Indian J Phys (May 2024) 98(6):2093–2103
https://doi.org/10.1007/s12648-023-02976-6

ORIGINAL PAPER

Supersymmetric approach to approximate analytical solutions of the Klein-Gordon equation: application to a position-dependent mass and a hyperbolic cotangent vector potential

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Received: 16 March 2023 / Accepted: 25 September 2023 / Published online: 12 December 2023

Abstract: In this paper, we study the approximate analytical solutions for bound states of the l-wave Klein-Gordon equation with a position-dependent mass subjected to a hyperbolic cotangent vector potential by using the concept of the supersymmetric quantum mechanics approach. Within the framework of the proper approximation of the centrifugal term, we obtain the bound state energy eigenvalues and the corresponding normalized wavefunctions written down in terms of the Jacobi polynomials. Furthermore, it is found that the solutions in the case of constant mass for nonzero l-values are identical to the ones obtained in the literature. Among these cases, Hulthén potential, Coulomb potential, and nonrelativistic limit are discussed

Keywords: Klein-Gordon equation; Position-dependent mass; Hyperbolic cotangent vector potential; Supersymmetric quantum mechanics; Bound states

1. Introduction

There has been a growing interest to investigate the analytical solutions of the wave equation with certain types of exactly solvable potentials which have become an important area of research in different branches of physics. This is due to the fact that the solutions of the wave equation encompass all necessary information for the quantum system. However, the relativistic wave equation plays an important role in the description of physical phenomena at high energies of a relativistic particle such as Klein-Gordon and Dirac equations.

The exact or approximate analytical solutions of the relativistic wave equations with various potential models have received much attention recently. These solutions are very essential in many aspects of modern physics, chemical physics, molecular spectroscopy and molecular physics. In particular, the Klein-Gordon equation is the most suitably used wave equation for the treatment of spinless particles in relativistic quantum mechanics.

Many authors have investigated the solutions of the Klein-Gordon equation of the spin-zero particle having a constant mass, subject to certain potential models by using different methods. These methods include the path integral approach [1–4], asymptotic iteration method [5, 6], Nikiforov-Uvarov method [7–10], supersymmetric quantum mechanics approach [11, 12], Laplace transform method [13, 14], and other methods [15–19].

The position-dependent mass formalism has been used to describe the physical properties of various microstructures, such as semiconductor heterostructure [20], quantum liquids [21], quantum dots [22], ^{3}He clusters [23], metal clusters [24], compositionally graded crystals [25], and electronic transport [26]. The ordering ambiguity of the mass and momentum operators disappears in a relativistic case, unlike their presence in a non-relativistic case. Therefore, many authors have used different methods to solve the Schrö dinger [27–30], Klein-Gordon [31–35], and Dirac [36–38] wave equations with various kinds of Hermitian potentials in the framework of the position-dependent mass system. In addition, relativistic quantum problems with position-dependent mass in the context of Klein-Gordon and Dirac equations with $\mathcal{PT}$-symmetric

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non-Hermitian potentials have been discussed in several works [39–44]. Moreover, the position-dependent effective mass Klein-Gordon equation for each potential model can be transformed into the constant mass Schrödinger-like equation with energy-dependent potential, which is easier to solve.

In this paper, our aim is to study the approximate analytical solution of any l-wave bounded states for the Klein-Gordon equation of the spinless particle with position-dependent mass distribution function in the presence of a hyperbolic cotangent vector potential by using the approach of supersymmetric quantum mechanics (SUSYQM). These solutions are presented for the arbitrary angular momentum by using the usual approximation scheme of the centrifugal potential term to acquire small values of the screening parameter [45]. The SUSYQM approach is very easy to implement and the results are sufficiently accurate for practical purposes.

2. Review of SUSYQM approach

The introduction of the SUSYQM approach [46] generated renewed interest in solvable problems of relativistic and nonrelativistic quantum mechanics. This approach has been used to determine the bound states solutions of stationary standard one-dimensional Schrödinger [47] or Schrödinger-like equations [35] for certain kinds of potentials in the context of a system with constant mass or position-dependent mass. Moreover, the description of SUSYQM approach can start with the one-dimensional stationary Schrödinger equation [46] for a particle with mass m_0 subject to a potential $U(x)$ which we write in a reduced form as

$$H\varphi_n(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \upsilon(x)\right)\varphi_n(x) = \mathcal{E}_n\varphi_n(x) \text{ for } x \in (a,b), \tag{1}$$

where $\upsilon(x)$ and the bound-states eigenvalues $\mathcal{E}_n$ are linked to the physical potential $U(x)$ and the energy levels E_n by

$$\upsilon(x) = \frac{2m_0}{\hbar^2}U(x) \text{ and } \mathcal{E}_n = \frac{2m_0}{\hbar^2}E_n. \tag{2}$$

For the purposes of the SUSYQM approach, the potential $U(x)$ is assumed to be hermitian and independent of the energy [46].

The SUSYQM approach is based on the definition of two new Hamiltonians, denoted $H^{(-)}(a_1)$ and $H^{(+)}(a_1)$ and called supersymmetric partner Hamiltonians, by means of a function $W(x,a_1)$, called superpotential, as

$$H^{(-)}(a_1) = A^+(a_1)A(a_1) = -\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \upsilon^{(-)}(x,a_1), \tag{3}$$

$$H^{(+)}(a_1) = A(a_1)A^+(a_1) = -\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \upsilon^{(+)}(x,a_1), \tag{4}$$

where $A(a_1)$ and $A^+(a_1)$ are first-order differential operators defined by

$$A(a_1) = \frac{\mathrm{d}}{\mathrm{dx}} + W(x,a_1), \tag{5}$$

$$A^+(a_1) = -\frac{\mathrm{d}}{\mathrm{dx}} + W(x,a_1), \tag{6}$$

and a_1 is a set of parameters (often only one parameter).

From (3), (4), (5) and (6), the potentials $\upsilon^{(\mp)}(x,a_1)$, called also partner potentials, are then given by

$$\upsilon^{(\mp)}(x,a_1) = W^2(x,a_1) \mp \frac{\mathrm{dW(x,a_1)}}{\mathrm{dx}}. \tag{7}$$

The goal is now to obtain $W(x,a_1)$ in such a way as to satisfy the equality

$$H^{(-)}(a_1) = H - \mathcal{E}_0, \tag{8}$$

or equivalently

$$\upsilon^{(-)}(x,a_1) = W^2(x,a_1) - \frac{\mathrm{dW(x,a_1)}}{\mathrm{dx}} = \upsilon(x) - \mathcal{E}_0, \tag{9}$$

where $\mathcal{E}_0$ is the ground-state eigenvalue of H.

