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ORIGINAL PAPER

The relation between different definitions of electromagnetic field tensor and Maxwell's equations in stationary spacetimes

H Ramezani-Aval*

Department of Physics, University of Gonabad, Gonabad, Iran

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Abstract: We state three different familiar definitions of electromagnetic field tensor in curvilinear coordinates and curved spacetimes, and we show the relation between these definitions. As two explicit examples, we give expressions for the electromagnetic field tensor and electromagnetic field equations for a general comoving observer in the Schwarzschild and Kerr backgrounds in terms of noncoordinate components of fields. We show that ignoring differentiation between coordinate and noncoordinate bases could lead to inconsistent and confusing results. Finally, because of its practical aspects, we obtain the electromagnetic field tensor and field equations in terms of noncoordinate components in the spacetime of a Galilean rotating observer.

Keywords: Electromagnetic field tensor; Maxwell equations; Stationary spacetimes

1. Introduction

In the presence of gravitational fields, as well as in non-inertial reference frames, one should find and employ the electromagnetic field tensor, the Maxwell equations, and constitutive equations in a curved spacetime or a curved 3-space with the corresponding line element expressed in a curvilinear coordinate system adapted to its symmetries [1–3]. An example that clearly demonstrates such a necessity is to obtain the electromagnetic field tensor and the Maxwell equations in a rotating frame. [4–11]. To encounter this problem, one needs to specify what is meant by space and spatial metric in a curved spacetime or a spatially non-Euclidean flat spacetime. When dealing with stationary spacetimes such as the spacetime of a rotating source, the decomposition of the line element into spatial and temporal sections is not a trivial task, and one has to choose a decomposition formalism. Here, we show how the electromagnetic field equations defined by this decomposition are consistent with those based on coordinate and non-coordinate bases in a 4-dimensional treatment

In this paper, we will express our final results in terms of the noncoordinate components of the fields. We remind that noncoordinate (physical) components of tensors are the component values that can be measured by experiment with standard physical instruments. But coordinate components are used in tensor analysis in a coordinate system and vary with the coordinate chosen. For the particular case where our metric is Minkowski metric in Cartesian coordinates, these components are the same, i.e. coordinate components equal the physical components we measure. But in other cases, we need to know how to find physical components from coordinate components. Hence, calculating physical components from coordinate components is essential for comparing experiments with theory. Of course, we know invariant quantities are the same for any coordinate system. They are not components of tensors, and they are equivalent to physically measured values [12–14].

The familiar covariant forms of electromagnetic field equations in curved spacetime are given by

$$\frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}F^{\mu\nu}) = j^\nu,\;\; F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \qquad (1)$$

These equations are usable in a coordinate basis ("Appendix A"). To have the explicit form of electromagnetic field equations, we should write the electromagnetic field tensor in a coordinate system and then insert the result in(1). But we usually encounter electromagnetic field tensor in flat spacetime in Cartesian coordinate, which is given by

*Corresponding author, E-mail: h.ramazani.a@gmail.com; ramezani@gonabad.ac.ir

$$F_{0i} = -E_i, \quad F_{ij} = \epsilon_{ijk} B^k \tag{2}$$

and the explicit forms of electromagnetic tensor and field equations in curved spacetimes or even in curvilinear coordinates in flat spacetimes are less familiar. For example, if we want to obtain electromagnetic field equations in cylindrical coordinates in flat spacetime or Schwarzschild spacetime, a more general definition than (2) is needed. In these coordinates and spacetimes, unlike Cartesian coordinate in flat spacetime, the coordinate and non-coordinate components of electromagnetic field tensor are not the same. We need to have a general definition in the coordinate basis to use (1). But there are different definitions of electromagnetic field tensor in curved spacetime in the literature. The relationship between these definitions is usually unclear, and each leads to different field equations. Moreover, the distinction between coordinate and non-coordinate components of the field is essential, and ignoring it leads to some confusion. Sometimes we encounter different field tensor and field equations for an observer. For example, in the case of the Galilean rotating observer, which is more interested, electromagnetic field tensor and equations are given in different forms in [15, 16, 18], without mentioning the relation between them or explaining differences. Here, we will show that these definitions can have the same explicit form for general comoving observers if we state them in terms of non-coordinate basis components of fields. In addition, in some cases, there are errors in the final form of the equations relating to the Galilean rotating observer. We will present the explicit form of these equations in Sect. 4.

In summary, this article is written for the following purposes:

1. In this paper, three different definitions of field tensors are expressed together, and the relationship between them is explained. Usually, only one of these definitions is used in the literature, and the relationship between them is unclear.

2. The relationship between the noncoordinate (physical) and coordinate components of electromagnetic fields is considered, and all final results are expressed in terms of physical components. In some papers, the relationship between coordinate and noncoordinate components of electromagnetic field is unclear. Sometimes the noncoordinate form of electromagnetic field tensor is used in a coordinate analysis, which is ambiguous. Some articles in electrical engineering use Landau formalism for electromagnetic field tensor and field equations. Although the determination of noncoordinate components is vital in this area, the relationship between field components in Landau formalism and noncoordinate components is not specified in these articles.

3. In this paper, the electromagnetic field equations in Schwarzschild and Kerr spacetimes, as well as in the rotating observer spacetime, are explicitly expressed in terms of noncoordinate components of electromagnetic field. In previous papers, these results are usually expressed in terms of coordinate components.

4. It is a straightforward translation among different conventions. It would be helpful for a student trying to use sources with different conventions.

The plan of the paper is as follows. In the next section, we review three definitions of electromagnetic field tensor in curvilinear coordinates and curved spacetimes in the literature. In Sect. 3, we show the relation between these definitions. As explicit examples, we give expressions for the electromagnetic field tensor and Maxwell equations for the general comoving observer in the Schwarzschild and Kerr backgrounds in terms of noncoordinate components of fields. We show that these results are the same as those obtained using coordinate tetrad or vielbein. We also explain the relationship between the components of the field in Landau formalism and the noncoordinate components of the field. In Sect. 4, we obtain the electromagnetic field tensor and field equations in terms of non-coordinate components in the spacetime of a Galilean rotating observer. In the final section, we will present a summary of the discussion.

2. Three definitions of electromagnetic field tensor

We can find three definitions for electromagnetic field tensor in the literature. Here we outline them:

Definition 1 Using the relation between the tensor components in the coordinate and noncoordinate bases via the properties of coordinate tetrad [15, 19];

First, the noncoordinate field tensor is defined as

$$F_{(0i)} = -E^{(i)}, \quad F_{(ij)} = \epsilon_{ijk} B^{(k)} \tag{3}$$

then the coordinate field tensor is obtained using the relation between the components of tensor in the coordinate and noncoordinate bases

$$F^{\mu\nu} = h^{\mu}_{(a)} h^{\nu}_{(b)} F^{(ab)} \tag{4a}$$

$$F_{\mu\nu} = h^{(a)}_{\mu} h^{(b)}_{\nu} F_{(ab)} \tag{4b}$$

in which $h^{(a)}_{\mu}$ (with mixed indices) are called coordinate tetrad or vielbein and their inverse denoted by $h^{\mu}_{(a)}$. These tetrads can be obtained from the metric (see "Appendix A"). The advantage of this definition is that its final form is expressed in terms of non-coordinate (physical [20]) components of fields.

