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ORIGINAL PAPER

Relativistic particles in electromagnetic field with confining scalar potential in doubly special relativity

D Seffai[1,2], M Merad[1]* and B Hamil[3]

[1]Faculté des sciences exactes, Université de Oum El Bouaghi, 04000 Oum El Bouaghi, Algeria

[2]Département of Material Science, Faculty of Science, University of Amar Telidji, BP 37G, 03000 Laghouat, Algeria

[3]Département de TC de SNV, Université Hassiba Benbouali, Chlef, Algeria

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Abstract: The present research paper attempts to study the relativistic Klein–Gordon and Dirac equations which are subjected to the action of a uniform electromagnetic field which is added to a confining scalar potential within the context Magueijo–Smolin model and in momentum space. In both cases, the energy spectrum of the mentioned equations and their corresponding eigenfunctions are obtained. The limiting cases are then deduced for a small parameter of deformation; in addition to that, a numerical study of the energy is presented.

Keywords: Bound states; Scattering states; MS Model

1. Introduction

All approaches that are allowed to construct the deformed algebra theory in physics have been continuously developed and became of great interest to prospect of research principally in physics and mathematical physics. Consequently, a considerable amount of literature and a class of models have been presented in this regard. Among which: Snyder model, the latter is proposed as the first example of a noncommutative geometry [1, 2]. Furthermore, the generalized uncertainty principle GUP has been proposed to incorporate gravity into quantum mechanics [3–6]. Another specific model is called the (anti)-de Sitter background, and it is associated with the topology of the physical space used for this purpose. This approach is known as the Extended Uncertainty Principle EUP [7–14]. Following that, it is important in this regard to highlight that these models are conducted through a deformed algebra that is characterized by certain physical parameters, and defined in their associated commutation relation and reflecting the effects of certain phenomena observed at different scales in the physical world.

Approximately ten years ago, Amelino-Camelia [15] followed by Magueijo–Smolin [16] suggested a new theory with two observed independent parameters: the speed of light c and an energy scale identified with Planck energy κ. This proposal is motivated by the consideration that the Planck energy sets a limit above which causes quantum gravity effects to become important, alongside and its rate. Therefore, it must not depend on the specific observer, and the same goes for special relativity case. On that account, this postulate should be introduced in such a manner that the relativity principle, i.e., the equivalence of all inertial observers, stays valid. The idea based on this supposition is called doubly special relativity (DSR). The majority of the work on the DSR has been performed in the context of algebraic construction based on the deformation of the standard Poincaré algebra of special relativity, and their principal physical effects are the modification of the standard dispersion relations and the existence of a nonlinear addition law for the momenta.

DSR is closely related to noncommutative (NC) geometry [17], where the standard Poisson brackets of phase space variables is replaced by a more complicated algebra. This is satisfying due to the existence of a fundamental length scale needed to deal with quantum gravity theory. This fundamental length scale can be introduced by an NC space-time via generalized uncertainty relation [3–5]. The study of theoretical and physical implications of the DSR is still a controversial topic. Various problems with great physical interest have been studied in connection with the

*Corresponding author, E-mail: meradm@gmail.com

DSR. As examples, we can mention quantum uncertainty in DSR [18], noncommutative space-time of DSR [19], the Magueijo–Smolin model of DSR from five dimensions [20], DSR and photons at the Planck scale [21] and cosmological constant and Planck scale phenomenology [22], the black hole thermodynamics in DSR-GUP [23, 24] and the relativistic oscillators in the context of DSR noncommutative model [25, 26].

In relativistic and nonrelativistic quantum mechanics, the solutions of the wave equation with external fields play a central role in various domains of physics, since they contain all the necessary information to understand the quantum behavior of physical models [27–31]. Moreover, the solutions of the wave equations have been used to study the behavior and dynamics of some physical systems such as thermodynamic properties [32, 33], Shannon entropy and Fisher information [34, 35].

Based on the above, the present research paper aims to study the relativistic Klein–Gordon KG and Dirac particles within the context of the DSR model in the presence of orthogonal electric and magnetic fields with confining scalar potentials. Additionally, it is worth noting that the same current work in the ordinary case without DSR was reviewed in [36].

This paper is organized as follows: Sect. 2 contains a review of the MS model and some relations necessary to the calculation. Section 3 is devoted to the determination of the energy eigenvalues and the corresponding eigenvectors of the KG equation in the momentum space within the framework of the MS model in the presence of orthogonal electric and magnetic fields with confining scalar potentials. Following the same method, in Sect. 4, the exact solutions of the Dirac equation are determined. Section 5 is given to results and discussion. Finally, in Sect. 6, conclusions are going to be presented.

