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ORIGINAL PAPER

An efficient technique for generalized conformable Pochhammer–Chree models of longitudinal wave propagation of elastic rod

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Abstract: In this article, we introduce analytical-approximate solutions of time-fractional generalized Pochhammer-Chree equations for wave propagation of elastic rod by means of the q-homotopy analysis of the transform method (q-HATM). In the Caputo sense, basic concepts for fractional derivatives are defined. Several examples are given and the results are illustrated via some surface plots to present the physical representation. The results show that the current methodology is productive, powerful, efficient, easy to use, and ready to incorporate a wide variety of partial fractional differential equations.

Keywords: Pochhammer–Chree model; q-homotopy analysis transform method; Laplace transform; Caputo derivative

1. Introduction

The fractional analysis is defined as an enhancement of the principles of classical order integral and derivative part of a conventional investigation to fractional order. Fractional analysis has been widely applied over the last century in the fields of engineering, physics, and biology in association with mathematics. The key explanation for this is that many phenomena such as propagation and wave motion, chaos, viscoelasticity and damping, filtering and irreversibility, controller architecture can be modeled and explained more specifically using fractional analysis. As a consequence, nonlinear partial fractional differential equations (NPFDEs) have been inspired by many scientists and researchers in recent years and have been thoroughly researched and applied in various branches of science and engineering for many real-life problems. As a result, many science and engineering applications can be found and applied via fractional NPFDEs [1–8]. Consequently, in these listed areas, the discovery of numerical and analytical-approximate solutions of NPFDEs has a hugely significant and special role. The most significant distinction between classical and fractional analysis is that there is no single derivative concept as in classical analysis. The occurrence of diverse derivative description with fractional calculus (FC) opens the door for considering the most suitable one for the model's form and thus obtaining the best solution to the problem. We have diverse and pioneering notions for FC and while there are transitions between them, they vary in terms of their meanings and their physical representations of their definitions. In this article, we preferred to apply Caputo derivative definition, which is the most popular one, for its practicality and compatibility with classical initial conditions.

Some of the classical approximate and analytical techniques for fractional differential equations (FDEs) have been developed to date are optimal homotopy asymptotic method that was implemented to Klein-Gordon, diffusion, wave-diffusion and telegraph equations [9, 10], differential transform method for system of equations and finance equation [11, 12], homotopy perturbation method for system of partial equations and Riccati equation [13, 14], finite difference method for Burger–Fisher equation [15], homotopy analysis method for Burgers–Korteweg–de Vries equation and Whitham–Broer–Kaup equation

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[16, 17], reproducing kernel Hilbert space method for linear and nonlinear partial and Robin equations [18, 19], residual power series method for Burgers–Kadomtsev–Petviashvili and Burgers' type equations [20, 21], q-homotopy analysis method for Lax's Korteweg–de Vries and Sawada–Kotera equations and Noyes–Field model [22, 23] and so on.

In this article, our goal is to study the Pochhammer–Chree (PC) equation using the q-homotopy analysis transform method (q-HATM) [26–32]. The PC model is given as

$$\mathcal{D}_t^{\mu} u(x,t) = \frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1\, u(x,t) + \beta_2\, u^{m+1}(x,t) + \beta_3\, u^{2m+1}(x,t)\right), \quad 1 < \mu \le 2, \ \ t > 0, \tag{1}$$

where m is a positive number and β_1, β_2, and β_3 are constants. This is a model of an elastic rod's longitudinal vibrations. It is a simplified version of the well-known equation called the regularized Boussinesq equations, an important model that basically explains the propagation of long waves in regimes where water is shallow and waves, like the other Boussinesq equations, have small amplitudes. The (q-HATM is the result of combining the (q-HAM and the Laplace transform method. This method employs two convergence control parameters, n and $\hbar$, to help modify and regulate the solution's convergence region. Many scholars have recently used the proposed approach to find solutions for differential equations illustrating various models and phenomena associated with fractional calculus and have presented numerical simulations to verify the accuracy of the method.

The paper structure is proposed as follows. The basic theory of fractional calculus is presented in Section 2. The projected algorithm for the considered equation is devoted to Section 3. Solutions for the considered equation are presented in Section 4. Finally, the paper is ended with a conclusion section.

2. Preliminaries

We recall some essential notions of FC and Laplace transform.

Definition 1 The Riemann–Liouville fractional integral of $f(t) \in C_\delta (\delta \ge -1)$ is defined as

$$J^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\vartheta)^{\mu-1} f(\vartheta)\mathrm{d}\vartheta. \tag{2}$$

Definition 2 In the Caputo sense, the fractional derivative of $f \in C_{-1}^{\eta}$ is presented as

$$\mathcal{D}_t^{\mu} f(t) = \begin{cases} \frac{d^{\eta} f(t)}{dt^{\eta}}, & \mu = \eta, \\ \frac{1}{\Gamma(\eta-\mu)} \int_0^t (t-\vartheta)^{\eta-\mu-1} f^{(\eta)}(\vartheta)\mathrm{d}\vartheta, & \eta - 1 < \mu < \eta, \quad \eta \in \mathbb{N}. \end{cases} \tag{3}$$

Definition 3 The Laplace transform (LT) of $\mathcal{D}_t^{\mu} f(t)$ is given by

$$\mathscr{L}\left[\mathcal{D}_t^{\mu} f(t)\right] = s^{\mu} F(s) - \sum_{r=0}^{\eta-1} s^{\mu-r-1} f^{(r)}(0^+), \quad \eta - 1 < \mu \le \eta, \tag{4}$$

where $F(s)$ is symbolize the LT of $f(t)$.

3. Projected algorithm for considered equation

Here, we illustrate procedure of hired scheme [33–37] for P-C model defined in Eq. (1) for the case when $n = 1$ given as

$$\mathcal{D}_t^{\mu} u(x,t) = \frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1 u(x,t) + \beta_2 u^2(x,t) + \beta_3 u^3(x,t)\right), \ t > 0, \ \ 1 < \mu \le 2, \tag{5}$$

subjected to

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x), \tag{6}$$

where $\mathcal{D}_t^{\mu} u(x,t)$ represents the Caputo fractional derivative of $u(x,t)$. Here, $u(x,t)$ is a bounded function. Now, we obtain the following equation by applying LT on Eq. (5)

$$\mathscr{L}\left[u(x,t)\right] - \frac{1}{s}\left[f(x)\right] - \frac{1}{s^2}\left[g(x)\right] - \frac{1}{s^{\mu}} \mathscr{L}\left\{ \frac{\partial^4 u}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1 u(x,t) + \beta_2 u^2(x,t) + \beta_3 u^3(x,t)\right)\right\} = 0. \tag{7}$$