Definition 2 The electromagnetic field tensor in 1+3 formalism [21, 22];

From theories of relativity, both special and general, we know that physical phenomena happen in an arena called spacetime, a new entity that was introduced by fusing the concepts of space and time. But when it comes to the measurements of these phenomena, it is their spatial and temporal characters which are being analyzed. So a decomposition of spacetime into space and time is a necessity if we are going to measure 3-dimensional quantities such as spatial distances and velocities. There are two well known formalisms called $3+1$ or foliation decomposition and $1+3$ or threading decomposition. Here, we are interested in the second approach to obtain electromagnetic field tensor and field equations. Ellis [22] formulate the electromagnetic field tensor in curved spacetimes based on the 4-velocity of a timelike observer. It is also called field tensor relative to a fundamental observer [23] and is given by

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu + \eta_{\mu\nu\alpha\beta} u^\alpha B^\beta \tag{5}$$

in which u^α is a timelike tangent vector not necessarily orthogonal to the surface of constant time, and the 1+3 splitting is made relative to it. The projection tensor h, projecting into the three-dimensional tangent plane orthogonal to u^α, is given by

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \tag{6}$$

and E and B are defined to be orthogonal to u^α.

Definition 3 Landau definition;

Landau and Lifshitz [24] in their very physical approach to spacetime decomposition, which among other things enables one to express Einstein field equations in the so-called *quasi-Maxwell* form in the context of the so-called *gravitoelectromagnetism* [25], introduce a three-dimensional form of Maxwell equations and an alternative form for the electromagnetic field tensor in stationary spacetimes. Their definitions, which are seemingly ad hoc, are as follows,

$$\begin{aligned}
&E_{Li} = F_{0i}, \;\; {D_L}^i = -\sqrt{g_{00}} F^{0i}, \;\; B_{ij} = F_{ij}, \\
&H^{ij} = \sqrt{g_{00}} F^{ij} \\
&{B_L}^i = -\frac{1}{2\sqrt{\gamma}} \epsilon^{ijk} B_{jk}, \\
&H_{Li} = -\frac{1}{2}\sqrt{\gamma}\,\epsilon_{ijk} H^{jk} \;\; (i,j,k = 1,2,3)
\end{aligned} \tag{7}$$

in which E_L and B_L are vectors in three-dimensional space, and the index L refers to Landau definition. Although we sometimes refer to this definition as 1+3 definition, there are some differences between this definition and definition 2 that we will mention later. The electromagnetic field tensor in Landau definition can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_{L1} & -E_{L2} & -E_{L3} \\ E_{L1} & 0 & \sqrt{\gamma}B_L^3 & -\sqrt{\gamma}B_L^2 \\ E_{L2} & -\sqrt{\gamma}B_L^3 & 0 & \sqrt{\gamma}B_L^1 \\ E_{L3} & \sqrt{\gamma}B_L^2 & -\sqrt{\gamma}B_L^1 & 0 \end{pmatrix} \tag{8}$$

Substituting this definition into (1), we will have usual three-dimensional form of Maxwell equations

$$\begin{aligned}
&div\mathbf{B} = 0, \;\; curl\mathbf{E} = -\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\,\mathbf{B}) \\
&div\mathbf{D} = 0, \;\; curl\mathbf{H} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\,\mathbf{D}).
\end{aligned} \tag{9}$$

in which *div* and *curl* are three-dimensional operators and γ is space metric were defined in "Appendix B". For example $div\mathbf{D} = 0$ and (81) give

$$\frac{\partial}{\partial x^i}(\sqrt{\gamma}\,D^i) = 0 \tag{10}$$

and substituting $D^i = -\sqrt{g_{00}}F^{0i}$ and (82) in (10), one ends up with the same equation obtained from(1) and (8) for $\nu = 0$. It seems that obtaining this three-dimensional form provided the motivation for Landau to define such a different definition of electromagnetic tensor.

These definitions have been used and have led to different explicit forms for electromagnetic tensor and electromagnetic field equations. For example, in [15] (using the first definition) and [16] (using third definition), we find two different forms for the electromagnetic tensor in the spacetime correspond to a rotating observer who uses Galilean rotational transformation. But the relationship between these two forms is unclear. We will show the relationship between them as well as the final corrected forms in Sect. 4.

As mentioned in [26], the covariant form of Maxwell's equations does not uniquely determine the three-vector explicit forms of electromagnetic tensor and field equations in curved spacetimes or accelerated frames. So there are few applications of these three-vector formulations. However, some applications of this formalism have been discussed by some authors. In [27], using definition 1, the electrostatic field of a point charge at rest in the Schwarzschild metric is given in algebraic form, and the magnetostatic field of a current loop surrounding a black hole is given in the integral form. In [28], a procedure for calculating the Debye potential corresponding to a stationary axisymmetric distribution of charges and currents in the Kerr metric is given. In [29], using definition 2, the authors derive the equations of motion of charged particles around a magnetized rotating star. In [30], using definition 2, Rezzolla et al. present analytic solutions of Maxwell equations in the internal and external background spacetime of a slowly rotating magnetized neutron star.

The most obvious application of these definitions is in electrodynamics of accelerated media. Electrical engineers are often confronted with configurations involving rotating bodies, and the solution of electromagnetic problems involving rotating bodies is of fundamental importance for those concerned with rotating machinery [17]. For example in [16], using definition 3, Bladel illustrates the solution of Maxwell's equations by evaluating the fields scattered by a rotating cylinder. In [15], using definition 1, a plane wave scattered by a rotating sphere is solved in the laboratory frame.

3. The relation between different definitions of electromagnetic field tensor

First, to distinguish physical quantities from coordinate quantities, it is necessary to express the final form of the field tensor and field equations in terms of noncoordinate components of fields. So the noncoordinate definition (3) is our basic definition, and all other definitions are expressed in terms of its components. In the first definition, the final form is expressed in terms of noncoordinate components of fields. But in the second and third definitions, the components of fields are not noncoordinate components. So to be able to compare the final forms, we have to express all the components of the field in terms of non-coordinate components.

3.1. The relation between definitions 1 and 2

Definition 2 depends on the timelike tangent vector u^μ. In general, if we choose the timelike tangent vector u_μ in the second definition as the timelike basis vector $h_\mu^{(0)}$ of tetrad in the first definition (Or equivalently u^μ as inverse timelike basis vector $h^\mu_{(0)}$), then the electromagnetic field tensor obtained from the two methods will have the same explicit form. In other words, in the first definition we set $h^\mu_{(0)} = u^\mu$ and then we define the three spacelike unit vectors upon it. We will then see that the results obtained from the two definitions are exactly the same. This is because in Ellis formalism, the velocity u^μ on which the decomposition is performed acts as a timelike tetrad vector. If we use the general comoving observer velocity in definition 2, the tetrad based on which it is made is the same as the coordinate tetrad (vielbein) obtained directly from the metric according to the relations (46). Therefore, if we use the general comoving observer velocity in definition 2 and the coordinate tetrad in definition 1, the results of the two approaches will be the same. This similarity arises from the properties of the general comoving observer. It can be said that the metric of a curved spactime (such as Schwarzschild spacetime) is written from the point of view of the general comoving observer. Therefore, the results obtained for field tensor from the point of view of a general comoving observer based on Ellis's formalism are the same as those obtained by the first definition with the help of coordinate tetrad.

As important examples, we obtain the electromagnetic field tensor for the general comoving observer in the Schwarzschild and Kerr backgrounds. In [30] another example for zero angular momentum observers (ZAMO) in a slow rotation metric is given by some approximations.