2. Review of MS model

The modified dispersion relation in the DSR model can be collectively expressed in the following form :

$$f_1^2E^2 - f_2^2P_i^2 = M^2 \tag{1}$$

in which functions f_1 and f_2 are specified by various DSR models [16, 37–39]:

DSR model	Function f_1	Function f_2
κ-Poincaré [37]	$f_1^2 = \frac{2\kappa^2\cosh(E/\kappa)}{E^2}$	$f_2^2 = \exp(E/\kappa)$
Magueijo–Smolin [16]	$f_1 = (1 - E/\kappa)^{-1}$	$f_2 = (1 - E/\kappa)^{-1}$
Herranz [38]	$f_1 = \frac{\kappa(\exp(E/\kappa)-1)}{E}$	$f_2 = 1$
Heuson [39]	$f_1 = (1 - P_i^2/\kappa^2)^{-1/2}$	$f_2 = (1 - P_i^2/\kappa^2)^{-1/2}$

where κ is the Planck energy ($\frac{1}{\kappa} \sim 10^{-35}m$) and the special relativity is recovered in the limit $\kappa \to \infty$. Recently, Ghosh and Pal [40] have shown that the operators of the position X_μ and the momentum P_μ in the MS model can be represented on a standard Hilbert space of functions of a canonical momentum variable as

$$X_\mu = (1 + \frac{E}{\kappa})x_\mu = i\left(1 + \frac{E}{\kappa}\right)\frac{\partial}{\partial p_\mu}, \tag{2}$$

$$P_\mu = \frac{p_\mu}{(1 + \frac{E}{\kappa})}, \tag{3}$$

where the operators x_μ, p_μ abide by the usual canonical commutation relations $[x_\mu, p_\mu] = i\eta_{\mu\nu}$. This representation leads to the following commutation relations [41, 42]:

$$\begin{aligned} &[X_i, X_j] = 0, \quad [P_0, P_i] = [P_i, P_j] = 0, \quad [X_i, P_0] = 0, \\ &[X_i, X_0] = \frac{i}{\kappa}X_i, \quad [P_i, X_0] = -\frac{i}{\kappa}P_i, \quad [X_i, P_j] = i\delta_{ij}, \\ &[X_0, P_0] = -i\left(1 - \frac{P_0}{\kappa}\right), \end{aligned} \tag{4}$$

In this model, the measure for which the operators X_μ are symmetric is given by

$$D_p = \frac{dp}{1 + \frac{p_0}{\kappa}}. \tag{5}$$

3. Solution of Klein–Gordon equation

3.1. Bound states

The movement of charged KG particles in $(3+1)$ dimensions in the presence of an electromagnetic field represented by the four-vector $A_\mu(V, A_i)$ and the scalar potential $S(X)$, is described by the following equation :

$$\left[(P_i - eA_i(X))^2 + (m + S(X))^2 - (P_0 - V(X))^2\right]\psi = 0. \tag{6}$$

Preliminary to proceeding with studying the equation above in the framework of DSR theory, which advocates Planck scale modifications of the energy–momentum dispersion relation, it is worth noting that the classical and quantum studies are established on this model in a series of papers [40, 43] that prove the equivalence between the dispersion relation and the deformed free Klein–Gordon equation in the considered DSR scheme. It is shown through the action formulation of the deformed free particle κ. It is also demonstrated that the solutions meet the set expectations of the correct dispersion relation. Previous studies can be extended to include interactions directly following different approaches.

On the same line of thoughth, we are interested in the following choice:

$$A_i = \lambda(0, X, 0); \quad S = \mu X; \quad V = \gamma X, \tag{7}$$

where A_i, S and V vary linearly (with) in x. A_i is the electromagnetic potential describing a uniform magnetic field $\overrightarrow{B}$ along the z axis. The linear potential is an important quantum mechanical model, it allows a quark-confining as it describes motion in a uniform gravitational or electrical field [44].

Now to derive the differential equation governing the motion of a system in question, this could be done by substituting the expression of $A_\nu(V, A)$, $S(X)$ in Eq. (6), using the representation for X_ν Eq. (2) and P_ν Eq. (3), as it has been mentioned before. We then obtain the following differential equation in momentum space:

$$\Big[\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)\frac{d^2}{dp_1^2} - 2i(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2) \frac{d}{dp_1} - \varpi^4 p_1^2 + \mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2\Big]\psi = 0, \tag{8}$$

using the following ansatz $\psi = \widetilde{\psi}(p_1)\exp[i(p_2 y - p_3 z)]$; hence, both p_2 and p_3 become constants of motion, where the coefficients $\mathcal{M}$, $\widehat{p}_2$, $\widehat{p}_3$, and $\mathcal{E}$ are given by:

$$\mathcal{E} = \frac{E}{\left(1 + \frac{p_0}{\kappa}\right)^2}; \quad \widehat{p}_2 = \frac{p_2}{\left(1 + \frac{p_0}{\kappa}\right)^2}; \quad \widehat{p}_3 \to \frac{p_3}{\left(1 + \frac{p_0}{\kappa}\right)^2}; \quad \mathcal{M} = \frac{m}{\left(1 + \frac{p_0}{\kappa}\right)}; \quad \varpi = \frac{1}{\left(1 + \frac{p_0}{\kappa}\right)}. \tag{9}$$

