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ORIGINAL PAPER

Fisher information entropies and the strength of an oscillator under a mixed hyperbolic Pöschl–Teller potential function

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Abstract: The Fisher information-theoretic measurement and the oscillator strength are studied with a mixed hyperbolic Pöschl–Teller potential (MHPTP). Using the total energy eigenvalue equation obtained via the super-symmetric WKB approach as well as the total wave function. We obtained the oscillator strength for the atomic transitions $1s-2p$ and $1s-3p$, where the strength length decreased with increasing potential screening parameters. Also, the Fisher information entropies for both position and momentum spaces are obtained numerically. We studied the correlation between potential parameters and the energy spectra graphically. Our results for the information-theoretic measures obey the Fisher's uncertainty product and the Cramer–Rao inequality in position space. The analytical result for the N-dimensional energy eigenvalue obtained by the super-symmetric WKB approach is the same as the result obtained by a different analytic approach in the existing literature. The oscillator strength conforms to the ones reported in existing literature using different potential energy functions.

Keywords: Super-symmetric WKB method; Pöschl–Teller potential; Oscillator strengths; Fisher information entropies; Cramer–Rao inequalities

1. Introduction

Quantum information-theoretic measures have applications in engineering as well as physical and chemical sciences. Information measurements have applications in quantum computing which is the basis for the technological development of some quantum and signal processing devices, and thus provide an in-depth understanding of the internal structure of atoms [1–4]. Fisher and Shannon information entropies happened to be two complementary information-theoretic measures that characterize the spread and measure of the probability distribution expressed in terms of hypergeometric functions, Jacobi, Laguerre polynomials as well as spherical harmonics [5]. Quantum information theory has a direct relationship with the Heisenberg uncertainty principle which plays a significant role in the simultaneous measurement of the position and momentum of quantum mechanical particles. Different potential energy functions have been used to study the information entropies [4, 6–11]. Dehesa et.al [12] studied the Fisher information-based uncertainty relation, Cramer–Rao inequalities and the kinetic energy for the D-dimensional central problem. Considering the effects of a large dimensional number on quantum mechanical systems, Puertas-Centeno et al. [13] obtained the Renyi entropies in conjugated spaces (position and momentum) for the hydrogenic states using a constructive technique. Also, their results attained the saturation of the known position–momentum Renyi-entropy-based uncertainty relations using the condition for small hyper-quantum numbers.

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On the other hand, the oscillator strength occurs due to the emission or the absorption of electromagnetic radiation as electrons transit between energy levels [14]. Hibbert [14] stated that the oscillator strength can be used to describe the electric dipole emission using dipole approximation and the selection rule. Equally, it can be used to study the spectra of stars due to the transition of atoms from a lower quantum state to an upper state either through the absorption or the radiation of energy [14]. Ikot et al.[15] studied the oscillator strength of a particle confined by the improved molecular Manning–Rosen potential. Their findings revealed that the oscillator strength length decreased with increasing potential parameters. Varshni [16] examined the energy levels and oscillator strengths for the transitions $1s-2p, 1s-3p$ and $2p-3d$ quantum states under the Hulthen potential function where the oscillator strengths decrease with an increase in the potential screening parameter. Hassanabadi et al. [17] investigated the expectation values and the oscillator strength of a generalized Pöschl–Teller potential where their results show that the oscillator strength increased as the potential parameters increased. In this present work, we present the analytical N-dimensional energy spectra using the super-symmetric WKB (SWKB) approximation method with a centrifugal approximation under a mixed hyperbolic Pöschl–Teller potential (MHPTP). The approximate eigensolutions, thermodynamic properties and expectation values of the mixed hyperbolic Pöschl–Teller potential were recently studied by the parametric Nikiforov–Uvarov method and WKB approach [18]. Presently, we are proposing its use to study the Fisher information-theoretic measurement in coordinate spaces and the oscillator strength for the first time to our knowledge best. The MHPTP is given as [18]

$$U(r) = \frac{A}{\mathrm{Cos}h^2(\alpha r)} + \frac{B}{\mathrm{Sin}h^2(\alpha r)} + C\mathrm{Tan}h^2(\alpha r) + \frac{D}{\mathrm{Tan}h^2(\alpha r)} + E \tag{1}$$

where A, B, C, D and E are parameters that could be controlled by proper adjustment. The notations r and α are the respective internuclear distance and the potential screening parameter. The retrieval of several Pöschl–Teller type potentials from Eq. (1) was shown in [18].

