Indian J Phys (November 2021) 95(11):2523–2529
https://doi.org/10.1007/s12648-020-01900-6

ORIGINAL PAPER

Exact solutions of Fokker–Planck equation via the Nikiforov–Uvarov method

H Heidari, H Motavalli* and M R Keramati

Faculty of Physics, University of Tabriz, 5167618949 Tabriz, Iran

Received: 14 November 2019 / Accepted: 20 July 2020 / Published online: 29 October 2020

Abstract: We propose a powerful approach to provide the exact solutions of the time-dependent Fokker–Planck equation (FPE) for a given pair of drift and diffusion functions in stochastic phenomena. First, we briefly review Nikiforov–Uvarov mathematical method and then apply it to consider three important examples. Subsequently, the probability distribution functions of FPE are obtained in terms of special orthogonal functions for three cases, as well as the corresponding eigenvalues are derived. Several applications are proposed and it is shown that the results of our approach are in good agreement with those obtained by other methods.

Keywords: Fokker–Planck; Nikiforov–Uvarov; Probability distribution; Stochastic phenomena

1. Introduction

The study of stochastic phenomena has been extensively increased in recent years, due to their applications in various fields such as physics, chemistry, biology, circuit theory, and even finance [1–4]. In such phenomena, one of the most popular differential equations, the so-called FPE, naturally arises [4]. Actually, it has been historically presented not only to the investigation of stochastic systems but also the Brownian motion of particles [4]. Mathematically, FPE is a linear partial differential equation that describes the time evolution of the probability density function associated with a dynamical system with random features [5]. Depending on the dynamics of drift and diffusion functions of such systems, the resulting FPE may be extremely complicated or quite simple, and accordingly, numerical algorithms or analytical solutions are proposed. Among analytical methods, various approaches have been presented, such as supersymmetry [4], Lie algebra [6], Laplace transformation [7], Fourier analysis [8], variational methods [9], Ornstein–Uhlenbeck process [10], etc. For instance, Lo [6] solved FPE for time-dependent nonlinear drift and diffusion coefficients using the Lie-algebraic approach analytically. Likewise, Grain et al. [11] has used a dynamical system technique to consider a stochastic inflationary cosmological model in phase-space.

On the other hand, one of the most commonly proposed approaches is the transformation of FPE into the Schrodinger equation formally, to get an analytical solution of the first one. In this regard, Zarrinkamar et al. [7] considered harmonic oscillator, Morse potential, and free particle using Laplace transformation. They showed that the resulting FPE of these potentials are, respectively, equivalent to quadratic, exponential, and logarithmic drift forces with a constant diffusion. In a similar work, Brics et al. [12] found exact solutions of FPE for tangent hyperbolic and constant drift forces with constant diffusion, in analogy to the Schrödinger equation for shifted Poschl–Teller and constant potentials, respectively. Additionally, Anjos et al. [4] provided analytical solutions of generalized Morse and Hulthén potentials by using the supersymmetry technique in quantum mechanics. However, the FPE-Schrödinger analogy method, which is based on the solvability of the latter equation, is limited only to consider confined potentials, which restricts its applicability. Obviously, in this analogy, each potential in the Schrodinger equation corresponds to a pair of drift and diffusion functions in FPE. This correspondence impose a specific mathematical form to FPE naturally, that may have undesirable physical properties in the framework of stochastic phenomena.

However, the existence of equivalence between FPE and the differential equation of type (3), allows one to use Nikiforov–Uvarov (NU) formalism, as a powerful

*Corresponding author, E-mail: Motavalli@Tabrizu.ac.ir

© 2020 IACS

approach to solve FPE [13]. This method is based on reducing the second-order linear differential equation to a generalized equation of the hypergeometric type. The solutions of the latter equation are presented in terms of special orthogonal functions, as well as the corresponding eigenvalues are obtained [13, 14]. One of the motivations for the present work is to demonstrate the merit of the NU method to get an analytical solution of FPE for a given pair of drift and diffusion functions. We will realize this aim in three important examples.