For the particular case $e^2\lambda^2 + \mu^2 = \gamma^2$, Eq. refsps8) will be reduced to a differential equation of the first order:

$$\Big[2i(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)\frac{d}{dp_1} + \varpi^4 p_1^2 - \left(\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2\right)\Big]\widetilde{\psi} = 0, \tag{10}$$

whose solution is

$$\widetilde{\psi}(p_1) = \widetilde{\psi}(0)\exp\left[ip_1\frac{\varpi^4 p_1^2 - 3\left(\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2\right)}{6(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)}\right], \tag{11}$$

when we take the limit $(\kappa \to \infty)$, we obtain, $\lim_{\kappa\to\infty}\widetilde{\psi}(p_1) = \widetilde{\psi}(0)e^{ip_1\frac{p_1^2 + 3\left(p_3^2 + p_2^2 + m^2 - E^2\right)}{6(m\mu + E\gamma - e\lambda p_2)}}$, which is exactly the ordinary result [45]. Now, in the case when $e^2\lambda^2 + \mu^2 > \gamma^2$, the electrostatic potential becomes weak compared to the scalar and the magnetic potentials.

By introducing the following change of variables:

$$u = \frac{\varpi}{\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{1/4}}p_1. \tag{12}$$

Equation (8) can be written in the following new form:

$$\left[\frac{d^2}{du^2} - 2i\zeta\frac{d}{du} - u^2 + \frac{\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2}{\varpi^2\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{1/2}}\right]\widetilde{\psi}(u) = 0, \tag{13}$$

where $\zeta = \frac{\left(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2\right)}{\varpi\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{3/4}}$.

In order to solve Eq. (13), we make the following substitution,

$$\widetilde{\psi}(u) = e^{i\zeta u - \frac{u^2}{2}}\Phi(u), \tag{14}$$

and a straightforward calculation gives the following differential equation:

$$\left[\frac{d^2}{du^2} - 2u\frac{d}{du} + \frac{\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2}{\varpi^2\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{1/2}} + \frac{\left(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2\right)^2}{\varpi^2\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{3/2}} - 1\right]\Phi = 0. \tag{15}$$

We identify this differential equation as the Hermite equation whose solution can be expressed in terms of the Hermite polynomials

$$\Phi = H_{n-1}(u), \tag{16}$$

with n which is a nonnegative integer:

$$2n - 2 = \frac{\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2}{\varpi^2\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{1/2}} + \frac{\left(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2\right)^2}{\varpi^2\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{3/2}} - 1, \tag{17}$$

or, equivalently:

$$(\mathcal{E}e\lambda - \widehat{p}_2\gamma)^2 + (\mathcal{E}\mu + \mathcal{M}\gamma)^2 - (\mathcal{M}e\lambda + \widehat{p}_2\mu)^2 - \left(e^2\lambda^2 + \mu^2 - \gamma^2\right)\widehat{p}_3^2 = (2n-1)\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)^{3/2}. \tag{18}$$

Then, solving Eq. (18) for the energy eigenvalues, we obtain the following:

$$E_n = \frac{\kappa\left[m^2\left(e^2\lambda^2-\gamma^2\right)+m\mu e\lambda p_2+(2n-1)\Omega^{\frac{3}{2}}\right]-\kappa^2\gamma(m\mu-e\lambda p_2)\pm\sqrt{\Omega}\sqrt{\kappa^4\triangle_1+\kappa^3\triangle_2+\kappa^2\triangle_3}}{\kappa^2\left(e^2\lambda^2+\mu^2\right)+2\kappa m\mu\gamma-m^2\left(e^2\lambda^2-\gamma^2\right)-(2n-1)\Omega^{\frac{3}{2}}}, \tag{19}$$

which the terms $\triangle_1$, $\triangle_2$ and $\triangle_3$, Ω are given by:

$$\begin{aligned}
\triangle_1 &= (me\lambda+\mu p_2)^2+\left(e^2\lambda^2+\mu^2\right)\left[p_3^2+(2n-1)\sqrt{\Omega}\right],\\
\triangle_2 &= 2m\mu\gamma\left(p_2^2+p_3^2\right)+2\gamma e\lambda p_2\left[m^2+(2n-1)\sqrt{\Omega}\right],\\
\triangle_3 &= p_2^2\left[\gamma^2m^2-\left(\mu^2-\gamma^2\right)(2n-1)\sqrt{\Omega}\right]+p_3^2\\
&\quad\left[m^2\left(e^2\lambda^2-\gamma^2\right)+(2n-1)\Omega^{\frac{3}{2}}\right],\\
\Omega &= e^2\lambda^2+\mu^2-\gamma^2,
\end{aligned} \tag{20}$$

