Indian J Phys (September 2021) 95(9):1917–1921
https://doi.org/10.1007/s12648-020-01825-0

ORIGINAL PAPER

Generalising certain cosmological solutions sourced by a stiff perfect fluid

B B Hazarika*

Department of Physics, Gauhati University, Guwahati 781014, India

Received: 20 August 2019 / Accepted: 02 March 2020 / Published online: 7 September 2020

Abstract: In this article, we show that a number of well-known time-dependent, spherically symmetric solutions of the Einstein's field equations sourced by a stiff, perfect fluid with a cosmological constant can be generalised into a solution with arbitrary metric functions. This metric can be applied to construct stiff, perfect fluid metrics with (or without) a cosmological constant. We also explore the possibility that this metric may allow us to generate a singularity-free, spherically symmetric cosmological model.

Keywords: Spherically symmetric metric; Stiff perfect fluid; Cosmological constant; Singularity-free

1. Introduction

The study of spherically symmetric solutions is an important sub-field of the general theory of relativity. We note that although it is commonly believed that a large number of spherically symmetric solutions of the Einstein's field equations are known, the actual situation is quite the opposite. In fact, as stated in [1], "Most known solutions are static or shearfree, and only very few of them satisfy the physical demands of having a plausible equation of state or being free from singularities". There are a number of time-dependent, stiff fluid, spherically symmetric solutions in the literature. In this article, our aim is to write down a metric for time-dependent perfect fluid solutions, with a stiff equation of state and a cosmological constant, which generalises some of these solutions and other solutions which are slight variations of some other solutions of this type. Here, the metric functions are products of arbitrary functions of the radial coordinate and time. Substituting appropriate known functions allows us to reproduce a number of well-known metrics. We also show that the spacetime is sourced by a stiff fluid that has non-zero expansion, shear and acceleration. The metric is algebraically special belonging to the type D in the Petrov classification scheme. Finally, we use this general metric to show that it could be possible to generate a spherically symmetric, singularity-free cosmological model sourced by a stiff perfect fluid.

*Corresponding author, E-mail: bidyutbikash116@gmail.com

A perfect fluid has the energy-momentum tensor of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1}$$

where ρ is the energy density, p, the isotropic pressure and u_μ its four velocity. For fluids with a linear equation of state, the relation between pressure and energy density is usually written as

$$p = (\gamma - 1)\rho \tag{2}$$

where $1 < \gamma < 2$ is a constant. The case $\gamma = 2$, which corresponds to $p = \rho$, is known as a stiff fluid. A physical characteristic of such a fluid is that the speed of sound equals that of light in the medium.

There have been several prior attempts to generalise perfect fluid solutions. Two notable papers in this connection are the ones by Lake [2] and by Maharaj et al. [3]. In [2], the functions μ, ν and λ in the spherically symmetric metric (in comoving coordinates), namely

$$\mathrm{d}s^2 = -e^{\nu(r,t)}\mathrm{d}t^2 + e^{\lambda(r,t)}\mathrm{d}r^2 + e^{\mu(r,t)}\mathrm{d}\Omega^2 \tag{3}$$

(where $\mathrm{d}\Omega^2$ is the metric of the 2-sphere $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$), are taken to be separable functions of r and t. In addition to the further assumptions of the vanishing of heat flux and the coefficient of shear viscosity, μ and λ were constrained by $\dot{\mu} = A\dot{\lambda}$ where A is a constant. It was found that under these assumptions, there exist two classes of solutions: with $A = 1$, the metric reduces to the Robertson–Walker metric and with $A = 0$, a set of four solutions with non-vanishing shear, acceleration and expansion are found. All the $A = 0$ solutions are singular at the origin or degenerate into spaces of constant curvature.

In [3], the authors start with the simplifying assumptions that $\dot{\nu} = 0 = \dot{\lambda}$. Taking μ to be separable, they reduce (3) to the form

$$\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\lambda(r)}\mathrm{d}r^2 + r^2T^2(t)\mathrm{d}\Omega^2 \tag{4}$$

By adopting the above line-element, a number of previously found metrics were shown to be special cases of it. The earlier work of Hajj-Boutros [7] was also extended by them.