The velocity of the general comoving observer is given by

$$u^\mu = (1/\sqrt{g_{00}}, 0, 0, 0) \tag{11}$$

and the Schwarzschild metric in spherical coordinates (t, r, θ, ϕ) is given by

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{12}$$

Using equations (51) and choosing the general comoving observer velocity (11) as $h^\mu_{(0)}$, the coordinate tetrad for this metric is obtained as follows (As we mentioned before, these tetrads can be obtained directly from the metric (12) according to the relations (46))

$$h_\mu^{(a)} = \begin{pmatrix} \sqrt{1-\frac{2m}{r}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\frac{2m}{r}}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$
$$h^\mu_{(a)} = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{2m}{r}}} & 0 & 0 & 0 \\ 0 & \sqrt{1-\frac{2m}{r}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix} \tag{13}$$

The electromagnetic tensor in a general noncoordinate basis is given by equation (3). To transform it into spherical coordinates, we choose

$$q_0 = t,\ q_1 = r\ q_2 = \theta,\ q_3 = \phi, \tag{14}$$

so that

$$E^{(1)} = E_r,\ E^{(2)} = E_\theta,\ E^{(3)} = E_\phi. \tag{15}$$

then using (4a) and (4b), the electromagnetic field tensor in a Schwarzschild background is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_r & \frac{E_\theta}{r\sqrt{1-\frac{2m}{r}}} & \frac{E_\phi}{r\sin\theta\sqrt{1-\frac{2m}{r}}} \\ -E_r & 0 & \frac{B_\phi\sqrt{1-\frac{2m}{r}}}{r} & \frac{-B_\theta\sqrt{1-\frac{2m}{r}}}{r\sin\theta} \\ -\frac{E_\theta}{r\sqrt{1-\frac{2m}{r}}} & \frac{-B_\phi\sqrt{1-\frac{2m}{r}}}{r} & 0 & \frac{B_r}{r^2\sin\theta} \\ \frac{-E_\phi}{r\sin\theta\sqrt{1-\frac{2m}{r}}} & \frac{B_\theta\sqrt{1-\frac{2m}{r}}}{r\sin\theta} & \frac{-B_r}{r^2 sin\theta} & 0 \end{pmatrix} \tag{16}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_r & -r\sqrt{1-\frac{2m}{r}}E_\theta & -r\sin\theta\sqrt{1-\frac{2m}{r}}E_\phi \\ E_r & 0 & \frac{rB_\phi}{\sqrt{1-\frac{2m}{r}}} & -\frac{r\sin\theta B_\theta}{\sqrt{1-\frac{2m}{r}}} \\ r\sqrt{1-\frac{2m}{r}}E_\theta & \frac{-rB_\phi}{\sqrt{1-\frac{2m}{r}}} & 0 & r^2\sin\theta B_r \\ r\sin\theta\sqrt{1-\frac{2m}{r}}E_\phi & \frac{r\sin\theta B_\theta}{\sqrt{1-\frac{2m}{r}}} & -r^2\sin\theta B_r & 0 \end{pmatrix} \tag{17}$$

On the other hand for general comoving observe by (5), (11) and (12) we have

$$F_{0i} = \sqrt{g_{00}}E_i,\ \ F_{ij} = \sqrt{\gamma}\epsilon_{ij0k}B^k \tag{18}$$

in which $g_{00}\gamma = g$ and

$$\eta_{\mu\nu\alpha\beta} = \sqrt{-g}\epsilon_{\mu\nu\alpha\beta} \tag{19}$$

But the field components in (5) are not noncoordinate components, and we can write them in terms of noncoordinate components as below

$$E_i = h_i^{(a)}E_{(a)},\ \ B^i = h^i_{(a)}B^{(a)} \tag{20}$$

Then we can see that the field tensor is the same as (17). As an example

$$F_{12} = \sqrt{\gamma}B^3 = \sqrt{\gamma}h^3_{(3)}B^{(3)} = \frac{1}{\sqrt{1-\frac{2m}{r}}}B^\varphi \tag{21}$$

Now we obtain the electromagnetic field equations. By working on a coordinate basis, that is, using equations (1) and (16), we have the following *vacuum* electromagnetic field equations in the Schwarzschild metric

$$\sin\theta\,\partial_r(r^2E_r) + \frac{r}{\sqrt{1-\frac{2m}{r}}}\partial_\theta(\sin\theta\,E_\theta) + \frac{r}{\sqrt{1-\frac{2m}{r}}}\partial_\phi E_\phi = 0 \tag{22a}$$

$$r\sin\theta\,\partial_t E_r - \sqrt{1-\frac{2m}{r}}\partial_\theta(\sin\theta\,B_\phi) + \sqrt{1-\frac{2m}{r}}\partial_\phi B_\theta = 0 \tag{22b}$$

$$\frac{r\sin\theta}{\sqrt{1-\frac{2m}{r}}}\partial_t E_\theta + \sin\theta\,\partial_r\left(r\sqrt{1-\frac{2m}{r}}B_\phi\right) - \partial_\phi B_r = 0 \tag{22c}$$

$$\frac{r}{\sqrt{1-\frac{2m}{r}}}\partial_t E_\phi - \partial_r\left(r\sqrt{1-\frac{2m}{r}}B_\theta\right) + \partial_\theta B_r = 0 \tag{22d}$$

for $\nu = 0, 1, 2, 3$, respectively. For the source free equations by using (1) and (17) we have

$$\sin\theta\partial_r(r^2B_r) + \frac{r}{\sqrt{1-\frac{2m}{r}}}\partial_\theta(\sin\theta\,B_\theta) + \frac{r}{\sqrt{1-\frac{2m}{r}}}\partial_\phi B_\phi = 0 \tag{23a}$$

$$r\sin\theta\,\partial_t B_r + \sqrt{1-\frac{2m}{r}}\partial_\theta(\sin\theta\,E_\phi) - \sqrt{1-\frac{2m}{r}}\partial_\phi E_\theta = 0 \tag{23b}$$

$$\frac{r\sin\theta}{\sqrt{1-\frac{2m}{r}}}\partial_t B_\theta - \sin\theta\,\partial_r\left(r\sqrt{1-\frac{2m}{r}}E_\phi\right) - \partial_\phi E_r = 0 \tag{23c}$$

$$\frac{r}{\sqrt{1-\frac{2m}{r}}}\partial_t B_\phi + \partial_r\left(r\sqrt{1-\frac{2m}{r}}E_\theta\right) - \partial_\theta E_r = 0 \tag{23d}$$

It can easily be seen that by setting $m = 0$, all the equations reduce to those expected in flat spacetime in spherical coordinates. Ignoring the difference between coordinate and noncoordinate bases has led to confusing expressions in the literature. As an example, in [27], without mentioning that equations (1) are expressed in a coordinate basis, the definition of the field tensor in a non-coordinate basis has been used to obtain the electromagnetic field equations in the Schwarzschild background. Therefore, the results presented for the field equations there (relations 7 and 11 in [27]) are not the same as Equations (22) and (23) here.