where the eigenvalues of energy depend on the parameters λ, γ and μ, in addition to the index n which represents the principal quantum number. It must be emphasized that the energy spectrum contains an additional correction term that depends on the deformation parameter κ and is not symmetrical. This effect is due to the modification of the standard Heisenberg algebra. So, for $n\rightarrow\infty$, it is remarked that the energy spectrum becomes bounded:

$$\lim_{n\rightarrow\infty}E_n=-\kappa. \tag{21}$$

Thus, the spectrum energy in the MS model is not allowed to decrease indefinitely, but approaches a finite value. However, if we remove the deformation by setting $\kappa\rightarrow\infty$, the energy in the large n becomes $E_n=\infty$.

In addition, in the limit case $\kappa\rightarrow\infty$, comparing the expression of the energy spectrum

$$E_n=\frac{-\gamma(m\mu-e\lambda p_2)\pm\sqrt{\Omega}\sqrt{(me\lambda+\mu p_2)^2+\left(e^2\lambda^2+\mu^2\right)\left[p_3^2+(2n-1)\sqrt{\Omega}\right]}}{\left(e^2\lambda^2+\mu^2\right)} \tag{22}$$

it goes hand in hand with Dominguez findings [36]. By expanding (19) to the first order in $\frac{1}{\kappa}$, we get

$$E_n=E_n+\frac{1}{\kappa}(\Delta E)_n \tag{23}$$

The first term can be obtained from Eq. (22), while the second term represents the correction due to the DSR effect, it takes the following form:

$$\begin{aligned}
\triangle E_n &= \frac{m^2\left(e^2\lambda^2-\gamma^2\right)+m\mu e\lambda p_2+(2n-1)\Omega^{\frac{3}{2}}}{e^2\lambda^2+\mu^2}\\
&\quad+\frac{2m\mu\gamma^2(m\mu-e\lambda p_2)}{\left(e^2\lambda^2+\mu^2\right)^2}\\
&\quad\pm\left[\frac{\sqrt{\Omega}\triangle_2}{2\sqrt{\triangle_1}\left(e^2\lambda^2+\mu^2\right)}-\frac{2m\mu\gamma^2\sqrt{\Omega\triangle_1}}{\left(e^2\lambda^2+\mu^2\right)^2}\right].
\end{aligned} \tag{24}$$

We can analyze the Eq. (19) according to different limits taken by the parameters (μ,λ,γ).

1. In the absence of electromagnetic fields, $\lambda=\gamma=0$, the energy level for a linear scalar potential is

$$E_n^{\mu}=\frac{\frac{(2n-1)}{\kappa}\pm\sqrt{p_2^2+p_3^2+\left(1+\frac{p_3^2-p_2^2}{\kappa^2}\right)(2n-1)|\mu|}}{1-\frac{(2n-1)}{\kappa^2}|\mu|}, \tag{25}$$

note that the energy levels become independent of particle mass. If we remove the deformation of the spectrum, $\lim_{\kappa\rightarrow\infty}E_n^{\mu}=\pm\sqrt{p_2^2+p_3^2+(2n-1)|\mu|}$, it can be noticed that the same result given in [36], is obtained.

2. In the presence of a uniform magnetic field, $\lambda\neq 0$, Eq. (19) is reduced to:

$$E_n^{\lambda}=\frac{\frac{m^2+(2n-1)|e\lambda|}{\kappa}\pm\sqrt{m^2+p_3^2\left(1+\frac{m^2}{\kappa^2}\right)+\left(1+\frac{p_3^2}{\kappa^2}\right)(2n-1)|e\lambda|}}{1-\frac{m^2+(2n-1)|e\lambda|}{\kappa^2}}, \tag{26}$$

in the case of $\kappa\rightarrow\infty$ the energy level becomes

$$E_n^{\lambda}\rightarrow\pm\sqrt{m^2+p_3^2+(2n-1)|e\lambda|} \tag{27}$$

the result expressed by the last Eq. (27) seems to be corresponding with the results given in [36] as mentioned earlier.

3.2. Scattering states

In order to calculate the pair creation rate, we have at our disposal many different methods at our disposal such as the Feynman path-integral method [45, 46], the Hamiltonian diagonalization technique [47, 48], the Schwinger method [49, 50] and the "in" and "out" formalism [51] which is used in this subsection.

