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ORIGINAL PAPER

Effect of Landau damping on ion acoustic solitary waves in a collisionless unmagnetized plasma consisting of nonthermal and isothermal electrons

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Abstract: A Korteweg–de Vries (KdV) equation including the effect of linear Landau damping of electrons is derived to study the propagation of weakly nonlinear and weakly dispersive ion acoustic waves in a collisionless unmagnetized plasma consisting of warm adiabatic ions and two species of electrons at different temperatures. It is found that the coefficient of the nonlinear term of this KdV-like evolution equation vanishes along different family of curves in different parameter planes. In this context, a modified KdV (MKdV) equation including the effect of linear Landau damping of electrons describes the nonlinear behaviour of ion acoustic waves. Again, the coefficients of the nonlinear terms of the KdV and MKdV-like evolution equations are simultaneously equal to zero along a family of curves in the parameter plane. In this situation, we have derived a further modified KdV (FMKdV) equation including the effect of linear Landau damping of electrons. The multiple time scale method has been applied to obtain the solitary wave solution of the evolution equations having the nonlinear term $\left(\phi^{(1)}\right)^r \frac{\partial \phi^{(1)}}{\partial \xi}$, where $\phi^{(1)}$ is the first-order perturbed electrostatic potential and $r = 1, 2, 3$. The amplitude of the ion acoustic solitary wave decreases with time for all $r = 1, 2, 3$.

Keywords: Nonthermal electrons; Ion acoustic wave; Landau damping; Modified Korteweg–de Vries equation; Solitary wave solution

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1. Introduction

The observations of electric field structures by the Freja Satellite [1] in the auroral zone of the upper ionosphere, the FAST [2–6] satellite and the Viking Satellite [7, 8] in the auroral zone indicate the presence of cooler and hotter electron species. The cooler electron species can be modelled by the Maxwell–Boltzmann velocity distribution, whereas the hotter electron species can be described by considering Cairns [9] distributed nonthermal electrons. The existence of different species of electrons at different temperatures has already been reported by Dalui et al. [10]. In the present paper, we have considered the effect of linear Landau damping of electrons on ion acoustic (IA) solitary wave in a collisionless unmagnetized electron–ion plasma consisting of warm adiabatic ions, isothermal and nonthermal electrons.

Several authors [10–29] investigated different linear and nonlinear properties of IA waves in a plasma consisting of one or two ion species and one or two electron species. In the present paper, we have investigated the effect of linear Landau damping of electrons of two different populations at different temperatures on IA solitary waves. But, Yu and Luo [30] reported that for phenomena on long-time scales, one can consider electrons into two different species if the electrons are physically separated in space/time domain of interest. So, Maxwell–Boltzmann distributed electrons and Cairns [9] distributed nonthermal electrons can be considered as two different electron species only when those electron species are physically separated in the phase space by external or self-consistent fields. On the basis of the assumption that the two groups of electrons occupy different regions of phase space, several authors [16, 22, 27]

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considered two populations of electrons at different temperatures.

Longitudinal electron plasma oscillations are damped during the propagation through a collisionless plasma. In particular, Vlasov [31] used the linearized Boltzmann equation to investigate the small amplitude steady-state longitudinal electron plasma oscillations. Shortly afterwards, Landau [32] pointed out that these oscillations are damped. This damping of longitudinal electron plasma waves in a collisionless plasma is known as linear electron Landau damping. For the first time, Ott and Sudan [33] investigated the effect of linear Landau damping of electrons on IA solitary waves in a collisionless plasma. Several authors investigated the effect of Landau damping on IA solitary waves in unmagnetized or magnetized plasmas theoretically [34–41] and experimentally [42]. In particular, Tajiri and Nishihara [36] investigated the effect of Landau damping on finite amplitude IA solitary waves in a collisionless unmagnetized electron–ion plasma consisting of cold ions and two distinct populations of isothermal electrons at different temperatures by considering a KdV-like evolution equation including the effect of Landau damping. Bandyopadhyay and Das [37] derived a Korteweg–de Vries–Zakharov–Kuznetsov (KdV–ZK) and a modified KdV–ZK equations including the effect of linear Landau damping of electrons to investigate the nonlinear behaviour of IA waves in a magnetized plasma consisting of warm adiabatic ions and nonthermal electrons. Recently, Ghai et al. [43] investigated the dust acoustic solitary and shock structures under the influence of Landau damping in a dusty plasma containing two different temperature ion species.