2. Fokker–Planck equation

The one-dimensional FPE of the probability distribution $P(x, t)$ is given by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}(F(x)P(x,t)) + \frac{\partial^2}{\partial x^2}(D(x)P(x,t)), \tag{1}$$

where $F(x)$ and $D(x)$ are the drift and diffusion functions, representing the deterministic and the stochastic part of the equation, respectively. In mathematical literature, this equation is often referred to as the forward Kolmogorov equation [15]. For analysis of this equation, one may use separation of variables method and assume that $P(x,t) = T(t)\psi(x)$. It allows one to rewrite (1) as

$$\psi''(x) + \left\{\frac{2D'(x) + F(x)}{D(x)}\right\}\psi'(x) + \left\{\frac{D''(x) + F'(x) + \lambda}{D(x)}\right\}\psi(x) = 0, \tag{2}$$

with $T(t) = T_0 e^{-\lambda t}$ in which T_0 and $\lambda > 0$ are constants. To solve this equation for a given pair of drift and diffusion coefficients, let us consider the NU method.

3. The Nikiforov–Uvarov method

In this section, we briefly review the mathematical NU method, which is based on reducing the second-order linear differential equation to a generalized equation of the hypergeometric type [13, 14]. In this formalism, solutions are given in terms of special orthogonal functions. Here, we apply this method to solve FPE for a given drift and diffusion functions. By introducing an appropriate coordinate transformation $s = s(x)$, we rewrite this equation in the following form

$$\psi_n''(s) + \frac{\widetilde{\tau}(s)}{\Gamma(s)}\psi_n'(s) + \frac{\widetilde{\sigma}(s)}{\Gamma^2(s)}\psi_n(s) = 0, \tag{3}$$

where $\widetilde{\tau}(s)$ is a first-degree polynomial, also $\Gamma(s)$ and $\widetilde{\sigma}(s)$ are polynomials at most of second-degree. Depending on the form of these polynomials, solutions of Eq. 3 are presented in terms of several classes of orthogonal functions, such as classical polynomials (Hermite, Jacobi, and Laguerre), spherical harmonics, Bessel, and hypergeometric functions. These functions, $\psi_n(s)$, are often referred to as the special functions of mathematical physics. To obtain the particular solution of this equation, one usually takes

$$\psi_n(s) = \phi_n(s)y_n(s). \tag{4}$$

This factorization leads to the following hypergeometric type of equation

$$\Gamma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{5}$$

where the new function $\tau(s)$ is defined as

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \tag{6}$$

in which

$$\pi(s) = \frac{\Gamma'(s) - \widetilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\Gamma'(s) - \widetilde{\tau}(s)}{2}\right)^2 - \widetilde{\sigma}(s) + k\Gamma(s)}. \tag{7}$$

Here, k is a parameter that plays an essential role in the calculation of $\pi(s)$, and it is simply obtained by setting the discriminant of the square root equal to zero. The values of k are also used to determine the eigenvalues via

$$\lambda = k + \pi'(s) = -n\tau'(s) - \frac{n(n-1)}{2}\Gamma''(s), \quad n = 0, 1, 2, \ldots, \tag{8}$$

where λ is a constant determined by n, and is often referred to as the eigenvalue. Besides, each λ corresponding to $n = 0, 1, 2, \ldots$, introduces a particular polynomial of degree n, which is a special solution of FPE for a given pair of drift and diffusion functions. It must be pointed out that there are usually more than one $\tau(s)$ functions in relation (6), corresponding to different values of $\pi(s)$ in (7) for various k, where the suitable one satisfies the constraint $\tau'(s) < 0$ based on the NU formulation. Polynomial solutions of Eq. 5 are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\Gamma^n(s)\rho(s)], \tag{9}$$

where B_n is the normalization constant, and the weight function $\rho(s)$ is obtained as follows [14]

$$\rho(s) = \exp\left\{\int \frac{\tau(s) - \sigma'(s)}{\sigma(s)} ds\right\}. \tag{10}$$

Furthermore, the other part of the solution is given by [14]

$$\phi_n(s) = exp\left\{\int \frac{\pi(s)}{\sigma(s)} ds\right\}. \tag{11}$$

4. Solving some examples of FPE by NU method

A comparison between Eqs. 2 and 3 implies that the former may be solved using the NU method, for a given pair of drift and diffusion coefficients. For this purpose, let us consider three examples.