In the case of $\gamma^2 > e^2\lambda^2 + \mu^2$, the situation is quite different, and therefore, the Klein–Gordon equation can be written as:

$$\left[\frac{d^2}{dp_1^2} + \frac{(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^2} + \frac{\varpi^4}{(\gamma^2 - e^2\lambda^2 - \mu^2)} p_1^2 - \frac{\left(\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2\right)}{(\gamma^2 - e^2\lambda^2 - \mu^2)}\right] F(p_1) = 0, \tag{28}$$

where we have used

$$\psi = e^{ip_1 \frac{\left(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2\right)}{(\gamma^2 - e^2\lambda^2 - \mu^2)}} F. \tag{29}$$

Now, to reduce the Eq. (28) to a class of known differential equations, the change of variable is introduced as follows:

$$\rho = \sqrt{\frac{2}{i}} \frac{\varpi}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{1/4}} p_1, \tag{30}$$

Then, the new form of Eq. (28) is

$$\left[\frac{d^2}{d\rho^2} - \frac{1}{4}\rho^2 + \chi + \frac{1}{2}\right] F = 0, \tag{31}$$

where

$$\chi = -\frac{1}{2} + \frac{i}{2\varpi^2}\left[\frac{(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{3/2}} + \frac{\mathcal{M}^2 + \widehat{p}_2^2 + \widehat{p}_3^2 - \mathcal{E}^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{1/2}}\right]. \tag{32}$$

According to [52], the exact solutions of the above differential equation can be written in terms of parabolic cylinder functions as

$$F(\rho) = D_\chi(\rho); \quad D_\chi(-\rho); \quad D_{-\chi-1}(i\rho); \quad D_{-\chi-1}(-i\rho). \tag{33}$$

These four solutions are linearly dependent. Now, according to [53] the classification of these solutions as "in" and "out" states is as follows

$$\psi_{\text{in}}^+ = D_\chi(\rho), \tag{34}$$

$$\psi_{\text{in}}^- = D_{-\chi-1}(i\rho), \tag{35}$$

$$\psi_{\text{out}}^+ = D_\chi(-i\rho), \tag{36}$$

$$\psi_{\text{out}}^- = D_{-\chi-1}(-\rho). \tag{37}$$

The positive frequency mode ψ_{in}^+ can be expressed in terms of the positive ψ_{out}^+ and negative ψ_{out}^- frequency modes via the Bogoliubov transformation

$$\begin{cases} \psi_{\text{in}}^+ = \alpha\psi_{\text{out}}^+ + \beta\psi_{\text{out}}^- \\ \psi_{\text{in}}^- = \alpha^*\psi_{\text{out}}^- + \beta^*\psi_{\text{out}}^+ \end{cases}, \tag{38}$$

where α and β are the Bogoliubov coefficients. In order to find the relation between $\psi_{\text{in}}^\pm$ and $\psi_{\text{out}}^\pm$ states, we use the relation between parabolic cylinder functions

$$D_\chi(\rho) = e^{i\pi\chi} D_\chi(-\rho) + \frac{\sqrt{\pi 2}}{\Gamma(-\chi)} e^{\frac{i\pi}{2}(\chi+1)} D_{-\chi-1}(-i\rho), \tag{39}$$

which results in:

$$\psi_{\text{in}}^+ = e^{i\pi\chi}\psi_{\text{out}}^- + \frac{\sqrt{\pi 2}}{\Gamma(-\chi)} e^{\frac{i\pi}{2}(\chi+1)} \psi_{\text{out}}^+. \tag{40}$$

On this basis, the Bogoliubov coefficients are then

$$\beta = e^{i\pi\chi}; \quad \alpha = \frac{\sqrt{\pi 2}}{\Gamma(-\chi)} e^{\frac{i\pi}{2}(\chi+1)}. \tag{41}$$

We can derive the density of created particles as

$$\mathcal{N} = |\beta|^2 = \exp\left\{-\frac{\pi}{\varpi^2}\left[\frac{(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{3/2}} + \frac{\mathcal{M}^2 + \widehat{p}_2^2 + \widehat{p}_3^2 - \mathcal{E}^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{1/2}}\right]\right\}. \tag{42}$$

We see that the expression of the density of created particles depends on the deformation parameters κ; in addition to that, when $\kappa \to \infty$, we obtain the result associated with the constant electromagnetic field:

$$\mathcal{N}_{\kappa\to\infty} = \exp\left\{-\pi\left[\frac{(\mu m + \gamma E - \lambda e p_2)^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{3/2}} + \frac{m^2 + p_2^2 + p_3^2 - E^2}{(\gamma^2 - e^2\lambda^2 - \mu^2)^{1/2}}\right]\right\}, \tag{43}$$

as it has been noticed, the effects of deformation have disappeared. When $\gamma = 0$, we have $\mathcal{N} = 0$. On the other hand, the density of created particles will be equal to zero when γ is equal to zero. This confirms that the magnetic fields cannot influence the creation of the particles. In addition, in the presence of constant electric fields, the density of the created particles is reduced to

$$\mathcal{N} = \exp\left\{-\pi\frac{\mathcal{M}^2 + \widehat{p}_2^2 + \widehat{p}_3^2}{\gamma\varpi^2}\right\}. \tag{44}$$