Now we check our idea for another example. The weak field, slow rotation limit of the Kerr metric representing the spacetime of a spherically symmetric rotating mass m with angular momentum per unit mass a in spherical (Schwarzschild-type) coordinates is given by,

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1+\frac{2m}{r}\right)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + \frac{4ma}{r}\sin(\theta)^2 d\phi dt \tag{24}$$

Again we choose the general comoving observer velocity (11) as $h^\mu_{(0)}$ and so we have

$$h \equiv h_\mu^{(a)} = \begin{pmatrix} \sqrt{1-\frac{2m}{r}} & 0 & 0 & 0 \\ 0 & \sqrt{1+\frac{2m}{r}} & 0 & 0 \\ 0 & 0 & r\sqrt{1+\frac{2m}{r}} & 0 \\ \frac{2am\sin^2(\theta)}{r\sqrt{1-\frac{2m}{r}}} & 0 & 0 & r\sin(\theta)\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}} \end{pmatrix} \quad (25a)$$

$$h^{-1} \equiv h^\mu_{(a)} = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{2m}{r}}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+\frac{2m}{r}}} & 0 & 0 \\ 0 & 0 & \frac{1}{r\sqrt{1+\frac{2m}{r}}} & 0 \\ \frac{-2am\sin(\theta)}{r(r-2m)\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}} & 0 & 0 & \frac{1}{r\sin(\theta)\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}} \end{pmatrix} \quad (25b)$$

and the electromagnetic field tensor in this background, using $F^{\mu\nu} = {h^{-1}}^T F^{(ab)} h^{-1}$, is given by

$$\begin{aligned} F^{01} =& \frac{1}{\sqrt{1-\frac{4m^2}{r^2}}}E_r - \frac{2am\sin(\theta)}{r^2\left(1-\frac{2m}{r}\right)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}}B_\theta \\ F^{02} =& \frac{1}{r\sqrt{1-\frac{4m^2}{r^2}}}E_\theta + \frac{2am\sin(\theta)}{r^3\left(1-\frac{2m}{r}\right)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}}B_r \\ F^{03} =& \frac{1}{\sin(\theta)\sqrt{1-\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}}E_\phi, \\ F^{12} =& \frac{1}{r(1+\frac{2m}{r})}B_\phi \\ F^{13} =& -\frac{1}{r\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}}B_\theta \\ F^{23} =& \frac{1}{r^2\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}}B_r \end{aligned} \quad (26)$$

and

$$\begin{aligned} F_{01} =& -\sqrt{1-\frac{4m^2}{r^2}}E_r,\ F_{02} = -r\sqrt{1-\frac{4m^2}{r^2}}E_\theta \\ F_{03} =& -r\sin(\theta)\sqrt{1-\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}E_\phi, \\ F_{12} =& r\left(1+\frac{2m}{r}\right)B_\phi \\ F_{13} =& \frac{2am\sin^2(\theta)\sqrt{1-\frac{4m^2}{r^2}}}{r-2m}E_r - r\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}B_\theta \\ F_{23} =& \frac{2amr\sin^2(\theta)\sqrt{1-\frac{4m^2}{r^2}}}{r-2m}E_\theta + r^2\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}B_r \end{aligned} \quad (27)$$

On the other hand, for general comoving observe in this stationary spacetime, we have

$$u_\mu = \frac{g_{\mu 0}}{\sqrt{g_{00}}} \quad (28)$$

by substituting this in (5) and using (24), we repeat what was done for the Schwarzschild metric, and we can again see that definition 2 leads to the same result as definition 1 for the electromagnetic field tensor in Kerr background. Here we obtain two components of field tensor by definition 2:

$$\begin{aligned} F_{02} =& u_0 E_2 = u_0 h_2^{(2)} E_{(2)} = r\sqrt{1-\frac{4m^2}{r^2}}E_\theta \\ F_{13} =& -u_3 E_1 + \eta_{1302} u^0 B^2 = -u_3 h_1^{(1)} E_{(1)} + \sqrt{\gamma} h^2_{(2)} B^{(2)} \\ =& \frac{2am\sin^2(\theta)\sqrt{1-\frac{4m^2}{r^2}}}{r-2m}E_r - r\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}B_\theta \end{aligned} \quad (29)$$

Electromagnetic field equations in Kerr background can also be obtained using (1), (26) and (27). For example setting $\nu = 0$ in the left hand side of equation (1) gives

$$\partial_r\left[-\frac{(1+\frac{2m}{r})sin(\theta)\sqrt{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}}{\sqrt{1-\frac{4m^2}{r^2}}}E_r+2am\sin^2(\theta)\sqrt{\frac{r+2m}{r-2m}}B_\theta\right]$$
$$+\partial_\theta\left[-\frac{(1+\frac{2m}{r})sin(\theta)\sqrt{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}}{r\sqrt{1-\frac{4m^2}{r^2}}}E_\theta-\frac{2am\sin^2(\theta)}{r}\sqrt{\frac{r+2m}{r-2m}}B_r\right]$$
$$+\partial_\phi\left[r^2(1+\frac{2m}{r})E_\phi\right]=0 \tag{30}$$

in the *vacuum*. And setting $\mu,\nu,\lambda=1,2,3$ in the right-hand side of equation (1), we have

$$\partial_r\left[\frac{2amr\sin^2(\theta)\sqrt{1-\frac{4m^2}{r^2}}}{r-2m}E_\theta+r^2\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}B_r\right]$$
$$-\partial_\theta\left[\frac{2am\sin^2(\theta)\sqrt{1-\frac{4m^2}{r^2}}}{r-2m}E_r-r\sin(\theta)\sqrt{1+\frac{2m}{r}}\sqrt{\frac{r^4-4r^2m^2+4m^2a^2\sin^2(\theta)}{r^3(r-2m)}}B_\theta\right]$$
$$+\partial_\phi\left[r\left(1+\frac{2m}{r}\right)B_\phi\right]=0 \tag{31}$$

Again it can be easily seen that if we set $m=0$, all the equations reduce to those expected in flat spacetime in spherical coordinates.

3.2. The relation between definitions 2 and 3

Landau and Lifshitz introduce the definitions of the electromagnetic field in a curved background in terms of a field tensor without referring to a specified frame. In this subsection, we show how their definitions are related to those given by Ellis [22] in terms of the 4-velocity of a timelike observer in its comoving frame. Definitions 2 and 3 are both based on 1+3 decomposition. In [22] an approach to this decomposition is given that focuses on applications of the 1 + 3 decomposition in relativistic cosmology. We refer to it as Ellis formalism. Another approach that we refer to as Landau formalism has been discussed in "Appendix B". It is easy to show that if we use Ellis formalism for general comoving observers, then we will have Landau formalism. If we substitute the velocity of the general comoving observer (11) into projecting tensor (6), we obtain the spatial tensor introduced by Landau

$$\gamma_{00}=0,\ \gamma_{0i}=0,\ \gamma_{ij}=g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}} \tag{32}$$

So we can say that Landau formalism is a special case of a general definition in Ellis formalism. But the interesting point is that by substituting the general comoving observer velocity in definition(5), the Landau definition for electromagnetic field tensor (8) is not obtained. In other words, although inserting the velocity of the general comoving observer into (6) gives the Landau projecting tensor, we can not obtain Landau definition for electromagnetic tensor(8) from Ellis definition (5) in the same way. So Landau definition is a particular case and we should classify it as a different definition. Using this definition, Landau and Lifshitz introduce a three-dimensional form of Maxwell equations and an alternative form for the electromagnetic field tensor in stationary spacetimes [24]. It seems that the purpose of Landau definition is to introduce electromagnetic field equations in a way that is similar to the usual three-dimensional form of Maxwell equations, and so the start point of Landau is (9).