The plan of the paper is as follows: in the next section, we begin with an ansatz for the spherically symmetric metric and impose certain relations amongst the metric functions to obtain a stiff fluid model. Section 3 consists of a discussion on the properties of the metric including local geometry and global properties. In Sect. 4, we discuss a few well-known metrics and show that the metric written in Sect. 2 is a general form for them. The last section, which is the concluding one, also contains a discussion on singularity-free models and future work on the possibility of generating such a model from the metric discussed in this paper.

2. Spacetime for stiff perfect fluid with a cosmological constant

We choose the metric functions such that (3) acquires the form

$$\mathrm{d}s^2 = -s(t)f^2(r)\,\mathrm{d}t^2 + g(r)\,\mathrm{d}r^2 + q(t)f^2(r)\,\mathrm{d}\Omega^2 \tag{5}$$

The line element (5) is such that the resulting Einstein tensor is diagonal. The following constraints are imposed on the metric functions $g(r)$ and $s(t)$ (where primes denote differentiation with respect to the argument),

$$g(r) = \frac{[f'(r)]^2}{1 + \beta f^2(r)} \tag{6}$$

and

$$s(t) = \frac{[q'(t)]^2}{\alpha - 4\,q(t) + 4\,q^2(t)} \tag{7}$$

Substituting (6) and (7) in (5) yields the line element

$$\mathrm{d}s^2 = -\frac{[q'(t)]^2}{\alpha - 4\,q(t) + 4\,q^2(t)}f^2(r)\,\mathrm{d}t^2 + \frac{[f'(r)]^2}{1 + \beta f^2(r)}\,\mathrm{d}r^2 + q(t)f^2(r)\,\mathrm{d}\Omega^2 \tag{8}$$

where α and β are constants.

The metric (8) is an exact solution of Einstein's field equations with a cosmological constant, namely

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \quad \text{or} \quad R_{\mu\nu} - \frac{1}{2}R + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{9}$$

where Λ is the cosmological constant. In (9), the units are chosen such that $c = 1$ and $8\pi G = 1$. The field equations are solved by a diagonal energy-momentum tensor with components

$$T^r_{\ r} = T^\theta_{\ \theta} = T^\phi_{\ \phi} = p + 3\,\beta \tag{10}$$

and

$$T^t_{\ t} = -\rho + 3\,\beta \tag{11}$$

where p is the isotropic pressure of the fluid and ρ its density. The explicit form of the pressure and density are

$$p = \rho = \frac{\alpha}{4f^2(r)\,q^2(t)} \tag{12}$$

The above result is obtained from calculating the mixed form of the Einstein tensor which yields the components

$$G^r_{\ r} = G^\theta_{\ \theta} = G^\phi_{\ \phi} = \frac{\alpha}{4f^2(r)\,q^2(t)} \tag{13}$$

and

$$G^t_{\ t} = -\frac{\alpha}{4f^2(r)\,q^2(t)} \tag{14}$$

Comparing (10), (11), (13) and (14) with the field Eq. (9), we find $3\,\beta = \Lambda$, the cosmological constant. We also note that Eqs. (6) and (7) are, in fact, equivalent to the requirement of a stiff fluid with a cosmological constant 3β, so that they are uniquely the stiff matter solutions of the ansatz (5).

3. Properties of the spacetime

The energy-momentum tensor of this spacetime represents a perfect fluid with velocity vector

$$\mathbf{u} = \frac{1}{f(r)\sqrt{s(t)}}\,\partial_t \tag{15}$$

where $s(t)$ is given by (6) with an equation of state

$$p = \rho + 6\,\beta \tag{16}$$

where p and ρ represent the pressure and energy density of the fluid, respectively. We may interpret the matter as a stiff, perfect fluid with a cosmological constant. In order to ensure that $\alpha - 4\,q(t) + 4\,q^2(t)$ remains positive, we impose on α the condition $\alpha \geq 1$. We also point out that setting $\beta = 0$ reduces (8) to a spacetime with stiff, perfect fluid as the matter content. On the other hand, if we choose $\alpha = 0$ and $\beta \neq 0$, we obtain a spherically symmetric form of the anti-de Sitter (or the de Sitter) metric.

From a calculation of the Weyl scalars based on the natural null tetrad for the metric (8) with (6) and (7), we find that the only non-zero one is

$$\Psi_2 = \frac{\alpha}{12 f^2(r)\, q^2(t)} \tag{17}$$

Thus, the spacetime in (8) is of type D in the Petrov classification scheme. This is appropriate as all spherically symmetric metrics are either of type D or conformally flat.

