational action, this theory can reproduce the observed cosmic acceleration without the need for an additional dark energy component. Compatibility with Observations: The $f(R, \sum, T)$-gravity has been tested against various cosmological and astrophysical observations. It has been proved promising in explaining the cosmic expansion history, structure formation, and other phenomena. However, further observational and experimental constraints are still needed to fully validate or refine the theory. The $f(R, \sum, T)$-gravity can explain three areas: gravity, strong-gravity and anti-gravity.

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