Thus, the superpotential $W(x,a_1)$ satisfies a Riccati nonlinear equation, which is often difficult to solve. In practice, we have just to propose a suitable function and proceed by identifying the similar terms between the two sides. This lets us fix $W(x,a_1)$, i.e., the parameters a_1, and $\mathcal{E}_0$ in terms of the constituents of $\upsilon(x)$.

Moreover, if we denote by $\mathcal{E}_n^{(-)}(a_1)$ and $\varphi_n^{(-)}(x,a_1)$, respectively, the eigenvalues and eigenfunctions of $H^{(-)}(a_1)$, then, by virtue of Eq. (8), it is clear that they are linked to those of the Hamiltonian H by

$$\varphi_n(x) = \varphi_n^{(-)}(x,a_1), \tag{10}$$

and

$$\mathcal{E}_n = \mathcal{E}_n^{(-)}(a_1) + \mathcal{E}_0. \tag{11}$$

In other words, the bound states of H are deduced from those of $H^{(-)}(a_1)$ by a simple substitution of the parameters a_1 by their expressions obtained from (9). Furthermore, it is clear that the ground-state eigenvalue of the partner $H^{(-)}(a_1)$ is zero

$$\mathcal{E}_0^{(-)}(a_1) = 0, \tag{12}$$

and form the Schrödinger equation for this state, one easily deduce that the corresponding unnormalized eigenfunction, $\varphi_0^{(-)}(x,a_1)$ is given by

$$\varphi_0^{(-)}(x,a_1) \sim \exp\left(-\int^x W(y,a_1)\mathrm{d}y\right). \tag{13}$$

Thus, the goal is then to solve the Schrödinger equation for $H^{(-)}(a_1)$.However, for certain potentials, called supersymmetric potentials, it is not necessary thanks to Gendenshtein's property of shape invariance between the corresponding partner potentials [48]. Indeed, it is proved that if $\upsilon(x)$ is supersymmetric, then equivalently its corresponding partner potentials satisfy the so-called shape invariance relation [48], given by

$$\upsilon^{(+)}(x;a_1) = \upsilon^{(-)}(x;a_2) + R(a_1), \tag{14}$$

where the set of parameters a_2 are function of a_1, namely, $a_2 = f(a_1)$, and the remainder $R(a_1)$ is independent of x, so that the eigenvalues $\mathcal{E}_n^{(-)}(a_1)$ are simply given by

$$\mathcal{E}_n^{(-)}(a_1) = \sum_{k=1}^{n} R(a_k) \text{ for } n = 1,2,\ldots n_{\max}, \tag{15}$$

where

$$a_k = f^{k-1}(a_1) \equiv \underbrace{f \circ f \circ \ldots \circ f}_{(k-1)\text{ times}}(a_1), \tag{16}$$

and $n_{\max}$ is the maximum number of accepted eigenvalues, that are corresponding to normalized eigenfunctions $\varphi_n^{(-)}(x,a_1)$, i.e.,

$$\int_a^b \left|\varphi_n^{(-)}(x,a_1)\right|^2 \mathrm{dx} < \infty. \tag{17}$$

The eigenfunctions of the excited sates of the Hamiltonian partner $H^{(-)}(a_1)$, namely $\varphi_n^{(-)}(x,a_1)$ for $1 \le n \le n_{\max}$, can be evaluated in two different ways. The first consists in using the recurrence relation [49]

$$\varphi_n^{(-)}(x;a_1) = \left[\prod_{i=1}^{n} A^+(a_i)\right]\varphi_0(x,a_{n+1}) \text{ for } 1 \le n \le n_{\max}, \tag{18}$$

or equivalently [50]

$$\varphi_n^{(-)}(x;a_1) = A^+(a_1)\varphi_{n-1}^{(-)}(x,a_2) \text{ for } 1 \le n \le n_{\max}, \tag{19}$$

which make it possible to express step by step all the eigenfunctions of the excited states. However, in this way, it is often difficult to obtain a general formula allowing to express all the eigenfunctions as a function of the quantum number n.

The second way, which is widely used in the literature, consists in substituting into Eq. (1) the eigenvalues $\mathcal{E}_n$ by their expressions obtained by the supersymmetry and then solving the resulting equation by the standard approach. This method is more manageable.

3. Klein Gordon equation with position-dependent mass

The time-independent Klein-Gordon equation for a spinless particle subjected to a mixture of central potentials, vector and scalar, $V(r)$ and $S(r)$, can be written as an equation for a particle whose mass $M(r)$ depends on the position, subject only to the vector potential $V(r)$.

Indeed, let

$$\left[-\nabla^2 + \beta^2 c^4 M^2(r) - \beta^2 (E - V(r))^2\right]\Psi(\mathbf{r}) = 0, \tag{20}$$

with

$$c^2 M(r) = \left(m_0 c^2 + S(r)\right), \tag{21}$$

where ∇^2 is the Laplacian operator, $\Psi(\mathbf{r}) \equiv \Psi(r,\theta,\varphi)$ is the wavefunction and E, the corresponding relativistic energy, and $\beta = 1/\hbar c$; $\hbar = h/2\pi$, h is the constant of Planck, c is the speed of light. Since, the potentials are radial functions, the wavefunction $\Psi(\mathbf{r})$ is written as

$$\Psi(\mathbf{r}) = \frac{u(r)}{r} Y_{lm}(\theta,\varphi), \tag{22}$$

where $u(r)$ is a radial function, and $Y_{lm}(\theta,\varphi)$ for $l \in \mathbb{N}$ and $-l \le m \le l$ are the spherical harmonic functions. Thus, by using the method of separation of variables, we obtain the second-order differential equation for the radial function $u(r)$ in the following form

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \beta^2 c^4 M^2(r) - \beta^2 (E - V(r))^2\right\} \times u(r) = 0. \tag{23}$$

Obviously, Eq. (23) is not an eigenvalues equation of the Schrödinger type. It is often written in several works of the literature in the form

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \beta^2\left(c^4 M^2(r) + V^2(r) + 2EV(r)\right)\right\} \times u(r) = \widetilde{E}u(r), \tag{24}$$

with $\widetilde{E} = \beta^2 E^2$. However, even in this form, this equation is no longer an eigenvalue equation of the Schrödinger type since the energy E exists both in the eigenvalue $\widetilde{E}$ and in the effective potential through the term $2\beta^2 EV(r)$. Hence, this equation cannot be treated by the approach of SUSYQM without some ambiguity.