- **Case 1**

As a first example, we assume an exponential form drift force with a constant diffusion coefficient as follows

$$F(x) = \alpha e^{-ax} - \beta, \quad D(x) = \sigma^2, \tag{12}$$

where a, α, $\beta > 0$ and σ are constant. With this choice, the corresponding Eq. 2 yields

$$\psi''(x) - \left\{\frac{\alpha e^{-ax} - \beta}{\sigma^2}\right\}\psi'(x) + \left\{\frac{\alpha a e^{-ax} + \lambda}{\sigma^2}\right\}\psi(x) = 0. \tag{13}$$

By introducing the auxiliary variable $s = e^{-ax}$, it is easily transformed into

$$\psi''(s) + \left\{\frac{\mu s + v}{s}\right\}\psi'(s) + \left\{\frac{\mu s + \eta}{s^2}\right\}\psi(s) = 0, \tag{14}$$

with $\mu = \frac{\alpha}{a\sigma^2}$, $v = 1 - \frac{\beta}{a\sigma^2}$ and $\eta = \frac{\lambda}{a^2\sigma^2}$. Now, by comparing the last equation with Eq. 3, one obtains

$$\widetilde{\tau}(s) = \mu s + v, \quad \widetilde{\sigma}(s) = \mu s + \eta \quad \Gamma(s) = s. \tag{15}$$

By inserting them into relation (7) we get π function as follows

$$\pi(s) = -\frac{1}{2}(\mu s + v - 1) \pm \sqrt{\left(\frac{\mu}{2}\right)^2 s^2 + \left[k - \frac{1}{2}\mu(3 - v)\right]s + \frac{1}{4}\zeta^2} \tag{16}$$

where $\zeta = \sqrt{(1 - v)^2 - 4\eta}$. Now, by setting the discriminant of the square root equal to zero we find

$$\pi(s) = \begin{cases} (1 - v - \zeta)/2 - \mu s & \text{for} \quad k_+ = \mu(3 - v + \zeta)/2 \\ (1 - v + \zeta)/2 & \text{for} \quad k_+ = \mu(3 - v + \zeta)/2 \\ (1 - v + \zeta)/2 - \mu s & \text{for} \quad k_- = \mu(3 - v - \zeta)/2 \\ (1 - v - \zeta)/2 & \text{for} \quad k_- = \mu(3 - v - \zeta)/2. \end{cases} \tag{17}$$

In this stage, by choosing a suitable value for k which satisfies the condition $\tau'(s) < 0$ in relation (6), and considering a physical solution one achieves

$$\pi(s) = \frac{1}{2}\{1 - v - \zeta\} - \mu s, \quad \text{for} \quad k_+ = \frac{1}{2}\mu(3 - v + \zeta) \tag{18}$$

with $\tau(s) = 1 - \zeta - \mu s$. By inserting it into relation (10), the weight function is given by

$$\rho(s) = \exp\left\{-\int\left(\mu + \frac{\zeta}{s}\right)\mathrm{d}s\right\}, \\ = \text{Const} \quad s^{-\zeta}e^{-\mu s}. \tag{19}$$

Substituting $\rho(s)$ into the Rodrigues relation (9) leads to

$$y_n(s) = B_n s^{\zeta} e^{\mu s}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n\left[s^{n-\zeta}e^{-\mu s}\right] \\ = L_n^{(n-\zeta)}(s), \tag{20}$$

where $L_n^{(n-\zeta)}(s)$ is the associated Laguerre polynomial. On the other hand, the other part of the solution is easily obtained from (11) as

$$\phi_n(s) = \exp\left\{\int\left(\frac{1 - \zeta - v}{2s} - \mu\right)\mathrm{d}s\right\}, \\ = \text{Const} \quad s^{-\frac{1}{2}(\zeta + v - 1)}e^{-\mu s}. \tag{21}$$

By multiplying the two parts, the final solution yields

$$\psi_n(x) = N\exp\{(\zeta + v - 1)ax/2 - \mu e^{-ax}\}L_n^{(n-\zeta)}(e^{-ax}), \tag{22}$$

where N is the normalization constant. The quantization condition (8) becomes $\frac{1}{2}(3 - v + \zeta) - 1 = n$, or explicitly

$$\lambda = an(\beta - an\sigma^2), \quad n = 0, 1, 2, \ldots \tag{23}$$

The behavior of the probability distribution of $P(x, t)$ is illustrated in Fig. 1 in terms of x. Accordingly, the figure exhibits a Gaussian form, and its values decrease with increasing time. The main features of this solution are associated with pair functions (12), which is corresponding to Morse potential in the framework of non-relativistic quantum mechanics [3]. This potential has various applications, such as a description of, atomic interactions in physics [16], structure of the DNA in genetics [17], elastic and fracture behaviors in engineering mechanics [18], etc. It is also worthwhile to mention that Eq. 14 has already been solved based on the formal analogy with the Schrodinger equation using the supersymmetry approach [3], which confirms our results.

