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Interacting two fluid models in modified theories of gravitation

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Abstract: This research focuses on the cosmic evolution of a scenario with two fluids pressure-less dark matter and dark energy-interacting inside a flat FRW universe within the context of some modified gravitational theories. Additionally, in light of the power law form of the scale factor, the properties of the flat FRW cosmological model are addressed using a number of cosmological parameters, including energy density, equation of state parameter, squared sound speed, and $(\omega_T - \omega'_T)$-plane. For various positive nonadditivity interaction parameters η, this scale factor produces a purely accelerating universe and the value of the deceleration parameters $q = -0.54$. Finally, a graphic explanation of the physical behavior of the universe is given for various values of the coefficient m.

1. Introduction

The different cosmological observations like type-Ia supernovae [1, 2], cosmic microwave background radiation (CMBR) [3, 4], baryon acoustic oscillations (BAO) [5, 6], galaxy redshift survey (GRS) [7] and large-scale structure (LSS) [8, 9] intensely put forward that our Universe is at present experiencing a phase of accelerated expansion by mentioning that the matter existing in the Universe is affected by an exotic forms of energy so-called dark energy. Such an accelerating expansion of the Universe essentially there are two probable access: one is to introduce an exotic forms of energy in the right-hand side of the Einstein field equations in the framework of general theory of relativity which acquire a negative equation of state parameter which is the ratio of the dark energy pressure (p) to its energy density (ρ) as ($\omega = \frac{p}{\rho}$). The simplest dark energy aspirant is the vacuum energy ($\omega = -1$) which is mathematically comparable to the cosmological constant (Λ). The other straight course of action, which can be described by minimally coupled scalar fields, is quintessence field ($\omega > -1$), phantom energy field ($\omega < -1$) and Quintom (that can across from phantom region to quintessence region as evolved) and $\omega \ll -1$ is directed toward an existing cosmological data also have time-dependent equation of state parameter. The observational results that come from SNe-Ia data and a mixture of SNe-Ia data with CMBR and GRS, respectively, acquired $-1.66 < \omega < -0.62$ and $-1.33 < \omega < -0.79$. Also, the most recent outcome in 2009 restrained the dark energy equation of state parameters to $-1.44 < \omega < -0.92$. The present Planck data tells that there is 68.3% dark energy of the total energy contents of the universe. A variety of dark energy models have been discussed in which cosmological constant is the primary dark energy candidate for describing dark energy phenomena but it has some serious problems. Due to some serious problems, several dark energy models like a family of scalar fields such as quintessence, phantom, quintom, tachyon, K-essence along with various Chaplygin gas models like generalized Chaplygin gas, extended Chaplygin gas and modified Chaplygin gas has been produced.

The second one is to modify the left-hand side of Einstein's equation, leading to a modified theories of gravity. There are a numbers of modified theory of gravity are present say *f(R)* obtained by the replacement of Ricci scalar

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with an arbitrary function of the Ricci scalar, Capozziello et al. [1] developed a general formalism in the metric framework by considering a point-like $f(R)$-Lagrangian. However, Azadi et al. [2] studied static cylindrically symmetric vacuum solutions in Weyl coordinates in the context of the metric $f(R)$ theories of gravity. Sharif and Yousaf [3] analyzed the role of the electromagnetic field and a viable $f(R) = R + \delta R^2$ model on the range of dynamical instability along with Shamir [4], Chirde and Shekh [5], Moraes et al. [6], Sahoo and Bhattacharjee [7], and Godani [8, 9] are the authors who have studied $f(R)$ theory of gravity for various space-times in the different contexts. Next, $f(T)$ is obtained by replacing torsion with curvature, Bengochea and Ferraro [10], Bamba et al. [11], Capozziello et al. [12], Bhatti et al. [13], Bhoyer et al. [14], Chirde and Shekh [15], Shekh and Chirde [16] have explored a few highlights of cosmological models within the framework of $f(T)$ gravity. Next, $f(R, T)$ is obtained by the replacement of Ricci scalar with an arbitrary function of both Ricci scalar and trace of energy tensor [17–24]. Further, $f(G)$ in which $f(G)$ is a generic function of the Gauss–Bonnet invariant G, Nojiri and Odintsov [25], Sharif and Fatima [26], Shekh et al. [27] have taken remarkable effort in $f(G)$ gravity. Next, $f(R, G)$ which combines Ricci scalar and Gauss–Bonnet scalar [28, 29]. After that $f(T, B)$ which involves torsion and boundary term [30]. Further, $f(Q)$ in which nonmetricity term Q is responsible for the gravitational interaction. Recently, Lazkoz et al. [31] have maintained the altered form of $f(Q)$ gravity model and checked their validity by proposing various polynomial forms of $f(z)$ including additional terms which causes the deviation from ΛCDM model while checking the stability of assumed $f(Q)$ gravity cosmological models Mandal et al. [32], scrutinized the energy conditions and constrained the model parameters with the present values of cosmological parameters. Under the title of testing $f(Q)$ gravity with redshift space distortions Barros et al. [33], analyzed the linear matter fluctuations are numerically evolved and the study of the growth rate of structures and predicted that the best fit parameters reveal that the tension between Planck and LSS data can be alleviated within this framework. Frusciante [34] investigated the impact on cosmological observables of $f(Q)$-gravity and focused on a specific model which is indistinguishable from the ΛCDM model at the background level.

Inciting with above discussion, in this work, the author was interested in investigating an interacting scenario between two fluids, pressureless dark matter and dark energy in $f(Q)$, $f(T)$, $f(G)$ and $f(R)$ theories of gravitation toward FRW Universe. Here, the author focuses on the dark energy models with mixed form of modified theories of gravitation. The present work is organized as follows: in Sec. 2, the author presented spatially homogeneous and isotropic space-time with interacting source. The isotropization of the Universe using constant deceleration parameter is presented in Sec. 3 while in Secs. 4 to 7 of some cosmographic parameters are analyzed with a brief review of $f(Q)$, $f(T)$, $f(G)$ and $f(R)$. The final concluding remarks are conferred in Sec. 8.

2. Homogeneous space-time with interacting source

Consider the spatially homogeneous and isotropic Friedmann–Robertson–Walker (FRW) space-time in the form

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right], \quad (1)$$

where a be the scale factor of the universe.

The angle θ and ϕ are the usual azimuthal and polar angles of spherical coordinates, with $0 \leq \theta \leq \phi$ and $0 \leq \phi \leq \phi$. The coordinates (t, r, θ, ϕ) are called comoving coordinates. The homogeneity of the universe fixes a special frame of reference, the cosmic rest frame given by the above coordinate system. Also, k is a constant which represent the curvature of the space-time. If $k = 1$, then this corresponds to a closed universe, the flat universe is obtained for $k = 0$ and $k = -1$ corresponds to an open universe. In this work, the author deliberate on the flat universe taken after $k = 0$ with infinite radius.

The energy momentum tensor for two fluids say pressureless dark matter and dark energy are defined as

$$\mathcal{T} = \mathcal{T}_{\mu\nu} + \bar{\mathcal{T}}_{\mu\nu} \quad (2)$$

where $\mathcal{T}_{\mu\nu}$ and $\bar{\mathcal{T}}_{\mu\nu}$ are, respectively, the energy momentum tensors for pressureless dark matter and dark energy, defined as $\mathcal{T}_{\mu\nu} = \rho_m u_\mu u_\nu$ and $\bar{\mathcal{T}}_{\mu\nu} = (\rho_d + p_d)u_\mu u_\nu - p_d g_{\mu\nu}$ in which ρ_m, ρ_d are the energy densities of dark matter and dark energy, respectively, p_d is the pressure of dark energy and u_i is the four velocity of the fluid. Also, $u_i = \delta_4^i$ is satisfied four-velocity vector.

$$g_{\mu\nu} = u^\mu u_\nu = -x^\mu x_\nu = 1 \quad \text{and} \quad u^\mu x_\nu = 0 \quad (3)$$

For the Universe where dark energy and dark matter are interacting to each other the total energy density satisfies the continuity equation as follows

$$\dot{\rho}_D + 3H(\rho_D + p_D) = 0 \quad (4)$$

where ρ_D is the combine energy density of two fluids of the form $\rho_D = \rho_m + \rho_d$.

Consider the curvature energy density ρ_k and the critical energy density ρ_{cr} as usual:

$$\rho_k = \frac{3k}{8\pi G a^2} \quad \text{and} \quad \rho_{cr} = 3H^2 \tag{5}$$

The fractional energy densities Ω_D, Ω_m and Ω_k are defined as:

$$\Omega_D = \frac{\rho_D}{\rho_{cr}} = \frac{\rho_D}{3H^2}, \tag{6}$$

$$\Omega_m = \frac{\rho_m}{\rho_{cr}} = \frac{\rho_m}{3H^2} \tag{7}$$

$$\Omega_k = \frac{\rho_k}{\rho_{cr}} = \frac{k}{H^2 a^2} \tag{8}$$

Then from the above Eqs. (6, 7 and 8), the Friedmann equation can then be written as:

$$1 + \Omega_k = \Omega_D + \Omega_m. \tag{9}$$

Consider the interaction between two fluids. Thus, the energy densities of two fluids do not conserve separately, the continuity of matter of two fluids yields

$$\dot{\rho}_D + 3H(\rho_D + p_D) = -\Gamma \tag{10}$$

where Γ represents the interaction between dark matter and dark energy. In general Γ should be a function with units of inverse of time. For convenience, I choose the following form of interaction term

$$\Gamma = 3\eta H(\rho_m + \rho_D) = 3\eta H \rho_D (1 + u) \tag{11}$$

where η is the coupling parameter. Considering $\eta = 0$, the equation of continuity reduces to the non-interacting case. Here u is defined as

$$u = \frac{\rho_m}{\rho_D} = \frac{\Omega_m}{\Omega_D} = \frac{1 - \Omega_D}{\Omega_D} \tag{12}$$

Under the above-defined parameters, the equation of state parameter for dark energy can be derived as:

$$\omega_D = -\frac{1}{2 - \Omega_D}\left(1 - \frac{\Omega_k}{3} + \frac{2\eta}{\Omega_D}(1 + \Omega_k)\right) \tag{13}$$

For flat Universe after taking $k = 0$, the equation of state parameter for dark energy from Eq. (13) can be rewritten as [35]:

$$\omega_D = -\frac{1}{2 - \Omega_D}\left(1 + \frac{2\eta}{\Omega_D}\right) \tag{14}$$

3. Isotropization with constant deceleration parameter

The deceleration parameter is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{15}$$

The sign of q indicates whether the model accelerates or decelerates. The negative sign of q indicates acceleration whereas positive value stands for deceleration. Also, recent observations of SNe-Ia expose that the present Universe is accelerating for $H > 0$ and $q < 0$. In this article, we use the well-known relation between Hubble parameter and scale factor as

$$H = ba^{-\alpha}, \tag{16}$$

where $b > 0$ and $\alpha \geq 0$ are constants. The above relation yields a constant value of deceleration parameter.

Considering the same relation recently Shekh et al. [30], analyzed the Tsallis holographic dark energy in the modified theory of gravity which involve torsion as well as boundary term toward the standard form of the model and observed that for the said relation not only second law of thermodynamics is valid for the same temperature between inside the horizon but for the change in temperature. Also the observation shows that the expansion of the quintessence field dominated model is accelerated due to violation of strong and fulfillment of weak and null energy conditions.

Using the definition of Hubble parameter, one can get

$$\dot{a} = ba^{-\alpha+1} \tag{17}$$

$$\ddot{a} = -b^2(\alpha - 1)a^{-2\alpha+1}. \tag{18}$$

Consequently the deceleration parameter q takes the form

$$q = -1 + \alpha \tag{19}$$

which is a constant. Integration of Eq. (16) yields

$$a = (\alpha bt + c)^{\frac{1}{\alpha}} = (a_1 t + c)^{\gamma}, \tag{20}$$

provided $\gamma = \frac{1}{\alpha} = \frac{1}{1+q}$ where $q \neq -1$, $a_1 = \alpha b \neq 0$ and c are constants of integration.

From Eq. (19), it is observed that the scale factor of the model is the functions of cosmic time, which increase with time at $q > -1$, decreases with time at $q < -1$, and does not exist at $q = -1$. Moreover, let us consider the Hubble's parameters, whose form is

$$H = \frac{\dot{a}}{a}. \tag{21}$$

For the relation of a and z as $a = \frac{1}{1+z}$, from the above Eq. (20) the expression for Hubble's parameters is obtained as

$$H = \alpha b(1 + z)^{\alpha}. \tag{22}$$

The behavior of Hubble's parameter of interacting dark energy model versus redshift with the appropriate choice of fix value of constants $b = 1$ and $\alpha = 0.45$ is shown in Fig. 1 and observed that it is positive increasing function of redshift. At all Universes, i.e., from past to present, the
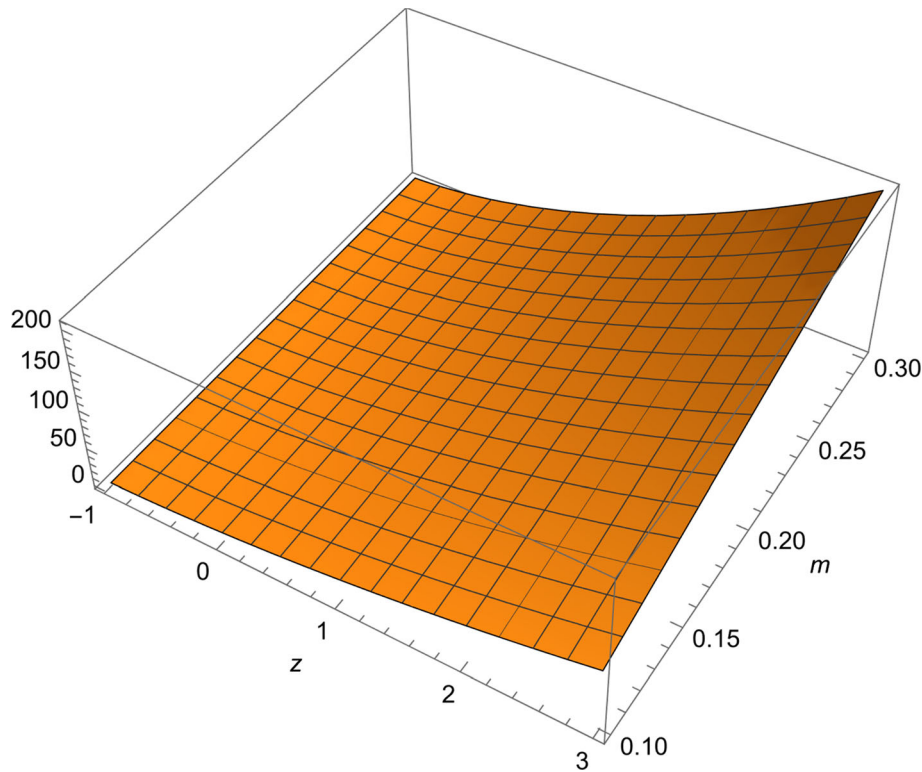
Fig. 1 Behavior of Hubble's parameter of dark energy model versus redshift with the appropriate choice of constants $b = 1, 0.10 \leq m \leq 0.30$ and $\alpha = 0.45$

Hubble's parameter is rigorously positive and increases with the rise in the value of redshift. The increasing behavior of Hubble's parameter of interacting dark energy model is clearly shown in Fig. 1.