In the limit $\kappa \to \infty$, we obtain the usual result associated with the constant electric field:

$$\mathcal{N}_{\kappa\to\infty} = e^{-\pi\frac{m^2+p_2^2+p_3^2}{\gamma}}. \tag{45}$$

Mathematically, the Klein–Gordon equation case study is the easiest model, which opens the way for the study of the Dirac equation, where the results of the latter lead to similar results to the case of the former equation. In the next section, we will discuss the Dirac equation in the effect of the MS model.

4. Solution of Dirac equation

In order to illustrate the effect of the MS model on the Dirac particle in a constant electromagnetic field and scalar potential, we will proceed in the same way as in the case of the KG equation. The stationary Dirac equation is given by

$$[\alpha_i(P_i - eA_i) + \beta(m+S) - (P_0 - eA_0)]\psi = 0, \tag{46}$$

where the matrices α and β are represented by:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{47}$$

And I_2 indicates the 2×2 identity matrix and $\sigma = (\sigma_x; \sigma_y; \sigma_z)$ are the Pauli matrices. In order to solve Eq. (46), it is more convenient to use the squared Dirac equation

$$\left[\left(e^2\lambda^2 + \mu^2 - \gamma^2\right)\frac{d^2}{dp_1^2} - 2i(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)\frac{d}{dp_1} - \varpi^4 p_1^2 + \mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2 - \Xi\right]\Psi = 0 \tag{48}$$

where the fermion field ψ can be obtained as

$$\psi = [\alpha_i(P_i - eA_i) + \beta(m+S) + (P_0 - V)]\Psi, \tag{49}$$

and Ξ is a 4×4 matrix defined as:

$$\Xi = i\alpha_1\alpha_2 e\lambda - i\alpha_1\gamma + \beta\alpha_1 i\mu = i\begin{pmatrix} ie\lambda & 0 & 0 & \gamma+\mu \\ 0 & -ie\lambda & \gamma+\mu & 0 \\ 0 & \gamma-\mu & ie\lambda & 0 \\ \gamma-\mu & 0 & 0 & -ie\lambda \end{pmatrix}, \tag{50}$$

whose eigenvalues are $s\sqrt{e^2\lambda^2 + \mu^2 - \gamma^2}$ with $s = \pm 1$. So, Ψ can be decomposed as

$$\Psi = v_s\varphi_s(p_1), \tag{51}$$

where

$$v_+ = \begin{pmatrix} i\frac{\lambda e + s\sqrt{\Omega}}{\gamma-\mu}\chi_1 \\ \chi_2 \end{pmatrix}; \quad v_- \begin{pmatrix} -i\frac{\lambda e - s\sqrt{\Omega}}{\gamma-\mu}\chi_1 \\ \chi_2 \end{pmatrix}; \quad \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad s = \pm 1 \tag{52}$$

and φ_s meet the goals of the equations

$$\left[\frac{d^2}{dp_1^2} - 2i\frac{(\mathcal{M}\mu + \gamma\mathcal{E} - \lambda e\widehat{p}_2)}{(e^2\lambda^2 + \mu^2 - \gamma^2)}\frac{d}{dp_1} + \frac{\mathcal{E}^2 - \mathcal{M}^2 - \widehat{p}_2^2 - \widehat{p}_3^2 - \varpi^4 p_1^2}{(e^2\lambda^2 + \mu^2 - \gamma^2)} - \frac{s}{\sqrt{e^2\lambda^2 + \mu^2 - \gamma^2}}\right]\varphi_s = 0. \tag{53}$$

This equation is similar to the case discussed in Sect. 3, and consequently, the obtained results are the same as those of Eq. (13); then, the solution of Eq. (48) can be written

$$\Psi \sim e^{i\zeta u - \frac{u^2}{2}} H_n(u) v_s, \tag{54}$$

and the corresponding energy spectrum is

$$E_n = \frac{\kappa\left[m^2\left(e^2\lambda^2 - \gamma^2\right) + m\mu e\lambda p_2 + (2n+1-s)\Omega^{\frac{3}{2}}\right] - \kappa^2\gamma(m\mu - e\lambda p_2) \pm \sqrt{\Omega}\left(\kappa^4\widetilde{\triangle}_1 + \kappa^3\widetilde{\triangle}_2 + \kappa^2\widetilde{\triangle}_3\right)^{\frac{1}{2}}}{\kappa^2\left(e^2\lambda^2 + \mu^2\right) + 2\kappa m\mu\gamma - m^2\left(e^2\lambda^2 - \gamma^2\right) - (2n+1-s)\Omega^{\frac{3}{2}}} \tag{55}$$