In order to overcome this drawback and use the SUSYQM approach unambiguously, we propose the following trick. Instead of solving Eq. (23), we solve the auxiliary Schrödinger-like eigenvalues equation given by

$$\left\{-\frac{d^2}{dr^2}+V_E(r)\right\}\psi(r)=\mathcal{E}\psi(r), \tag{25}$$

where the effective potential $V_E(r)$, which reads

$$V_E(r)=\frac{l(l+1)}{r^2}+\beta^2\left(c^4M^2(r)-(E-V(r))^2\right), \tag{26}$$

is explicitly dependent on the energy E, considered as a real parameter.

Then, Eq. (25) can now be solved for bound states by SUSYQM approach without ambiguity if $V_E(r)$ is supersymmetric. Hence, when the eigenvalues, denoted by $\mathcal{E}_{nl}\equiv\mathcal{E}_{nl}(E)$ (where the indices n and l denote the radial and angular quantum numbers, respectively) are obtained, the energy levels E_{nl} of Eq. (23) are deduced by the real solutions of the equation

$$\mathcal{E}_{nl}(E)=0. \tag{27}$$

The corresponding radial wavefunctions $u_{nl}(r)$ may be obtained from the eigenfunctions $\psi_{nl}(r;E)$ by

$$u_{nl}(r)=\psi_{nl}(r;E)|_{\mathcal{E}_{nl}(E)=0}. \tag{28}$$

The procedure to follow to do this is to replace E by E_{nl} in Eq. (23) and then solve the differential equation by the standard method, for the radial wavefunctions $u_{nl}(r)$.

4. Bound state solutions by means of SUSYQM approach for an appropriate model

The goal is to solve Eq. (25) for bound states by means of SUSYQM for an interesting model characterized by a position-dependent mass and a vector potential, as a central hyperbolic functions, defined by

$$M(r)=\sqrt{m_0^2+\frac{\left(\frac{\lambda}{c}\right)^2}{\sinh^2\alpha r}}, \tag{29}$$

and

$$V(r)=-\eta c\coth\alpha r, \tag{30}$$

where $m_0,\alpha,\ \lambda$ and η are real positive parameters with $[m_0]=\mathrm{M},\ [\alpha]=\mathrm{L}^{-1}$and $[\lambda]=[\eta]=\mathrm{MLT}^{-1}$.

Note that the speed of light c is explicitly included in the expressions of $M(r)$ and $V(r)$ only by convenience of calculations, so that λ and η have the same dimension. However, in concrete examples, they can be considered as order 0 and order 1, compared to c^{-1}, respectively, i.e.,

$$\lambda=\lambda_0+O\left(c^{-1}\right)\ \text{and}\ \eta=\eta_0c^{-1}+O\left(c^{-2}\right). \tag{31}$$

Furthermore, because of the centrifugal term in the effective potential, Eq. (25) cannot be solved exactly for states with $l\neq 0$. In order to get analytical solutions for any l-states, we use the usual approximation scheme of the centrifugal potential term [45], which we write in the following form

$$\frac{1}{r^2}\approx\frac{\alpha^2}{\sinh^2\alpha r}, \tag{32}$$

which is satisfied for $\alpha r<<1$. Substituting Eqs. (29), (30) and (32) into (26), the E-dependent effective potential takes the following form

$$V_E(r)=\frac{\Lambda_1}{\sinh^2\alpha r}-2\Lambda_2\coth\alpha r+\Lambda_3, \tag{33}$$

which is a shifted hyperbolic Eckart-type potential, where the parameters Λ_1, Λ_2 and Λ_3 are given by

$$\Lambda_1\equiv\Lambda_1(l)=\alpha^2l(l+1)+\beta^2c^2\left(\lambda^2-\eta^2\right), \tag{34}$$

$$\Lambda_2\equiv\Lambda_2(E)=\beta^2c\eta E, \tag{35}$$

and

$$\Lambda_3\equiv\Lambda_3(E)=\beta^2\left(m_0^2c^4-\eta^2c^2-E^2\right). \tag{36}$$

Substituting (33) into Eq. (25), we obtain the following Schrödinger-like eigenvalues equation as

$$\begin{aligned}H_E\psi_{nl}(r)&=\left(-\frac{d^2}{dr^2}+\frac{\Lambda_1}{\sinh^2\alpha r}-2\Lambda_2\coth\alpha r+\Lambda_3\right)\psi_{nl}(r)\\&=\mathcal{E}_{nl}\psi_{nl}(r),\end{aligned} \tag{37}$$

where the eigenvalues and eigenfunctions are depending on the parameter E, i.e., $\mathcal{E}_{nl}\equiv\mathcal{E}_{nl}(E)$ and $\psi_{nl}(r)\equiv\psi_{nl}(E,r)$.

4.1. Bound states eigenvalues $\mathcal{E}_{nl}$

In order to solve the differential equation (37) for the bound states eigenvalues $\mathcal{E}_{nl}$ by means of SUSYQM approach, we chose the superpotential $W_E(r,a_1)$ as

$$W_E(r,a_1)=-a_1\coth\alpha r+\frac{\Lambda_2}{a_1}. \tag{38}$$

Following the recipe in Sect. 2, the partner Hamiltonians read

$$H_E^{(\mp)}(a_1)=-\frac{d^2}{dr^2}+V_E^{(\mp)}(r,a_1), \tag{39}$$

and we denote by $\mathcal{E}_{nl}^{(\mp)}(a_1)\equiv\ \mathcal{E}_{nl}^{(\mp)}(E,a_1)$ and

$\psi_{nl}^{(\mp)}(r,a_1) \equiv \psi_{nl}^{(\mp)}(r,E,a_1)$, respectively, their eigenvalues and eigenfunctions. The partner potentials are given by

$$V_E^{(-)}(r,a_1) = \frac{a_1(a_1-\alpha)}{\sinh^2\alpha r} - 2\Lambda_2\coth\alpha r + a_1^2 + \frac{\Lambda_2^2}{a_1^2}, \quad (40)$$

$$V_E^{(+)}(r,a_1) = \frac{a_1(a_1+\alpha)}{\sinh^2\alpha r} - 2\Lambda_2\coth\alpha r + a_1^2 + \frac{\Lambda_2^2}{a_1^2}. \quad (41)$$