Now the question arises as to what is the relation between field components in Landau definition and the corresponding components in a noncoordinate basis. For example, what is the relation between E_{Li} and E_r? To answer this question, we take equation (8) as the electromagnetic field tensor in the coordinate basis. As an example, in the case of Schwarzschild metric, we set equations (8) and (17) equal to each other and find the following relations

$$E_{L1}=-F_{01}=E_r,\ \ E_{L2}=-F_{02}=r\sqrt{1-\frac{2m}{r}}E_\theta$$
$$E_{L3}=-F_{03}=r\sin\theta\sqrt{1-\frac{2m}{r}}E_\phi$$
$$B_L{}^1=\frac{1}{\sqrt{\delta}}F_{23}=\sqrt{1-\frac{2m}{r}}B_r,\ \ B_L{}^2=-\frac{1}{\sqrt{\delta}}F_{31}=\frac{B_\theta}{r}$$
$$B_L{}^3=\frac{1}{\sqrt{\delta}}F_{12}=\frac{B_\phi}{r\sin\theta} \tag{33}$$

along with using equations (7) to get,

$$
\begin{aligned}
D^1{}_L &= -\sqrt{g_{00}}\,F^{01} = \sqrt{1-\frac{2m}{r}}E_r,\\
D^2{}_L &= -\sqrt{g_{00}}\,F^{02} = \frac{E_\theta}{r}\\
D^3{}_L &= -\sqrt{g_{00}}F^{03} = \frac{E_\phi}{r\sin\theta}\\
H_{1L} &= \sqrt{\gamma}\sqrt{g_{00}}\,F^{23} = B_r,\\
H_{2L} &= \sqrt{\gamma}\sqrt{g_{00}}\,F^{31} = r\sqrt{1-\frac{2m}{r}}B_\theta\\
H_{3L} &= \sqrt{\gamma}\sqrt{g_{00}}\,F^{12} = r\sin\theta\sqrt{1-\frac{2m}{r}}B_\phi
\end{aligned} \tag{34}
$$

in which for the Schwarzschild metric we have used:

$$\sqrt{\gamma} = \frac{r^2\sin\theta}{\sqrt{1-\frac{2m}{r}}} \tag{35}$$

Now setting from (33) and (34) into (9) and using (81), one can obtain the electromagnetic field equations in Schwarzschild metric, which are the same as the equations we obtained before. For example, by using these equations, one gets

$$
\begin{aligned}
&\partial_r H_2 - \partial_\theta H_1 = \partial_t(\sqrt{\gamma}D^3) \;\Rightarrow\\
&\partial_r\left(r\sqrt{1-\frac{2m}{r}}B_\theta\right) - \partial_\theta(B_r) = \partial_t\left(\frac{r}{\sqrt{1-\frac{2m}{r}}}E_\phi\right)
\end{aligned} \tag{36}
$$

which is the same as (22d), i.e. the two formalisms are equivalent, leading to the same field equations. This is a consistency check for the applicability of $1+3$ formulation and its definition of 3-space as the arena where the 3-objects **E** and **B** play their physical role.

4. Electromagnetic field tensor and field equations for Galilean rotating observer

The electrodynamics of rotating frames has been studied because of its practical aspects, especially in electrical engineering [1, 10, 15–18, 31, 32]. In these works, the Galilean rotational transformation (GRT) and the three-dimensional form of the field equations are usually used. In [33, 34] Nouri-Zonoz et al. have shown the properties, limitations and problems of GRT. It has been demonstrated that GRT can only be used for the centric rotating observers. However, here we obtain electromagnetic field tensor and field equations in the Galilean rotating observer's frame, as a particular case of a stationary spacetime. We use definition 1 introduced in 2. It should be noted that these equations are obtained in [16, 17] based on definition 3, but like most works in this field, the relationship between their field components and the non-coordinate field components is not clear.

Due to the axial symmetry of the problem, one naturally employs cylindrical coordinates, assigning events in the non-rotating frame with coordinates(t', ρ', ϕ', z') and in the one rotating with constant angular velocity Ω around their common axis with (t, ρ, ϕ, z). In the simplest rotational transformation, these are related through the Galilean rotational transformation [24],

$$t = t', \quad \rho = \rho', \quad \phi = \phi' - \Omega t, \quad z = z' \tag{37}$$

or in its differential form

$$dt = dt', \quad d\rho = d\rho', \quad d\phi = d\phi' - \Omega dt, \quad dz = dz'. \tag{38}$$

So that the line element from the spatially Euclidean flat spacetime in cylindrical coordinates, i.e.

$$ds^2 = dt'^2 - d\rho'^2 - \rho'^2 d\phi'^2 - dz^2, \tag{39}$$

transforms into

$$ds^2 = (1-\Omega^2\rho^2)dt^2 - 2\Omega\rho^2 dt d\phi - d\rho^2 - \rho^2 d\phi^2 - dz^2, \tag{40}$$

which is the spatially non-Euclidean flat spacetime in a rotating frame. The corresponding coordinate tetrad is given by

$$h \equiv h_\mu^{(a)} = \begin{pmatrix} \sqrt{1-\rho^2\Omega^2} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \frac{-\rho^2\Omega}{\sqrt{1-\rho^2\Omega^2}} & 0 & \frac{\rho}{\sqrt{1-\rho^2\Omega^2}} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{41a}$$

$$h^{-1} \equiv h^\mu_{(a)} = \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2\Omega^2}} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} & 0 & \frac{\sqrt{1-\rho^2\Omega^2}}{\rho} & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41b}$$

so that the electromagnetic field tensor in this background, using $F^{\mu\nu} = {h^{-1}}^T F^{(ab)} h^{-1}$, is given by

$$
\begin{aligned}
F^{01} &= \frac{1}{\sqrt{1-\rho^2\Omega^2}} E_\rho - \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} B_z \\
F^{02} &= \frac{1}{\rho} E_\phi \\
F^{03} &= \frac{1}{\sqrt{1-\rho^2\Omega^2}} E_z + \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} B_\rho, \\
F^{12} &= \frac{1}{\sqrt{1-\rho^2\Omega^2}} B_z \\
F^{13} &= -B_\phi \\
F^{23} &= \frac{1}{\sqrt{1-\rho^2\Omega^2}} B_\rho
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
F_{01} &= -\sqrt{1-\rho^2\Omega^2} E_\rho, \quad F_{02} = -\rho E_\phi \\
F_{03} &= -\sqrt{1-\rho^2\Omega^2} E_z, \\
F_{12} &= \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} B_z - \frac{\rho^2\Omega}{\sqrt{1-\rho^2\Omega^2}} E_\rho \\
F_{13} &= -B_\phi \\
F_{23} &= \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} B_\rho + \frac{\rho^2\Omega}{\sqrt{1-\rho^2\Omega^2}} E_z
\end{aligned}
\tag{43}
$$

in the coordinate basis, by using left-hand side equation of (1) and (42), we have the following *vacuum* electromagnetic field equations

$$
\partial_\rho \left[\frac{\rho}{\sqrt{1-\rho^2\Omega^2}} (E_\rho - \rho\Omega B_z) \right] + \partial_\phi E_\phi + \frac{\rho}{\sqrt{1-\rho^2\Omega^2}} \partial_z (E_z + \rho\Omega B_\rho) = 0 \tag{44a}
$$

$$
\frac{1}{\sqrt{1-\rho^2\Omega^2}} \partial_\phi B_z - \partial_z B_\phi = \frac{1}{\sqrt{1-\rho^2\Omega^2}} \partial_t (E_\rho + \rho\Omega B_z) \tag{44b}
$$

$$
\partial_\rho \left(\frac{\rho}{\sqrt{1-\rho^2\Omega^2}} B_z \right) - \frac{\rho}{\sqrt{1-\rho^2\Omega^2}} \partial_z B_\rho = -\partial_t E_\phi \tag{44c}
$$

$$
\frac{\sqrt{1-\rho^2\Omega^2}}{\rho} \partial_\rho (\rho B_\phi) - \partial_\phi B_\rho = \partial_t (E_z + \rho\Omega B_\rho) \tag{44d}
$$

for $\nu = 0, 1, 2, 3$, respectively. For the source free equations by using right-hand side equation of (1) and (43) we have

$$
\partial_\rho \left[\frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} (B_\rho + \rho E_z) \right] + \partial_\phi B_\phi + \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} \partial_z (B_z - \rho E_\rho) = 0 \tag{45a}
$$

$$
\sqrt{1-\rho^2\Omega^2} \partial_\phi E_z - \rho \partial_z E_\phi = -\frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} \partial_t (B_\rho + \rho E_z) \tag{45b}
$$

$$
\partial_\rho (\sqrt{1-\rho^2\Omega^2} B_z) - \sqrt{1-\rho^2\Omega^2} \partial_z E_\rho = \partial_t B_\phi \tag{45c}
$$

$$
\partial_\rho (\rho E_\phi) - \sqrt{1-\rho^2\Omega^2} \partial_\phi E_\rho = \frac{\rho\Omega}{\sqrt{1-\rho^2\Omega^2}} \partial_t (-B_z + \rho E_\rho) \tag{45d}
$$

It can easily be seen that by setting $m = 0$ all the equations reduce to those expected in flat spacetime in cylindrical coordinates.