The nonlinear is contracted as follows:

$$\mathcal{N}[\varphi] = \mathscr{L}[\varphi] - \frac{1}{s}\left[f(x)\right] - \frac{1}{s^2}\left[g(x)\right] - \frac{1}{s^{\mu}} \mathscr{L}\left\{ \frac{\partial^4 \varphi}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1 \varphi + \beta_2 \varphi^2 + \beta_3 \varphi^3\right)\right\}, \tag{8}$$

where $q \in \left[0, \frac{1}{n}\right]$. Then, the homotopy is construct as

$$(1 - nq)\mathscr{L}\left[\varphi - u_0(x,t)\right] = \hbar q \mathcal{N}[\varphi], \tag{9}$$

where $\mathscr{L}$ is signifying LT. For $q = 0$ and $q = \frac{1}{n}$, the following conditions satisfies;

$$\varphi(x, t; 0) = u_0(x, t) \qquad \varphi\left(x, t; \frac{1}{n}\right) = u(x, t). \tag{10}$$

By using Taylor theorem, we consider

$$u(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m, \tag{11}$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m}|_{q=0}. \tag{12}$$

For the proper chaise of $u_0(x, t)$, n and $\hbar$, the series Eq. (12) converges at $q = 1/n$. Then

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m. \tag{13}$$

After differentiating Eq. (13) m-times with q and multiplying by 1/m! and substituting $q = 0$, one can get

$$\mathscr{L}\left[u_m(x, t) - k_m u_{m-1}(x, t)\right] = \hbar \mathcal{R}_m(\overrightarrow{u}_{m-1}), \tag{14}$$

with

$$\overrightarrow{u}_m = \left\{u_0, u_1, \ldots, u_m\right\}. \tag{15}$$

Eq. (12) reduces after employing inverse LT to

$$u_m(x, t) = k_m u_{m-1}(x, t) + \hbar \mathscr{L}^{-1}\left[\mathcal{R}_m(\overrightarrow{u}_{m-1})\right], \tag{16}$$

where

$$\begin{aligned} \mathcal{R}_m(\overrightarrow{u}_{m-1}) =& \mathscr{L}\left[u_{m-1}(x, t)\right] - \left(1 - \frac{k_m}{n}\right)\left[\frac{1}{s}\{f(x)\} + \frac{1}{s^2}\{g(x\}\right] \\ &- \frac{1}{s^\mu} \mathscr{L}\left\{\frac{\partial^4 u_{m-1}}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1 u_{m-1}\right.\right. \\ &\left.\left. + \beta_2 \sum_{j=1}^{m-1} u_j u_{m-1-j} + \beta_3 \sum_{j=1}^{m-1} \sum_{i=1}^{j} u_i u_{j-i} u_{m-1-j}\right)\right\}, \end{aligned} \tag{17}$$

and

$$k_m = \begin{cases} 0 & m \leqslant 1, \\ n & otherwise. \end{cases} \tag{18}$$

From Eq. (16), in addition to Eqs. 17 and 18, we have

$$\begin{aligned} u_m =& (k_m + \hbar) u_{m-1} - \hbar\left(1 - \frac{k_m}{n}\right) u_0(x, t) \\ &- \hbar \mathscr{L}^{-1}\left[\frac{1}{s^\mu} \mathscr{L}\left\{\frac{\partial^4 u_{m-1}}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1 u_{m-1}\right.\right.\right. \\ &\left.\left.\left. + \beta_2 \sum_{j=1}^{m-1} u_j u_{m-1-j} + \beta_3 \sum_{j=1}^{m-1} \sum_{i=1}^{j} u_i u_{j-i} u_{m-1-j}\right)\right\}\right]. \end{aligned} \tag{19}$$

Here,

$$u_0(x, t) = \mathscr{L}^{-1}\left[\frac{1}{s}\{f(x)\} + \frac{1}{s^2}\{g(x\}\right]. \tag{20}$$

Finally, the q-HATM solution is

$$u(x, t) = \sum_{m=0}^{\infty} u_m \left(\frac{1}{n}\right)^m. \tag{21}$$

Here, we illustrate the convergence analysis of considered scheme for projected model.

Theorem 3.1 (*Uniqueness theorem*) *By the help of considered scheme, the achieved solution for the PC Eq.* (5) *is unique wherever* $0 < \lambda < 1$, *where* $\lambda = (k_m + \hbar) + \hbar\{\delta_1^4 + \delta_2^2(\beta_1 + \beta_2(P + Q) + \beta_3(P^2 + Q^2 + PQ))\}\mathcal{T}$.

Proof The solution for consider equation defined in Eq. (5) is presented as

$$u(x, t) = \sum_{m=0}^{\infty} u_m \left(\frac{1}{n}\right)^m, \tag{22}$$

where u_m is defined in Eq. (19). If possible, let u and w be the two distinct solutions for Eq. (5) such that $|u| \leq P$ and $|w| \leq Q$, then

$$\begin{aligned} |u - w| =& \left|(k_m + \hbar)(u - w) - \hbar \mathscr{L}^{-1}\left\{\frac{1}{s^\mu} \mathscr{L}\left[\frac{\partial^4 u}{\partial t^2 \partial x^2} - \frac{\partial^4 w}{\partial t^2 \partial x^2}\right.\right.\right. \\ &\left.\left.\left. + \frac{\partial^2}{\partial x^2}\left(\beta_1(u - w) + \beta_2(u^2 - w^2) + \beta_3(u^3 - w^3)\right)\right]\right\}\right|. \end{aligned} \tag{23}$$

Now, we get the following result by applying convolution theorem for LT

$$\begin{aligned} |u - w| =& (k_m + \hbar)|u - w| + \hbar \int_0^t \left(\left|\frac{\partial^4 u}{\partial t^2 \partial x^2} - \frac{\partial^4 w}{\partial t^2 \partial x^2}\right.\right. \\ &\left.\left. + \frac{\partial^2}{\partial x^2}\left(\beta_1(u - w) + \beta_2(u^2 - w^2) + \beta_3(u^3 - w^3)\right)\right|\right) \frac{(t - \xi)^\mu}{\Gamma(\mu + 1)} d\xi \\ \leq& (k_m + \hbar)|u - w| + \hbar \int_0^t \left(\frac{\partial^4}{\partial t^2 \partial x^2}|u - w| + \frac{\partial^2}{\partial x^2}\left(\beta_1|u - w|\right.\right. \\ &\left.\left. + \beta_2|(u - w)(u + w)| + \beta_3|(u - w)(u^2 + uw + w^2)|\right.\right. \frac{(t - \xi)^\mu}{\Gamma(\mu + 1)} d\xi \\ \leq& (k_m + \hbar)|u - w| + \hbar \int_0^t \left(\delta_1^4|u - w| + \delta_2^2\{\beta_1|u - w|\right. \\ &\left. + \beta_2|u - w|(P + Q) + \beta_3|u - w|(P^2 + PQ + Q^2)\}\right) \frac{(t - \xi)^\mu}{\Gamma(\mu + 1)} d\xi, \end{aligned}$$

where $\delta_1^4 = \frac{\partial^4}{\partial t^2 \partial x^2}$ and $\delta_2^2 = \frac{\partial^2}{\partial x^2}$. By the help of integral mean value, the above equation reduces to

$$\begin{aligned} |u - w| \leq& |u - w|\left((k_m + \hbar) + \hbar\{\delta_1^4 + \delta_2^2(\beta_1 + \beta_2(P + Q)\right. \\ &\left. + \beta_3(P^2 + PQ + Q^2))\}\mathcal{T}\right) \leq |u - w|\lambda. \end{aligned}$$