Moreover, substituting (38) into (13), it is easy to express the unnormalized ground-state eigenfunction of $H_E^{(-)}(a_1)$ in the form

$$\psi_{0l}^{(-)}(r,a_1) \sim (\sinh\alpha r)^{\frac{a_1}{\alpha}} e^{-\frac{\Lambda_2}{a_1}r}, \quad (42)$$

which corresponds to a null eigenvalue

$$\mathcal{E}_{0l}^{(-)}(a_1) \equiv 0. \quad (43)$$

To ensure that this function is normalizable on the interval $r \in]0,\infty[$, in the sense of (17), it must satisfy the boundary conditions

$$\lim_{r\to 0}\psi_{0l}^{(-)}(r,a_1) = \lim_{r\to\infty}\psi_{0l}^{(-)}(r,a_1) = 0, \quad (44)$$

which yield the following constraints on the parameters a_1 and Λ_2, given by

$$a_1 > 0, \quad (45)$$

and

$$\Lambda_2 = \beta^2 c\eta E > a_1^2. \quad (46)$$

From (40) and (41), we see that the shape invariance between $V_E^{(+)}(r,a_1)$ and $V_E^{(-)}(r,a_1)$ is satisfied, namely

$$V_E^{(+)}(r,a_1) = V_E^{(-)}(r,a_2) + R(a_1), \quad (47)$$

with

$$a_2 = a_1 + \alpha, \quad (48)$$

and the reminder $R(a_1) \equiv R(E,a_1)$ is given by

$$R(a_1) = \left(a_1^2 + \frac{\Lambda_2^2}{a_1^2}\right) - \left((a_1+\alpha)^2 + \frac{\Lambda_2^2}{(a_1+\alpha)^2}\right). \quad (49)$$

Consequently, by virtue of (15) and (16), the eigenvalues $\mathcal{E}_{nl}^{(-)}(a_1)$, of the partner Hamiltonian $H^{(-)}(a_1)$, are given by

$$\mathcal{E}_{nl}^{(-)}(a_1) = \left(a_1^2 + \frac{\Lambda_2^2}{a_1^2}\right) - \left((a_1+n\alpha)^2 + \frac{\Lambda_2^2}{(a_1+n\alpha)^2}\right) \text{ for } n = 1,2,\ldots,n_{\max}, \quad (50)$$

where $n_{\max}$ is given by the constraint

$$(a_1+n\alpha)^2 < \Lambda_2 \Longrightarrow n_{\max} = \left\{\frac{1}{\alpha}\left(\sqrt{\Lambda_2} - a_1\right)\right\}. \quad (51)$$

In order to obtain the eigenvalues $\mathcal{E}_{nl}$, we have first to fix the ground state eigenvalue $\mathcal{E}_{0l}$ and the parameter a_1 by means of Eq. (9), which reads

$$\frac{a_1(a_1-\alpha)}{\sinh\alpha r} - 2\Lambda_2\coth\alpha r + a_1^2 + \frac{\Lambda_2^2}{a_1^2} = \frac{\Lambda_1}{\sinh^2\alpha r} - 2\Lambda_2\coth\alpha r + \Lambda_3 - \mathcal{E}_{0l}. \quad (52)$$

By identifying identical terms and taking into account the constraint (45), we easily find

$$\mathcal{E}_{0l} = \Lambda_3 - \left(a_1^2 + \frac{\Lambda_2^2}{a_1^2}\right), \quad (53)$$

and

$$a_1 \equiv a_1(l) = \alpha\nu_{0l}, \quad (54)$$

where

$$\nu_{0l} = \sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{\lambda^2-\eta^2}{\hbar^2\alpha^2}} + \frac{1}{2}, \quad (55)$$

with the new constraint on the parameters λ and η, given by

$$\left(l+\frac{1}{2}\right)^2 + \frac{\lambda^2-\eta^2}{\hbar^2\alpha^2} \geq 0 \Longrightarrow \eta^2 \leq \lambda^2 + \frac{\hbar^2\alpha^2}{4}. \quad (56)$$

Using now (11) with (50), (53) and (55) leads

$$\mathcal{E}_{nl}(E) = \beta^2\left(m_0^2c^4 - \eta^2c^2 - E^2\right) - \left(\alpha^2\nu_n^2 + \frac{\beta^4c^2\eta^2E^2}{\alpha^2\nu_n^2}\right), \quad (57)$$

with

$$\nu_{nl} = \nu_{0l} + n. \quad (58)$$

4.2. Energy levels E_{nl}

Solving the equation $\mathcal{E}_{nl}(E) = 0$ leads, after a straightforward calculation, to

$$E_{nl}^2 = m_0^2c^4\left(\frac{\hbar^2\alpha^2\nu_{nl}^2}{\hbar^2\alpha^2\nu_{nl}^2+\eta^2} - \frac{\hbar^2\alpha^2\nu_{nl}^2}{m_0^2c^2}\right). \quad (59)$$

Now, knowing that under the constraint (46), the parameter E must be positive, we deduce that the energy levels E_{nl} must also be positive quantities and then we have

$$E_{nl} = m_0c^2\sqrt{\frac{\hbar^2\alpha^2\nu_{nl}^2}{\hbar^2\alpha^2\nu_{nl}^2+\eta^2} - \frac{\hbar^2\alpha^2\nu_{nl}^2}{m_0^2c^2}} \text{ for } n = 0,1,\ldots,\overline{n}_{\max}, \quad (60)$$

where $\overline{n}_{\max}$ is the number of bound states, which will be given later from the condition of normalization of the corresponding wavefunctions $u_{nl}(r)$.

4.3. Radial wavefunctions $u_{nl}(r)$

As stated before, the radial wavefunctions $u_{nl}(r)$ can in principle be derived using Eq. (28) if the eigenfunctions $\psi_{nl}(E,r)$ are explicitly dependent of the eigenvalues $\mathcal{E}_{nl}$, but unfortunately it is not. For that, we will obtain the $u_{nl}(r)$ straightforwardly by solving Eq. (37) after taking $\mathcal{E}_{nl}=0$ and replacing E by E_{nl}.

Thus, after some algebraic manipulations and taking into account the notation (54), the equation to solve reads

$$\left[\frac{d^2}{dr^2}-\frac{\alpha^2 v_{0l}(v_{0l}-1)}{\sinh^2\alpha r}+\frac{2\eta E_{nl}}{c\hbar^2}\coth\alpha r-\alpha^2 v_{nl}^2-\frac{\eta^2E_{nl}^2}{c^2\hbar^4\alpha^2 v_{nl}^2}\right]u_{nl}(r)=0. \tag{61}$$

To solve this equation with the standard method, we first make the following point transformation

$$z=1-e^{-2\alpha r}\in\,]0,1[\text{ and } u_{nl}(r)=\overline{u}_{nl}(z), \tag{62}$$

and with straightforward manipulations and with simple manipulations, it turns out that the new function $\overline{u}_{nl}(z)$ satisfies the following hypergeometric differential equation

$$z(1-z)\overline{u}''_{nl}(z)-z\overline{u}'_{nl}(z)-\frac{1}{z(1-z)}\left[\frac{1}{4}\left(\frac{\eta E_{nl}}{\hbar^2 c\alpha^2 v_{nl}}-v_{nl}\right)^2 z^2 -\left(\frac{\eta E_{nl}}{\alpha^2 c\hbar^2}+v_{0l}(v_{0l}-1)\right)z+v_{0l}(v_{0l}-1)\right]\overline{u}_{nl}(z)=0. \tag{63}$$