- **Case 2**

As a second example, we consider the following set

$$F(x) = \frac{\alpha e^{-2ax}}{1 - \beta e^{-2ax}} + \gamma, \quad D(x) = \sigma^2. \tag{24}$$

For this choice, Eq. 2 can be rewritten as

$$\psi''(x) - \frac{1}{\sigma^2}\left\{\frac{\alpha e^{-2ax}}{(1 - \beta e^{-2ax})} + \gamma\right\}\psi'(x) \\ + \frac{1}{\sigma^2}\left\{\frac{2a\alpha e^{-2ax}}{(1 - \beta e^{-2ax})^2} + \lambda\right\}\psi(x) = 0. \tag{25}$$

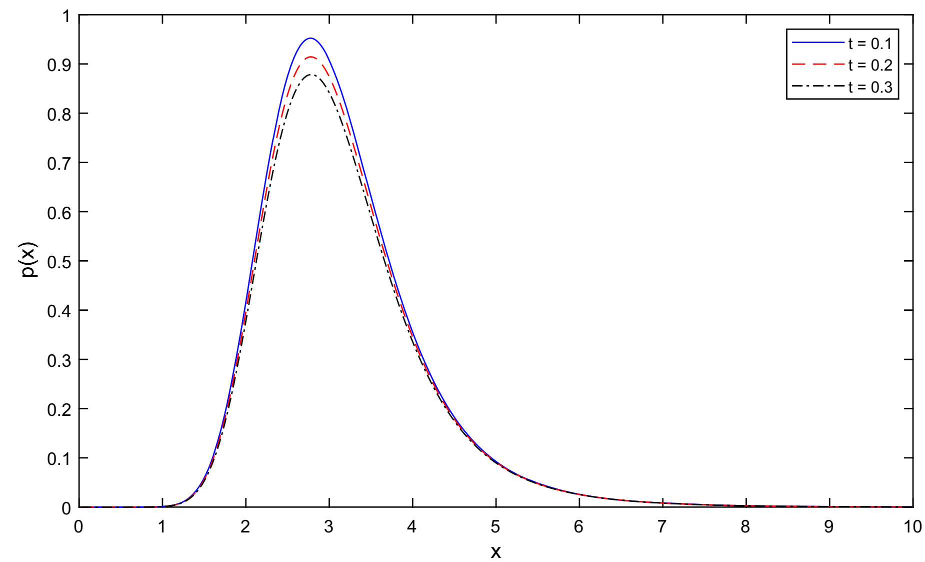

Fig. 1 Variation of the probability distribution $P(x, t)$ for $F(x) = \alpha e^{-ax} - \beta$ and $D(x) = \sigma^2$ in terms of x for $a = 1$, $\alpha = 0.37$, $\beta = 0.4$, and $\sigma = 0.16$

It can be easily shown that, by introducing the auxiliary variable $s = e^{-2ax}$, this equation is transformed into

$$\psi''(s) + \left\{\frac{(\alpha - \beta\gamma - 2a\beta\sigma^2)s + \gamma + 2a\sigma^2}{2a\sigma^2[s(1-s)]}\right\}\psi'(s) + \left\{\frac{\lambda\beta^2 s^2 + 2(a\alpha - \lambda\beta)s + \lambda}{4a^2\sigma^2[s(1-s)]^2}\right\}\psi(s) = 0. \tag{26}$$

Now, by comparing the last equation with Eq. 3, one obtains

$$\begin{aligned}\tilde{\tau}(s) &= \frac{(\alpha - \beta\gamma - 2a\beta\sigma^2)s + \gamma + 2a\sigma^2}{2a\sigma^2},\\ \tilde{\sigma}(s) &= \frac{\lambda\beta^2 s^2 + 2(a\alpha - \lambda\beta)s + \lambda}{4a^2\sigma^2}\\ \Gamma(s) &= s(1-s).\end{aligned} \tag{27}$$