5. Conclusions

Clarifying the roles of coordinate (holonomic) and noncoordinate (anholonomic) bases in the presentation of the electromagnetic field tensor and Maxwell equations, we stated that three different often-used definitions of electromagnetic field tensor in curvilinear coordinates and curved spacetimes can have the same explicit final forms. If we choose tetrad in definition 1 (the definition uses tetrad formalism) so that its timelike basis vector $h^{(0)}_\mu$ be equal to the timelike tangent vector u_μ in definition 2 (via Ellis formalism), and then state them in terms of noncoordinate basis components of fields, the final form of the equations will be the same. Moreover, the results obtained for field tensor from the point of view of general comoving observer based on Ellis's formalism are the same as those obtained by the first definition with the help of coordinate tetrad. Definition 3 (via Landau formalism) is a particular definition that its components can be obtained in terms of noncoordinate components by comparison with other definitions. We checked our idea for the general comoving observer in two backgrounds: Schwarzschild and weak Kerr spacetimes. Finally, we obtained the electromagnetic field tensor and field equations in terms of noncoordinate components in the spacetime of a Galilean rotating observer.

It should be noted that from the physical point of view, the components of the fields that are measured in an experiment are those components introduced in the noncoordinate basis. For example in a cylindrically symmetric distribution of charges, the azimuthal electromagnetic fields in the noncoordinate basis, e.g. $E_\phi \equiv E^{(2)} = h^{(2)}_2 E^2$

and $B_\phi \equiv B^{(2)} = h_2^{(2)} B^2$, are the so-called *physical or ordinary* components [20, 35]. Some of the results of this article could be found in different books and papers, and one of our goals is to gather them in a systematic way in one place so that they could be easily accessible to those interested. Also, it is a straightforward translation among different conventions.

Appendix A: Coordinate and noncoordinate bases

In this appendix, we give a brief account of coordinate and noncoordinate bases and the relation between the components of a tensor in the two bases.

A.1: Coordinate or holonomic basis

In a coordinate basis (also called **holonomic** basis) [35–37] the basis vectors and their duals are denoted by $\hat{e}_\mu$ and $\hat{e}^\mu$, respectively. They are unit vectors and satisfy the following inner product relations,

$$\hat{e}_\mu \cdot \hat{e}_\nu = g_{\mu\nu} \tag{46a}$$

$$\hat{e}^\mu \cdot \hat{e}^\nu = g^{\mu\nu} \tag{46b}$$

$$\hat{e}_\mu \cdot \hat{e}^\nu = \delta_\mu^\nu \tag{46c}$$

where $g_{\mu\nu}$ and $g^{\mu\nu}$ are the metric components. In this basis, the vector **V** and the second rank tensor **T**, in terms of their covariant and contravariant components, are given by

$$\mathbf{V} = V^\mu \hat{e}_\mu = V_\mu \hat{e}^\mu, \quad \mathbf{T} = T^{\mu\nu} \hat{e}_\mu \otimes \hat{e}_\nu = T_{\mu\nu} \hat{e}^\mu \otimes \hat{e}_\nu \tag{47}$$

raising and lowering of the indices are done with the help of the general metric components $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively.

A.2: Non-coordinate or anholonomic basis

In a noncoordinate basis (also called **orthonormal or anholonomic** basis) [35–37], the basis vectors and their duals, denoted by $\hat{\epsilon}_{(a)}$ and $\hat{\epsilon}^{(a)}$, respectively, are given by tangent and cotangent vectors of unit length satisfying the following inner product relations

$$\hat{\epsilon}_{(a)} \cdot \hat{\epsilon}_{(a)} = 1 \tag{48a}$$

$$\hat{\epsilon}_{(a)} \cdot \hat{\epsilon}_{(b)} = \eta_{(ab)} \tag{48b}$$

$$\hat{\epsilon}^{(a)} \cdot \hat{\epsilon}^{(b)} = \eta^{(ab)} \tag{48c}$$

$$\hat{\epsilon}^{(a)} \cdot \hat{\epsilon}_{(b)} = \delta_{(b)}^{(a)} \tag{48d}$$

where $\eta_{(ab)}$ is the flat spacetime metric in Cartesian coordinates. For example, in such a basis, a vector **V** and a second rank tensor **T** are written as as follows

$$\begin{aligned} \mathbf{V} &= V^{(a)} \epsilon_{\hat{(a)}} = V_{(a)} \hat{\epsilon}^{(a)}, \\ \mathbf{T} &= T^{(ab)} \hat{\epsilon}_{(a)} \otimes \hat{\epsilon}_{(b)} = T_{(ab)} \hat{\epsilon}^{(a)} \otimes \hat{\epsilon}^{(b)} \end{aligned} \tag{49}$$

Raising and lowering of the indices are done with the Minkowski metric elements $\eta^{(ab)}$ and $\eta_{(ab)}$, respectively. A very important point that should be emphasized here is that the components of **T** and **V** in a noncoordinate basis (i.e $V_{(a)}$, $V^{(a)}$, $T^{(ab)}$ and $T_{(ab)}$) all are scalars under coordinate transformation. Later we will obtain the components of the same tensors **T** and **V** in other bases in terms of these components.

One can express coordinate basis vectors in terms of the noncoordinate basis vectors as follows

$$\hat{e}_\mu = h_\mu^{(a)} \hat{\epsilon}_{(a)} \tag{50}$$

where $h_\mu^{(a)}$ (with mixed indices) are called tetrads and with their inverses denoted by $h_{(a)}^\mu$ satisfy the following relations

$$g_{\mu\nu} = h_\mu^{(a)} h_\nu^{(b)} \eta_{(ab)} \tag{51a}$$

$$h_{(a)}^\mu h_\nu^{(a)} = \delta_\nu^\mu \tag{51b}$$

$$h_\mu^{(a)} h_{(b)}^\mu = \delta_{(b)}^{(a)} \tag{51c}$$

$$\eta_{(ab)} = g_{\mu\nu} h_{(a)}^\mu h_{(b)}^\nu \tag{51d}$$

$$h_{(a)}^\mu = g^{\mu\nu} \eta_{(ab)} h_\nu^{(b)} \tag{51e}$$

A.3: Covariant and contravariant components of tensors in the two bases

According to the definitions given in the previous subsections, we have the following relations between contravariant and covariant components in the two bases