Setting now

$$\overline{u}_{nl}(z)=z^{v}(1-z)^{\mu}\chi_{nl}(z), \tag{64}$$

and searching for v and μ such that

$$\mathrm{Re}\, v>0 \text{ and } \mathrm{Re}\,\mu>0, \tag{65}$$

and $\chi_{nl}(z)$ satisfies Gauss hypergeometric differential equation. Thus, by inserting (64) into (63), we obtain straightforwardly the following differential equation

$$z(1-z)\chi''_{nl}(z)+(2v-(2(v+\mu)+1)z)\chi'_{nl}(z)+\frac{Pz^2+Qz+R}{z(1-z)}\chi_{nl}(z)=0, \tag{66}$$

with

$$P=(v+\mu)^2-\frac{\eta E_{nl}}{2\alpha^2 c\hbar^2}-\frac{1}{4}\left(v_{nl}^2+\frac{\eta^2E_{nl}^2}{c^2\hbar^4\alpha^4 v_{nl}^2}\right), \tag{67}$$

$$Q=v-2v(v+\mu)+\frac{\eta E_{nl}}{\alpha^2 c\hbar^2}+v_{0l}(v_{0l}-1), \tag{68}$$

$$R=v^2-v-v_{0l}(v_{0l}-1). \tag{69}$$

Choosing v and μ such that $R\equiv 0$ and $P+Q\equiv 0$, Eq. (66) reduces to the canonical Gauss hypergeometric equation, given by

$$z(1-z)\chi''_{nl}(z)+(2v_{0l}-(2(v_{0l}+\mu_{nl})+1)z)\chi'_{nl}(z) -(v_{0l}+\mu_{nl}-\xi_{nl})(v_{0l}+\mu_{nl}+\xi_{nl})\chi_{nl}(z)=0, \tag{70}$$

where

$$v=v_{0l}, \tag{71}$$

$$\mu\equiv\mu_{nl}=\frac{1}{2}\left(\frac{\eta E_{nl}}{c\hbar^2\alpha^2 v_{nl}}-v_{nl}\right), \tag{72}$$

and

$$\xi_{nl}=\frac{1}{2}\left(\frac{\eta E_{nl}}{c\hbar^2\alpha^2 v_{nl}}+v_{nl}\right). \tag{73}$$

Taking into account (65), it is easy to show that the constraint $\mu_{nl}>0$ induces a restriction on the number of allowed bound states, namely

$$0\le n\le \overline{n}_{\max}=\left\{\frac{\sqrt{\eta(m_0c-\eta)}}{\hbar\alpha}-v_{0l}\right\}, \tag{74}$$

with $\{k\}$ stands for the largest integer inferior to k. Moreover, the parameter η must be strictly positive.

Consequently $\chi_{nl}(z)$ is none other than the hypergeometric Gauss function

$$\chi_{nl}(z)\sim {}_2F_1(v_{0l}+\mu_{nl}-\xi_{nl},v_{0l}+\mu_{nl}+\xi_{nl},2v_{0l};z), \tag{75}$$

From Eqs. (71), (72) and (73) it follows that

$$v_{0l}+\mu_{nl}-\xi_{nl}=-n, \tag{76}$$

and therefore $\chi_{nl}(z)$ is a polynomial of degree n as it should be.

Taking into account Eqs. (62), (64) and the relation (76), the radial wavefunctions $u_{nl}(r)$ are then given by

$$u_{nl}(r)=N_{nl}e^{-2\alpha\mu_{nl}r}\left(1-e^{-2\alpha r}\right)^{v_{0l}} \times{}_2F_1\left(-n,n+2(v_{0l}+\mu_{nl}),2v_{0l};1-e^{-2\alpha r}\right), \tag{77}$$

where N_{nl} are normalization constants an n satisfying (74).

For the purpose of evaluating the normalization constants N_{nl} directly, it is best to write the Gauss hypergeometric polynomials in terms of Jacobi polynomials. So, by using the following relationship (see formula (8.962.1) in Ref. [51])

$$
{}_2F_1\left(-n, n+a+b+1, b+1; \frac{1+x}{2}\right) = \frac{(-1)^n n!\Gamma(b+1)}{\Gamma(n+b+1)} P_n^{(a,b)}(x), \tag{78}
$$

Equation (77) becomes

$$
u_{nl}(r) = N_{nl} \frac{(-1)^n n!\Gamma(2v_{0l})}{\Gamma(n+2v_{0l})} e^{-2\alpha\mu_{nl} r}\left(1-e^{-2\alpha r}\right)^{v_{0l}} \times P_n^{(2\mu_{nl}, 2v_{0l}-1)}\left(1-2e^{-2\alpha r}\right). \tag{79}
$$

The normalization condition reads

$$
\int_0^{+\infty} |u_{nl}(r)|^2 dr = 1, \tag{80}
$$

and by using the change of variable as $x = 1 - 2e^{-2\alpha r}$, leads to

$$
N_{nl}^{-2} = \frac{\alpha^{-1}}{2^{a+b+2}}\left[\frac{n!\Gamma(2v_{0l})}{\Gamma(n+2v_{0l})}\right]^2 \int_{-1}^{1} (1-x)^{2\mu_{nl}-1}(1+x)^{2v_{0l}} \times \left[P_n^{(2\mu_{nl}, 2v_{0l}-1)}(x)\right]^2 \mathrm{dx}. \tag{81}
$$

The integral in (81) is forwardly evaluated by setting

$$
1 + x = 2 - (1 - x), \tag{82}
$$

and making use of the following two integrals (see formula (7.391.5) in [51])

$$
\int_{-1}^{1} (1-x)^{a-1}(1+x)^b \left[P_n^{(a,b)}(x)\right]^2 \mathrm{dx} = \frac{2^{a+b}\Gamma(n+a+1)\Gamma(n+b+1)}{n! a \Gamma(n+a+b+1)}, \tag{83}
$$

where $\mathrm{Re}\,(a) > 0$ and $\mathrm{Re}\,(b) > -1$, and (see formula (7.391.1) in [51])

$$
\int_{-1}^{1} (1-x)^{a}(1+x)^b \left[P_n^{(a,b)}(x)\right]^2 \mathrm{dx} = \frac{2^{a+b+1}\Gamma(n+a+1)\Gamma(n+b+1)}{n!(2n+a+b+1)\Gamma(n+a+b+1)}, \tag{84}
$$

which are valid for $\mathrm{Re}\,(a) > -1$ and $\mathrm{Re}\,(b) > -1$. The final result is

$$
N_{nl} = \sqrt{\frac{4\alpha\mu_{nl}(n+\mu_{nl}+v_{0l})\Gamma(n+2v_{0l})\Gamma(n+2\mu_{nl}+2v_{0l})}{n!(n+v_{0l})\Gamma(n+2\mu_{nl}+1)\Gamma^2(2v_{0l})}}. \tag{85}
$$

5. Special cases

The aim of this section is to deduce in special cases the relativistic bound states for a spinless particle of constant mass and zero scalar potential for two models of vector potentials, namely the Hulthén and Coulomb potentials. Also, we show how to obtain their counter parts in the non-relativistic limit.