By inserting them into relation (7) we get π function as follows

$$\pi = \pm\sqrt{[k-K]s + \left[-\beta\left(k + \frac{\beta\gamma^2}{16a^2\sigma^4}\right) + \frac{1}{4}\left(\beta + \frac{\alpha - \beta\gamma}{2a\sigma^2}\right)\right]s^2} - \frac{1}{2}\left[\frac{2a\beta\sigma^2 + \alpha - \beta\gamma}{2a\sigma^2}s + \frac{\gamma}{2a\sigma^2}\right] \tag{28}$$

where $K = \frac{4a\alpha\sigma^2 - \beta\gamma^2 + \gamma(2a\beta\sigma^2 + \alpha - \beta\gamma)}{8a^2\sigma^4}$, and we have assumed $\gamma^2 = 4\lambda\sigma^2$ for simplicity. By setting the discriminant of the square root equal to zero, an appropriate choice yields

$$\pi = -\frac{\gamma}{4a\sigma^2} - \frac{2a\beta\sigma^2 + \alpha - \beta\gamma \pm \zeta}{4a\sigma^2}s \quad \text{for} \quad k = -\frac{\alpha\gamma + 2a\beta\gamma\sigma^2 - 4a\alpha\sigma^2}{8a^2\sigma^4}, \tag{29}$$

where $\zeta = \sqrt{(2a\beta\sigma^2 + \alpha - \beta\gamma)^2 + \beta[2\alpha\quad\gamma + 4a\beta\gamma\sigma^2 - \beta\gamma^2 - 8a\alpha\sigma^2]}$. Since the derivative of the τ function in relation (6) should be negative; one must select a positive sign for ζ in the last relation of π. Accordingly, $\tau(s)$ is

$$\tau(s) = -\left(2\beta + \frac{\zeta}{2a\sigma^2}\right)s + 1. \tag{30}$$

Now, by simple manipulation, integration of (10) yields

$$\begin{aligned}\rho(s) &= \exp\left\{-\frac{\zeta}{2a\sigma^2}\int\frac{ds}{1-\beta s}\right\},\\ &= \text{Const }(1-\beta s)^{\frac{\zeta}{2a\beta\sigma^2}}.\end{aligned} \tag{31}$$

By substituting this into the Rodrigues relation (9) we have

$$\begin{aligned}y_n(s) &= B_n(1-\beta s)^{-\frac{\zeta}{2a\beta\sigma^2}}\left(\frac{d}{ds}\right)^n\left[s^n(1-\beta s)^{n+\frac{\zeta}{2a\beta\sigma^2}}\right],\\ &= B_n P_n^{\left(\frac{\zeta}{2a\beta\sigma^2},0\right)}(2\beta s - 1)\end{aligned} \tag{32}$$

where $P_n^{(l,m)}(s)$ denotes the generalized Jacobi-polynomial. The other part of the wave function in relation (11) simply becomes

$$\begin{aligned}\phi_n(s) &= \exp\left\{-\frac{1}{4a\sigma^2}\int\frac{\gamma + [\alpha + 2a\beta\sigma^2 + \zeta - \beta\gamma]s}{s(1-\beta s)}ds\right\},\\ &= \text{Const}\quad s^{-\frac{\gamma}{4a\sigma^2}}(1-\beta s)^{\frac{\alpha+\zeta}{4a\beta\sigma^2}+1/2}.\end{aligned}$$

Finally, by multiplying the two parts, the total function is given by

$$\begin{aligned}\psi_n(x) &= N\, e^{\frac{\gamma x}{2\sigma^2}}(1-\beta e^{-2ax})^{\frac{\alpha+\zeta}{4a\beta\sigma^2}+1/2}\\ &\quad P_n^{\left(\frac{\zeta}{2a\beta\sigma^2},0\right)}(2\beta e^{-2ax} - 1)\end{aligned} \tag{33}$$

where N is the normalization constant. Now, from the relation (8) we calculate the quantization condition as

$$-\frac{\alpha\gamma + 2a\beta\gamma\sigma^2 - 4a\alpha\sigma^2}{8a^2\sigma^4} - \frac{\zeta + \alpha - \beta\gamma + 2a\beta\sigma^2}{4a\sigma^2} = n\left(\frac{\zeta + 4a\beta\sigma^2}{2a\sigma^2}\right)$$

or explicitly

$$\lambda = \frac{\gamma^2}{4\sigma^2} = 4[(n+1)\alpha - 2a\beta\sigma^2(3n+1)]^2 a^2\sigma^2/\alpha^2, \quad n = 0, 1, 2, \ldots.$$