Using the natural null tetrad of the line-element to write the components of the kinematic tensors associated with the fluid velocity vector $\mathbf{u}$, we obtain the expansion as

$$\Theta = \frac{\sqrt{\alpha - 4\,q(t) + 4\,q^2(t)}}{f(r)\,q(t)} \tag{18}$$

and the non-zero components of the shear tensor are

$$\sigma_{11} = \frac{f'^2(r)\sqrt{\alpha - 4\,q(t) + 4\,q^2(t)}}{3 f(r)\,q(t)(1 + \beta f^2(r))}, \tag{19}$$

$$\sigma_{22} = \frac{f(r)\sqrt{\alpha - 4\,q(t) + 4\,q^2(t)}}{6} \tag{20}$$

and

$$\sigma_{33} = \sin^2\theta\,\sigma_{22}. \tag{21}$$

The vorticity of $\vec{u}$ vanishes, and its acceleration is

$$\vec{a} = \frac{1 + \beta f^2(r)}{f(r) f'(r)}\partial_r \tag{22}$$

The spacetime is Lorentzian with determinant of the metric tensor $g_{\mu\nu}$ given by

$$det\, g = -\frac{f^6(r)\,q^2(t)[f'(r)]^2\,[q'(t)]^2\,\sin^2\theta}{[1 + \beta f^2(r)][\alpha - 4\,q(t) + 4\,q^2(t)]} \tag{23}$$

A few of the scalar curvature invariants are

$$R = R^{\mu}{}_{\mu} = -12\beta - \frac{\alpha}{f^2(r) q^2(t)} \tag{24}$$

$$R^{\mu\nu}R_{\mu\nu} = 36\beta^2 + \frac{\alpha[\alpha + 12\beta f^2(r) q^2(t)]}{4 f^4(r) q^4(t)} \tag{25}$$

and the Kretschmann scalar

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 24\,\beta^2 + \frac{\alpha(3\,\alpha + 8\,\beta f^2(r)\,q^2(t))}{4 f^4(r)\,q^4(t)} \tag{26}$$

We note that there appears to be a singularity in the metric function g_{rr} at $\sqrt{-1/\beta}$. However, the curvature invariants as well as the pressure and density are regular there. This blow-up of the metric function is an artefact of the coordinate system and can be removed by a coordinate transformation.

4. Generating known solutions

The metric written in Sect. 2 can be used to generate a number of known solutions by substitution of the appropriate functions for $f(r)$ and $q(t)$. In what follows, we obtain a number of well-known solutions with, as well as without, a cosmological constant.

Case 1

Choosing $s(t) = 1$, and then solving (7) for $q(t)$ yields

$$q(t) = a + (1 - \alpha) C_1\, e^{-2t} + \frac{1}{16\, C_1} e^{2t} \tag{27}$$

Setting $f(r) = r$ and redefining the constants as $a = \epsilon/2$, $\epsilon = 1$, $(1 - \alpha)C_1 = A$ and $1/16C_1 = B$, we obtain the following metric from (8)

$$\mathrm{d}s^2 = -r^2\,\mathrm{d}t^2 + \frac{1}{1 + \beta r^2}\mathrm{d}r^2 + r^2\left[\frac{\epsilon}{2} + Ae^{-2t} + Be^{2t}\right]\mathrm{d}\Omega^2 \tag{28}$$

which is one of the metrics in equation (16.66) of [1] found by Leibovitz [4] and in [5] with an equation of state $p = \rho + 6\beta$.

If $\beta = -k$, $f(r) = r$ and $2\,q(t) = (1 + a_1\,e^{2t} + a_2\,e^{-2t})$, where $a_1 = \frac{1}{8\,C_1}$ and $a_2 = 2\,(1 - \alpha)\,C_1$, then from (8), we get the metric

$$\mathrm{d}s^2 = -r^2\,\mathrm{d}t^2 + \frac{1}{1 - k\,r^2}\,\mathrm{d}r^2 + \frac{1}{2}\,r^2\left[1 + a_1\,e^{2t} + a_2\,e^{-2t}\right]\mathrm{d}\Omega^2 \tag{29}$$

which is the well-known form of the metrics found by Lake [2] with the equation of state $p = \rho - 6\,k = -3\,k + \alpha\,r^{-2}\,(1 + a_1\,e^{2t} + a_2\,e^{-2t})^{-2}$.