Thus, $(1-\lambda)|u-w|\leq 0$. Since $0<\lambda<1$, we conclude that $|u-w|=0\Rightarrow u=w$. This completes the required result. $\square$

Theorem 3.2 (*Convergence theorem*) *Suppose $\wp$ is a Banach space and $F:\wp\to\wp$ is mapping. Suppose for $\forall u,v\in\wp$*

$$\|F(u)-F(v)\|\leq\lambda\|u-v\|, \tag{24}$$

then there is a fixed point for F [24, 25]. Further, the sequence converges to fixed point of F with an arbitrary choice of $u_0, v_0\in\wp$ and

$$\|u_m-u_n\|\leq\frac{\lambda^n}{1-\lambda}\|u_1-u_0\|. \tag{25}$$

Proof Let $(\wp[J],\|\cdot\|)$ be a Banach space of J with $\|g(t)\|=\max_{t\in J}|g(t)|$. First, we prove $\{u_n\}$ is Cauchy sequence in $(\wp[J],\|\cdot\|)$. Now, consider

$$\begin{aligned}
\|u_m-u_n\| &= \max_{t\in J}|u_m-u_n|\\
&= \max_{t\in J}\Bigg|(k_m+\hbar)(u_{m-1}-u_{n-1})-\hbar\mathscr{L}^{-1}\Bigg[\frac{1}{s^\mu}\mathscr{L}\Bigg[\frac{\partial^4u_{m-1}}{\partial t^2\partial x^2}\\
&\quad-\frac{\partial^4u_{n-1}}{\partial t^2\partial x^2}+\frac{\partial^2}{\partial x^2}\left(\beta_1(u_{m-1}-u_{n-1})+\beta_2(u_{m-1}^2-u_{n-1}^2)\right.\\
&\quad\left.+\beta_3(u_{m-1}^3-u_{n-1}^3)\right)\Bigg]\Bigg]\Bigg|\\
&\leq\max_{t\in J}\Bigg\{(k_m+\hbar)|u_{m-1}-u_{n-1}|-\hbar\mathscr{L}^{-1}\Bigg[\frac{1}{s^\mu}\mathscr{L}\Bigg[\Bigg|\frac{\partial^4u_{m-1}}{\partial t^2\partial x^2}\\
&\quad-\frac{\partial^4u_{n-1}}{\partial t^2\partial x^2}\Bigg|+\frac{\partial^2}{\partial x^2}\left(\beta_1|u_{m-1}-u_{n-1}|+\beta_2|u_{m-1}^2-u_{n-1}^2|\right.\\
&\quad\left.+\beta_3|u_{m-1}^3-u_{n-1}^3|\right)\Bigg]\Bigg]\Bigg\}.
\end{aligned} \tag{26}$$

With the help of convolution theorem for LT, we obtain

$$\begin{aligned}
\|u_m-u_n\| &\leq\max_{t\in J}\Bigg\{(k_m+\hbar)|(u_{m-1}-u_{n-1})|\\
&\quad-\hbar\int_0^t\left(\left|\frac{\partial^4u_{m-1}}{\partial t^2\partial x^2}-\frac{\partial^4u_{n-1}}{\partial t^2\partial x^2}\right|+\frac{\partial^2}{\partial x^2}\left(\beta_1|u_{m-1}-u_{n-1}|\right.\right.\\
&\quad\left.\left.+\beta_2|u_{m-1}^2-u_{n-1}^2|+\beta_3|u_{m-1}^3-u_{n-1}^3|\right)\right\}\frac{(t-\xi)^\mu}{\Gamma(\mu+1)}\mathrm{d}\xi\\
&\leq\max_{t\in J}\Bigg\{(k_m+\hbar)|u_{m-1}-u_{n-1}|-\hbar\int_0^t\left(\delta_1^4|u_{m-1}-u_{n-1}|\right.\\
&\quad+\delta_2^2\left(\beta_1|u_{m-1}-u_{n-1}|+\beta_2|u_{m-1}-u_{n-1}|(P+Q)\right.\\
&\quad\left.\left.+\beta_3|u_{m-1}-u_{n-1}|(P^2+PQ+Q^2)\right)\right\}\frac{(t-\xi)^\mu}{\Gamma(\mu+1)}\mathrm{d}\xi.
\end{aligned} \tag{27}$$

By using integral mean value theorem [24, 25], the Eq. (27) simplifies to

$$\begin{aligned}
\|u_m-u_n\| &\leq\max_{t\in J}\Big\{(k_m+\hbar)|u_{m-1}-u_{n-1}|\\
&\quad+\hbar(\delta_1^4|u_{m-1}-u_{n-1}|+\delta_2^2\left(\beta_1|u_{m-1}-u_{n-1}|\right.\\
&\quad+\beta_2|u_{m-1}-u_{n-1}|(P+Q)+\beta_3|u_{m-1}\\
&\quad\left.-u_{n-1}|(P^2+PQ+Q^2)\right)\mathcal{T}\Big\}\\
\|u_m-u_n\| &\leq\lambda\|u_{m-1}-u_{n-1}\|.
\end{aligned} \tag{28}$$

For $m=n+1$;

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leq\lambda\|u_n-u_{n-1}\|\leq\lambda^2\|u_{n-1}-u_{n-2}\|\leq\cdots\\
&\leq\lambda^n\|u_1-u_0\|.
\end{aligned} \tag{29}$$

On using triangular inequality, we have

$$\begin{aligned}
\|u_m-u_n\| &\leq\|u_{n+1}-u_n\|+\|u_{n+2}-u_{n+1}\|+\ldots+\|u_m-u_{m-1}\|\\
&\leq\left(\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-1}\right)\|u_1-u_0\|\\
&\leq\lambda^n\left(1+\lambda+\lambda^2+\cdots+\lambda^{m-n-1}\right)\|u_1-u_0\|\\
&\leq\lambda^n\left(\frac{1-\lambda^{m-n-1}}{1-\lambda}\right)\|u_1-u_0\|.
\end{aligned} \tag{30}$$

Since $0<\lambda<1$, we have $1-\lambda^{m-n-1}<1$, then

$$\|u_m-u_n\|\leq\frac{\lambda^n}{1-\lambda}\|u_1-u_0\|. \tag{31}$$

But $\|u_1-u_0\|<\infty$ consequently as $m\to\infty$ than $\|u_m-u_n\|\to 0$, and which gives $\{u_n\}$ is Cauchy sequence in $\wp[J]$. Hence, $\{u_n\}$ is convergent sequence which completes the required result. $\square$

4. Solutions for the considered equation

Here, we consider three cases of Eq. (5) to confirm the applicability and efficiency of the hired algorithm.