5.1. Vector potential as Hulthén potential and constant mass

For Hulthén potential [52]

$$
V^{(\mathcal{H})}(r) = -\frac{\bar{\eta}\delta e^{-\delta r}}{1-e^{-\delta r}}, \tag{86}
$$

we have just to replace in Eq. (26) $V(r)$ by $V^{(\mathcal{H})}(r)$. But since the Hulthén potential may be written

$$
V^{(\mathcal{H})}(r) = -\frac{\bar{\eta}\delta e^{-\delta r}}{1-e^{-\delta r}} = -\frac{\bar{\eta}\delta}{2}\coth\frac{\delta}{2}r + \frac{\bar{\eta}\delta}{2}, \tag{87}
$$

the approximate bound states corresponding to Hulthén potential in (86) with constant mass may be obtained straightforwardly by making the following substitutions

$$
E_{nl} \longrightarrow E_{nl}^{(\mathcal{H})} - \frac{\bar{\eta}\delta}{2}, \alpha \longrightarrow \frac{\delta}{2}, \eta \longrightarrow \frac{\bar{\eta}\delta}{2c} \text{ and } \lambda = 0, \tag{88}
$$

in (60), (74), (77) and (85). After laborious algebraic manipulations, and by defining new parameters $\overline{v}_{nl}$ and $\overline{\mu}_{nl}$ as

$$
\overline{v}_{nl} = \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{\bar{\eta}^2}{\hbar^2c^2}} + \frac{1}{2} + n, \tag{89}
$$

and

$$
\overline{\mu}_{nl} = \frac{1}{2}\left(\frac{2\bar{\eta}m_0}{\hbar^2\delta}\sqrt{\frac{1}{\overline{v}_{nl}^2 + \frac{\bar{\eta}^2}{\hbar^2c^2}} - \frac{\hbar^2\delta^2}{4m_0^2c^2}} - \overline{v}_{nl}\right), \tag{90}
$$

we get

$$
E_{nl}^{(\mathcal{H})} = \frac{\bar{\eta}\delta}{2} + m_0c^2\sqrt{\frac{\overline{v}_{nl}^2}{\overline{v}_{nl}^2 + \frac{\bar{\eta}^2}{\hbar^2c^2}} - \frac{\hbar^2\delta^2}{4m_0^2c^2}\overline{v}_{nl}^2}, \tag{91}
$$

and

$$
u_{nl}^{(\mathcal{H})}(r) = N_{nl}^{(\mathcal{H})} e^{-\overline{\mu}_{nl}\delta r}\left(1-e^{-\delta r}\right)^{\overline{v}_{0l}} \times {}_2F_1\left(-n, n+2(\overline{v}_{0l}+\overline{\mu}_{nl}), 2\overline{v}_{0l}; 1-e^{-\delta r}\right), \tag{92}
$$

with

$$N_{nl}^{(\mathcal{H})}=\sqrt{\frac{2\delta\overline{\mu}_{nl}(n+\overline{\mu}_{nl}+\overline{\nu}_{0l})\Gamma(n+2\overline{\nu}_{0l})\Gamma(n+2\overline{\mu}_{nl}+2\overline{\nu}_{0l})}{n!(n+\overline{\nu}_{0l})\Gamma(n+2\overline{\mu}_{nl}+1)\Gamma^2(2\overline{\nu}_{0l})}}, \tag{93}$$

for $n=1,2,\ldots,n_{\max}^{(\mathcal{H})}$, and

$$n_{\max}^{(\mathcal{H})}=\left\{\sqrt{\frac{2\overline{\eta}m_0}{\delta\hbar^2}}\sqrt{1-\frac{\overline{\eta}\delta}{2m_0c^2}}-\overline{\nu}_{0l}\right\}, \tag{94}$$

where $\{k\}$ stands for the largest integer inferior to k.

By adjusting the notations of $\overline{\nu}_{nl}$ and $\overline{\mu}_{nl}$ to those of the literature, it is easy to see that our results are in perfect agreement with, for example, those of the references [16, 35].

5.1.1. Nonrelativistic limit

In order to obtain the approximate bound states energy levels for Hulthén potential in the non-relativistic limit, denoted here by $E_{nl}^{(\mathcal{H}_{NR})}$, it suffices to subtract the rest energy m_0c^2 from $E_{nl}^{(\mathcal{H})}$ and then take the limit $c\to\infty$, i.e.,

$$E_{nl}^{(\mathcal{H}_{NR})}=\frac{\overline{\eta}\delta}{2}+\lim_{c\longrightarrow\infty}m_0c^2\left(\sqrt{\frac{\overline{\nu}_{nl}^2}{\overline{\nu}_{nl}^2+\frac{\overline{\eta}^2}{\hbar^2c^2}}}-\frac{\hbar^2\delta^2\overline{\nu}_{nl}^2}{4m_0^2c^2}-1\right). \tag{95}$$

Let us write the right hand side of Eq. (95) up to first order in $x=\frac{\overline{\eta}^2}{\hbar^2c^2}$. Since

$$\overline{\nu}_{nl}^2\underset{c\longrightarrow\infty}{\sim}(n+l+1)^2-\frac{n+l+1}{l+\frac{1}{2}}\frac{\overline{\eta}^2}{\hbar^2c^2}, \tag{96}$$

it comes that

$$\frac{\overline{\nu}_{nl}^2}{\overline{\nu}_{nl}^2+\frac{\overline{\eta}^2}{\hbar^2c^2}}\underset{c\longrightarrow\infty}{\sim}1-\frac{1}{(n+l+1)^2}\frac{\overline{\eta}^2}{\hbar^2c^2}, \tag{97}$$

and

$$\frac{\hbar^2\delta^2\overline{\nu}_{nl}^2}{4m_0^2c^2}\underset{c\longrightarrow\infty}{\sim}\frac{\hbar^4\delta^2(n+l+1)^2}{4\overline{\eta}^2m_0^2}\frac{\overline{\eta}^2}{\hbar^2c^2}. \tag{98}$$

Substituting (97) and (98) into (95) leads straightforwardly to the approximate nonrelativistic eigenvalues of Hulthén potential, given by

$$E_{nl}^{(\mathcal{H}_{NR})}=-\frac{\hbar^2}{2m_0}\left(\frac{\overline{\eta}m_0}{\hbar^2(n+l+1)}-\frac{\delta(n+l+1)}{2}\right)^2. \tag{99}$$

By putting $\overline{\eta}=Ze^2$, we see that this result coincides exactly with what is obtained in the literature [53, 54].