where

$$\widetilde{\triangle}_1 = (me\lambda + \mu p_2)^2 + (e^2\lambda^2 + \mu^2)\left[p_3^2 + (2n+1-s)\Omega^{\frac{1}{2}}\right],$$
$$\widetilde{\triangle}_2 = 2m\mu\gamma(p_2^2 + p_3^2) + 2\gamma e\lambda p_2\left[m^2 + (2n+1-s)\Omega^{\frac{1}{2}}\right],$$
$$\widetilde{\triangle}_3 = p_2^2\left[\gamma^2 m^2 - (\mu^2 - \gamma^2)(2n+1-s)\Omega^{\frac{1}{2}}\right] + p_3^2\left[m^2(e^2\lambda^2 - \gamma^2) + (2n+1-s)\Omega^{\frac{3}{2}}\right]. \quad (56)$$

The following remark can be applied to the Dirac equations, and the first-order corrections in $\frac{1}{\kappa}$, on the energy spectrum, take the same form as the corrections to the Klein–Gordon spectrum Eqs. (22), (23) and (24). Only the quantum number n will be changed by $n - \frac{s}{2}$. In the limit $\kappa \to \infty$, one recovers the results of [36]. So, the expression (55) of the energy spectrum is written by:

$$2\left(n + \frac{1}{2} - \frac{s}{2}\right)\Omega^{\frac{3}{2}} = (Ee\lambda - \gamma p_2)^2 + (E\mu + m\gamma)^2 - (e\lambda m + \mu p_2)^2 - p_3^2\Omega \quad (57)$$

and is independent of particle mass, whereas for $n \to \infty$, one obtains the energy spectrum

$$\lim_{n\to\infty} E_n = -\kappa \quad (58)$$

Let us consider the following particular cases:

1. In the absence of electromagnetic fields, $\lambda = \gamma = 0$, the energy levels for a linear scalar potential are:

$$E_n = \frac{\frac{|\mu|}{\kappa}(2n+1-s) \pm \sqrt{p_3^2 + p_2^2 + (2n+1-s)|\mu| + \frac{1}{\kappa^2}(p_3^2 - p_2^2)(2n+1-s)|\mu|}}{1 - \frac{|\mu|}{\kappa^2}(2n+1-s)}, \quad (59)$$

in the limit $\kappa \to \infty$, one recovers the results of [36],

$$E_n = \pm\sqrt{p_3^2 + p_2^2 + (2n+1-s)|\mu|}, \quad (60)$$

2. In the case of the particle in a uniform magnetic field ($\lambda \neq 0$ and $\mu = \gamma = 0$), the energy spectrum becomes:

$$E_n = \frac{\frac{m^2 + (2n+1-s)|e\lambda|}{\kappa} \pm \sqrt{m^2 + p_3^2 + (2n+1-s)|e\lambda| + \frac{p_3^2}{\kappa^2}[m^2 + (2n+1-s)|e\lambda|]}}{1 - \frac{m^2 + (2n+1-s)|e\lambda|}{\kappa^2}}$$

3. In the limit $\kappa \to \infty$, we have

$$E_n = \pm\sqrt{m^2 + p_3^2 + (2n+1-s)|e\lambda|}. \quad (61)$$

5. Results and discussion

In the present research paper, we study the Klein–Gordon and Dirac equations in the electromagnetic field with confining scalar potential in doubly special relativity and in momentum space . The analytical expressions for the relativistic energy eigenvalues and the corresponding eigenfunctions are given in Eqs. (25) and (26), respectively. Numerical results of the positive energies corresponding to positive particles with arbitrary quantum numbers are presented.

The effects of quantum number n on the bound state energy eigenvalues for the particle are given in Figs. 1 and 2. In Fig. 1, we plot the energy levels for a linear scalar potential Eq. (25) with quantum numbers n for different values of deformation parameters κ. From the results shown in Fig. 1, it is seen that the energy eigenvalues $\left(E_n^{\mu}\right)^+$ increase monotonically as n increases for various values of deformation parameters κ. We also note that for a fixed value of n, the energy levels increase when the deformation parameter κ decreases. This trend is also shown in Fig. 2, as the bound state energy eigenvalues vary with n for various values of κ.

Furthermore, Fig. 3 shows the variation of the density of created particles as a function of the variable γ for various values of $x = \frac{E}{\kappa}$. As a result, we observe that the density of created particles increases monotonically with variable γ in all cases. Moreover, for a fixed value of κ, the free energy function decreases when the deformation parameter κ grows.