Example 1 Consider Eq. (5) at $\beta_1\neq 0$, $\beta_2\neq 0$ and $\beta_3=0$. Then, we have

$$\mathcal{D}_t^\mu u=\frac{\partial^4u}{\partial t^2\partial x^2}+\frac{\partial^2}{\partial x^2}\left(\beta_1u+\beta_2u^2\right), \tag{32}$$

subjected to

$$u(x,0)=f(x)=-\frac{2\beta_1}{\beta_2}\,\mathrm{sech}^2(x), \tag{33}$$

$$u_t(x,0)=g(x)=\frac{4\sqrt{-\beta_1}\beta_1}{\sqrt{3}\beta_2}\tanh(x)\,\mathrm{sech}^2(x). \tag{34}$$

By the help of projected algorithm, we have

$$u_m(x,t) = (k_m+\hbar)u_{m-1} - \hbar\left(1-\frac{k_m}{n}\right)u_0(x,t) - \hbar\mathscr{L}^{-1}\left[\frac{1}{s^{\mu}}\mathscr{L}\left\{\frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2} + \beta_1\frac{\partial^2 u_{m-1}}{\partial x^2} + \beta_2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{m-1}u_i u_{m-1-i}\right)\right\}\right], \tag{35}$$

where

$$u_0(x,t) = -\frac{2\beta_1}{\beta_2}\,\mathrm{sech}^2(x) + \frac{4\beta_1 t}{\beta_2}\sqrt{-\frac{\beta_1}{3}}\tanh(x)\,\mathrm{sech}^2(x). \tag{36}$$

From Eqs. 35 and 36, we have

$$\begin{aligned} u_1(x,t) = &\frac{\hbar\, t^{\mu}\beta_2^3}{(972\beta_3^{\frac{7}{2}}(9\beta_1\beta_3-2\beta_2^2))\Gamma(\alpha+1)}\,\mathrm{sech}^4\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\\ &\times\left(i\sqrt{2}\beta_2^5 t^3\left(3\cosh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)-4\right)\mathrm{sech}^4\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right.\\ &+6\sqrt{\beta_3}\beta_2^4 t^2\left(13\sinh\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)-2\sinh\left(3\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right)\\ &\times\mathrm{sech}^3\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)-54i\sqrt{2}\beta_1\beta_2\beta_3^2 t\left(\cosh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)-2\right)\\ &+36i\sqrt{2}\beta_2^3\beta_3 t\left(2\cosh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)-3\right)\mathrm{sech}^2\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\\ &\left.-162\beta_1\beta_3^{\frac{5}{2}}\sinh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)+216\beta_2^2\beta_3^{\frac{3}{2}}\tanh\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right). \end{aligned} \tag{37}$$

With the help of Eq. (35) other terms can be achieve. The analytical result for the corresponding equation is

$$u(x,t) = -\frac{2\beta_1}{\beta_2}\sec h^2\left(x-\sqrt{-\frac{\beta_1}{3}}t\right). \tag{38}$$

Example 2 Consider Eq. (5) at $\beta_1 \neq 0$, $\beta_2 = 0$ and $\beta_3 \neq 0$ and then we have

$$D_t^{\mu}u(x,t) = \frac{\partial^4 u}{\partial t^2\partial x^2} + \frac{\partial^2}{\partial x^2}\left(\beta_1 u + \beta_3 u^3\right), \tag{39}$$

subjected to the initial conditions

$$u(x,0) = f(x) = \frac{\sqrt{2\beta_1}}{\sqrt{-\beta_3}(x+\phi)} \text{ and } u_t(x,0) = \frac{\sqrt{2}\beta_1}{\sqrt{-\beta_3}(x+\phi)^2}. \tag{40}$$

By the help of projected algorithm, we have

$$u_m(x,t) = (k_m+\hbar)u_{m-1}(x,t) - \hbar\left(1-\frac{k_m}{n}\right)u_0(x,t) - \hbar\mathscr{L}^{-1}\left[\frac{1}{s^{\mu}}\mathscr{L}\left[\frac{\partial^4}{u_{m-1}}\partial t^2\partial x^2 + \frac{\partial^2}{\partial x^2}\left(\beta_1 u_{m-1} + \beta_3\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}u_j u_{i-j}\right)u_{m-1-i}\right)\right]\right], \tag{41}$$

where

$$u_0(x,t) = \frac{\sqrt{2\beta_1}}{\sqrt{-\beta_3}(x+\phi)} + t\frac{\sqrt{2}\beta_1}{\sqrt{-\beta_3}(x+\phi)^2}. \tag{42}$$

On simplifying the forgoing equations, we get

$$\begin{aligned} u_1(x,t) = &-\frac{\hbar\, t^{\alpha}}{\Gamma(\alpha+1)(\sqrt{-\beta_3}(x+\phi)^8)}\Big(2\sqrt{2}\beta_1^{\frac{3}{2}}(-42\beta_1^{\frac{3}{2}}t^3 - 90\beta_1 t^2(x+\phi)\\ &+3\sqrt{\beta_1}t(x+\phi)^2(x^2+2x\phi+\phi^2-20)+(x+\phi)^3(x^2+2x\phi+\phi^2-12))\Big), \end{aligned} \tag{43}$$

$$\begin{aligned} u_2(x,t) = &(n+\hbar)u_1(x,t) + \frac{24\sqrt{2}\beta_1^{\frac{3}{2}}\hbar^2 t^{2\alpha-2}}{\sqrt{-\beta_3}\Gamma(2\alpha+1)(x+\phi)^{14}}\big(3276\beta_1^{\frac{7}{2}}t^7 + 11484\beta_1^3 t^6(x+\phi)\\ &-30\beta_1^{\frac{5}{2}}t^5(x+\phi)^2(12x^2+24x\phi+12\phi^2-517)-2\beta_1^2 t^4(x+\phi)^3(308x^2+616x\phi\\ &+308\phi^2-4995)+\beta_1^{\frac{3}{2}}t^3(x+\phi)^4(-252\mu^2-1260\mu+5x^4+20x^3\phi+15x^2(2\phi^2-21)\\ &+10x\phi(2\phi^2-63)+5\phi^4-315\phi^2+1512)+\beta_1 t^2(x+\phi)^5(-420\alpha^2-1260\alpha+x^4+4x^3\phi\\ &+x^2(6\phi^2-45)+x(4\phi^3-90\phi)+\phi^4-45\phi^2-504)+5\alpha(\mu+1)\sqrt{\beta_1}t(x+\phi)^6(x^2+2x\phi\\ &+\phi^2-42)+(\mu-1)\mu(x+\phi)^7(x^2+2x\phi+\phi^2-30)\big). \end{aligned} \tag{44}$$