With the help of Eq. (21), the time derivatives of H observed as

$$\dot{H} = -b^2\alpha(1+z)^{2\alpha}, \quad \ddot{H} = 2b^3\alpha^2(1+z)^{3\alpha},$$
$$\dddot{H} = -6b^4\alpha^3(1+z)^{4\alpha}, \quad \ddddot{H} = 24b^5\alpha^4(1+z)^{5\alpha} \tag{23}$$

Throughout the analysis, fix value of constants like $b = 1$ and $\alpha = 0.45$.

4. $f(Q)$ gravity

In this section, I shall work on symmetric teleparallel gravity in which the gravitational interaction is completely described by the nonmetricity term defined by Q by way of torsion and curvature free geometry.

Let us consider the action for $f(Q)$ gravity given by [36]

$$S = \left(\frac{1}{2\kappa^2}\int f(Q) + \int \pounds_m\right)\sqrt{-g}\mathrm{d}^4x, \tag{24}$$

where $f(Q)$ is a general function of Q, $\mathcal{L}_m$ is the matter Lagrangian density and g is the determinant of metric $g_{\mu\nu}$.

The nonmetricity tensor and its traces are such that

$$Q_{\gamma\mu\nu} = \nabla_\gamma g_{\mu\nu}, \tag{25}$$

$$Q_\gamma = Q_\gamma{}^\mu{}_\mu, \qquad \widetilde{Q}_\gamma = Q^\mu{}_{\gamma\mu}. \tag{26}$$

Moreover, the superpotential as a function of nonmetricity tensor is given

$$4P^\gamma{}_{\mu\nu} = -Q^\gamma{}_{\mu\nu} + 2Q_{(\mu}{}^\gamma{}_{\nu)} - Q^\gamma g_{\mu\nu} - \widetilde{Q}^\gamma g_{\mu\nu} - \delta^\gamma{}_{(\gamma}Q_{\nu)}, \tag{27}$$
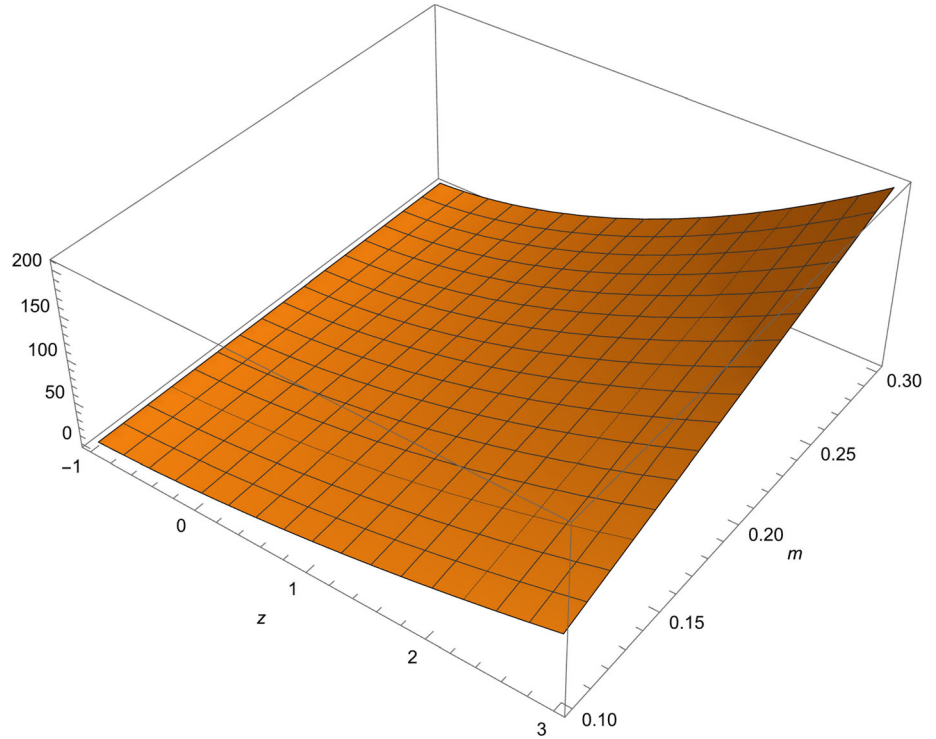
where the trace of nonmetricity tensor has the form

$$Q = -Q_{\gamma\mu\nu}P^{\gamma\mu\nu}. \tag{28}$$

Another relevant ingredient for our approach is the energy-momentum tensor for the matter, whose definition is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{29}$$

Taking the variation of action (23) with respect to metric tensor, one can find the field equations

Fig. 2 Behavior of energy density of dark energy $f(Q)$ gravity model versus redshift with the appropriate choice of constants $n = 2$, $0.10 \leq m \leq 0.30$ and $\kappa = 1$

$$\frac{2}{\sqrt{-g}} \nabla_\gamma \left(\sqrt{-g} f_Q P^\gamma{}_{\mu\nu} \right) + \frac{1}{2} g_{\mu\nu} f$$
$$+ f_Q \left(P_{\mu\gamma i} Q_\nu{}^{\gamma i} - 2 Q_{\gamma i\mu} P^{\gamma i}{}_\nu \right) = -\kappa^2 \mathcal{T}_{\mu\nu}, \tag{30}$$

where $f_Q = \frac{df}{dQ}$. Besides, can take the variation of (23) with respect to the connection, yielding to

$$\nabla_\mu \nabla_\gamma \left(\sqrt{-g} f_Q P^\gamma{}_{\mu\nu} \right) = 0. \tag{31}$$

The components of field equation for spatially homogeneous and isotropic FRW space-time for $f(Q)$ gravity is given by

$$3H^2 = \frac{1}{2f_Q} \left(-\kappa^2 \rho_D + \frac{f}{2} \right) \tag{32}$$

$$\dot{H} + 3H^2 + \frac{\dot{f}_Q}{f_Q} H = \frac{1}{2f_Q} \left(\kappa^2 p_D + \frac{f}{2} \right) \tag{33}$$

Also, the modified Friedmann equations enable to write the density and the pressure for the Universe as

$$\kappa^2 \rho_D = \frac{f}{2} - 6H^2 f_Q \tag{34}$$

$$\kappa^2 p_D = \left(\dot{H} + 3H^2 + \frac{\dot{f}_Q}{f_Q} H \right) (2f_Q) - \frac{f}{2} \tag{35}$$

which are the density and the pressure for the $f(Q)$ gravity.

Very recently, in view of the relation between cosmic time and redshift as $t = \frac{1}{\alpha k} \log \left(1 + \frac{1}{(1+z)^\alpha} \right)$ by considering the cutoff as Hubble's and Granda-Oliveros Shekh [36] had

discussed the evolution trajectories of the equation of state parameter and stability parameters in an evolving $f(Q)$ gravity model toward the models of holographic dark energy and observed that $f(Q)$ gravity model is purely accelerating.

Analysis of some cosmographic parameters in $f(Q)$ gravity

The cosmographic parameters such as energy density, equation of state parameter, stability through squared velocity of sound and plane like $(\omega_T - \omega_T')$ in dark energy $f(Q)$ gravity toward $f(Q) = Q + mQ^n$ are derived as follows

4.1. Energy density

Making use of Eqs. (21) in (33), the energy density of dark energy in $f(Q)$ gravity model is obtained as

$$\rho_D = -\frac{1}{2\kappa^2} (6H^2) + \frac{m(1 - 2n)}{2\kappa^2} (6H^2)^n \tag{36}$$

The behavior of energy density of dark energy $f(Q)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m is described in Fig. 2. The energy density of dark energy $f(Q)$ gravity model is always positive and increasing function of redshift (see fig. 2).

Fig. 3 Behavior of equation of state parameter of dark energy $f(Q)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.01$
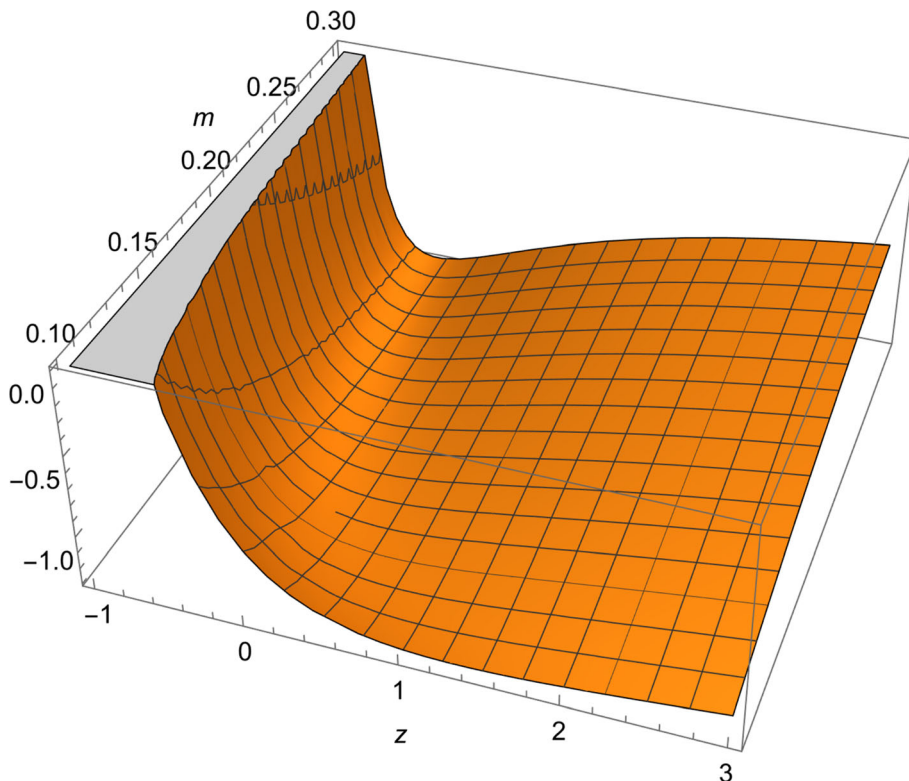
4.2. Equation of state parameter

From Eq. (14), the equation of state parameter of dark energy $f(Q)$ gravity model is obtained as

$$\omega_D = \frac{-1}{3 - m(1 - 2n)(6H^2)^{n-1}}$$
$$\left(1 - \frac{2\eta}{1 - m(1 - 2n)(6H^2)^{n-1}}\right) \quad (37)$$

The behavior of equation of state parameter of $f(Q)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 3 and 4, respectively.

From Fig. 3, the equation of state parameter of dark energy $f(Q)$ gravity model at $z = -1$, $z = 0$ has less than -1 and at $z > 1$, $\omega_D > -1$ which represents the dark energy $f(Q)$ gravity model at $z = -1$, $z = 0$ involve phantom field and goes toward quintessence field region at $z > 1$ by crossing the phantom divide line and always stays in the quintessence field region of the universe for the value of interacting parameter $\eta = 0.01$ while Fig. 4, described the behavior of equation of state parameter of dark energy $f(Q)$ gravity model toward the value of interacting parameter $\eta = 0.05$. According to Fig. 4, the evolution of the

equation of state parameter of dark energy $f(Q)$ gravity model represents the same behavior as that of $\eta = 0.01$. Thus, the analysis of an accelerating expansion of the dark energy $f(Q)$ gravity model is consistent with the recent observations for the values of interacting parameter toward $\eta = 0.01$, $\eta = 0.05$ along with the holographic dark energy with Hubble's cutoff in $f(Q)$ gravity studied by Shekh [36].

4.3. Squared velocity of sound

The stability of dark energy $f(Q)$ gravity model is analyzed through the squared velocity of sound ϑ_s^2. For the model of the Universe to be stable, the ϑ_s^2 must be larger than zero. By taking the time derivative of Eqs. (33) and (34), the stability parameter $\vartheta^2\left(=\frac{\partial p}{\partial \rho}\right)$ is obtained as

The behavior of squared velocity of sound of dark energy $f(Q)$ gravity model versus redshift toward the appropriate choice of fix constant $n = 2$ is described in Fig. 5. It is observed that the squared velocity of sound of dark energy $f(Q)$ gravity model for all universe at $z < -1$, $z = 0$ and $z > 0$, $\vartheta^2 > 0$ and approaches to small positive value which is ($\ll 1$). Hence, in all Universe the model is stable.

$$\vartheta_s^2 = \frac{2\left(1 + mn(6H^2)^{n-1}\right)\ddot{H} + 12\left(1 + mn(6H^2)^{n-1}\right)H\dot{H} + 2mn(n-1)6^{n-1}\left\{\ddot{H} + (6H^2 + 4\dot{H})\dot{H} + (2n-3)\dot{H}\right\}H^{2n-3}}{-6H\dot{H}\left(1 + mn(6H^2)^{n-1} + 2mn6^{n-1}(n-1)H^{2n}\right)}$$

(38)