Figure 2 shows the variations of the probability distribution of $P(x, t)$ in terms of x. The distribution spreads to the right side as time goes on, and the probability decreases before $x \approx 180$, and afterward it grows slowly. This solution of the FPE, which is associated with pair functions (24), is corresponding to Hulthen potential in the Schrodinger equation [4]. The exact solution of this equation has been obtained by Anjos et al. using the supersymmetric quantum mechanics approach [4], which is in agreement with our result. It is worthwhile to mention that Hulthen is one of the essential short-range potentials with interesting applications in nuclear, solid-state, and atomic physics [19, 20]. Furthermore, in the biological models, this potential is also used to describe the cellular differentiation process, dynamics of tumor growth, and cancer therapy [21–23].

- **Case 3**

As a last example, let us set a logarithmic drift function with a quadratic diffusion as follows

$$F(x) = \alpha x - \beta x Lnx, \quad D(x) = \sigma^2 x^2. \tag{34}$$

Substituting them into Eq. 2 leads to

$$\psi''(x) + \left\{\frac{4\sigma^2 - \alpha + \beta Lnx}{\sigma^2 x}\right\}\psi'(x) + \left\{\frac{2\sigma^2 + \lambda + \beta - \alpha + \beta Lnx}{\sigma^2 x^2}\right\}\psi(x) = 0. \tag{35}$$

By applying the transformation $s = Lnx$, we find the following simple form

$$\psi''(s) + \{\mu s + \nu\}\psi'(s) + \{\mu s + \eta\}\psi(s) = 0, \tag{36}$$

where $\mu = \beta\sigma^{-2}$, $\nu = 3 - \alpha\sigma^{-2}$ and $\eta = 2 + (\lambda + \beta - \alpha)\sigma^{-2}$. By comparing this equation with Eq. 3 we have

$$\tilde{\tau}(s) = \mu s + \nu, \quad \tilde{\sigma}(s) = \mu s + \eta \quad \Gamma(s) = 1, \tag{37}$$

and the corresponding π function becomes

$$\pi(s) = \begin{cases} -\mu s - \nu + 1 & \text{for } k = \frac{\lambda+\beta}{\sigma^2} \\ -1 & \text{for } k = \frac{\lambda+\beta}{\sigma^2}. \end{cases} \tag{38}$$

Here, we assume $\beta > 0$ and $\alpha > \sigma^2$ without loss of generality. Subsequently, from the relation (6) one should take $\tau(s) = -\mu s - \nu + 2$, to ensure that $\tau(s)$ has a negative derivative for $\mu > 0$. By inserting it into relation (10), the weight function is given by

$$\rho(s) = \text{ Const } e^{-\mu s^2/2 + (2-\nu)s}. \tag{39}$$

Further, polynomial solution $y_n(s)$ is obtained from the Rodrigues relation (9) as

$$y_n(s) = B_n e^{\mu s^2/2 - (2-\nu)s}\left(\frac{d}{ds}\right)^n e^{-\mu s^2/2 + (2-\nu)s}, = H_n\left(\sqrt{\mu/2}s + \frac{\nu - 2}{\sqrt{2\mu}}\right).$$

which is the Hermit function. On the other hand, the other part of the solution is easily obtained from (11) as