Moreover, we obtain the radial wavefunctions as

$$u_{nl}^{(\mathcal{H}_{NR})}(r)=N_{nl}^{(\mathcal{H}_{NR})}e^{-\overline{\mu}_{nl}^{NR}\delta r}\left(1-e^{-\delta r}\right)^{\overline{\nu}_{nl}^{NR}}\times {}_2F_1\left(-n,n+2\left(\overline{\nu}_{0l}^{NR}+\overline{\mu}_{nl}^{NR}\right),2\overline{\nu}_{0l}^{NR};1-e^{-\delta r}\right), \tag{100}$$

with

$$N_{nl}^{(\mathcal{H}_{NR})}=\sqrt{\frac{2\delta\overline{\mu}_{nl}^{NR}\left(n+\overline{\mu}_{nl}^{NR}+\overline{\nu}_{0l}^{NR}\right)\Gamma\left(n+2\overline{\nu}_{0l}^{NR}\right)\Gamma\left(n+2\overline{\mu}_{nl}^{NR}+2\overline{\nu}_{0l}^{NR}\right)}{n!\left(n+\overline{\nu}_{0l}^{NR}\right)\Gamma\left(n+2\overline{\mu}_{nl}^{NR}+1\right)\Gamma^2\left(2\overline{\nu}_{0l}^{NR}\right)}}, \tag{101}$$

where

$$\overline{\nu}_{nl}^{NR}=\lim_{c\longrightarrow\infty}\overline{\nu}_{nl}=n+l+1, \tag{102}$$

and

$$\overline{\mu}_{nl}^{NR}=\lim_{c\longrightarrow\infty}\overline{\mu}_{nl}=\frac{1}{2}\left(\frac{2\overline{\eta}m_0}{\hbar^2\delta(n+l+1)}-(n+l+1)\right). \tag{103}$$

The number of the energy levels, $n_{\max}^{(\mathcal{H}_{NR})}$, is given by

$$n_{\max}^{(\mathcal{H}_{NR})}=\lim_{c\longrightarrow\infty}n_{\max}^{(\mathcal{H})}=\left\{\frac{1}{\hbar}\sqrt{\frac{2\overline{\eta}m_0}{\delta}}-(l+1)\right\}. \tag{104}$$

Also, these results are in perfect agreement with those of Refs. [53, 54].

5.2. Vector potential as Coulomb potential and constant mass

Since Coulomb potential is obtained from Hulthén potential in the limit $\delta\longrightarrow0$, i.e.,

$$V^{(\mathcal{C})}(x)=\lim_{\delta\longrightarrow0}V^{(\mathcal{H})}(r)=-\frac{\overline{\eta}}{r}, \tag{105}$$

their approximate eigenstates are obtained by the same limit on those of Hulthén potential.

Thus,

$$\begin{aligned}E_{nl}^{(C)}&=\lim_{\delta\longrightarrow0}\left(\frac{\overline{\eta}\delta}{2}+m_0c^2\sqrt{\frac{\overline{\nu}_{nl}^2}{\overline{\nu}_{nl}^2+\frac{\overline{\eta}^2}{\hbar^2c^2}}-\frac{\hbar^2\delta^2}{4m_0^2c^2}\overline{\nu}_{nl}^2}\right)\\&=m_0c^2\left(1+\frac{\overline{\eta}^2}{\hbar^2c^2\overline{\nu}_{nl}^2}\right)^{-\frac{1}{2}}.\end{aligned} \tag{106}$$

where $n=0,1,2,\ldots\infty$, since

$$n_{\max}^{(C)}=\lim_{\delta\longrightarrow0}n_{\max}^{(\mathcal{H})}=\infty. \tag{107}$$

Moreover, the corresponding radial wavefunctions are given by formula

$$u_{nl}^{(C)}(r) = \lim_{\delta\to 0} N_{nl}^{(\mathcal{H})} e^{-\overline{\mu}_{nl}\delta r}\left(1-e^{-\delta r}\right)^{\overline{\nu}_{0l}} \times {}_2F_1\left(-n, n+2(\overline{\nu}_{0l}+\overline{\mu}_{nl}), 2\overline{\nu}_{0l}; 1-e^{-\delta r}\right). \quad (108)$$

We have

$$\lim_{\delta\to 0}\left(1-e^{-\delta r}\right)^{\overline{\nu}_{0l}} \sim (\delta r)^{\overline{\nu}_{0l}}, \quad (109)$$

and using (90) and (106), it is easy to find

$$\lim_{\delta\to 0}\overline{\mu}_{nl}\delta = \frac{\overline{\eta} m_0}{\hbar^2}\frac{1}{\overline{\nu}_{nl}^2+\frac{\overline{\eta}^2}{\hbar^2 c^2}} = \frac{\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}}, \quad (110)$$

and consequently

$$\lim_{\delta\to 0} e^{-\overline{\mu}_{nl}\delta r} = e^{-\frac{\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}} r}. \quad (111)$$

Moreover, by virtue of (109) and (110), we can write

$$\begin{aligned} &\lim_{\delta\to 0} {}_2F_1\left(-n, n+2(\overline{\nu}_{0l}+\overline{\mu}_{nl}), 2\overline{\nu}_{0l}; 1-e^{-\delta r}\right) \\ &\sim \lim_{\delta\to 0} {}_2F_1\left(-n, \frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}\delta}, 2\overline{\nu}_{0l}; \delta r\right) \\ &= {}_1F_1\left(-n, 2\overline{\nu}_{0l}; \frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}} r\right), \end{aligned} \quad (112)$$

where we used the known asymptotic formula [55]

$$\lim_{\delta\to 0} {}_2F_1\left(-n, \frac{\lambda}{\delta}, \gamma; \delta r\right) = {}_1F_1(-n, \gamma; \lambda r). \quad (113)$$

Substituting (109), (110) and (112) into (108), the approximate radial wavefunctions of the relativistic Coulomb potential read

$$u_{nl}^{(C)}(r) = N_{nl}^{(C)} r^{\overline{\nu}_{0l}} \exp\left(-\frac{\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}} r\right) {}_1F_1\left(-n, 2\overline{\nu}_{0l}; \frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}} r\right), \quad (114)$$

and the normalization constant $N_{nl}^{(\mathcal{C})}$ is given by

$$N_{nl}^{(\mathcal{C})} = \lim_{\delta\to 0} \delta^{\overline{\nu}_{0l}} N_{nl}^{(\mathcal{H})}. \quad (115)$$

The details of the calculations are as follows. We first need to get the behavior of each factor in $N_{nl}^{(\mathcal{H})}$ and then calculate the limit.