The Pöschl–Teller potential happens to be one of the potentials of significant interest over the past decades. It is a long-range potential applicable in atomic physics for the description of energy spectra of atoms and vibrations of diatomic molecules [19, 20]. The exact and approximate bound state solutions of the Pöschl–Teller potential have been obtained using the Schrödinger, Klein–Gordon and Dirac equations [21]. Ikhdair and Falaye [22] applied the asymptotic iteration method to obtain the bound state solution of the Schrödinger equation using the Pöschl–Teller potential. They also obtain the solution of the Dirac equation for the same potential under the condition of spin and pseudo-spin symmetries. Hamzavi and Ikhdair [23] used the trigonometric Pöschl–Teller potential to describe diatomic molecular vibrations where they obtained the approximate eigensolutions of the radial Schrödinger equation by the Nikiforov–Uvarov method. Saregar et al.[24] obtained the energy eigenvalues and eigenfunction of the trigonometric Pöschl–Teller plus Rosen–Morse potential using the super-symmetric quantum mechanics approach. They obtained the ground state wave function using the lowering operator and the excited state wave function using the raising operator. As a result of numerous applications of the Pöschl–Teller-type potentials especially in studying diatomic molecular vibrations, many researchers have also developed a keen interest in understanding the contribution of the Pöschl–Teller potential family to information-theoretic measurements. Ghafourian and Hassanabadi [25] studied the Shannon information entropies for the three-dimensional Klein–Gordon equation using Pöschl–Teller potential where position space information entropies were calculated for both the ground and excited states. Dehesa et al. [26] studied information-theoretic measures for the Morse and Pöschl–Teller potential where they established a general relationship between the variances and the Fisher information entropies in coordinate spaces. Sun et al. [27] studied the effects of the Pöschl–Teller-like function parameters on the information entropy densities and verified the Shannon sum entropies for different quantum states. Using hyperbolic potential energy, Valencia-Torres et al. [28] obtained the Shannon entropy in position and momentum spaces for the low-lying quantum states $(n=0, n=1)$, where their results obey the Bialynicki–Birula–Mycieski inequality. The Shannon information entropy for the position-dependent mass Schrödinger equation under a hyperbolic type potential has been investigated in [29].

The organization of the remaining parts of the paper is as follows: In Sect. 2, we shall obtain the analytical energy spectra solution of the Schrödinger equation under the MHPTP function using the SWKB approach. In Sect. 3, we present a brief review of the Fisher information-theoretic measures in coordinate spaces. In Sect. 4, we discuss the oscillator strength. The numerical analysis of the energy spectra, Fisher information entropies and oscillator strength are presented in Sect. 5. Finally, in Sect. 6, we give the concluding remarks.

2. Energy spectra of the Schrödinger equation under the MHPTP by SWKB approximation method

The SWKB approach has been used to obtain the exact energy spectra of quantum mechanical solvable potentials and also the potentials that obey the shape invariance condition. Specifically, the method is developed by combining the super-symmetric quantum mechanics approach and the zeroth-order WKB approximation. The approach involves the use of super-symmetric partner Hamiltonians $H_{1,2}$ and potentials $V_{1,2}(r)$ which are constructed from the Schrödinger equation.

$$H_{1,2} = \frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_{1,2} \tag{3}$$

The partner Hamiltonian H_1 has a zero ground state energy while H_2 has a nonzero ground state energy.

The eigenenergies of the Hamiltonians are given by the relation

$$E_{nl}^{(2)} = E_{nl}^{(1)} + E_{0l} \tag{4}$$

where E_{0l} is the ground state $(n = 0)$ energy of a particle under the MHPTP.