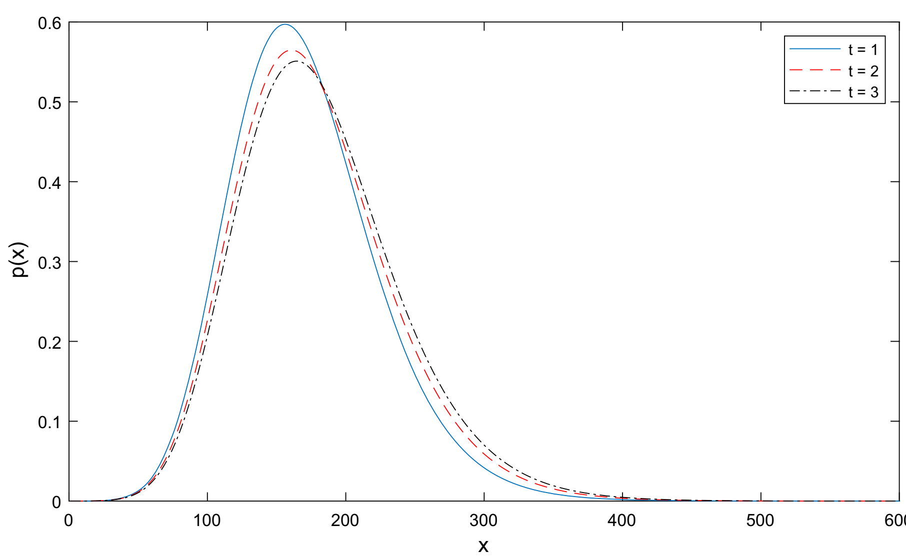

Fig. 2 Variation of the probability distribution $P(x, t)$ for $F(x) = \frac{\alpha e^{-2ax}}{1-\beta e^{-2ax}} + \gamma$ and $D(x) = \sigma^2$ in terms of x for $a = 0.5$, $\alpha = 0.3$, $\beta = 0.1$, $\sigma = 0.4$, and $\gamma = -1$

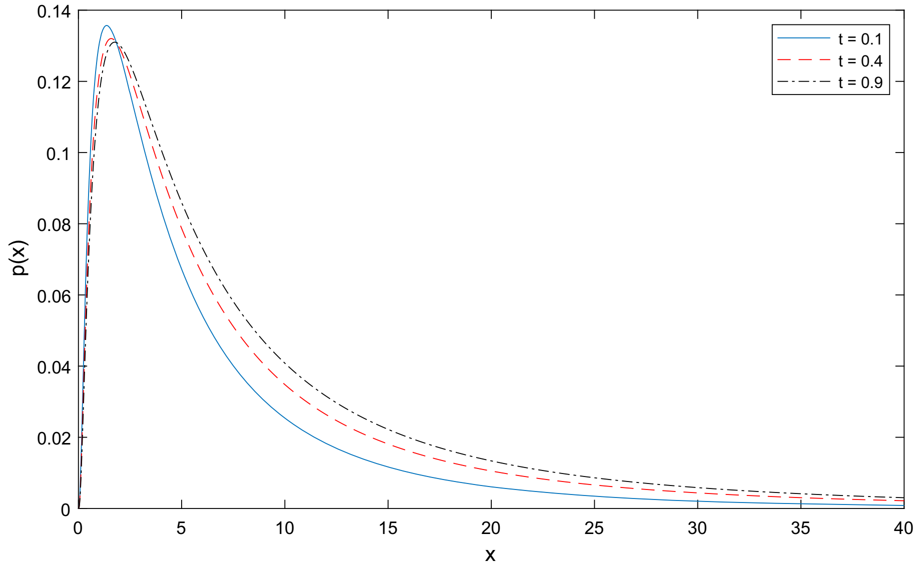

Fig. 3 Variation of the probability distribution $P(x, t)$ for $F(x) = \alpha x - \beta x Lnx$ and $D(x) = \sigma^2 x^2$ in terms of x for $\alpha = 8$, $\beta = 2.5$, and $\sigma = 1.8$

$$\phi_n(s) = \text{ Const } e^{-\mu s^2/2+(1-\nu)s}. \tag{40}$$

Finally, by multiplying the two parts, the total function $\psi_n(x)$ is obtained as follows

$$\psi_n(x) = Ne^{-\frac{\mu}{2}\left(Lnx+\frac{\nu-1}{\mu}\right)^2} H_n\left(\sqrt{\mu/2}Lnx + \frac{\nu-2}{\sqrt{2\mu}}\right), \tag{41}$$

where N is the normalization constant. Furthermore, the relation (8) leads to the quantization condition as

$$\lambda = n\beta, \quad n = 0, 1, 2, \ldots$$

Figure 3 shows the dynamics of the probability distribution of $P(x, t)$ in terms of x. According to this figure for three different time values, the probability grows sharply at the beginning, and then drops to small values, and eventually disappears. It shifts to the right side, by increasing the time. This model is usually used to predict the evolution of various stochastic biological phenomena in the literature [3, 10, 24]. For instance, Polotto et al. used this pattern to consider the evolution of the probability density function of the tumor growth by using the finite element method [3]. Similarly, Albano et al. simulated the effects of a time-dependent therapy for a parathyroid tumor, based on the Ornstein–Uhlenbeck process [10]. These works confirm our results well.