Fig. 4 Behavior of equation of state parameter of dark energy $f(Q)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.05$
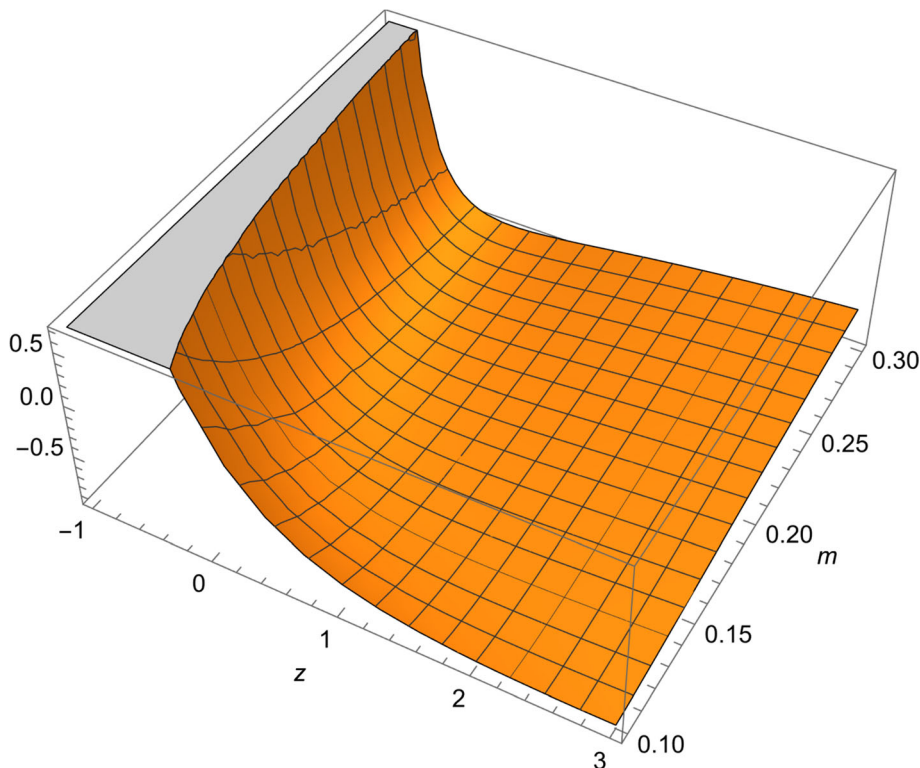

4.4. $(\omega_D - \omega_D')$-plane

To discuss the $\omega_D - \omega_D'$-plane, it is necessary to find ω_D'. By taking the variation of Eq. (36) with respect to $\ell n a$, the ω_D' is obtained as

The behavior of ω_D' of $f(Q)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 6 and 7, respectively.

$$\omega_D' = \frac{k}{\alpha H}\left\{\frac{12m(n-1)(1-2n)6^{n-2}H^{2n-4}H\dot{H}}{3 - m(1-2n)(6H^2)^{n-1}}\left(-2\eta + \frac{1 + 2\eta - m(1-2n)(6H^2)^{n-1}}{\left(-1 + m(1-2n)(6H^2)^{n-1}\right)\left(3 - m(1-2n)(6H^2)^{n-1}\right)}\right)\right\}$$

(39)

For all $\eta = 0.01$, $\eta = 0.05$ and varying m, the behavior of ω_D' is always negative decreasing with redshift, i.e.,

Fig. 5 Behavior of squared
velocity of sound of dark energy
$f(Q)$ gravity model versus
redshift with the appropriate
choice of constant $n = 2$ and
$0.10 \leq m \leq 0.30$
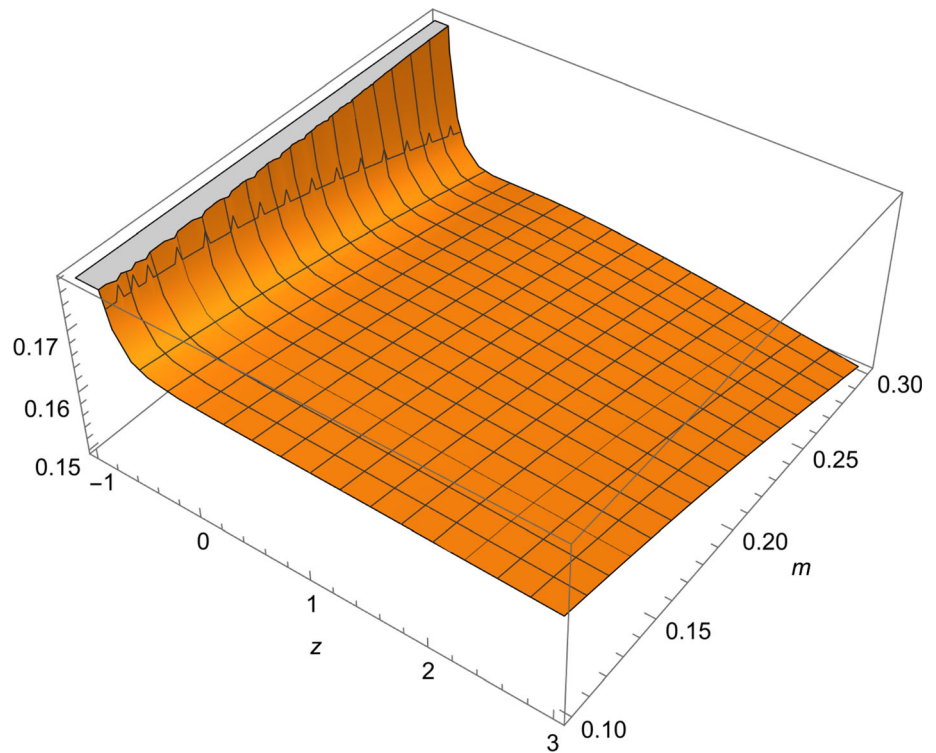
$\omega'_D < 0$. Hence from Figs. 6 and 7 together with 3 and 4, it is predicted that the dark energy $f(Q)$ gravity model has ω_D and ω'_D both are <0, which shows a freezing region and remain present in freezing region in all the Universes which described the consistency of an accelerated expansion of the universe.

Fig. 6 Behavior of equation of
state parameter of dark energy
$f(Q)$ gravity model with respect
to ℓna versus redshift with the
appropriate choice of constant
$n = 2$, $0.10 \leq m \leq 0.30$ and
$\eta = 0.01$
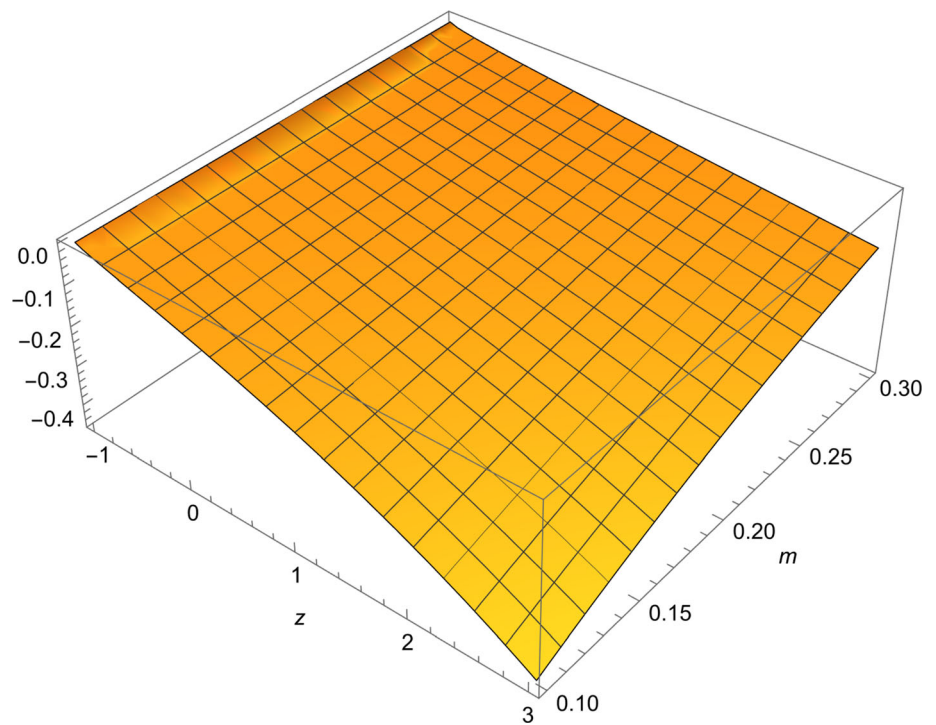
Fig. 7 Behavior of equation of state parameter of dark energy $f(Q)$ gravity model with respect to $\ell n a$ versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.05$
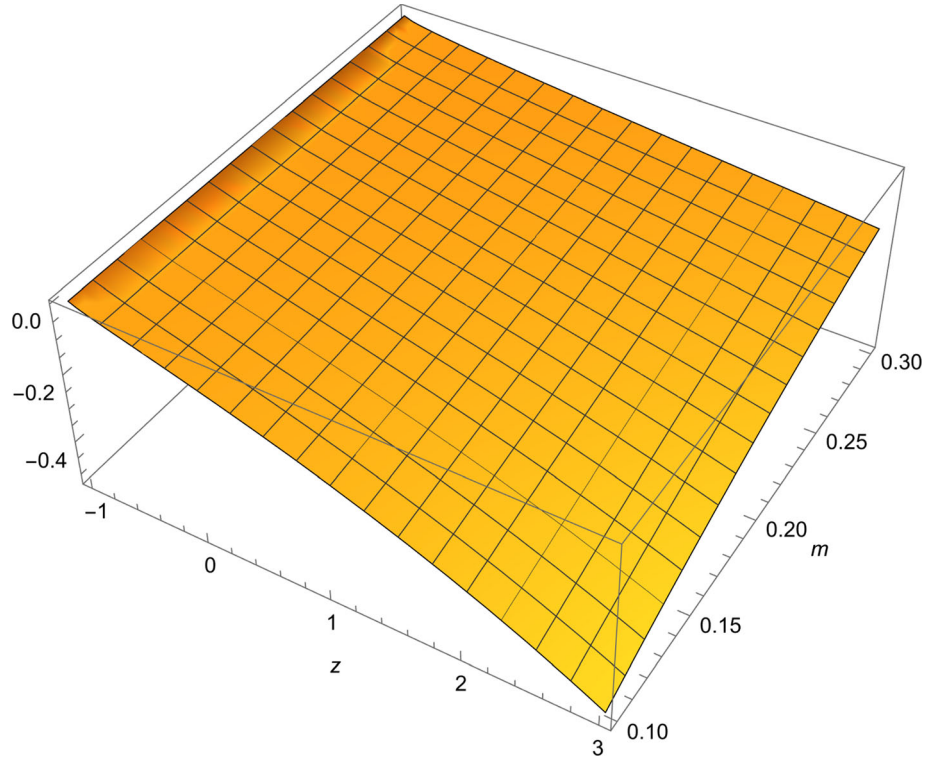
5. $\hat{f}(T)$ gravity

In this section, a brief description of $f(T)$ gravity and a detailed derivation of its field equations are given. Let us define the notations of the Latin subscript as these related to the tetrad field and the Greek one related to the space-time coordinates. For a general space-time metric, the line element is defined as

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{40}$$

where $g_{\mu\nu}$ are the components of the metric which is symmetric and possesses ten degrees of freedom. One can describe the theory in the space-time or in the tangent space, which allows us to rewrite the line element which can be converted to the Minkowski's description of the transformation called tetrad (which represent the dynamic fields of the theory), as follows

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ij}\theta^i\theta^j, \tag{41}$$

$$dx^\mu = e_i^\mu\theta^i, \quad \theta^i = e_\mu^i dx^\mu, \tag{42}$$

where η_{ij} is a metric on Minkowski space-time and $\eta_{ij} = diag[1, -1, -1, -1]$ and $e_i^\mu e_\nu^i = \delta_\nu^\mu$ or $e_i^\mu e_\mu^j = \delta_i^j$. The root of metric determinant is given by $\sqrt{-g} = \det[e_\mu^i] = e$.

Now, the action for $f(T)$ theory of gravity is [8–10]

$$S = \int [T + f(T) + L_m]\, e\, d^4x. \tag{43}$$

Here, $f(T)$ denotes an algebraic function of the torsion scalar T. Making the functional variation of the action (42) with respect to the tetrads, one can get the following equations of motion

$$S_\mu^{\nu\rho}\partial_\rho T f_{TT}$$
$$+ \left[e^{-1}e_\mu^i\partial_\rho\left(ee_i^\alpha S_\alpha^{\nu\rho}\right) + T^\alpha_{\lambda\mu}S_\alpha^{\nu\lambda}\right]f_T + \frac{1}{4}\delta_\mu^\nu(f) = \kappa^2 \mathcal{T}_\mu^\nu. \tag{44}$$

where $\hat{T}_\mu^\nu$ is the energy momentum tensor, $f_T = df(T)/dT$, $f_{TT} = d^2f(T)/dT^2$ and by setting $f(T) = a_0 = $ constant the equations of motion (43) are the same as that of the Teleparallel Gravity with a cosmological constant.

$$\kappa^2\rho_D = 3\frac{\dot{a}^2}{a^2}f_T + \frac{f}{4} \tag{45}$$

$$\kappa^2 p_D = \frac{\dot{a}}{a}\dot{T}f_{TT} + \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)f_T + \frac{f}{4} \tag{46}$$

Analysis of some cosmographic parameters in $f(T)$ gravity

The cosmographic parameters such as energy density, equation of state parameter, stability through squared speed of sound and plane like $(\omega_D - \omega_D')$ in dark energy $f(T)$ gravity toward $f(T) = T + mT^n$ are derived as follows