The difference between the usual WKB and SWKB methods is rooted in the proposition of a super-symmetric super-potential $W(r)$ in the SWKB approach which satisfies two first-order differential equations given by

$$V_{1,2}(r) = W^2(r) \mp \frac{\hbar}{\sqrt{2\mu}}\frac{dW(r)}{dr} \tag{5}$$

$$W(r) = \frac{-\hbar}{\sqrt{2\mu}}\frac{1}{\Psi_{0l}(r)}\frac{d\Psi_{0l}(r)}{dr} \tag{6}$$

If the ground state wave function is known, then the super-potential can be obtained from Eq. (6). It is noteworthy to state that the $V_{1,2}(r)$ was obtained from the factorization [30, 31] of Eq. (3). Another difference between the SWKB and WKB methods is that the former does not require modification of the orbital centrifugal barrier since the wave function is well behaved near the origin [32, 33]. Once the super-potential is obtained then the energy eigenvalue $E_{nl}^{(1)}$ belonging to H_1 can be obtained from the energy quantization integral given by [32, 33]

$$\sqrt{\frac{2\mu}{\hbar^2}}\int_{r_1}^{r_2}\sqrt{E_{nl}^{(1)} - W^2(r)}dr = n\pi, \quad n = 0, 1, 2, 3 \tag{7}$$

where the turning points $0 < r_1 < r_2$ are obtained from the solution $E_{nl}^{(1)} - W^2(r) = 0$.

We proposed a super-potential of the form

$$W(r) = f\ \mathrm{Tan}h(\alpha r) - \frac{g}{\mathrm{Tan}h(\alpha r)}, \quad g, f > 0 \tag{8}$$

where g and f can be determined from the partner potential $V_1(r)$ in Eq. (5).

In Eq. (5), we set $V_1(r) = V_{\mathrm{eff}}(r) - E_{0l}$ to preserve the super-symmetric ground state energy $E_{0l}^{(1)} = 0$.

$$V_{\mathrm{eff}}(r) - E_{0l} = W^2(r) - \frac{\hbar W'(r)}{\sqrt{2\mu}} \tag{9}$$

where the effective potential and notations for the super-potential $V_{eff}(r)$, $W^2(r)$ and $\frac{dW(r)}{dr}$ are given as

$$V_{\mathrm{eff}}(r) = \frac{A}{\mathrm{Cos}h^2(\alpha r)} + \frac{B}{\mathrm{Sin}h^2(\alpha r)} + C\mathrm{Tan}h^2(\alpha r) + \frac{D}{\mathrm{Tan}h^2(\alpha r)} + E + \frac{\hbar^2}{2\mu r^2}\left[\left(l + \frac{N-2}{2}\right)^2 - \frac{1}{4}\right] \tag{10}$$

$$W^2(r) = f^2\mathrm{Tan}h^2(\alpha r) - 2gf + \frac{g^2}{\mathrm{Tan}h^2(\alpha r)} \tag{11}$$

$$\frac{dW}{dr} = \alpha\left(\frac{f}{\mathrm{Cos}h^2(\alpha r)} + \frac{g}{\mathrm{Sin}h^2(\alpha r)}\right) \tag{12}$$

In Eq. (10), following the authors [18, 34], we used the centrifugal approximation for $1/r^2$ as

$$\frac{1}{r^2} = \alpha^2\left(\frac{d_0}{\mathrm{Cos}h^2(\alpha r)} + \frac{1}{\mathrm{Sin}h^2(\alpha r)}\right), \quad d_0 = 1/3 \tag{13}$$

If we insert Eqs. (10–12) into (9) and solve it completely, we obtained the following equations

$$f^2 + \alpha f\frac{\hbar}{\sqrt{2\mu}} - (C - A - Ld_0) = 0 \tag{14}$$

$$g^2 - \alpha g\frac{\hbar}{\sqrt{2\mu}} - (B + D + L) = 0 \tag{15}$$

$$\frac{\alpha\hbar}{\sqrt{2\mu}}(g - f) - 2gf = A - B + E + Ld_0 - L - E_{0l} \tag{16}$$

where $L = \frac{\alpha^2\hbar^2}{2\mu}\left[\left(l + \frac{N-2}{2}\right)^2 - \frac{1}{4}\right]$.