Further terms can be obtained with the aid of Eq. (41). The corresponding analytical result is

$$u(x,t)=\frac{\sqrt{2\beta_1}}{\sqrt{-\beta_3}\left(x+\sqrt{\beta_1}t+\phi\right)}. \tag{45}$$

Example 3 Consider Eq.(5) at $\beta_1\neq 0, \beta_2\neq 0$ and $\beta_3\neq 0$ and then

$$D_t^{\mu}u(x,t)=\frac{\partial^4 u}{\partial t^2\partial x^2}+\frac{\partial^2}{\partial x^2}\left(\beta_1 u+\beta_2 u^2+\beta_3 u^3\right), \tag{46}$$

subjected to

$$u(x,0)=f(x)=\frac{-\beta_2}{3\beta_3}(1-\tanh\left(x\sqrt{\frac{\beta_2^2}{2(2{\beta_2}^2-9\beta_1\beta_3)}}\right) \tag{47}$$

and

$$u_t(x,0)=\frac{\beta_2^2}{9\sqrt{-2}\beta_3^{\frac{3}{2}}}\ \mathrm{sech}^2\left(x\sqrt{\frac{{\beta_2}^2}{2(2{\beta_2}^2-9\beta_1\beta_3)}}\right). \tag{48}$$

$$\begin{aligned}u_m(x,t)&=(k_m+\hbar)u_{m-1}(x,t)-\hbar\left(1-\frac{k_m}{n}\right)u_0(x,t)\\&\quad-\hbar\mathscr{L}^{-1}\left[\frac{1}{s^{\mu}}\mathscr{L}\left\{\frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2}+\frac{\partial^2}{\partial x^2}\left(\beta_1 u_{m-1}\right.\right.\right.\\&\quad\left.\left.\left.+\beta_2\sum_{j=0}^{i}u_j u_{i-j}+\beta_3\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}u_j u_{i-j}\right)u_{m-1-i}\right)\right\}\right],\end{aligned} \tag{49}$$

where

$$\begin{aligned}u_0(x,t)&=-\frac{\beta_2}{3\beta_3}\left(1-\tanh\left(x\sqrt{\frac{{\beta_2}^2}{2(2\beta_2^2-9\beta_1\beta_3)}}\right)\right)\\&\quad+t\frac{\beta_2^2}{9\sqrt{-2}\beta_3^{\frac{3}{2}}}\ \mathrm{sech}^2\left(x\sqrt{\frac{\beta_2^2}{2(2\beta_2^2-9\beta_1\beta_3)}}\right).\end{aligned} \tag{50}$$

On solving the above equations, we can obtain

$$\begin{aligned}u_1(x,t)=&\frac{\hbar\, t^{\alpha}}{\left(972\beta_3^{7/2}(9\beta_1\beta_3-2\beta_2^2)\right)\Gamma(\alpha+1)}\left(\beta_2^3\,\mathrm{sech}^4\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right.\\&\times\left(i\sqrt{2}\beta_2^5t^3\left(3\cosh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)-4\right)\mathrm{sech}^4\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right.\\&+\ 6\sqrt{\beta_3}\beta_2^4t^2\left(13\sinh\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)-2\sinh\left(3\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right)\\&\times\ \ \mathrm{sech}^3\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)-54i\sqrt{2}\beta_1\beta_2\beta_3^2t\left(\cosh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)-2\right)\\&+\ 36i\sqrt{2}\beta_2^3\beta_3t\left(2\cosh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)-3\right)\mathrm{sech}^2\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\\&\left.\left.-\ 162\beta_1\beta_3^{5/2}\sinh\left(\sqrt{2}\sqrt{\frac{\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}x\right)+216\beta_2^2\beta_3^{3/2}\tanh\left(\sqrt{\frac{\beta_2^2}{4\beta_2^2-18\beta_1\beta_3}}x\right)\right)\right).\end{aligned} \tag{51}$$

By the help of projected algorithm, we have

Further terms can be obtained with the aid of Eq. (49). The corresponding exact results is

$$u(x,t)=\frac{-\beta_2}{3\beta_3}(1-\tanh\left(\frac{1}{2}\left(\sqrt{\frac{2\beta_2^2}{2\beta_2^2-9\beta_1\beta_3}}(x-ct)\right)\right), \tag{52}$$

where $c=\frac{1}{3}\sqrt{\frac{9\beta_1\beta_3-2\beta_2^2}{\beta_3}}$.

Table 1 Numerical stimulation for Example 1 at $\hbar = -1$, $n = 1$, $\beta_1 = -1.5$, $\beta_2 = 1$, $\mu = 2$ for different x and t

x/t	0.01	0.02	0.03	0.04	0.05
0	2.39536×10^{-3}	9.52573×10^{-3}	2.12238×10^{-2}	3.72103×10^{-2}	5.70930×10^{-2}
2.5	2.60790×10^{-5}	1.02801×10^{-4}	2.27659×10^{-4}	3.97830×10^{-4}	6.10184×10^{-4}
5	3.32879×10^{-7}	1.35988×10^{-6}	3.12339×10^{-6}	5.66558×10^{-6}	9.02838×10^{-6}
7.5	2.25085×10^{-9}	9.19657×10^{-9}	2.11260×10^{-8}	3.83266×10^{-8}	6.10843×10^{-8}
10	1.51664×10^{-11}	6.19675×10^{-11}	1.42349×10^{-10}	2.58249×10^{-10}	4.11594×10^{-10}