We have by virtue of (110)

$$\lim_{\delta\to 0}\sqrt{2\delta\overline{\mu}_{nl}(n+\overline{\mu}_{nl}+\overline{\nu}_{0l})} \sim \sqrt{2\delta\overline{\mu}_{nl}^2} \sim \left(\frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}}\right)\frac{1}{\sqrt{\delta}}. \quad (116)$$

For the other factors, i.e., the Gamma functions containing the parameter δ, we first write their behaviors as

$$\lim_{\delta\to 0}\Gamma(n+2\overline{\mu}_{nl}+2\overline{\nu}_{0l}) \sim \Gamma\left(\frac{A}{\delta}+\overline{\nu}_{0l}\right), \quad (117)$$

$$\lim_{\delta\to 0}\Gamma(n+2\overline{\mu}_{nl}+1) \sim \Gamma\left(\frac{A}{\delta}+1-\overline{\nu}_{0l}\right), \quad (118)$$

with

$$A = \frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}}, \quad (119)$$

Using Stirling's famous formula

$$\Gamma(X) \sim \sqrt{2\pi} X^{X-\frac{1}{2}} e^{-X}, \quad (120)$$

which is valid for large X, it is easy to obtain

$$\lim_{\delta\to 0}\Gamma\left(\frac{A}{\delta}+\zeta\right) \sim \sqrt{2\pi}\left(\frac{A}{\delta}\right)^{\frac{A}{\delta}+\zeta-\frac{1}{2}} e^{-\frac{A}{\delta}}, \quad (121)$$

for parameters A and ζ independent of δ. Thus, we deduce that

$$\lim_{\delta\to 0}\sqrt{\frac{\Gamma(n+2\overline{\mu}_{nl}+2\overline{\nu}_{0l})}{\Gamma(n+2\overline{\mu}_{nl}+1)}} = \lim_{\delta\to 0}\frac{\sqrt{\frac{A}{\delta}+\overline{\nu}_{0l}}}{\sqrt{\frac{A}{\delta}+1-\overline{\nu}_{0l}}} \sim \left(\frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}}\frac{1}{\delta}\right)^{\overline{\nu}_{0l}-\frac{1}{2}}. \quad (122)$$

Substituting (116) and (122) into (93) and taking the limit $\delta \to 0$ leads

$$\lim_{\delta\to 0} N_{nl}^{(\mathcal{H})} \sim \frac{1}{\Gamma(2\overline{\nu}_{0l})}\left(\frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}}\right)^{\overline{\nu}_{0l}+\frac{1}{2}}\sqrt{\frac{\Gamma(n+2\overline{\nu}_{0l})}{n!(n+\overline{\nu}_{0l})}}\delta^{-\overline{\nu}_{0l}}, \quad (123)$$

and by virtue of (115), we obtain the normalization constant of the relativistic Coulomb potential as

$$N_{nl}^{(C)} = \frac{1}{\Gamma(2\overline{\nu}_{0l})}\left(\frac{2\overline{\eta} E_{nl}^{(C)}}{\hbar^2 c^2 \overline{\nu}_{nl}}\right)^{\overline{\nu}_{0l}+\frac{1}{2}}\sqrt{\frac{\Gamma(n+2\overline{\nu}_{0l})}{n!(n+\overline{\nu}_{0l})}}. \quad (124)$$

These results are identical to those in [35, 56].

5.3. Nonrelativistic limit

As for the Hulthén potential, the nonrelativistic limit in the case of Coulomb potential may be obtained by subtracting the rest energy m_0c^2 from the energy levels $E_{nl}^{(C)}$ and then taking the limit $c \longrightarrow \infty$, in all the relativistic quantities. We got results identical to those in Refs [35, 56]. Indeed, we have

$$E_{nl}^{(\mathcal{C}_{NR})} = \lim_{c \longrightarrow \infty} m_0 c^2 \left(\left(1 + \frac{\overline{\eta}^2}{\hbar^2 c^2 \overline{v}_{nl}^2} \right)^{-\frac{1}{2}} - 1 \right). \tag{125}$$

Taking into account the result in (102), we easily obtain

$$E_{nl}^{(\mathcal{C}_{NR})} = -\frac{m_0 \overline{\eta}^2}{2\hbar^2 (n+l+1)^2} \text{ for } n = 0, 1, 2, \ldots. \tag{126}$$

In the same way, we obtain the corresponding nonrelativistic radial wavefunctions in the form

$$u_{nl}^{NR}(r) = N_{nl}^{(\mathcal{C}_{NR})} r^{l+1} e^{-\frac{\overline{\eta} m_0}{\hbar^2 (n+l+1)} r} \times {}_1F_1\left(-n, 2l+2; \frac{2\overline{\eta} m_0}{\hbar^2 (n+l+1)} r \right), \tag{127}$$

with

$$N_{nl}^{(\mathcal{C}_{NR})} = \frac{1}{\Gamma(2l+2)} \sqrt{\frac{\Gamma(n+2l+2)}{n!(n+l+1)}} \left(\frac{2\overline{\eta} m_0}{\hbar^2 (n+l+1)} \right)^{l+\frac{3}{2}}. \tag{128}$$

Note that all the quantities evaluated above are approximate. If we set $l = 0$, we recover their exact values corresponding to the s- wave states.

6. Conclusions

In this paper, we investigated the approximate bound state solutions of the l-wave Klein-Gordon equation for a spinless particle with position-dependent mass, subjected to a Hermitian vector potential within the framework of SUSYQM approach by approximating the centrifugal term in the effective potential.

We have considered a model with a vector potential of hyperbolic cotangent type and the position-dependent mass is chosen such that the effective potential is of the hyperbolic Eckart type, but explicitly energy dependent. This difficulty led us to use a trick to be able to apply the SUSYQM approach without ambiguity. We have thus determined the energy levels in a simple and elegant way and then we have obtained the corresponding wavefunctions. Also, we have clarified all the necessary constraints that the parameters of the problem must satisfy for the existence of bound states.

By way of application and also verification of our results, we deduced all the results known in the literature, in the context of a constant mass and vector potential of the Hulthén or Coulomb type, both in the relativistic and nonrelativistic framework.

Acknowledgments The authors would like to thank the Algerian government for the financial assistance allocated within the framework of PRFU project under the code B00L02UN250120220019.

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