$$V^{(a)} = h_\mu^{(a)} V^\mu \tag{52a}$$

$$V_{(a)} = h_{(a)}^\mu V_\mu \tag{52b}$$

$$T^{(ab)} = h_\mu^{(a)} h_\nu^{(b)} T^{\mu\nu} \tag{52c}$$

$$T_{(ab)} = h_{(a)}^\mu h_{(b)}^\nu T_{\mu\nu} \tag{52d}$$

The inverse relations are

$$V^\mu = h_{(a)}^\mu V^{(a)} \tag{53a}$$

$$V_\mu = h_\mu^{(a)} V_{(a)} \tag{53b}$$

$$T^{\mu\nu} = h_{(a)}^\mu h_{(b)}^\nu T^{(ab)} \tag{53c}$$

$$T_{\mu\nu} = h_\mu^{(a)} h_\nu^{(b)} T_{ab} \tag{53d}$$

The divergence of a vector in the coordinate basis with generalized coordinates q^μ can be written as follows

$$\nabla \cdot \mathbf{V} = (\hat{e}^i \nabla_i) \cdot (V^j \hat{e}_j) = \partial V^j / \partial q^j + V^j \Gamma^i{}_{ij} \tag{54}$$

where (46c) was used and $\Gamma^k{}_{ij}$ is also defined by

$$\nabla_i \hat{e}_j = \partial \hat{e}_j / \partial q^i = \hat{e}_k \Gamma^k_{ij}, \quad i,j,k = 1,2,3 \tag{55}$$

Using the relation $\frac{\partial g}{\partial q^k} = 2g\Gamma^i_{ik}$ in (54) we end up with the well known relation for the divergence of a vector

$$\nabla \cdot \mathbf{V} = V^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial q^k} (\sqrt{-g} V^k) \tag{56}$$

In the same way, for the gradient operator we have

$$\nabla \mathbf{V} = (\hat{e}^j \partial_j) \otimes (V^i \hat{e}_i) = (\partial_j V^j + \Gamma^i_{j\gamma} V^k) \hat{e}^j \otimes \hat{e}_i \tag{57}$$

And for the covariant derivative of a second rank tensor $\mathbf{T}$, we have

$$\begin{aligned} \nabla \mathbf{T} =& (\partial_\gamma \hat{e}^\gamma) \otimes (T^{\mu\nu} \hat{e}_\mu \otimes \hat{e}_\nu) \\ =& (\partial_\gamma T^{\mu\nu} + \Gamma^\mu_{\delta\gamma} T^{\delta\nu} + \Gamma^\nu_{\delta\gamma} T^{\mu\delta}) \hat{e}^\gamma \otimes \hat{e}_\mu \otimes \hat{e}_\nu \end{aligned} \tag{58}$$

The expression in parenthesis gives the components of the covariant derivative of $\mathbf{T}$. Finally for the divergence of a tensor $\mathbf{T}$ denoted by $\nabla \cdot \mathbf{T}$ we have:

$$\begin{aligned} \nabla \cdot \mathbf{T} =& (\partial_\mu T^{\mu\nu} + \Gamma^\mu_{\gamma\mu} T^{\gamma\nu} + \Gamma^\nu_{\gamma\mu} T^{\mu\gamma}) \hat{e}_\nu \\ =& \left[\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + \Gamma^\nu_{\gamma\mu} T^{\mu\gamma} \right] \hat{e}_\nu \end{aligned} \tag{59}$$

If $T^{\mu\nu} = -T^{\nu\mu}$ then

$$\nabla \cdot \mathbf{T} = \left[\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) \right] \hat{e}_\nu, \tag{60}$$

as expected. In other words, the familiar relations that are employed for covariant derivative or divergence of a tensor are obtained in a coordinate basis.

Example: Cylindrical coordinate and noncoordinate bases in flat 3-space

As an example of the above discussions in a curvilinear coordinate, we introduce cylindrical coordinate and noncoordinate bases in flat three-dimensional Euclidean space. Using ρ, ϕ and z as the coordinates, the line element of such a space is given by

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \tag{61}$$

Then by using equation (51a) we obtain the tetrad components

$$h_1^{(1)} = 1 \ \ h_2^{(2)} = \rho \ \ h_3^{(3)} = 1 \ \textit{and} \ h_i^{(j)} = 1 \ \textit{if} \ i \neq j \tag{62}$$

and their inverses

$$h_{(1)}{}^1 = 1 \ \ h_{(2)}{}^2 = 1/\rho \ \ h_{(3)}{}^3 = 1 \ \textit{and} \ h_{(i)}{}^j = 1 \ \textit{if} \ i \neq j. \tag{63}$$

Now the holonomic basis vectors are chosen to be

$$\hat{e}_1 = \hat{\rho} \ \ \hat{e}_2 = \hat{\phi} \ \ \hat{e}_3 = \hat{z} \tag{64}$$

and a vector $\mathbf{V}$ is represented by

$$\mathbf{V} = V^1 \hat{e}_1 + V^2 \hat{e}_2 + V^3 \hat{e}_3. \tag{65}$$

On the other hand, the noncoordinate basis vectors are given by,

$$\hat{\epsilon}_{(1)} = \hat{\rho} \ \ \hat{\epsilon}_{(2)} = \frac{\hat{\phi}}{\rho} \ \ \hat{\epsilon}_{(3)} = \hat{z} \tag{66}$$

and the contravariant vector components are

$$V^{(1)} \equiv V_\rho \ \ V^{(2)} \equiv V_\phi \ \ V^{(3)} \equiv V_Z. \tag{67}$$

Now using the above relations and employing (53a), (63) and (67) we have,

$$V^1 = V^{(1)} = V_\rho \ \ V^2 = \frac{1}{\rho}(V^{(2)}) = \frac{V_\phi}{\rho} \ \ V^3 = V^{(3)} = V_z \tag{68}$$

so that according to (56) the divergence of a vector is given by

$$\begin{aligned} \nabla \cdot \mathbf{V} =& \partial_\rho (V_\rho) + \partial_\phi \left(\frac{V_\phi}{\rho} \right) + \partial_z (V_z) + V_\rho \Gamma^i_{i1} \\ &+ \left(\frac{V_\phi}{\rho} \right) \Gamma^i_{i2} + V_z \Gamma^i_{i3} \\ =& \frac{1}{\rho} \partial_\rho (\rho V_\rho) + \frac{1}{\rho} \partial_\phi (V_\phi) + \partial_z (V_z) \end{aligned} \tag{69}$$

Appendix B: 1+3 (threading) formulation of spacetime decomposition (Landau formalism)

To define spatial metric and spatial distances in a given spacetime (metric), we choose the $1+3$ (threading) formulation of spacetime decomposition. Unlike the $3+1$ (or foliation) formulation of spacetime decomposition in which spacetime is foliated into constant-time hypersurfaces, in the $1+3$ formulation, it is decomposed into threads tracking history of each spatial point. This formulation of spacetime decomposition starts from the following general form for the spacetime metric [24]

$$\begin{aligned} ds^2 =& d\tau^2_{syn} - dl^2 = g_{00}(dx^0 - A_{g_i} dx^i)^2 \\ &- \gamma_{ij} dx^i dx^j \ \ ; \ \ i,j = 1,2,3 \end{aligned} \tag{70}$$

in which $d\tau_{syn} = \sqrt{g_{00}}(dx^0 - A_{g_i} dx^i)$ is the synchronized proper time, $A_{g_i} = -\frac{g_{0i}}{g_{00}}$ is the so-called gravitomagnetic potential and

$$dl^2 = \gamma_{ij} dx^i dx^j = \left(-g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}} \right) dx^i dx^j \tag{71}$$