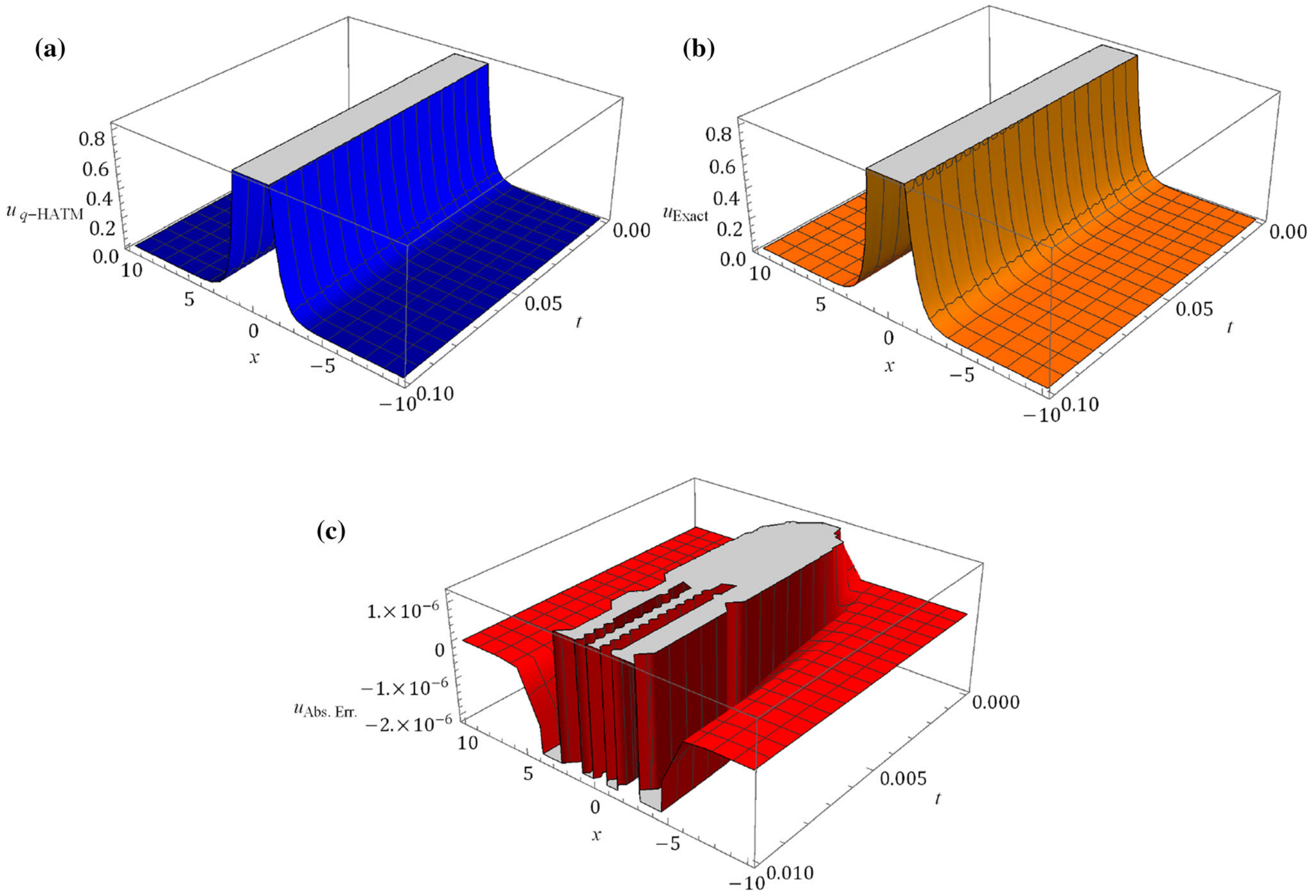

Fig. 1 Nature of (**a**) q-HATM, (**b**) exact results, and (**c**) absolute error for Example 1 at $n = 1$, $\hbar = -1$, $\beta_1 = -1.5$, $\beta_2 = 1$, and $\mu = 2$

5. Results and Discussion

In the present study, we find the solutions for the generalized Pochhammer-Chree equations with the assist of an efficient solution procedure (i.e., q-HATM) and captured corresponding consequences. The considered model illustrates the stimulating consequences of the wave propagation of elastic rod. Three cases are hired to illustrate the efficiency and applicability of the method with nonlinear differential equations without hiring any perturbation and dissertation. The numerical comparison study has been carried between exact and achieved results for Example 1 and cited in Table 1, and it confirms as space increases the accuracy of the attained series solution is also increase.

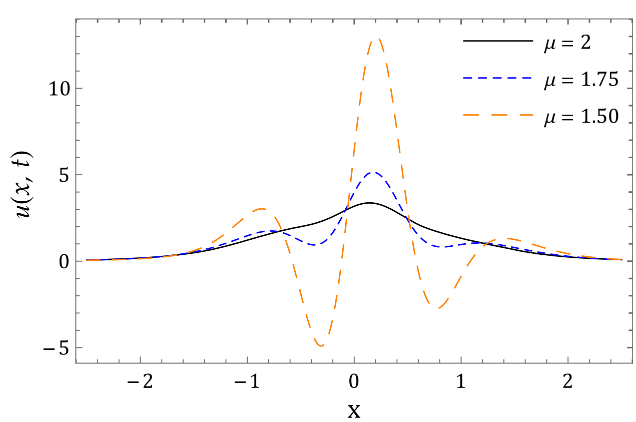

Fig. 2 Nature of achieved results for Example 1 for distinct μ at $t = 0.1$, $n = 1, \hbar = -1$, $\beta_1 = -1.5$, and $\beta_2 = 1$

Further, the nature of archived for three cases is captured for diverse fractional-order and with respect to parameters are associated with the method. For the fractional Pochhammer-Chree equation is studied in Example 1, the surfaces of the exact solution with achieved results in terms of absolute error are presented in Fig. 1 and the effect of fractional order in the obtained results, in this case, is presented in Figs. 2, 3 with the change in x. We can observe from Fig. 2 that, between the range [-1.5,2.5], nature is stimulating for fractional order ($\mu = 0.50$). For Examples 2, 3, the comparison of obtained consequences with analytical results are, respectively, cited in Figs. 4, 5,

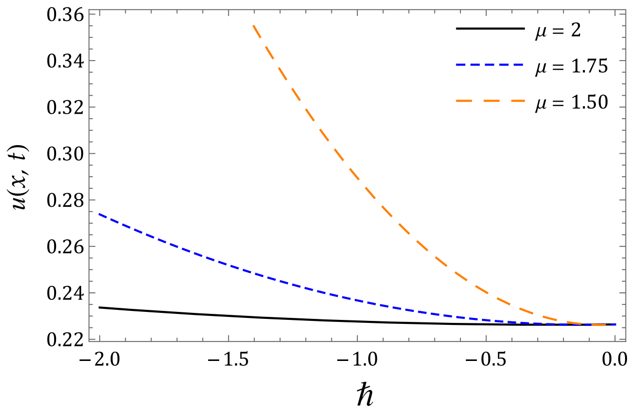

Fig. 3 $\hbar$-curves for Example 1 for different μ at $n = 1$, $\beta_1 = 1.5$, $\beta_2 = 1$, $t = 0.05$ and $x = 2$