It is easy to see that Eqs. (14) and (15) are quadratic in f and g, and their respective roots are obtained as

$$f = \frac{-\alpha\hbar}{\sqrt{2\mu}}\left(\frac{1}{2} \mp \sqrt{\frac{1}{4} + \frac{2\mu}{\alpha^2\hbar^2}(C - A - Ld_0)}\right) \tag{17}$$

$$g = \frac{\alpha\hbar}{\sqrt{2\mu}}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2\mu}{\alpha^2\hbar^2}(B + D + L)}\right). \tag{18}$$

Solving Eqs. (14)–(16), we found the ground state energy of the SE with the MHPTP as

$$E_{0l} = C + D + E - (g - f)^2. \tag{19}$$

To obtain the energy for excited states, we used the SWKB energy quantization formula in Eq. (7).

$$\sqrt{\frac{2\mu}{\hbar^2}}\int_{r_1}^{r_2}\sqrt{E_{nl}^{(1)}-f^2\mathrm{Tan}h^2(\alpha r)+2gf-\frac{g^2}{\mathrm{Tan}h^2(\alpha r)}}dr=n\pi. \tag{20}$$

Using the change of variable $z=\mathrm{Tan}h^2(\alpha r)$, Eq. (20) transforms as

$$\sqrt{\frac{\mu f^2}{2\alpha\hbar^2}}\int_{z_1}^{z_2}\frac{dz}{z(1-z)}\sqrt{-z^2+\Lambda z-\Gamma}=n\pi, \quad n=0,1,2,3\cdots \tag{21}$$

where $\Lambda=\frac{2gf+E_{nl}^{(1)}}{f^2}$, $\Gamma=\frac{g^2}{f^2}$.

Using the integration solution in [18], the energy spectra equation can be obtained from Eq. (21) as

$$E_{nl}^{(1)}=\Lambda_0-\Lambda_1(n+\Lambda_2)^2-E_{0l}. \tag{22}$$

where

$$\Lambda_0=C+D+E,\quad \Lambda_1=\frac{2\alpha^2\hbar^2}{\mu}.$$

$$\Lambda_2=\frac{1}{2}+\frac{1}{2}\left(\sqrt{\frac{2\mu(B+D)}{\alpha^2\hbar^2}+\left(l+\frac{(N-2)}{2}\right)^2}-\sqrt{\frac{2\mu(C-A)}{\alpha^2\hbar^2}-\left(l+\frac{(N-2)}{2}\right)^2d_0+\frac{1}{3}}\right). \tag{23}$$

The exact total energy of the system can be obtained from the identity $E_{nl}^{(2)}=E_{nl}^{(1)}+E_{0l}$. It is easy to see that the energy $E_{nl}^{(1)}$ gives the exact value for the ground state $(n=0)$. The normalized wave function of the MHPTP was obtained recently in [18] using the parametrized Nikiforov–Uvarov method.

$$\Psi_{nl}(z)=N_{nl}z^{\frac{1}{4}+Q_1}(1-z)^{Q_2}P_n^{(2Q_1,2Q_2)}(1-2z), z=\mathrm{Tan}h^2(\alpha r)\in(0,1) \tag{24}$$

where N_{nl} is the normalization constant given by

$$N_{nl}=\sqrt{\frac{2\alpha n!2Q_2\Gamma(2Q_1+2Q_2+n+1)}{\Gamma(2Q_2+n+1)\Gamma(2Q_1+n+1)}}$$

$$Q_2=\Lambda_2,\quad Q_1=\frac{1}{2}\sqrt{\frac{2\mu(B+D)}{\alpha^2\hbar^2}+\left(l+\frac{(N-2)}{2}\right)^2}$$

and

$P_n^{(2Q_1,2Q_2)}(1-2z)$ is the Jacobi polynomial of order n.

3. Fisher information-theoretic measurement

The Fisher information-theoretic measures of a quantum system in position and momentum space are given by the relations, respectively [12, 35]

$$I(\rho)=4\int_{\mathcal{R}^3}|\nabla\Psi_{nlm}(\boldsymbol{r})|^2d\boldsymbol{r}=\int_{\mathcal{R}^3}\frac{1}{\rho(\boldsymbol{r})}|\nabla\rho(\boldsymbol{r})|^2d\boldsymbol{r} \tag{25}$$

$$I(\gamma)=4\int_{\mathcal{R}^3}|\nabla\Psi_{nlm}(\boldsymbol{p})|^2d\boldsymbol{p}=\int_{\mathcal{R}^3}\frac{1}{\phi(\boldsymbol{p})}|\nabla\phi(\boldsymbol{p})|^2d\boldsymbol{p} \tag{26}$$