If $\beta = 0$, $f(r) = r$ and $2\,q(t) = (1 + a_1\,e^{2t} + a_2\,e^{-2t})$, then the metric (8) becomes

$$\mathrm{d}s^2 = -r^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + \frac{1}{2}\,r^2\left[1 + a_1\,e^{2t} + a_2\,e^{-2t}\right]\mathrm{d}\Omega^2 \tag{30}$$

where $a_1 = \frac{1}{8\,C_1}$ and $a_2 = 2\,(1 - \alpha)\,C_1$. The metric (30) was found by Gutman and Bespal'ko [6] with the equation of state $p = \rho = \alpha\,r^{-2}\,(1 + a_1\,e^{2t} + a_2\,e^{-2t})^{-2}$. Note that, the metric (30) can also be obtained directly from (28) by substituting $\beta = 0$ and redefining the constants.

Case 2 If $\alpha = 0$ and $\beta \neq 0$, then the spacetime (8) reduces to de Sitter (or anti-de Sitter) space. Setting $f(r) = r$ and $q(t) = \frac{\epsilon}{2} + Ae^{-2t} + Be^{2t}$, we obtain the metric

$$ds^2 = -r^2\,dt^2 + \frac{1}{\epsilon + \beta r^2}dr^2 + r^2\left[\frac{\epsilon}{2} + Ae^{-2t} + Be^{2t}\right]d\Omega^2 \tag{31}$$

where $A = C_1$ and $B = \frac{1}{16C_2}$.

Case 3 Let $\alpha = 1$ and $\beta \neq 0$. Setting $f(r) = r$ and $q(t)$ defined as below

$$q(t) = \frac{1}{2}\left[1 + \frac{t^2}{\left(2 - c\,t + 2\sqrt{b\,t^2 - c\,t + 1}\right)^2}\right], \tag{32}$$

we get the metric

$$ds^2 = -\frac{r^2}{t^2(b\,t^2 - c\,t + 1)}dt^2 + \frac{1}{1 + \beta r^2}dr^2 + r^2 q(t)d\Omega^2 \tag{33}$$

which is similar to the metric given in [8] (equation (16.67) of [1]) except for $q(t)$ in (32). The equation of state is $p = \rho + 6\beta$.

Case 4 Let $\alpha = 1$ and $\beta \neq 0$. Setting $f(r) = r$ and $q(t) = a + c\,e^{-t}$, where $a = \frac{1}{2}$, we find that the metric (8) becomes

$$ds^2 = -\frac{r^2}{4}dt^2 + \frac{1}{1 + \beta r^2}dr^2 + r^2(a + c\,e^{-t})d\Omega^2 \tag{34}$$

which is similar to the metric of equation (16.66) of [1] and has the equation of state $p = \rho + 6\beta$.

If $\beta = 0$, then we get

$$ds^2 = -\frac{r^2}{4}dt^2 + dr^2 + \frac{1}{2}r^2(k + a_2\,e^{-t})d\Omega^2, \tag{35}$$

a sub-case to the metric of equation (37.57) of [1] with the equation of state $p = \rho = r^{-2}(k + a_2\,e^{-t})^{-2}$, where $k = 1$ and $a_2 = 2c$.

Case 5 If $q(t) = t^2$ and $f(r) = r$, we get from (8)

$$ds^2 = -\frac{4\,t^2\,r^2}{(\alpha - 4\,t^2 + 4\,t^4)}dt^2 + \frac{1}{1 + \beta r^2}dr^2 + r^2\,t^2 d\Omega^2 \tag{36}$$

which is very similar to the metric found by Collins and Lang [9] (equation (15.75) of [1]) with equation of state $p = \rho$.

5. Conclusion and outlook

In this paper, we have attempted to write down a generalised metric admitting a stiff perfect fluid as its matter source. The fluid has non-zero acceleration, shear and expansion. We have shown that the metric can be used to obtain various known spacetimes by choosing the arbitrary functions $f(r)$ and $q(t)$. It is possible that the stiff fluid metric (5) (with (6) and (7)) can be employed to generate a cosmological model which is singularity-free. The study of singularity-free cosmological models was initiated by Senovilla in his seminal paper [10] where he constructed such a model with $\rho = 3p > 0$ which was followed by a number of metrics having the same property [11–13]. However, all these models are cylindrically symmetric rather than spherically symmetric except the one in [13] representing a spherically symmetric singularity-free universe filled with a non-adiabatic fluid with anisotropic pressure accompanied with heat flux along the radial direction. We choose $\beta = 0$ and the metric functions as $f(r) = e^{b\,r}$, with b real and positive, and $q(t) = \cosh t$ so that the metric (8) acquires the form