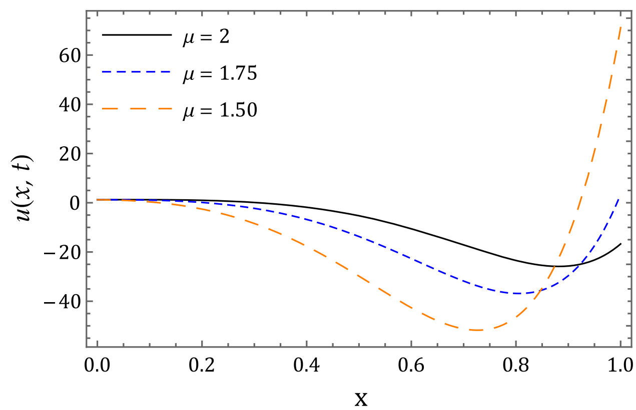

Fig. 5 Nature of achieved results for Example 2 for distinct μ at $n = 1$, $\hbar = -1$, $\beta_1 = 1.5$, $\beta_3 = -0.5$ and $\sigma = -1$, $\phi = 1$ and $x = 1$

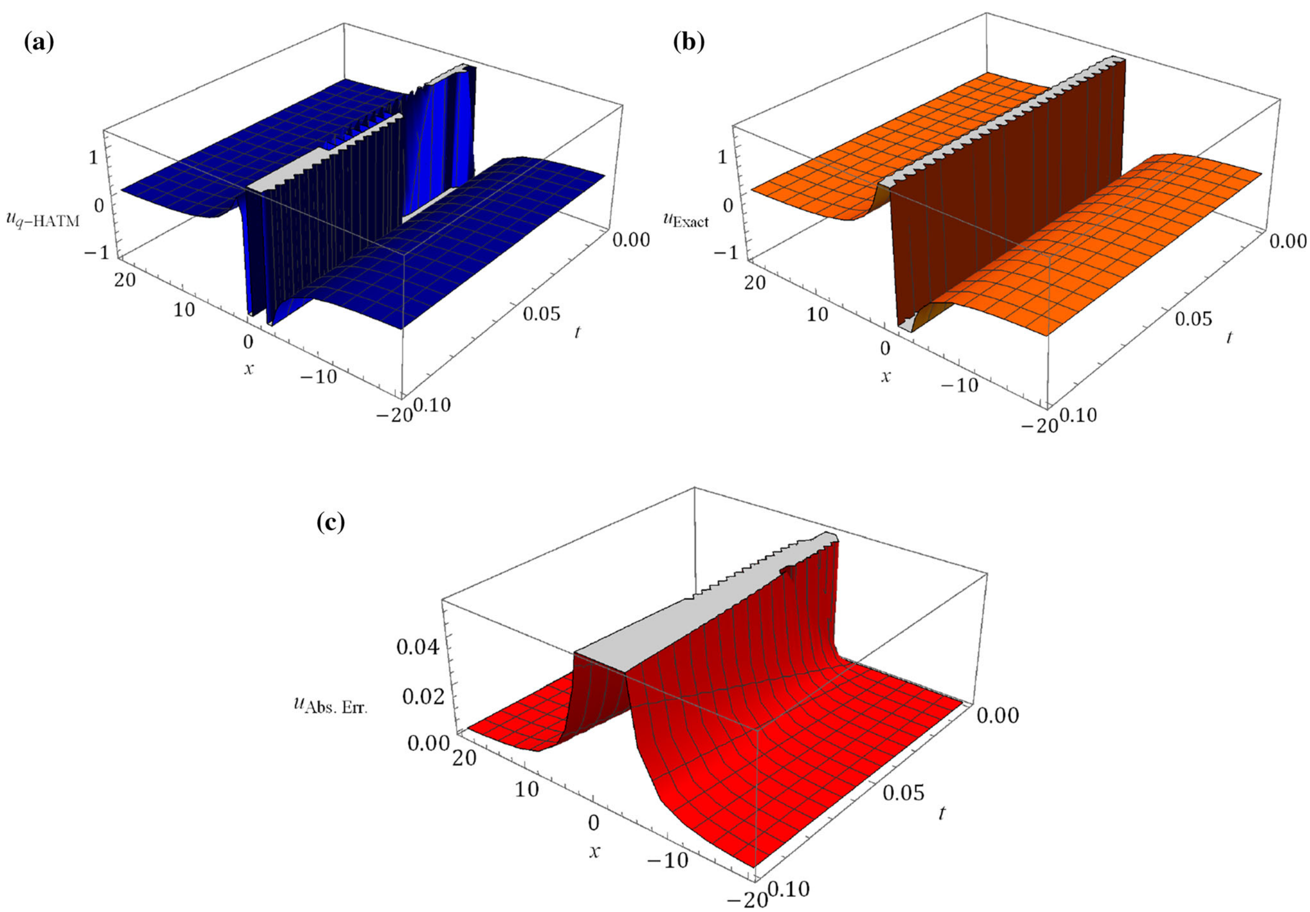

Fig. 4 Nature of (**a**) q-HATM, (**b**) exact results, (**c**) absolute error for the Example 2 at $n = 1$, $\hbar = -1$, $\beta_1 = -1.5$, $\beta_3 = -0.5$, $\sigma = -1$, $\phi = 1$ and $\mu = 2$

6, 7, and also the effect of fractional order is demonstrated in Figs. 5, 8. Specifically, we can observe from Fig. 7, the q-HATM results are accurate and confirmed with a small absolute region.

The considered algorithms offer parameters, which can assist for the converges region of the obtained solution with the assist of the axillary parameter or homotopy parameter $(\hbar)$ and which is presented in Fig. 3 for Example 1 with distinct order and it can assist us to adjust and control the convergence of the obtained results by modulating its value with n. Precisely, the flat line segment authorizes the

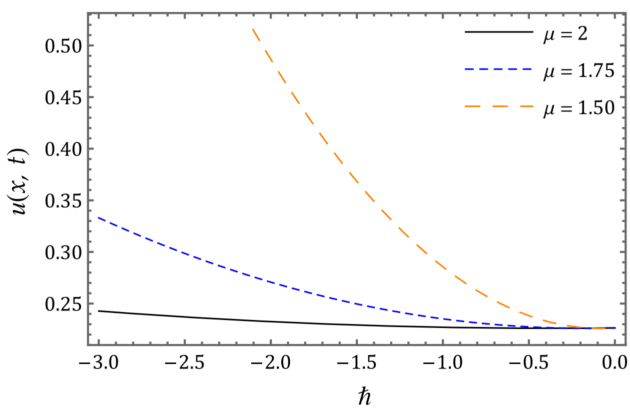

Fig. 6 $\hbar$-curves for Example 2 for different μ at $n = 1$, $\beta_1 = 1.5$, $\beta_3 = -0.5$, $\sigma = -1$, $\phi = 1$, $t = 0.05$, $x = 8$