Fig. 8 Behavior of energy density of dark energy $f(T)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$, $-1.2 \leq \gamma_1 \leq -3.6$ and $\kappa = 1$
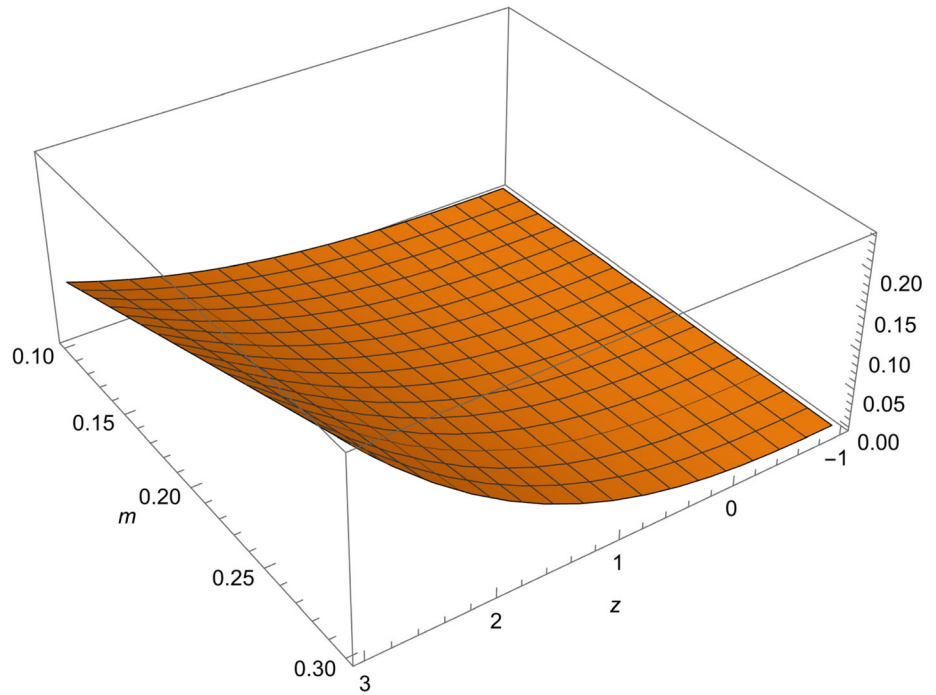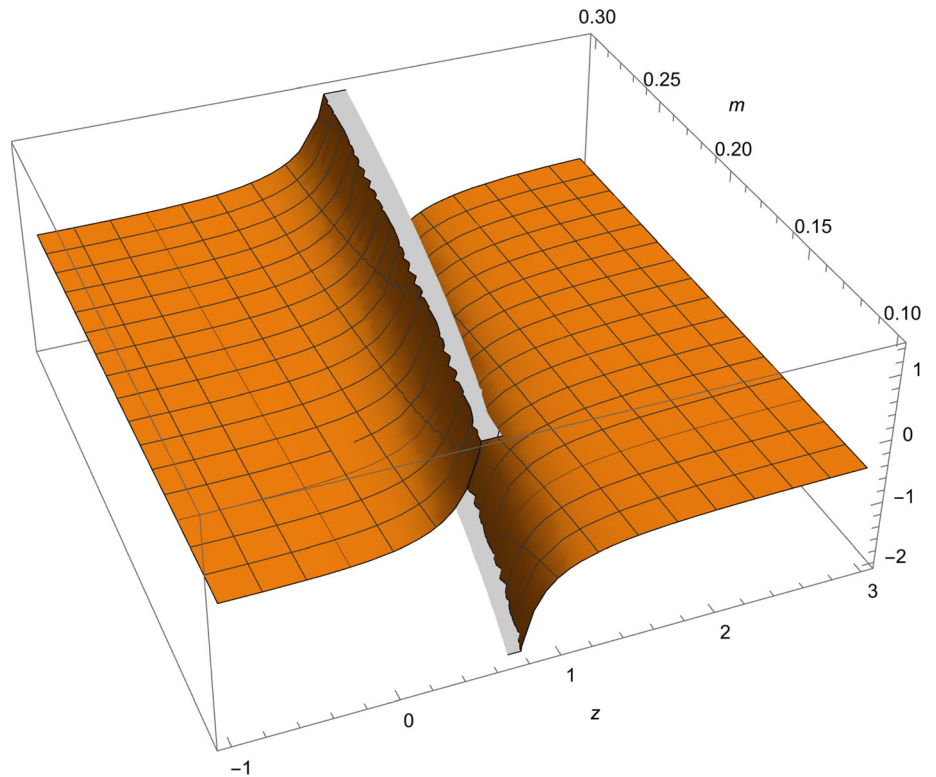

Fig. 9 Behavior of equation of state parameter of dark energy $f(T)$ gravity model versus redshift with the appropriate choice of constants $n = 2$, $\eta = 0.01$, $0.10 \leq m \leq 0.30$ and $-1.2 \leq \gamma_1 \leq -3.6$
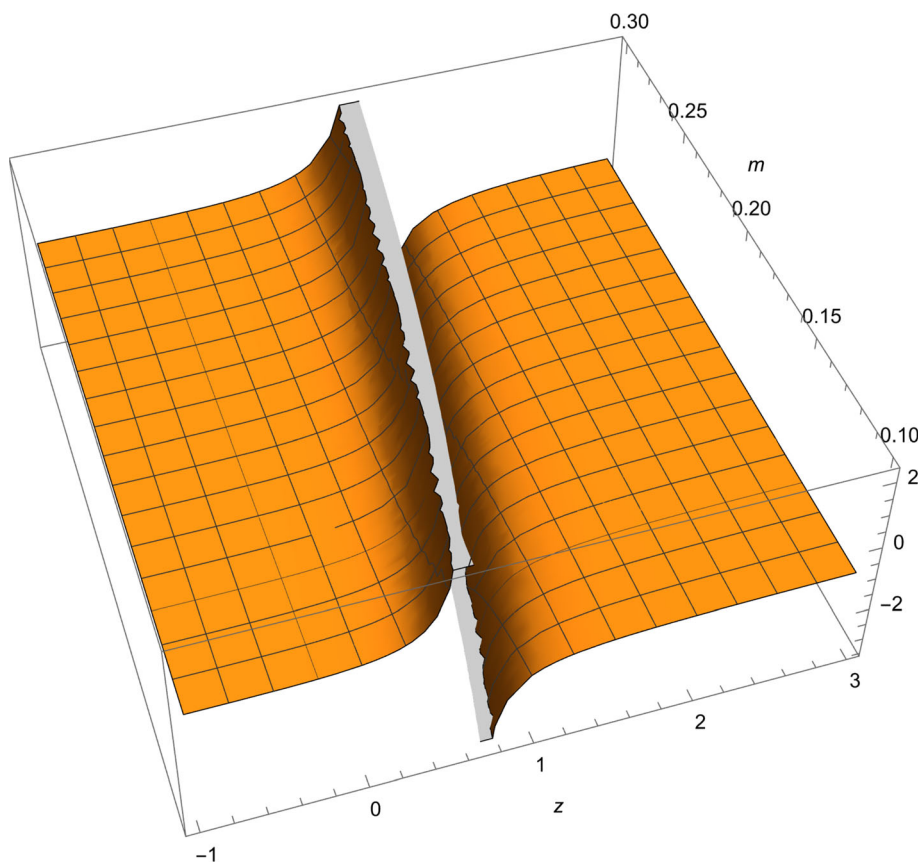

5.1. Energy density

Making use of Eq. (21) in Eq. (44), the energy density of dark energy $f(T)$ gravity model is obtained as

$$\kappa^2 \rho_D = \frac{3}{2}H^2 + \left(\frac{3\gamma_1(2n+1)}{2n}\right)H^{2n} \tag{47}$$

where $\gamma_1 = mn(-6)^{n-1}$.

The behavior of energy density of dark energy $f(T)$ gravity model versus redshift toward the appropriate choice

Fig. 10 Behavior of equation of state parameter of dark energy $f(T)$ gravity model versus redshift with the appropriate choice of constants $n = 2$, $\eta = 0.01$, $0.10 \leq m \leq 0.30$ and $-1.2 \leq \gamma_1 \leq -3.6$

of constant $n = 2$ and varying m is described in Fig. 8. It is observed that the energy density of dark energy $f(T)$ gravity model is always positive and increasing function of redshift (see Fig. 8).

5.2. Equation of state parameter

From Eq. (14), the equation of state parameter of dark energy $f(T)$ gravity model is obtained as

$$\omega_D = -\frac{1}{1 - \gamma_1(2n+1)H^{2n-2}}\left(1 - \frac{2\eta}{1 + \gamma_1(2n+1)H^{2n-2}}\right) \tag{48}$$

The behavior of equation of state parameter of dark energy $f(T)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 9 and 10, respectively.

Figure. 9 provides the value of equation of state parameter at $z = -1$ and $z = 0$ has greater than -1 while at $0.6 \leq z \leq 0.8$ it has less than -1 which represents the dark energy $f(T)$ gravity model at $z = -1$ and $z = 0$ involve a quintessence field region of the Universe for the value of interacting parameter $\eta = 0.01$ and goes to phantom field region at $0.6 \leq z \leq 0.8$. Also, the dark energy $f(T)$ gravity

model toward $\eta = 0.01$ crosses the phantom divide line and converges to the matter-dominated phase of the Universe as shown in Fig. 9. The $f(T)$ gravity model always stays in the matter-dominated phase of the universe whereas in the another Fig. 10, for $\eta = 0.05$, the evolution of the equation of state parameter of dark energy $f(T)$ gravity model at $z = -1$ and $z = 0$ represents the same behavior as that of $\eta = 0.01$, i.e., a quintessence field region and goes toward phantom field region at $0.4 \leq z \leq 0.6$, it also crosses the phantom divide line and converges to the matter-dominated phase of the Universe (see Fig. 10) and always stays in the matter-dominated phase of the universe. Thus, the analysis of an accelerating expansion of the dark energy $f(T)$ gravity model is consistent with the recent observations for the values of interacting parameter toward $\eta = 0.01$, $\eta = 0.05$.

5.3. Squared velocity of sound

By taking the time derivative of Eqs. (44) and (45), the stability parameter $\vartheta_s^2 \left(= \frac{\partial p}{\partial \rho}\right)$ is obtained as the behavior of squared velocity of sound of dark energy $f(T)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ is described in Fig. 11.

Fig. 11 Behavior of squared
velocity of sound of dark energy
$f(T)$ gravity model versus
redshift with the appropriate
choice of constants $n = 2$,
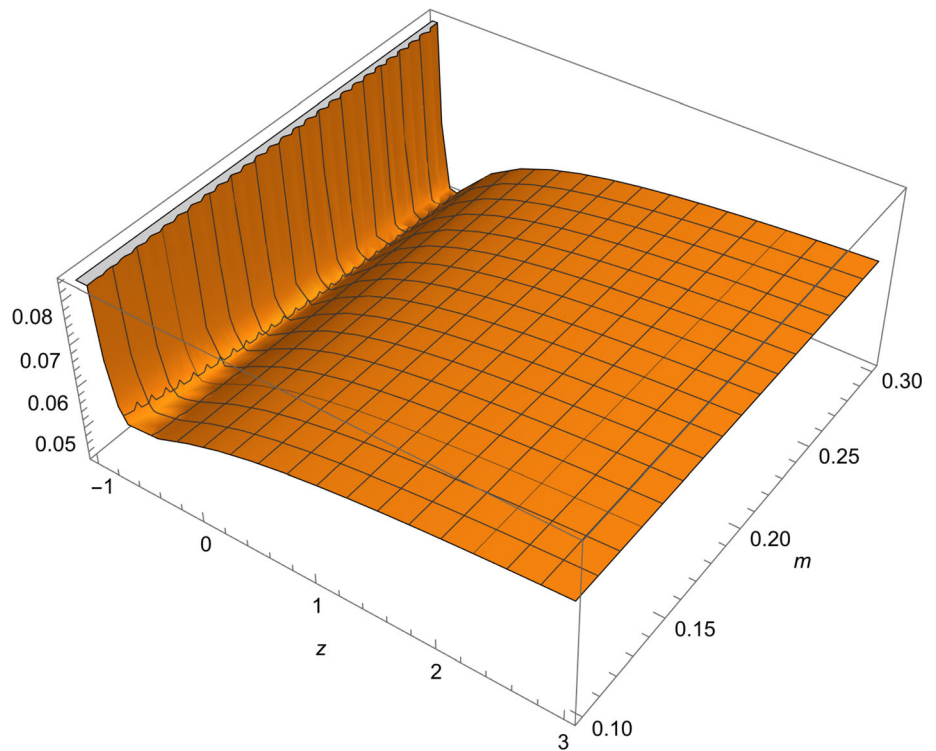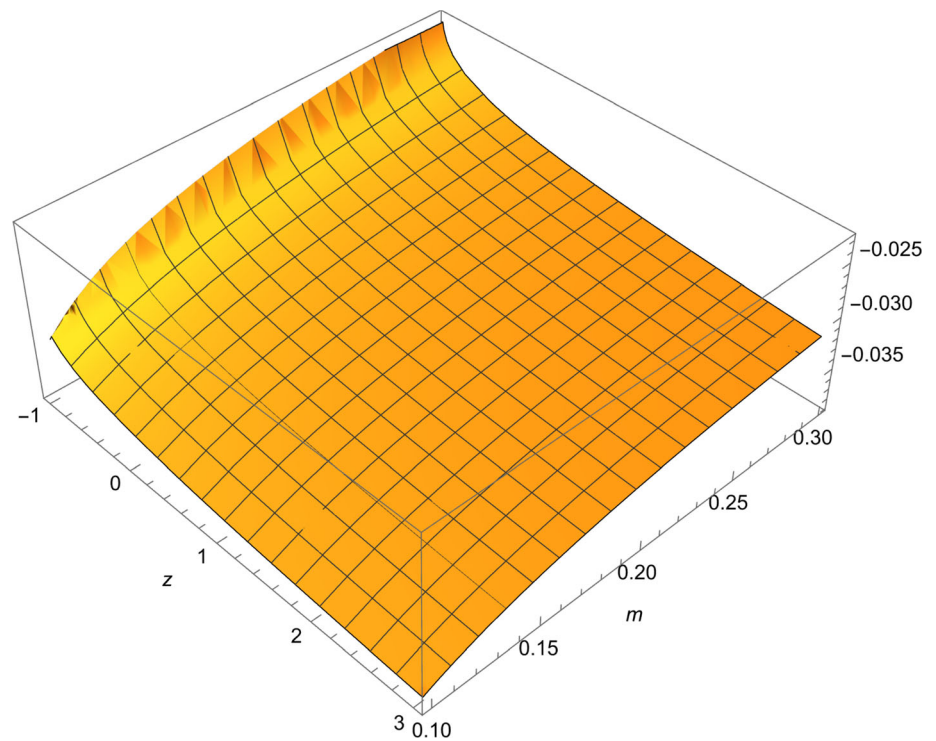$\eta = 0.01$, $0.10 \leq m \leq 0.30$ and
$-1.2 \leq \gamma_1 \leq -3.6$

Fig. 12 Behavior of equation of
state parameter of dark energy
$f(T)$ gravity model with respect
to ℓna versus redshift with the
appropriate choice of constants
$n = 2$, $\eta = 0.01$,
$0.10 \leq m \leq 0.30$ and
$-1.2 \leq \gamma_1 \leq -3.6$