$$ds^2 = -e^{2\,b\,r}\frac{\sinh^2 t}{\left(\alpha - 4\cosh t + 4\cosh^2 t\right)} \\ dt^2 + b^2\,e^{2\,b\,r}dr^2 + \cosh t\,e^{2\,b\,r}d\Omega^2 \tag{37}$$

Metric (37) describes a universe with matter having energy density and pressure

$$\rho = \frac{\alpha}{4}\,e^{-2\,b\,r}\ \operatorname{sech}^2 t \tag{38}$$

$$p = \frac{\alpha}{4}\,e^{-2\,b\,r}\ \operatorname{sech}^2 t \tag{39}$$

which means a stiff perfect fluid the energy density and pressure of which is non-singular in the full range of the coordinates r and t, *i.e.*, $0 \le r < \infty$ and $-\infty < t < \infty$. Calculating the kinematic parameters, we find for the expansion the expression

$$\Theta = e^{-b\,r}\sqrt{\alpha\ \operatorname{sech}^2 t - 4\ \operatorname{sech}\ t + 4} \tag{40}$$

The condition $\alpha \ge 1$ ensures that the expansion remains real. The acceleration vector is

$$\mathbf{a} = \frac{1}{b}\,e^{-2\,b\,r}\partial_r \tag{41}$$

while the shear is found to be

$$\sigma^2 = \frac{e^{-2\,b\,r}}{12}\left(\alpha\ \operatorname{sech}^2 t - 4\ \operatorname{sech}\ t + 4\right). \tag{42}$$

The scalar curvature invariants calculated in Sect. 3, acquire the explicit form

$$R^{\mu}{}_{\mu} = -\frac{1}{2}\alpha\,e^{-2\,b\,r}\ \operatorname{sech}^2 t \tag{43}$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{1}{4}\,\alpha^2\,e^{-4\,b\,r}\ \operatorname{sech}^4 t \tag{44}$$

and

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{3}{4}\,\alpha^2\,e^{-4\,b\,r}\ \ \mathrm{sech}^4 t \tag{45}$$

It is evident that all the parameters obtained above are free from singularities in the full range of the coordinates. Both the non-zero Weyl scalar for this metric Ψ_2, obtained for the general case in Eq. (17), as well as the scalar curvature invariants are regular and singularity-free.

Singularity-free models are required to be free from closed timelike curves, that is, they must be casually stable which is inferred from the existence of a time-function. In this case, it can be shown that the coordinate t itself is a time function for $\alpha \geq 1$. However, the singularity-free nature manifest here may be an artefact of the coordinate system. That a spacetime is free from singularities can only be ensured by the completeness of geodesics. We aim to show this in a future work.

References

[1] H Stephani, D Kramer, M MacCallum, C Hoenselaers and E Herlt *Exact Solutions to Einstein's Field Equations* (Cambridge: Cambridge Univ. Press) (2003)

[2] K Lake *Gen. Relat. Grav.* **15** 357 (1983)

[3] S D Maharaj, R Maartens and M S Maharaj *Il Nuov. Cim.* **108** 75 (1993)

[4] C Leibovitz *Phys. Rev.* **D4** 2949 (1971)

[5] N van den Bergh and P Wils *Gen. Relat. Grav.* **17** 223 (1985)

[6] I I Gutman and R M Bespal'ko *Sbornik Sovrem. Probl. Grav.*, Tbilisi (1967)

[7] J Hajj-Boutros *J. Math. Phys.* **26** 771 (1985)

[8] M Marklund and M Bradley *Class. Quant. Grav.* **16** 1577 (1999)

[9] C B Collins and J M Lang *Class. Quant. Grav.* **3** 1143 (1987)

[10] J M M Senovilla *Phys. Rev. Lett.* **64** 2219 (1990)

[11] E Ruiz and J M M Senovilla *Phys. Rev.* **D45** 1995 (1992)

[12] M Mars *Phys. Rev.* **D51** R3989 (1995)

[13] N K Dadhich *J. Astrophys. Astr.* **18** 343 (1997)