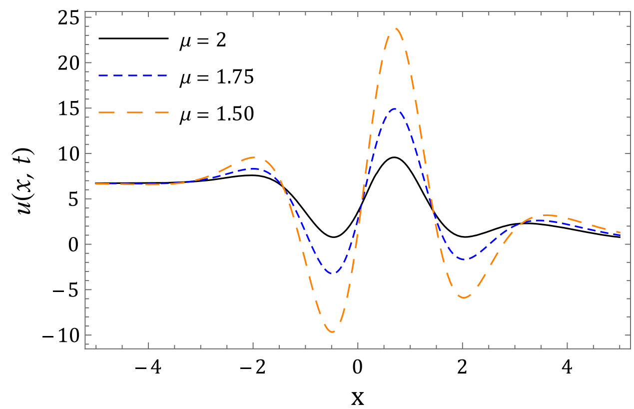

Fig. 8 Nature of achieved results for Example 3 for distinct μ at $n = 1$, $\hbar = -1$, $\beta_1 = 1.5$, $\beta_2 = 1$, $\beta_3 = -0.1$ and $t = 1$

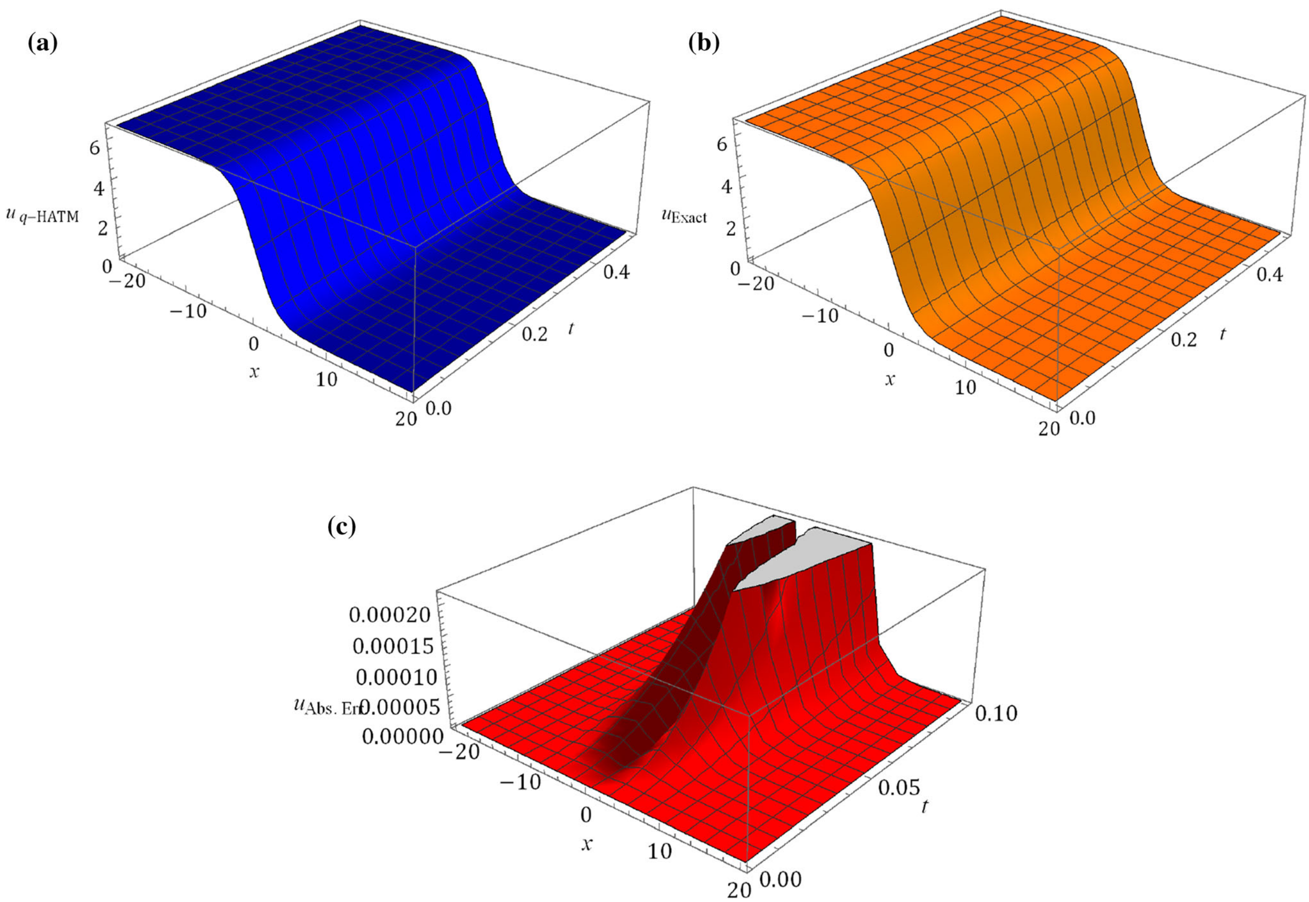

Fig. 7 Surfaces of (**a**) q-HATM result, (**b**) exact result, (**c**) absolute error for Example 3 with $\hbar = -1$, $n = 1$, $\beta_1 = 1.5$, $\beta_1 = 1$, $\beta_3 = -0.1$, and $\mu = 2$

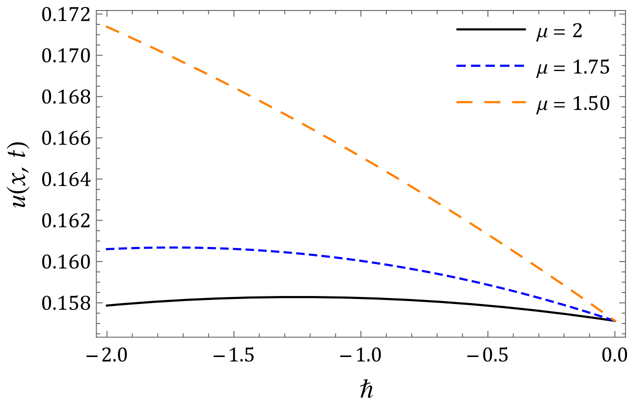

Fig. 9 $\hbar$-curves for Example 3 for distinct μ at $n=1$, $\beta_1=1.5$, $\beta_2=1$, $\beta_3=-0.1$, $\sigma=-1$, $\phi=1$, $t=0.01$, and $x=5$

convergence region with exact results. For Examples 2, 3, the ($\hbar$)-curves are, respectively, presented in Figs. 6, 9 with a small time for the particular values of the other parameters. Moreover, with seized records, we can realize that the considered equations exceptionally be contingent on the order and give a huge degree of freedom for generalizing with fractional order. Also, this investigation may support to illustrate miscellaneous types and assist to seizure more consequences.

6. Conclusions

In this research, the time-fractional generalized Pochhammer-Chree partial differential equation occurring in longitudinal vibrations of an elastic rod has been implemented with a reliable tool, namely the q-homotopy analysis transform method. Using the Caputo derivative definition, with three separate examples, effective approximate results of the equation have been obtained. Then, the surface plots are presented to illustrate approximate solutions. It is therefore shown that the approach is very powerful and could be used in various forms of FPDEs occurring in various fields of mathematical physics.

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