where $\rho(\boldsymbol{r})=|\Psi_{nl}(r)|^2|Y_{lm}(\theta,\varphi)|^2$, $\phi(\boldsymbol{p})=|\Psi(p)|^2|Y_{lm}(\theta,\varphi)|^2$ and $\Psi_{nlm}(\boldsymbol{p})$ are the respective probability densities in radial and momentum spaces and the wave function in momentum space. The differential operator $d\boldsymbol{x}$ $(x^{N-1}\mathrm{Sin}(\theta)dxd\theta d\varphi, x=r\,\mathrm{or}\,p)$ is the volume element. The notation ∇ is the gradient of a particle defined as

$$\nabla=\hat{\boldsymbol{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{1}{Sin(\varphi)}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial}{\partial\varphi} \tag{27}$$

The wave function in momentum space can be obtained from the Fourier transform given by

$$\Psi_{nlm}(\boldsymbol{p})=\frac{1}{(2\pi)^{3/2}}\int_{\mathcal{R}^3}e^{-i(\vec{p}\cdot\vec{r})}\Psi_{nml}(\boldsymbol{r})d\boldsymbol{r} \tag{28}$$

Also, the Fisher information in position and momentum spaces can be represented by the radial expectation value and momentum expectation value as [12]

$$I(\rho)\geq 4\left(1-\frac{2|m|}{2L'+1}\right)\langle p^2\rangle \tag{29}$$

$$I(\gamma)\geq 4\left(1-\frac{2|m|}{2L'+1}\right)\langle r^2\rangle \tag{30}$$

where $L'=l+\frac{N-3}{2}$ is the grand orbital quantum number, N,l and m are the respective dimensionality number, orbital and magnetic quantum numbers. The Fisher information uncertainty product can then be expressed as

$$I(\gamma)I(\rho)\geq 16\left(1-\frac{2|m|}{2L'+1}\right)^2\langle p^2\rangle\langle r^2\rangle \tag{31}$$

where the Heisenberg product is given as [12]

$$\langle p^2\rangle\langle r^2\rangle\geq\frac{9}{4} \tag{32}$$

Substituting Eq. (32) into (31) in the absence of magnetic interaction ($m=0$) results to

$$I(\gamma)I(\rho)\geq 36 \tag{33}$$

Another important uncertainty relation is the Cramer–Rao inequalities given as the products of the Fisher

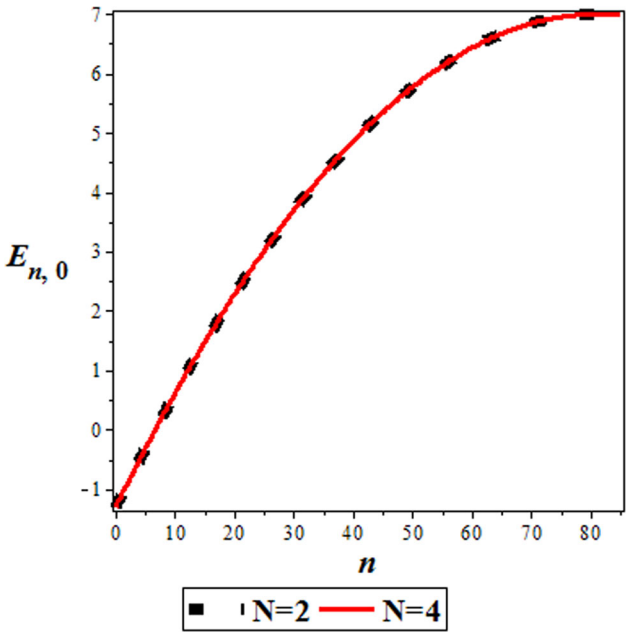

Fig. 1 Variation of energy level (in a.u) with the radial quantum number. We used the arbitrary constants in [18]($A=-20$, $B=3, C=4, E=2, D=1, l=0, \hbar=1, \mu=1, \alpha=0.025$)