is the spatial line element (also called the radar distance element) of the 3-space in terms of its three-dimensional spatial metric γ_{ij}. It is the integral of this line element between two spatial points which gives the spatial distance between two events,

$$L = \int_{x_I}^{x_F} dl \tag{72}$$

The 3-velocity is defined in terms of the synchronized proper time as follows

$$v^i = \frac{dx^i}{d\tau_{syn}} = \frac{cdx^i}{\sqrt{g_{00}}(dx^0 - A_i dx^i)}. \tag{73}$$

where now using (70) and (73) the spacetime line element could be written as follows

$$ds^2 = d\tau_{syn}^2\left(1 - \frac{v^2}{c^2}\right). \tag{74}$$

Now the components of the 4-velocity $u^\alpha = \frac{dx^\alpha}{ds}$, in terms of the components of the 3-velocity are given by

$$u^0 = \frac{1}{\sqrt{g_{00}}\sqrt{1-\frac{v^2}{c^2}}} + \frac{A_i v^i}{c\sqrt{1-\frac{v^2}{c^2}}}; \quad u^i = \frac{v^i}{c\sqrt{1-\frac{v^2}{c^2}}}, \tag{75}$$

where in the comoving frame, $v^i = 0$, it reduces to $u^\alpha = (\frac{1}{\sqrt{g_{00}}}, 0, 0, 0)$ as expected. It is this same formulation of spacetime decomposition which allows one to use the analogy with electromagnetism and define the so called *gravitoelectric* and *gravitomagnetic* fields as follows;

$$\mathbf{E}_g = -\frac{1}{2g_{00}}\nabla g_{00} \;\; \mathbf{B}_g = \nabla \times \mathbf{A}_g. \tag{76}$$

In terms of the above fields and in the context of the so-called *Gravitoelectromagnetism*, vacuum Einstein field equations could be rewritten in the following *quasi-Maxwell* form [25]

$$\nabla \times \mathbf{E}_g = 0, \;\; \nabla \cdot \mathbf{B}_g = 0 \tag{77}$$

$$\nabla \cdot \mathbf{E}_g = 1/2hB_g^2 + E_g^2 \tag{78}$$

$$\nabla \times (\sqrt{h}\mathbf{B}_g) = 2\mathbf{E}_g \times (\sqrt{h}\mathbf{B}_g)\mathbf{v}] \tag{79}$$

$${}^{(3)}R^{ij} = -E_g^{i;j} + \frac{1}{2}h(B_g^i B_g^j - B_g^2\gamma^{ij}) + E_g^i E_g^j. \tag{80}$$

where ${}^{(3)}R^{ij}$ is the 3-dimensional Ricci tensor of the 3-space constructed from the three-dimensional metric $\gamma_{\alpha\beta}$ in the same way that the usual 4-dimensional Ricci tensor $R^{\mu\nu}$ is made out of $g_{\mu\nu}$.

It should also be noted that in the above equations, all the differential operations are defined in the 3-space with metric γ_{ij} [24, 25], in particular, divergence and curl are defined as follows:

$$div\mathbf{V} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}(\sqrt{\gamma}\,V^i), \;\; (curl\mathbf{V})^i = \frac{1}{2\sqrt{\gamma}}\epsilon^{ijk}\left(\frac{\partial V_k}{\partial x^j} - \frac{\partial V_j}{\partial x^k}\right) \tag{81}$$

in which $\gamma = det\,\gamma_{ij}$ and one can show that

$$-g = g_{00}\gamma. \tag{82}$$

References

[1] G F T del Castillo and J Mercado-Perez *J. Math. Phys.* **40** 2882 (1999)
[2] G W Gibbons and M C Werner *Universe* **5** 88 (2019)
[3] U Leonhardt and T G Philbin *New J. Phys.* **8** 247 (2006)
[4] L Schiff *Proc. Nat. Acad. Sci.* **25** 391 (1939)
[5] J Ise and L Uretsky *Am. J. Phys.* **26** 431 (1958); Erratum: **29** 328 (1961)
[6] G E Modesitt *Am. J. Phys.* **38** 1487 (1970)
[7] H A Atwater *Nature* **228** 272 (1970); H A Atwater and T Shiozawa *Proc. IEEE* **63** 316 (1975)
[8] J F Corum *J. Math. Phys.* **21** 2360 (1980)
[9] G N Pellegrini and A R Swift *Am. J. Phys.* **63** 694 (1995)
[10] P Hillion *Turk J. Elec. Eng.* **18** 281 (2010)
[11] J C Hauck and B Mashhoon *Ann. Phys.* **12** 275 (2003)
[12] R D Klauber *Foundations of Physics* **37** 198 (2007)
[13] R D Klauber *Foundations of Physics Letters* **11** 405 (1998)
[14] R D Klauber in *Relativity in Rotating Frames: Relativistic Physics in Rotating Reference Frames* (*ed.*) G Rizzi and M L Ruggiero (Kluwer academic publishers) 2004
[15] Tse Chin Mo *J. Math. Phys.* **11** 2589 (1970)
[16] J Van Bladel *Proc. IEEE* **64** 301 (1976)
[17] J Van Bladel *Proc. IEEE* **67** 1654 (1979)
[18] J Van Bladel *Relativity and Engineering* (Berlin: Springer) Ch 9 (1984); J Van Bladel *Electromagnetic Fields* 2nd edn. (John-Wiley and Sons) Ch 17 (2007)
[19] W M Irvine *Physica* **30** 1160 (1964)
[20] S Weinberg *Gravitation and Cosmology: principles and applications of the general theory of relativity* (John wiley and Sons) (1972)
[21] Ehlers *General Relativity and Gravitation* **25** 1225 (1993)
[22] G F R Ellis R Maartens and M Maccallum *Relativistic cosmology* (Cambridge University Press) (2012); G F R Ellis *Cargese Lectures in Physics* **Vol VI** (*ed.*) E Schatzman (New York: Gordon and Breach) p 1 (1973) ; G F R Ellis *Relativistic Cosmology* in *General Relativity and Cosmology Proceedings of the XLVII Enrico Fermi Summer School* (*ed.*) R K Sachs (New York: Academic Press) (1971)
[23] C G Tsagas *Class.Quant.Grav.* **22** 393 (2005)
[24] L D Landau and E M Lifshitz *The classical theory of fields* (Pergamon Press) (1975)
[25] D Lynden-Bell and M Nouri-Zonoz *Rev. Mod. Phys,* **70** 427 (1998)
[26] O Gron *Int. J. Theor. Phys.* **23** 441 (1984)
[27] B J Linet *J.Phys. A: Math. Gen.* **9** 1081 (1976)
[28] B J Linet *J.Phys. A: Math. Gen.* **12** 839 (1979)
[29] N Sakai and S Shibata *ApJ* **584** 427 (2003)
[30] L Rezzolla B J Ahmedov and J. C. Miller *MNRAS.* **322** 723 (2001)
[31] T Shiozawa **Proc. IEEE 61** 1694 (1973)
[32] A Georgiou **Proc. IEEE 76** 1051 (1988)
[33] M Nouri-Zonoz H Ramezani-Aval and R. Gharechahi *Eur. Phys. J. C* **74** 3098 (2014)

[34] M Nouri-Zonoz and H Ramezani-Aval *Eur. Phys. J. C* **74** 3128 (2014)
[35] L Ryder *introduction to General relativity* (New York: Cambridge University Press) (2009)
[36] S M Carroll, *Spacetime and geometry* (Addison Wesley) (2004)
[37] C W Misner K S Thorne and J A Wheeler *Gravitation* (New York: W. H. Freeman and Company) (1995)