It is observed that the squared velocity of sound of dark
energy $f(T)$ gravity model for all universe at $z < -1$, $z =
0$ and $z > 0$ is always greater than zero and approaches to
small positive value which is $(\ll 1)$. Hence, in all the
Universe the model is stable.
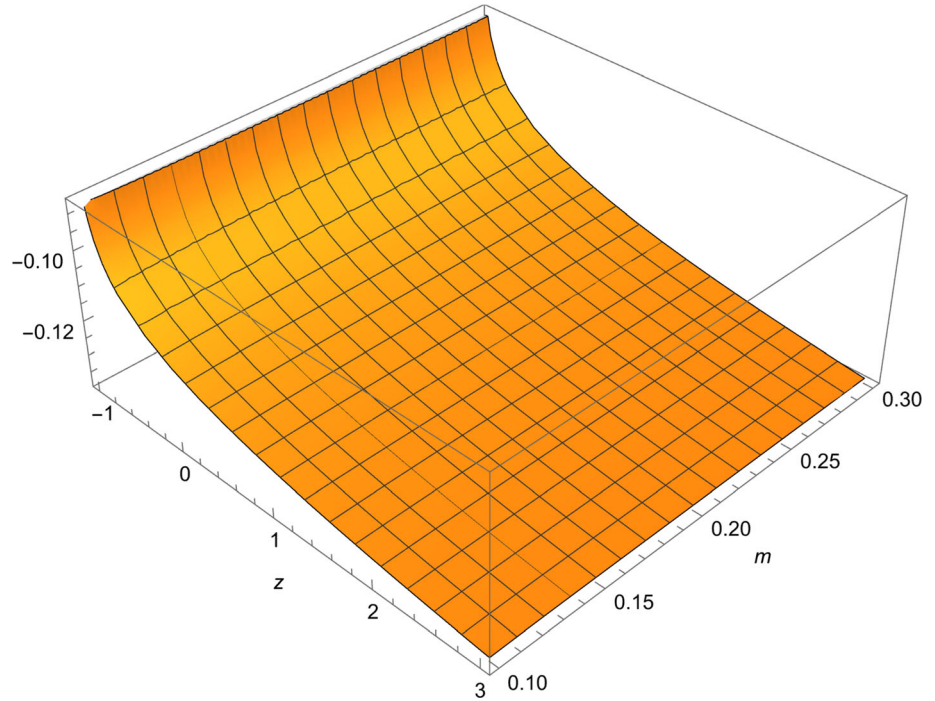
$$\vartheta_s^2 = \frac{3H\dot{H} + \dddot{H} - \gamma_1(5H\dot{H} + \ddot{H})H^{2n-2} + 2\gamma_1(n-1)(3H^3\dot{H} + H\dot{H}^2 + \dot{H}^2 + H\ddot{H})H^{2n-4} - 4\gamma_1(n-1)(n-2)\dot{H}^2H^{2n-4}}{3H\dot{H} + 3\gamma_1(2n-1)\dot{H}H^{2n-1}}$$

(49)

5.4. $\omega_D - \omega_D'$ plane

By taking the variation of Eq. (47) with respect to ℓna, the ω' for dark energy $f(T)$ gravity model is obtained as The behavior of ω_D' of $f(T)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 12 and 13, respectively.

For all $\eta = 0.01$, $\eta = 0.05$ and varying m, the behavior of ω' is always less than 0, i.e., $\omega' < 0$. Hence from Figs. 12 and 13 together with 9 and 10, it is predicted that the dark energy $f(T)$ gravity model showing a freezing region and remain present in freezing region in all the Universes

which described the consistency of an accelerated expansion of the Universe.

6. $\hat{f}(G)$ gravity

In this section, we review $\hat{f}(G)$ gravity and formulate the field equations as follows.

The modified Gauss–Bonnet gravity is described by the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + \hat{f}(G) \right] + S_m(g^{\mu\nu}, \psi),$$

(51)

where κ is the coupling constant, g is the determinant of the metric tensor $g_{\mu\nu}$, and $S_m(g^{\mu\nu}, \psi)$ is the matter action, in

$$\omega_D' = \frac{k}{\alpha H} \left\{ \frac{2\gamma_1(n-1)(2n+1)\dot{H}H^{2n-3}}{1 - \gamma_1(2n+1)H^{2n-2}} \left(-2\eta + \frac{-1 + 2\eta - \gamma_1(2n+1)H^{2n-2}}{(1 + \gamma_1(2n+1)H^{2n-2})(1 - \gamma_1(2n+1)H^{2n-2})} \right) \right\}$$

(50)

Fig. 14 Behavior of energy density of dark energy $\hat{f}(G)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \le m \le 0.30$ and $\kappa = 1$
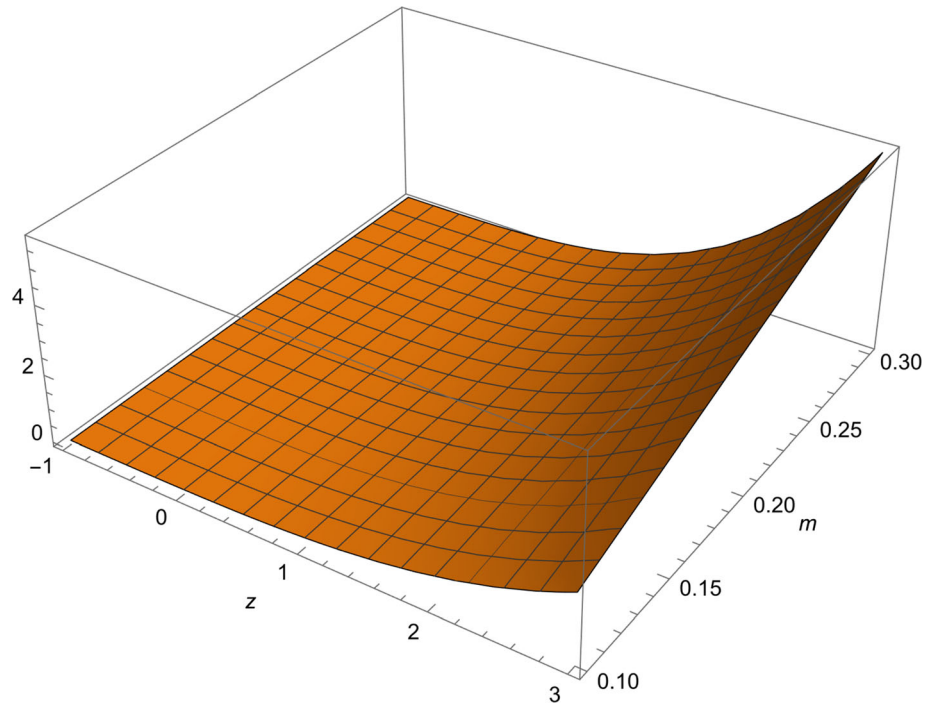
which matter is minimally coupled to the metric tensor and ψ denotes the matter fields. This coupling of matter to the metric tensor suggests that $\hat{f}(G)$ gravity is a purely metric theory of gravity where $\hat{f}(G)$ is an arbitrary function of the Gauss–Bonnet invariant G:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}, \tag{52}$$

where R is the Ricci scalar and $R_{\mu\nu}$, $R_{\mu\nu\sigma\rho}$ denote the Ricci and Riemann tensors.

The gravitational field equations are obtained by varying the action in Eq. (50) with respect to the metric tensor:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\big[R_{\mu\nu\sigma\rho} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu}$$
$$- R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho} + \frac{1}{2}R(R_{\mu\nu}g_{\sigma\rho} - R_{\rho\sigma}g_{\nu\mu})\big]\times \tag{53}$$
$$\nabla^\rho\nabla^\sigma F + (G\hat{f}_G - \hat{f})g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu},$$

where ∇^ρ denotes the covariant derivative and $\hat{f}_G$ represents the derivative of $\hat{f}$ with respect to G. From Eq. (10), (51) and (52) the construction of Gauss–Bonnet contribution in energy density and pressure are obtained as

$$\kappa^2 \rho_D = \frac{1}{2k^2}\Big\{6H^2 + 24H^3\dot{\hat{f}}_G - G\hat{f}_G + \hat{f}\Big\}, \tag{54}$$

$$\kappa^2 p_D = \frac{1}{2k^2}\Big\{-4\dot{H} - 6H^2 - 8H^2\ddot{\hat{f}}_G - 16H(\dot{H} + H^2)\dot{\hat{f}}_G + G\hat{f}_G - \hat{f}\Big\}, \tag{55}$$

For the different context of use recently the authors Cognola [37], Bamba et al. [38], Tsujikawa [39], Fayaz et al. [40], Abbas et al. [41] have analyzed the same gravity.

Analysis of some cosmographic parameters in $\hat{f}(G)$ gravity

The cosmographic parameters such as energy density, equation of state parameter, stability through squared speed of sound and plane like $(\omega_D - \omega'_D)$ in dark energy $\hat{f}(G)$ gravity toward $\hat{f}(G) = mG^n$ are derived as follows

6.1. Energy density

Making use of Eq. (21) in Eq. (53), the energy density of dark energy in $\hat{f}(G)$ gravity model is obtained as

$$\kappa^2 \rho_D = 6H^2 + 24\text{mn}(n-1)H^3\dot{G}G^{n-2} - m(n-1)G^n \tag{56}$$

where $\qquad G = 24(\dot{H} + H^2)H^2 \qquad$ and $\dot{G} = 24(\ddot{H}H + 2\dot{H}^2 + 4H^2\dot{H})H$.

The behavior of energy density of dark energy $\hat{f}(G)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m is described in Fig. 14. It is observed that the energy density of dark energy $\hat{f}(G)$ gravity model is always positive and increasing function of redshift (see Fig. 14).

6.2. Equation of state parameter

From Eq. (14), the equation of state parameter of dark energy $\hat{f}(G)$ gravity model is obtained as

$$\omega_D = \frac{-1}{(m(n-1)/3)H^{-2}G^n - 8mn(n-1)H\dot{G}G^{n-2}}$$
$$\left(1 + \frac{2\eta}{2 + 8mn(n-1)H\dot{G}G^{n-2} - (m(n-1)/3)H^{-2}G^n}\right) \tag{57}$$

The behavior of equation of state parameter of dark energy $\hat{f}(G)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 15 and 16, respectively.

The equation of state parameter of dark energy $\hat{f}(G)$ gravity model at $z = -1$, $z = 0$ and $z > 0$ represents a quintessence field region of the Universe for the value of interacting parameter $\eta = 0.01$ and always stays in the quintessence field region of the Universe as shown in Fig. 15 whereas in the another Fig. 16, for $\eta = 0.05$, the evolution of the equation of state parameter of dark energy $\hat{f}(G)$ gravity model at $z = -1$, $z = 0$ and $z > 0$ represents the same behavior as that of $\eta = 0.01$, i.e., represents a quintessence field region of the universe. Thus, the analysis of an accelerating expansion of the dark energy $\hat{f}(G)$ gravity model is consistent with the recent observations for the values of interacting parameter toward $\eta = 0.01$, $\eta = 0.05$.

6.3. Squared velocity of sound

By taking the time derivative of Eqs. (53) and (54), the stability parameter $\vartheta^2 \left(= \frac{\partial p}{\partial \rho}\right)$ is obtained as

It is observed that the squared velocity of sound of dark energy $\hat{f}(G)$ gravity model for the Universe at $z = -1$, the stability parameter $\vartheta_s^2 < 0$ which forecast that the dark energy $\hat{f}(G)$ gravity model is unstable while for $z = 0$ and $z > 0$ always greater than zero, i.e., $\vartheta_s^2 > 0$. Hence, the same dark energy $\hat{f}(G)$ gravity model of the Universe is stable.

6.4. $\omega_D - \omega_D'$ plane

By taking the variation of Eq. (56) with respect to $\ell n a$, the ω_D' for dark energy $\hat{f}(G)$ gravity model is obtained as

$$\omega_D' = \frac{k}{\alpha H}\left\{\frac{\dot{\Omega}_D'}{2 - \Omega_D'}\left(-2\eta + \frac{2\eta - \Omega_D'}{\Omega_D'(2 - \Omega_D')}\right)\right\} \tag{59}$$

where $\Omega_D' = 8mn(n-1)H\dot{G}G^{n-2} - (m(n-1)/3)H^{-2}G^n$ and $\dot{\Omega}_D' = 8mn(n-1) \times (\dot{H}\dot{G}G^{n-2} + H\ddot{G}\ G^{n-2} + (n-2)H\dot{G}^2G^{n-3}) - (m(n-1)/3) \times (nH^{-2}\dot{G}G^{n-1} - 2H^{-3}\dot{H}G^n)$.

The behavior of ω_D' of dark energy $\hat{f}(G)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 18 and 19, respectively.

For all $\eta = 0.01$, $\eta = 0.05$ and varying m, the behavior of ω_D' shows both $\omega_D' > 0$ as well as $\omega_D' < 0$. Figures 18 and 19 together with 15 and 16 show that it is predicted that the dark energy $\hat{f}(G)$ gravity model showing both a freezing and thawing region in all the Universes. Hence, due to involving freezing region the dark energy $\hat{f}(G)$ gravity model described the consistency of an accelerated expansion of the Universe.

$$\vartheta_s^2 = \frac{-4\ddot{H} - 4(3 - 8\ddot{f}_G)H\dot{H} - (1 + 2H)\ddot{f}_G - 16(\dot{H}^2 + H\ddot{H} + 3\dot{H}H^2 + G)\dot{f}_G + \dot{G}f_G - \dot{f}}{12H\dot{H} + 72H\dot{H}\dot{f}_G + 24H^3\ddot{f}_G - \dot{G}f_G - Gf_G + \dot{f}} \tag{58}$$

where $\dot{f} = mn\dot{G}G^{n-1}$, $f_G = mnG^{n-1}$, $\dot{f}_G = mn(n-1)\dot{G}G^{n-2}$, $\ddot{f}_G = m(n-1)\ddot{G}G^{n-2} + m(n-1)(n-2)\dot{G}^2G^{n-3}$ and $\ddot{G} = 24(\ddot{H}H^2 + 6H\dot{H}\ddot{H} + 2\dot{H}^3 + 12H^2\dot{H}^2 + 4\ddot{H}H^3)$.

The behavior of squared velocity of sound of dark energy $\hat{f}(G)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ is described in Fig. 17.