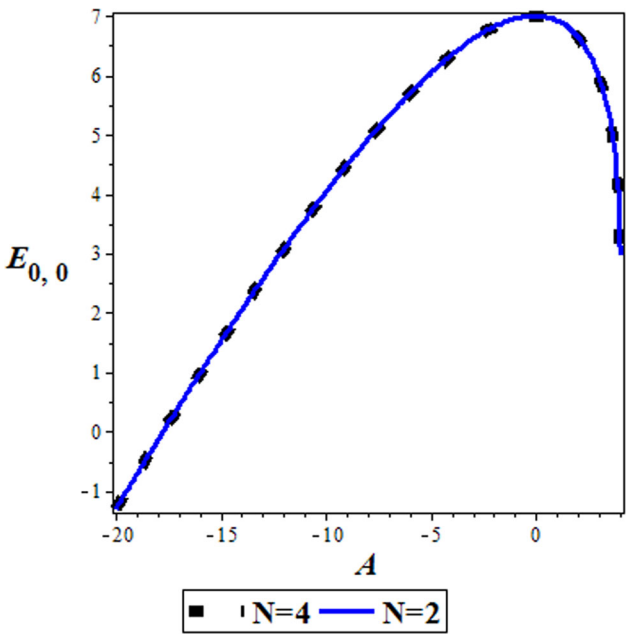

Fig. 2 Variation of energy level (in a.u) with potential depth.We used the arbitrary energy equation constants in[18] ($\alpha=0.025$, $B=3, C=4, E=2, D=1, l=0, n=0, \hbar=1, \mu=1$)

information measure and the expectation values in coordinate spaces [12].

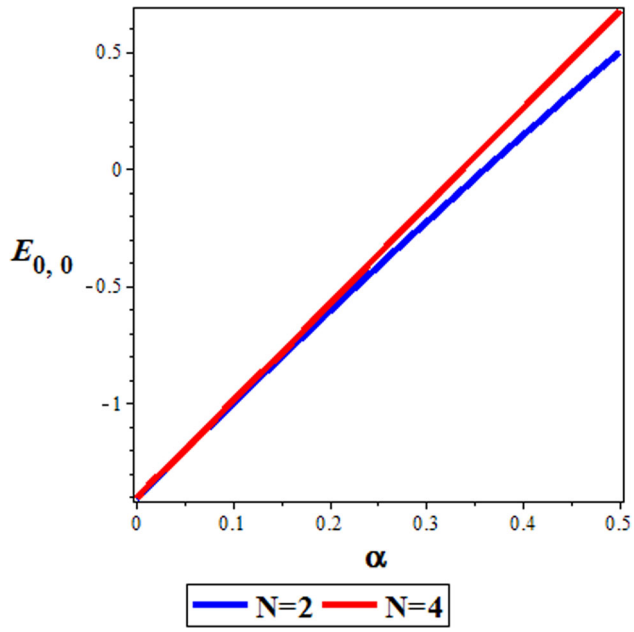

Fig. 3 Variation of energy level (in a.u) with screening parameter. We used the arbitrary energy equation constants in [18] ($A=-20, B=3, C=4, E=2, D=1, l=0, n=0, \hbar=1, \mu=1$)

$$I(\rho)\langle r^2\rangle \geq N^2 \tag{34}$$

$$I(\gamma)\langle p^2\rangle \geq N^2 \tag{35}$$

4. Oscillator strength

The oscillator strength of electrons that transit from a lower energy state to a higher state is given by the formula:

$$f_{ij}^{l} = \frac{2M}{3\hbar^2}\left(E_j - E_i\right)\left|\langle\Psi_j|r|\Psi_i\rangle\right|^2 \tag{36}$$

where E_j and Ψ_j are at a higher state than the respective E_i and Ψ_i. The M represents an electronic mass. The notation $\left|\langle\Psi_j|r|\Psi_i\rangle\right|$ is the matrix element and $\left(E_j - E_i\right)$ is the energy difference. Hibbert [14]stated that there are two sources of errors associated with the absorption oscillator strength, namely, the matrix element and the energy difference. According to Hibbert[14], the error associated with the matrix element can be removed using experimental energy difference, while the errors introduced by the energy can be isolated by taking into account the geometric mean of the oscillator strength.