5. Conclusion

In this work, we proposed a new and powerful approach to solve FPE, known as the Nikiforov–Uvarov mathematical method, for a given pair of drift and diffusion coefficients. By using this formalism, the probability distribution was found analytically for three examples, and their time-evolution were studied at different times. In the first two cases, we solved FPE for exponential form drift forces with constant diffusions, that led to the associated Laguerre (22) and the generalized Jacobi (33) polynomials. These functions correspond to Morse and Hulthen potentials, respectively, based on a formal analogy between FPE and the Schrodinger equation. In the third case, it was shown that a logarithmic force with the quadratic diffusion model provided the Hermit-polynomial, that was used to investigate the effects of time-dependent therapy of tumor growth based on the Gompertz law of growth [10, 24]. The comparison between the three studied cases indicates that they display different behaviors. Specifically, in the Morse potential, the probability distribution exhibits almost a Gaussian form, without spreading, and its values decrease with increasing time. While, in Hulthen potential, the distribution propagates to the right, and as time increases, it reduces before $x \approx 180$, and then rises gradually. In logarithmic force, the probability grows sharply at the beginning, and then drops to small values, and eventually disappears. By increasing the time, it shifts to the right side. Our results in these examples, were well consistent with the usual drift-diffusion dynamics in stochastic phenomena, and in good agreement with other reports. Finally, we demonstrated that the NU method is an efficient mathematical approach to obtain analytical solutions of FPE as well as the corresponding eigenvalues. Actually, the efficiency of this method depends on finding an appropriate coordinate transformation, which transforms FPE to the differential equation of type (3). However, further investigation is needed to develop a systematic way to obtain such transformation for a general type of FPE.

Acknowledgements This paper is published as a part of a research project supported by the Iran National Science Foundation (No. 96001905).

References

[1] T K Caughey *Adv. Appl. Mech.* **11** 209 (1971)
[2] M T Araujo and F E Drigo *J. Stat. Phys. Lett.* **146** 610 (2012)
[3] F Polotto, M T Araujo and D Filho *J. Phys. A. Math. Thor.* **43** 015207 (2010)
[4] R C Anjos, G B Freitas and C H Coimbra-Araujo *J. Stat Phys* **162** 387 (2016)
[5] B Oksendal *Stochastic Differential Equations*, 5th edn. (Berlin: Springer) (1988)
[6] C F Lo *Eur. Phys. J. B* **84** 131 (2011)
[7] S Zarrinkamar, H Panahi and F Hosseini *U.P.B. Sci. Bull. Series A* **79** 3 (2017)
[8] V Buslaev and V Grecchi *J. Phys. A* **36** 5541 (1993)
[9] E D Filho and R M Ricotta *Mod. Phys. Lett. A* **10** 1613 (1995)
[10] G Albano and V Giorno *J. Thor. Biology* **242** 329 (2006)
[11] J Grain and V Vennin *JCAP* **1705** 05 (2017)
[12] M Brics, J Kaupuzs and R Mahnke *Condensed Matter Physics* **16** 1 (2013)
[13] A F Nikiforov and V B Uvarov *Special Functions of Mathematical Physics* (Birkhauser) (1988)
[14] H Motavalli *Mod. Phys. Lett. A* **24** 1227 (2009)
[15] J Crow and M Kimura *An Introduction to Population Genetics Theory* (New York: Harper and Row) (1970)
[16] Z H Deng and Y P Fan *Shandong Univ. J.* **7** 162 (1957)
[17] M Peyrard and A R Bishop *Phys. Rev. Lett.* **62** 2755 (1989)
[18] L A Girifalco and V G Weizer *Phys. Rev.* **114** 687 (1959)
[19] T Roose, S J Chapman and P K Maini *SIAM Rev.* **49** 179 (2007)
[20] A R A Anderson and V Quaranta *Nat Rev. Cancer* **8** 227 (2008)
[21] L Preziosi *Cancer Modelling and Simulation* (Boca Raton: Chapman and Hall) (2003)
[22] R P Araujo and D L S McElwain *Bull. Math. Biol.* **66** 1039 (2004)
[23] H Feizi, M R Shojaei and A A Rajabi *Adv. Stud. Thor. Phys.* **6** 477 (2012)
[24] S Drakos *American J. of Biomedical Engineering* **6** 166 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.