7. $\hat{f}(R)$ gravity

As the theory of $\hat{f}(R)$ gravity is the generalization of the general theory of relativity. The $\hat{f}(R)$ gravity has three main approaches which are "Metric Approach", "Palatine formalism" and "affine $\hat{f}(R)$ gravity". In the metric approach, the connection is the Levi-Civita connection and variation of the action is done with respect to the metric

Fig. 15 Behavior of equation of state parameter of dark energy $\hat{f}(G)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.01$
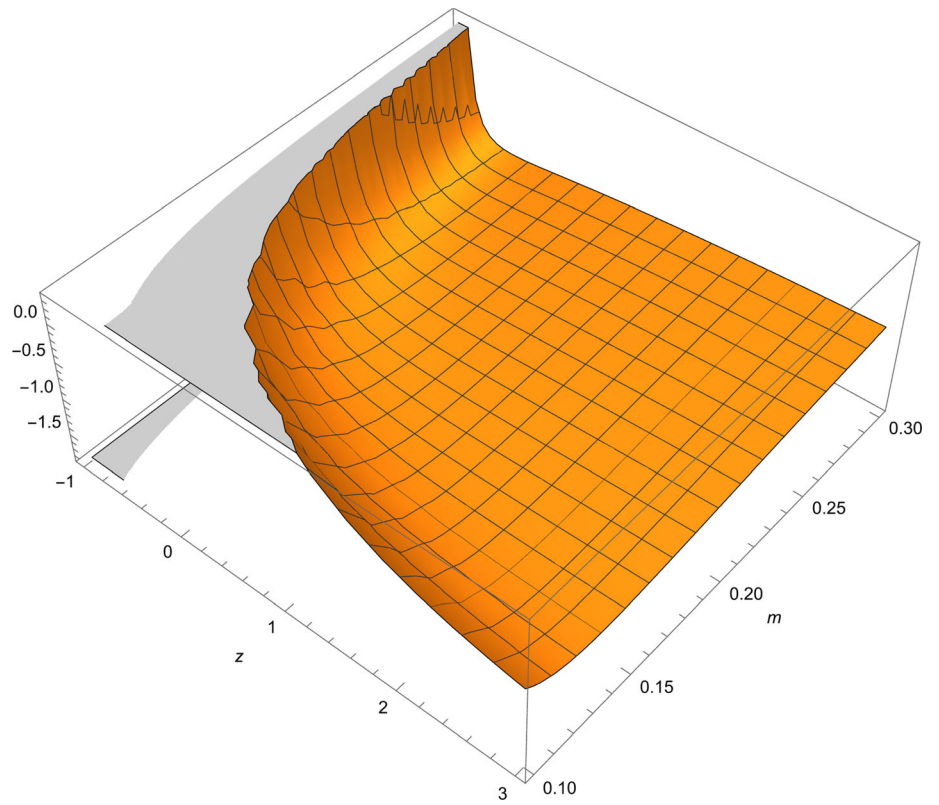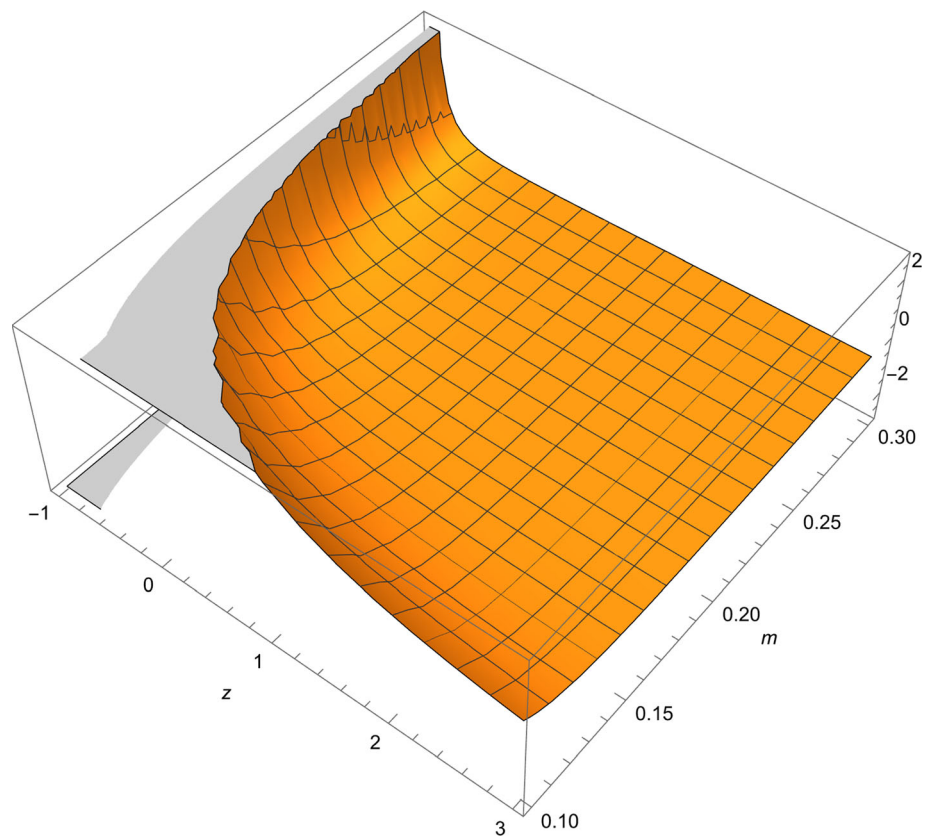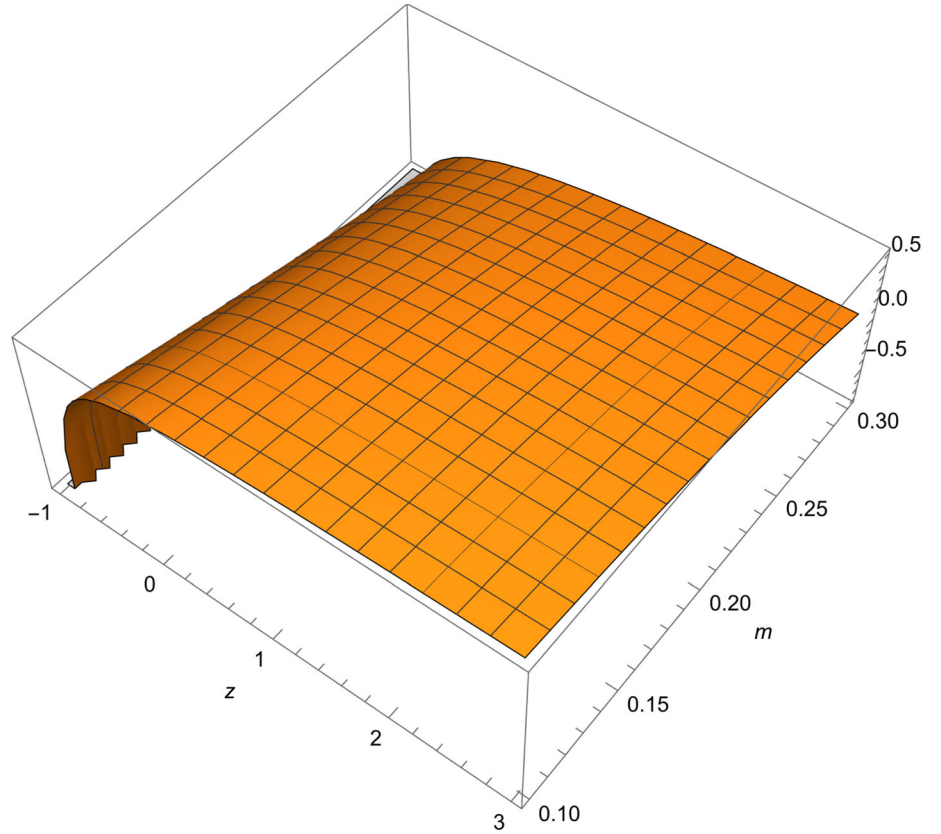

Fig. 16 Behavior of equation of state parameter of dark energy $\hat{f}(G)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.05$

Fig. 17 Behavior of squared velocity of sound of dark energy $\hat{f}(G)$ gravity model versus redshift with the appropriate choice of constants $n = 2$ and $0.10 \leq m \leq 0.30$

tensor. While, in Palatine formalism, the metric and the connection are independent of each other and variation is done for the two mentioned parameters independently. In metric-affine $\hat{f}(R)$ gravity, both the metric tensor and connection are treated independently and assuming the matter action depends on the connection as well.

The action for this theory is given by

$$S = \frac{1}{2\kappa^2} \int d^4 \sqrt{-g}\hat{f}(R) + \int d^4 x L_m(g_{\mu\nu}, \psi_m). \quad (60)$$

Here $\hat{f}(R)$ is a general function of the Ricci Scalar, g is the determinant of the metric $g_{\mu\nu}$ and L_m is the metric Lagrangian that depends on $g_{\mu\nu}$ and the matter field ψ_m. It is noted that this action is obtained just by replacing R by $\hat{f}(R)$ in the standard Einstein–Hilbert action.

The corresponding field equations are found by varying the action with respect to the metric,

$$F(R)R_{\mu\nu} - \frac{1}{2}\hat{f}(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R)) + g_{\mu\nu}\Box F(R) = \mathcal{T}_{\mu\nu} \quad (61)$$

where

$$\Box \equiv \nabla^\mu\nabla_\mu, F(R) \equiv \frac{d\hat{f}(R)}{dR}, \quad (62)$$

∇_μ is the covariant derivative and $\mathcal{T}_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m.

From the equation of motion (59), the Friedmann equation for two fluid scenarios reduces to the following set of field equations as:

$$\kappa^2 p_D = 2H^2 F - 2H\dot{F} - \ddot{F} - \frac{f}{2} \quad (63)$$

$$\kappa^2 \rho_D = 3H\dot{F} - 3(\dot{H} + H^2)F + \frac{f}{2} \quad (64)$$

Capozziello et al. [1] developed a general formalism in the metric framework by considering a point-like $f(R)$-Lagrangian while Azadi et al. [2] studied static cylindrically symmetric vacuum solutions in Weyl coordinates in the context of the metric $f(R)$ theories of gravity. Sharif and Yousaf [3] analyzed the role of the electromagnetic field and a viable $f(R) = R + \delta R^2$ model on the range of dynamical instability along with Shamir [4], Chirde and Shekh [5], Moraes et al. [6], Sahoo and Bhattacharjee [7] are the authors who have studied $f(R)$ theory of gravity for various space-times in different contexts.

Analysis of some cosmographic parameters in $\hat{f}(R)$ gravity

The cosmographic parameters such as energy density, equation of state parameter, stability through a squared

Fig. 18 Behavior of equation of state parameter of dark energy $\hat{f}(G)$ gravity model with respect to ℓna versus redshift with the appropriate choice of constant $n = 2$, $0.10 \le m \le 0.30$ and $\eta = 0.01$
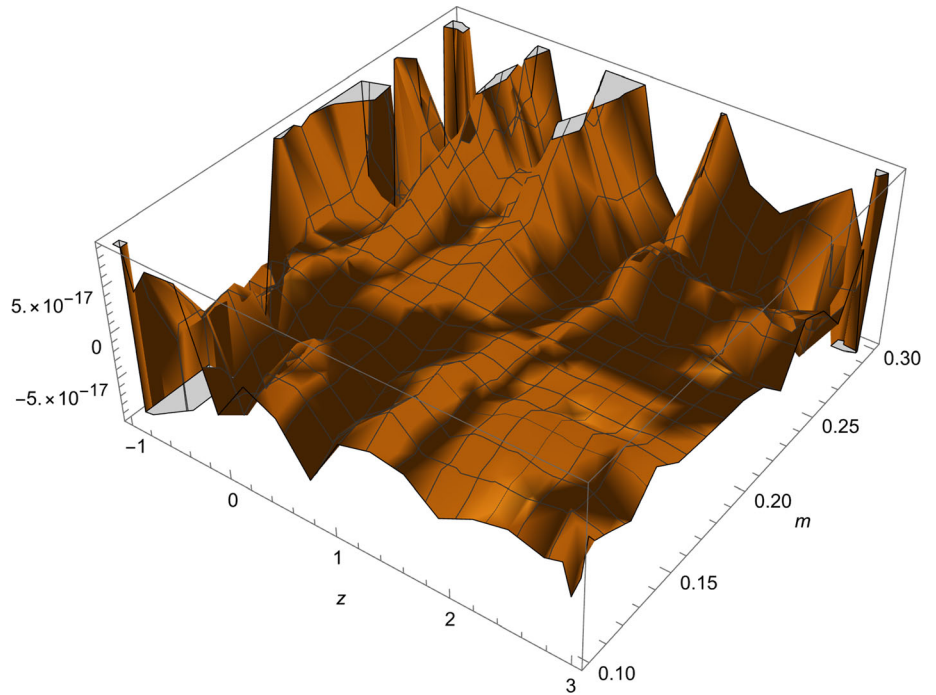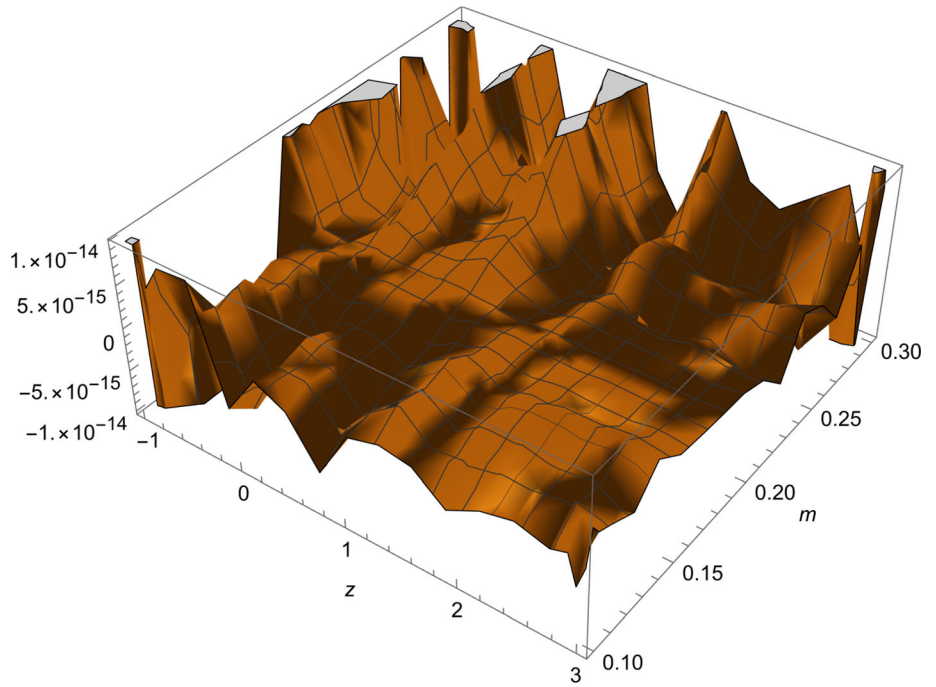
Fig. 19 Behavior of equation of state parameter of dark energy $\hat{f}(G)$ gravity model with respect to ℓna versus redshift with the appropriate choice of constant $n = 2$, $0.10 \le m \le 0.30$ and $\eta = 0.05$

speed of sound and plane like $(\omega_D - \omega'_D)$ in dark energy $\hat{f}(R)$ gravity toward $\hat{f}(R) = R + mR^n$ are derived as follows

7.1. Energy density

Making use of Eq. (21) in Eq. (65), the energy density of dark energy in $\hat{f}(R)$ gravity model is obtained as