5. Results and discussion

We obtained the non-relativistic N-dimensional energy spectrum of the MHPTP potential function in closed form using the SWKB approach. The analytical result of the

Table 1 Fisher information-theoretic measurement and Cramer–Rao inequality $(N = 3, A = 1, B = 1, C = 3, D = 1, l = 0, n = 0, \hbar = 1, \mu = 1, d_0 = 1/3)$

α	$I(\rho)$	$I(\gamma)$	$I(\rho)I(\gamma) \geq 36$	$I(\rho)\langle r^2\rangle \geq 9$
0.1	0.040045	2672.9	107.0373	26.75932
0.2	0.160665	516.966	83.05849	20.76462
0.3	0.3631	196.346	71.29314	17.82328
0.4	0.649065	98.6458	64.02755	16.00689
0.5	1.020566	57.8484	59.03813	14.75953
0.6	1.479774	37.4335	55.39315	13.84829
0.7	2.028944	25.9348	52.62032	13.15508
0.8	2.670354	18.8927	50.45009	12.61252
0.9	3.406259	14.3018	48.71559	12.1789
1	4.23886	11.1603	47.30712	11.82678

Table 2 Oscillator strength for the MHPTP. We used the arbitrary parameters $(A = -20, B = 3, C = 4, E = 2, D = 1, N = 3, \hbar = 1, \mu = 1)$ in Ref [18]

Transition	α	f_{ij}^{l}
$1s - 2p$	0.025	145.4991434
	0.050	144.7037977
	0.075	143.9715694
	0.100	143.3010130
	0.150	142.1412047
$1s - 3p$	0.025	2836.953785
	0.050	1407.239593
	0.075	930.2359023
	0.100	691.4132820
	0.150	446.0197072

energy levels coincides with the result obtained recently [18] using the Nikiforov–Uvarov method. The plots of the energy levels against the potential parameters for the two- and four-dimensional spaces are investigated. In Fig. 1, the energy levels overlay on each other and increase with the radial quantum number. At a maximum value of around $n = 81$, the energy reaches a maximum of 7 units. Also in Fig. 2, we graphed the energy levels as a function of potential depth. As can be seen, the energy increases from about $A = -20$ units to a maximum value of 7 units before decreasing and converging asymptotically at $A \geq 4$. In Fig. 3, the energy increases with the dimension number and the screening parameter of the potential. Also, there is an occurrence of energy overlap between $\alpha = 0 - 0.1$ while the splitting of the energy levels is noticeable as the screening parameter gain strength. We obtained the Fisher information-theoretic measure in coordinate spaces numerically by using Eq. (25) for the position and Eq. (30), for the momentum coordinate with $N = 3, m = 0, n = 0, l = 0$. The Fisher entropy in position space $I(\rho)$ increases monotonically with increasing screening parameters. The increment in $I(\rho)$ is compensated by the gradual decrease in $I(\gamma)$ as the screening parameter increases. Generally, our results obey the Fisher uncertainty product $(I(\rho)I(\gamma) \geq 36)$ and the Cramer–Rao inequality $(I(\rho)\langle r^2\rangle \geq 9)$ in position space as presented in Table 1. We calculated the oscillator strength for the transitions $1s - 2p$ and $1s - 3p$ states using both the total energy equation and wave function for arbitrary potential parameters. Our results presented in Table 2 revealed that the oscillator strengths decrease with increasing screening parameters of the MHPTP function. The current results are consistent with the ones reported in existing literature using different potential energy functions [15, 16].

6. Conclusions

In this work, using the SWKB method, we obtained the analytical N-dimensional energy eigenvalue of the Schrödinger wave equation under a mixed hyperbolic Pöschl–Teller potential function. The variations of the energy level for the two- and four-dimensional spaces are studied. We obtained the Fisher information numerically for both the position and momentum spaces. Using the energy transitions $1s - 2p$ and $1s - 3p$ with the matrix elements, we obtained the oscillator strength. Our results for the information uncertainty measures obey the inequality $(I(\rho)I(\gamma) \geq 36)$ as well as the Cramer–Rao inequality in position space $(I(\rho)\langle r^2\rangle \geq 9)$. The behavior of the oscillator strength for the MHPTP is similar to the ones reported in existing literature using different potential energy functions.

Data availability All data used in this paper are derived from the equations in the article. Therefore, no data were used in our paper.

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