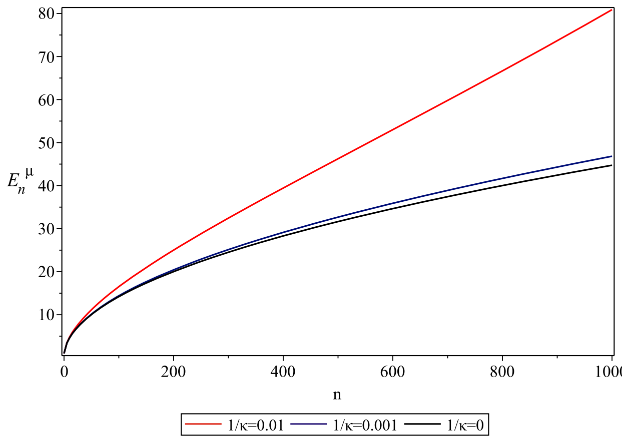

Fig. 1 E_n^μ vs n for some κ value

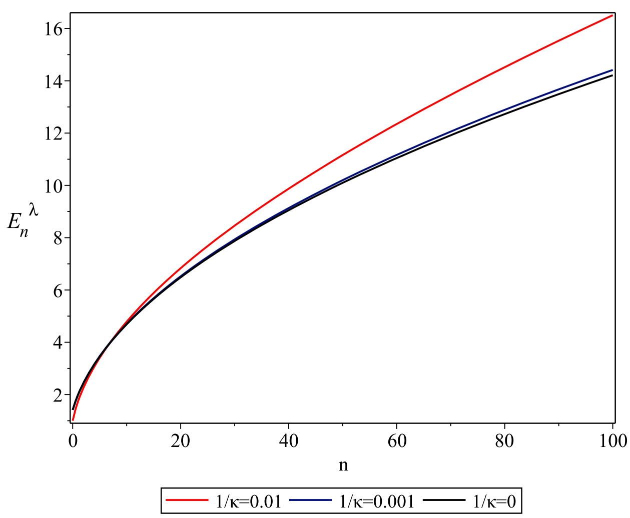

Fig. 2 E_n^λ vs n for some κ value

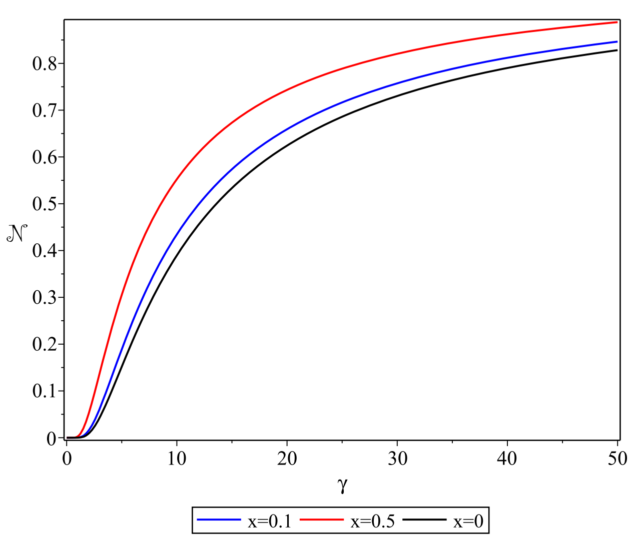

Fig. 3 $\mathcal{N}$ vs γ for various values of $x = \frac{E}{\kappa}$

As a general result, we observed that the effect of the DSR model is very significant, and in the limit $\kappa \to \infty$, all curves agree with the standard case.

6. Conclusions

Based on the aforementioned, we have solved exactly and analytically the relativistic KG and Dirac equations subjected to the action of a uniform electromagnetic field in addition to a linear scalar potential in the context of MS in the momentum space. For the KG equation case, the solutions for bound and states are determined. By using these scattering states we derived the density of created particles via the Bogoliubov transformations technique and by using these bound states the solution obtained is expressed by Hermite polynomials. Following the same method, the Dirac equation case is established. For both cases, the deformed expressions of energy spectra in the MS model are deduced and are not symmetrical, they contained corrections of all orders of $(\frac{1}{\kappa})$ and vary with the power of n. Some limiting cases are evaluated: For example, in the case when $e^2\lambda^2 + \mu^2 = \gamma^2$ there are no bound state and no confining; for $\kappa \to \infty$ we recover exactly the same result without deformed uncertainty relation, which has been done by Dominguez-Adame and Méndez [36]. Moreover, we mention that the close relation connecting the principles of relativity to the gauge invariance in the Klein–Gordon (6) and Dirac (46) equations have not been explored sufficiently in the existing literature and the difficulties and the ambiguities remain. Therefore, our work lies in the solution of the problem in question within the limit of an approximation model. Finally, the numerical study of the energy $E^\mu_{n,\kappa}$ and the density of created particles N for some κ value is exposed.

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