Fig. 20 Behavior of energy density of dark energy $\hat{f}(R)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.01$
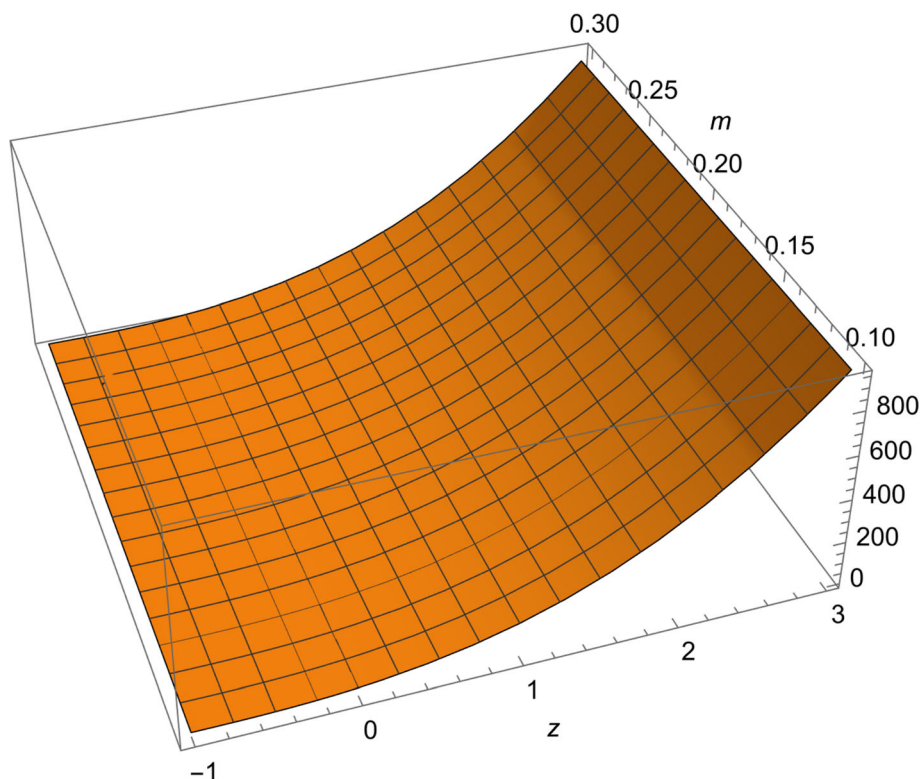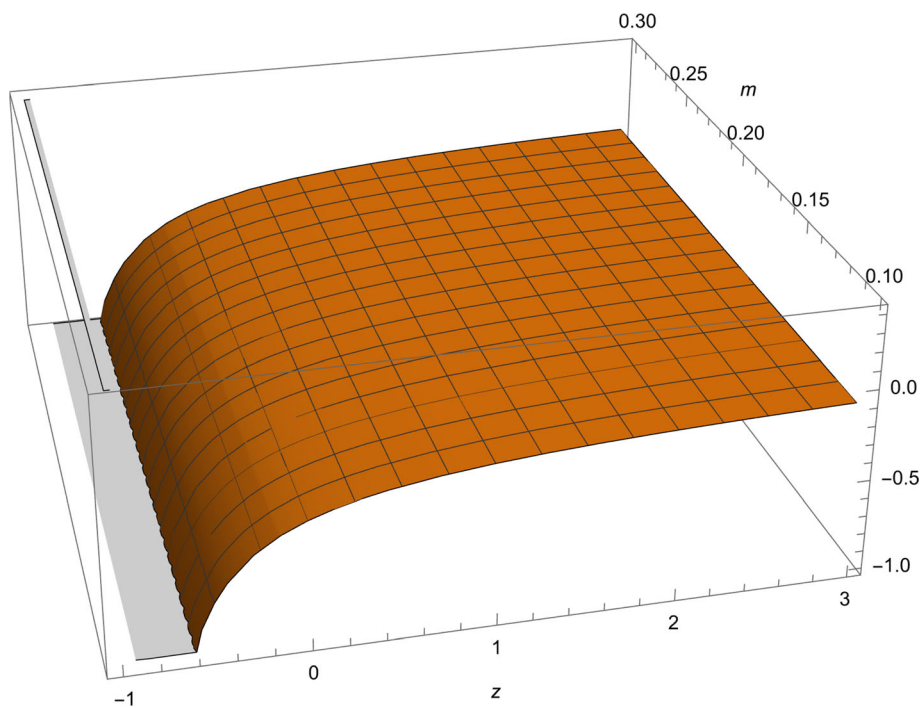

Fig. 21 Behavior of equation of state parameter of dark energy $\hat{f}(R)$ gravity model versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.01$


$$\kappa^2 \rho_D = -3\gamma_2 H \dot{R} R^{n-1} + 3mn(\dot{H} + H^2)R^{n-1} - (R + mR^n)/2 \tag{65}$$

where $R = 6(\dot{H} + 2H^2)$ and $\dot{R} = 6(\ddot{H} + 4H\dot{H})$.

The behavior of energy density of dark energy $\hat{f}(R)$ gravity model versus redshift toward the appropriate choice of fix constants $\alpha = 0.45$, $n = 2$ and varying m is described in Fig. 20. It is observed that the energy density of dark

Fig. 22 Behavior of equation of
state parameter of dark energy
$\hat{f}(R)$ gravity model versus
redshift with the appropriate
choice of constant $n = 2$,
$0.10 \leq m \leq 0.30$ and $\eta = 0.05$
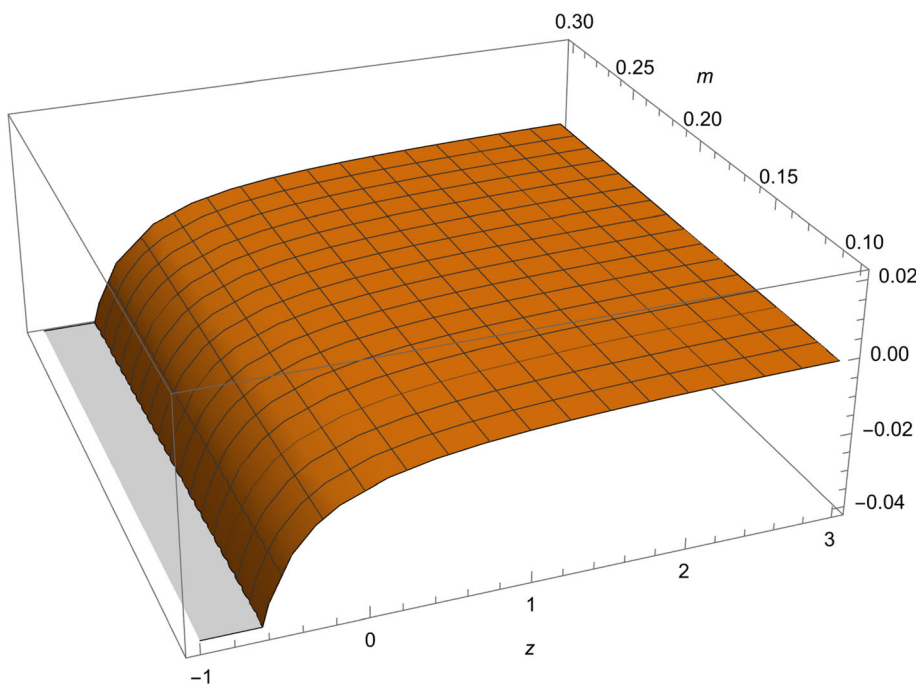
Fig. 23 Behavior of squared
velocity of sound of dark energy
$\hat{f}(R)$ gravity model versus
redshift with the appropriate
choice of constants $n = 2$ and
$0.10 \leq m \leq 0.30$
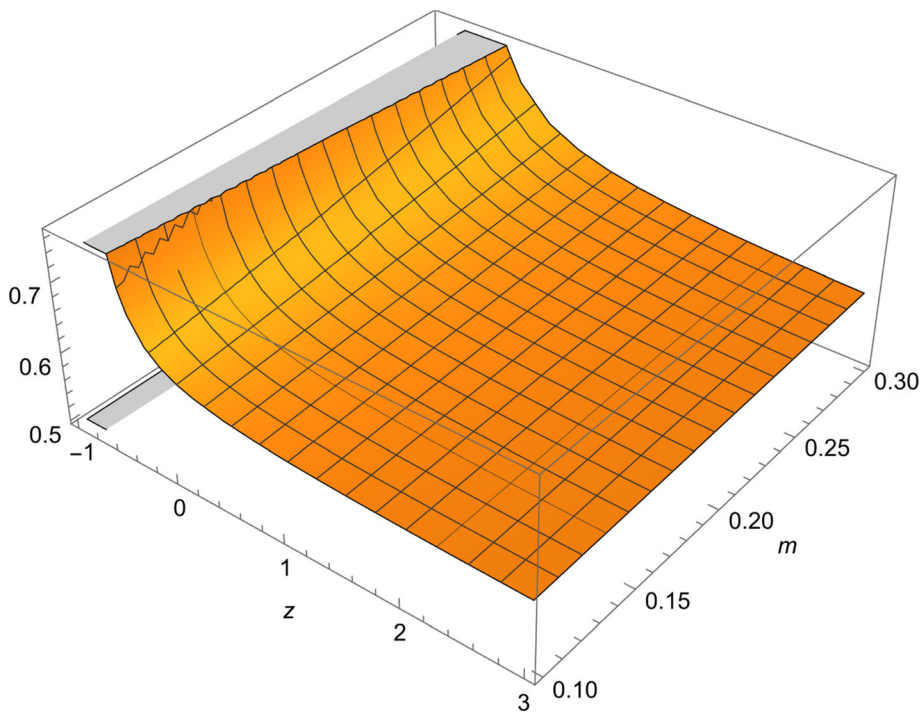
energy $\hat{f}(R)$ gravity model is always positive and increasing function of redshift (see Fig. 20).
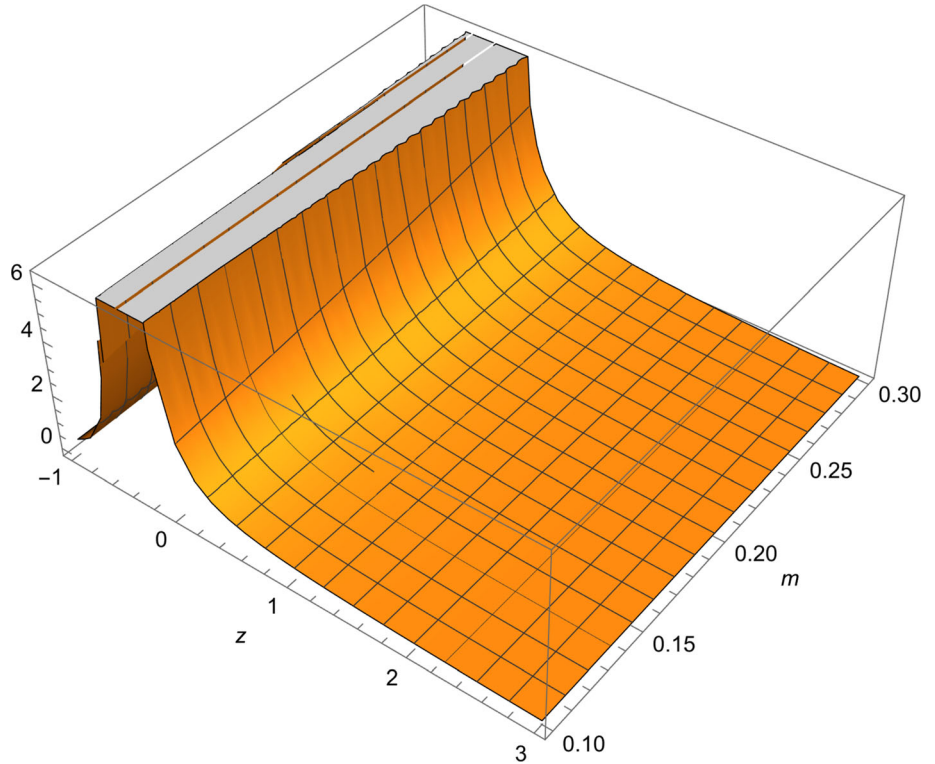
7.2. Equation of state parameter

From Eq. (14), the equation of state parameter of dark energy $\hat{f}(R)$ gravity model is obtained as

$$\kappa^2 \omega_D = \frac{3H^2}{3H^2(1-F) + 3(H\dot{F} + \dot{H}F) + (f/2)}$$

$$\left(1 + \frac{6\eta H^2}{3\gamma_2 H\dot{R}R^{n-1} - 3mn(\dot{H} + H^2)R^{n-1} + (R + mR^n)/2}\right)$$

$$(66)$$

where $F = mnR^{n-1}$ and $\dot{F} = \gamma_2 \dot{R}R^{n-1}$.

Fig. 24 Behavior of equation of state parameter of dark energy $\hat{f}(R)$ gravity model with respect to ℓna versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.01$

The behavior of equation of state parameter of dark energy $\hat{f}(R)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 21 and 22, respectively.

The equation of state parameter of dark energy $\hat{f}(R)$ gravity model at $z = -1$, $z = 0$ and $z > 0$ represents a quintessence field region of the universe for the value of interacting parameter $\eta = 0.01$ and always stays in the quintessence field region of the universe as shown in Fig. 21. Also, on the other hand, Fig. 22, for $\eta = 0.05$, the evolution of the equation of state parameter of dark energy $\hat{f}(R)$ gravity model at $z = -1$, $z = 0$ and $z > 0$ represents the same behavior as that of $\eta = 0.01$, i.e., represents a quintessence field region of the universe. Thus, the analysis of an accelerating expansion of the dark energy $\hat{f}(R)$ gravity model is consistent with the recent observations for the values of interacting parameter toward $\eta = 0.01$, $\eta = 0.05$.

7.3. Squared velocity of sound

By taking the time derivative of Eqs. (62) and (63), the stability parameter $\vartheta^2 \left(= \frac{\partial p}{\partial \rho} \right)$ is obtained as

$$v^2 = \frac{\dddot{F} + 2\dot{H}\dot{F} + 2H\ddot{F} + 4H\dot{H}F + 2H^2\dot{F} - \dot{f}/2}{-3H\ddot{F} + 3\dddot{F} + 3\dot{F}H^2 + 6H\dot{H}F - \dot{f}/2} \quad (67)$$

where $\dot{f} = \dot{R} + mn\dot{R}R^{n-1}$, $\ddot{F} = \gamma_2\ddot{R}R^{n-1} + (n-1)\gamma_2\dot{R}^2R^{n-2}$ and $\dddot{F} = \gamma_2 \dddot{R} R^{n-1} + 3(n-1)\gamma_2\dot{R}\ddot{R}R^{n-2} + (n-1)(n-2)\gamma_2\dot{R}^3R^{n-3}$.

The behavior of squared velocity of sound of dark energy $\hat{f}(R)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ is described in Fig. 23.

It is observed that the squared velocity of sound of dark energy $\hat{f}(R)$ gravity model for the Universe at $z = -1$, $z = 0$ and $z > 0$ is always greater than zero, i.e., $\vartheta_s^2 > 0$. Hence the same dark energy $\hat{f}(R)$ gravity model of the Universe is always stable.

7.4. $\omega_D - \omega_D'$ plane

By taking the variation of Eq. (65) with respect to ℓna, the ω_D' for dark energy $\hat{f}(R)$ gravity model is obtained as

$$\omega_D' = \frac{k}{\alpha H} \left\{ \frac{\dot{\Omega}_D''}{2 - \Omega_D''} \left(-2\eta + \frac{2\eta - \Omega_D''}{\Omega_D''(2 - \Omega_D'')} \right) \right\} \quad (68)$$

where $\Omega_D'' = \frac{-3H\dot{F}}{3H^2} + \frac{3(\dot{H}+H^2)F}{3H^2} - \frac{f}{6H^2}$ and $\dot{\Omega}_D'' = -\ddot{F} + H(\dot{F} - F) + \frac{\ddot{H}}{H}F + \left(2 - \frac{1}{H}\right)\dot{H}F + \left(\frac{\dot{H}}{H}\right)\dot{F} - \frac{1}{6H}(f - \dot{f})$.

The behavior of ω_D' of dark energy $\hat{f}(R)$ gravity model versus redshift toward the appropriate choice of constant $n = 2$ and varying m for different values of interacting

Fig. 25 Behavior of equation of state parameter of dark energy $\hat{f}(R)$ gravity model with respect to $\ell n a$ versus redshift with the appropriate choice of constant $n = 2$, $0.10 \leq m \leq 0.30$ and $\eta = 0.05$
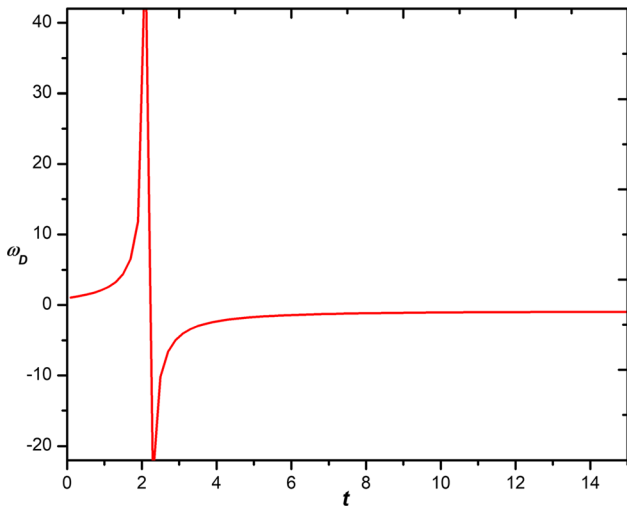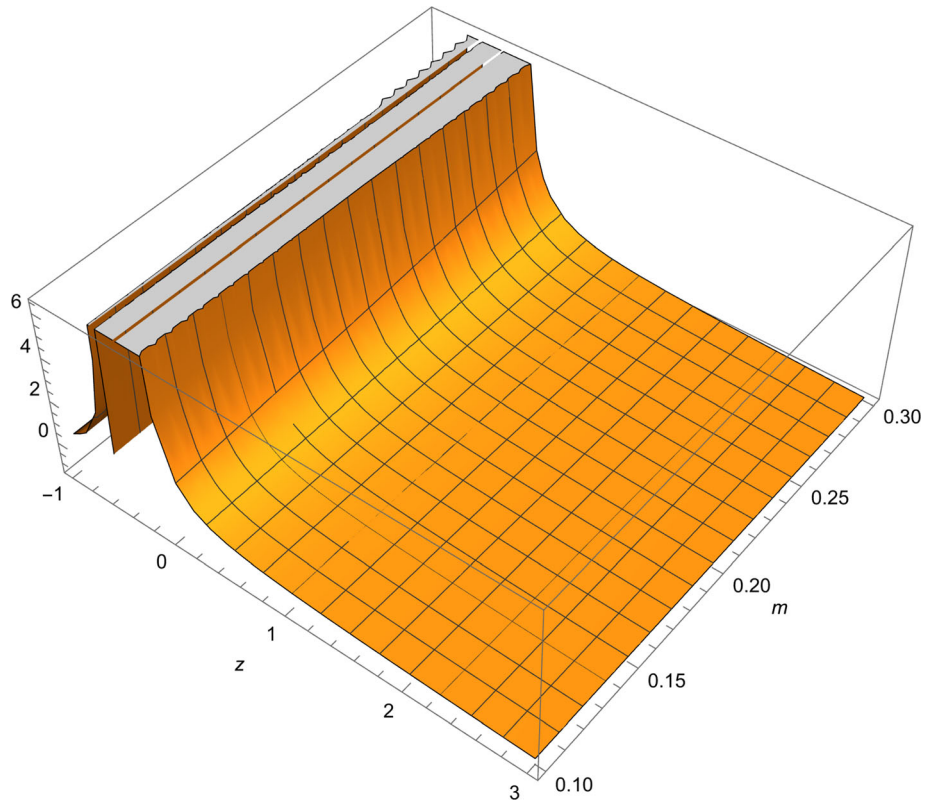

Fig. 26 Behavior of equation of state parameter of dark energy $f(Q)$ gravity model versus time with the appropriate choice of constant $n = 1.5, m = -1$ and $\eta = 1$
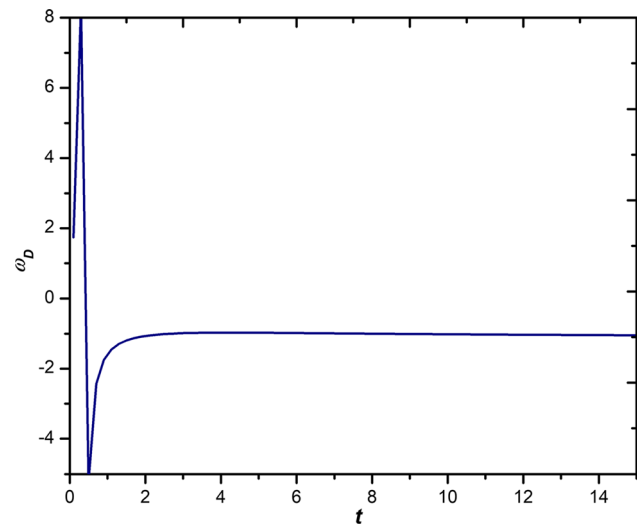
Fig. 27 Behavior of equation of state parameter of dark energy $f(T)$ gravity model versus time with the appropriate choice of constant $n_1 = 2.5, m_1 = 0.5$ and $\eta = 1$

parameter $\eta = 0.01$ and $\eta = 0.05$ is described in Figs. 24 and 25, respectively.

For all $\eta = 0.01$, $\eta = 0.05$ and varying m, the behavior of ω'_D is always greater than 0, i.e., $\omega'_D > 0$. Hence from
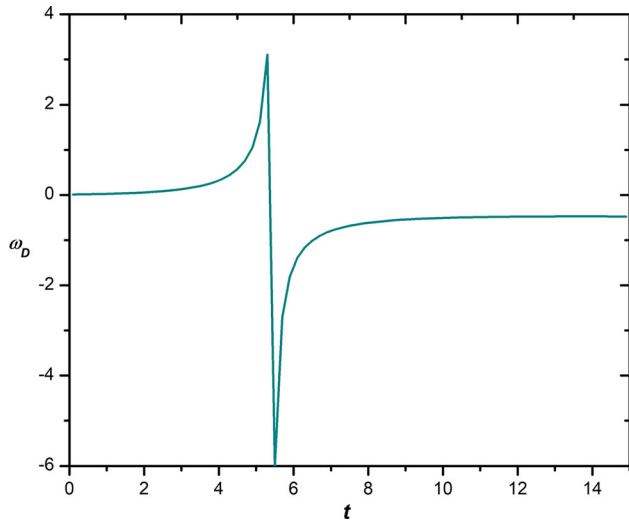
Fig. 28 Behavior of equation of state parameter of dark energy $\hat{f}(G)$ gravity model versus time with the appropriate choice of constant $n_2 = 1.5, m_2 = 1$ and $\eta = 1$
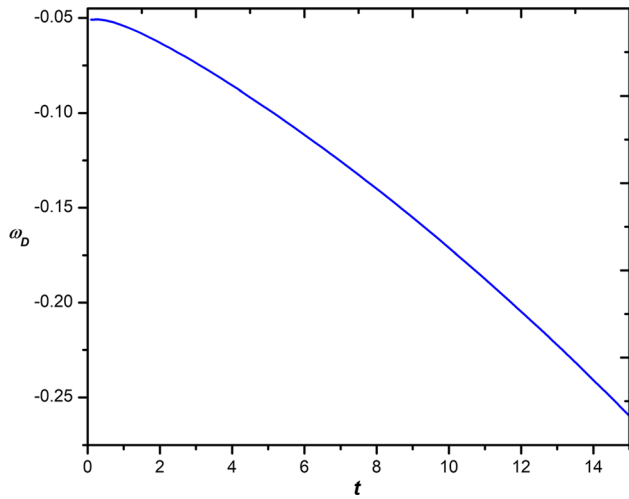

Fig. 29 Behavior of equation of state parameter of dark energy $\hat{f}(R)$ gravity model versus time with the appropriate choice of constant $n_3 = 2, m_3 = -1$ and $\eta = 1$

Figs. 24 and 25 together with 21 and 22, it is predicted that the dark energy $\hat{f}(R)$ gravity model showing a thawing region remains present in thawing region only in all the Universes.

8. Conclusions

In the present paper the analysis of dark energy using FRW line element with the help of well-known relation between Hubble parameter and scale factor which yields a purely

acceleration Universe in the context of modified theories of gravity say $f(Q)$, $f(T)$, $f(G)$ and $f(R)$ gravity is presented. The outcomes in this analysis are as follows:

The behavior of energy densities in all the dark energy models are the same as always positive and increasing function of redshift.

In all the dark energy models, the behavior of the equation of state parameter is discussed toward two different values of coupling parameters, say $\eta = 0.01$ and $\eta = 0.05$. For all the values of the coupling parameter at η at $z = -1$ to $z > 0$ the value of the equation of state parameter lies in -3 to -1.5 and for $z > 1$ it is greater than - 1. Hence, the dark energy $f(Q)$ gravity model for $z = -1$ to $z > 0$ represents a phantom field region of the Universe while crossing the phantom divide line at $z > 0$ and it is in the quintessence field region and always stays in the quintessence field region of the universe. For the value of the interacting parameter $\eta = 0.01$, the evolution of equation of state parameter at $z = -1$, $z = 0$ and $z > 0$, $\omega < 0$. Hence, the dark energy $f(T)$ gravity model at $z = -1$ and $z = 0$ represents a quintessence field region of the Universe and approaches to phantom field region of the Universe at $0.6 \le z \le 0.8$. While for $\eta = 0.05$, it represents a quintessence field and goes toward the phantom field region of the Universe at $0.4 \le z \le 0.6$. Also for both the values of η, the equation of state parameter of dark energy $f(T)$ gravity model crosses the phantom divide line and converges to the matter-dominated phase and always stays in the matter-dominated phase of the Universe. In dark energy $\hat{f}(G)$ gravity model for all the values of η at $z = -1$, $z = 0$ and $z > 0$, the value of equation of state parameter lies in between -0.8 to -0.2 that has $\omega > -1$. Hence, for all the Universe, the dark energy $\hat{f}(G)$ gravity model involves the quintessence field region and remains present in the quintessence field region of the Universe. The behavior of dark energy $\hat{f}(R)$ gravity model is same as that of dark energy $\hat{f}(G)$ gravity model in contrast with the range of equation of state parameter. The equation of state parameter of dark energy $\hat{f}(R)$ gravity model for all η at $z = -1$, $z = 0$ and $z > 0$ lies in between -0.5 to -0.1 that has $\omega > -1$ which represents dark energy $\hat{f}(R)$ gravity model quintessence field region and remain present in the quintessence field region of the Universe. Thus, the analysis of an accelerating expansion of the dark energy models is consistent with the recent observations for the values of interacting parameters toward $\eta = 0.01$, $\eta = 0.05$ and varies m from 0.1 to 0.3.

The stability of the model is discussed with the help of the stability parameter which is the form squared velocity of sound. The squared velocity of sound must be greater than zero for the stable model of the Universe. It is observed that for all universe at $z < -1$, $z = 0$ and $z > 0$,

the squared velocity of sound in dark energy $f(Q)$ gravity model, dark energy $f(T)$ gravity model and dark energy $\hat{f}(R)$ gravity model is always positive and approaches to small positive value which is ($\ll 1$). Hence, all the above dark energy models are stable alongside it is observed that the squared velocity of sound of dark energy $\hat{f}(G)$ gravity model for the Universe at $z = -1$ is less than zero for $z = 0$ and $z > 0$ are always greater than zero, which predicts that the dark energy $\hat{f}(G)$ gravity model at $z = -1$ is unstable and for $z = 0$ and $z > 0$ dark energy $\hat{f}(G)$ gravity model of the Universe is stable.

The phase space analysis of the dark energy models is discussed using $(\omega_D - \omega'_D)$-plane toward different values of coupling parameters. The evolution trajectory of the phase space analysis incorporate all $\eta = 0.01$, $\eta = 0.05$, the dark energy $f(Q)$ gravity model as well as dark energy $f(T)$ gravity model both shows a freezing region and remain present in freezing region in all the Universe. However, the dark energy $\hat{f}(G)$ gravity model shows both a freezing and thawing region in all the Universes whereas the dark energy $\hat{f}(R)$ gravity model involve a thawing region and remain present in thawing region in all the Universe. Hence, due to the violation of thawing region the dark energy $f(Q)$ gravity model, dark energy $f(T)$ gravity model and dark energy $\hat{f}(G)$ gravity model described the consistency of an accelerated expansion of the Universe. Now to check whether the models yield the convergence of the parameter $\omega \to -1$, practically at the present time that honestly improves the cosmic-coincidence problem which demands $\omega = -1$ at the present time. Note that all theories except $\hat{f}(R)$ support the observation, $\omega = -1$ at the present time (see Figs. 26, 27, 28 and 29), i.e., the coincidence problem in this context is restated as why is $\omega = -1$ now. In future, one can study in more details in this context.

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