To investigate the effect of linear Landau damping of electrons on IA solitary waves in a collisionless unmagnetized electron–ion plasma consisting of two distinct populations of electrons at different temperatures, we have considered coupled Vlasov–Poisson model for two different electron species along with the fluid model for ions. So, in the present plasma system, the kinetic effects of two different species of electrons at different temperatures have been investigated on IA solitary structures with special emphasis on the following cases:

Case-1: Using the reductive perturbation method, an evolution equation has been derived which describes the nonlinear behaviour of IA waves along with a correction due to the kinetic effects of two different species of electrons. This evolution equation reduces to a well-known Korteweg–de Vries (KdV) equation if electron-to-ion mass ratio is neglected.

Case-2: It is found that a factor (B_1) of the coefficient of the nonlinear term of the evolution equation derived in Case-1 vanishes along different family of curves in different parameter planes. In this situation, i.e. when $B_1 = 0$, a modified evolution equation including the effect of linear Landau damping of electrons describes the nonlinear behaviour of IA waves and this modified evolution equation becomes a modified KdV (MKdV) equation having the nonlinear term $(\phi^{(1)})^2 \frac{\partial \phi^{(1)}}{\partial \xi}$ if electron-to-ion mass ratio is neglected, where $\phi^{(1)}$ is the perturbed electrostatic potential and ξ is the stretched space variable.

Case-3: It has been observed that a factor (B_2) of the coefficient of the nonlinear term of the evolution equation derived in Case-2 vanishes along a family of curves in the parameter plane. In this context, a further modified evolution equation including the effect of linear Landau damping of electrons can describe the nonlinear behaviour of IA waves when the conditions $B_1 = 0$ and $B_2 = 0$ hold simultaneously and this further modified evolution equation reduces to a further modified KdV (FMKdV) equation having nonlinear term $(\phi^{(1)})^3 \frac{\partial \phi^{(1)}}{\partial \xi}$ if electron-to-ion mass ratio is neglected. For the first time, we have derived a FMKdV equation having nonlinear term $(\phi^{(1)})^3 \frac{\partial \phi^{(1)}}{\partial \xi}$ including the effect of linear Landau damping of electrons.

Case-4: Using the multiple time scale analysis, we have developed a general method to find the solitary wave solution of the evolution equation having nonlinear term $(\phi^{(1)})^r \frac{\partial \phi^{(1)}}{\partial \xi}$ including the effect of linear Landau damping of electrons.

Case-5: The amplitudes of the solitary wave solutions of the different evolution equations including the effect of linear Landau damping of electrons have been investigated for $r = 1, 2, 3$ and it is found that the amplitude of the solitary wave solution decreases with time for all $r = 1, 2, 3$.

2. Basic equations

In this paper, we have considered the effect of linear Landau damping of electrons on the IA solitary waves. So, to describe the nonlinear behaviour of IA waves including the effect of linear Landau damping of electrons, we take the Vlasov–Poisson model for two different electron species and the fluid model for ions. In this section, we have shown that if we neglect the electron-to-ion mass ratio or if we neglect the inertia of electrons, i.e. if we neglect the effect of linear Landau damping of electrons, then the system of equations reduces to a system of hydrodynamic equations. These hydrodynamic equations can describe the nonlinear behaviour of IA waves and small amplitude IA solitary waves can be described by usual KdV and different modified KdV equations. So, here Vlasov–Poisson model of electron species depends on the inertia of electrons only, i.e. if we neglect the inertia of electrons, then the system of equations reduces to a system of hydrodynamic equations. Therefore, to study the effect of linear Landau damping of electrons on IA solitary waves, we cannot neglect the inertia of electrons. In fact, considering Vlasov–Poisson model for electrons and the fluid model for ions, Ott and Sudan [33] derived a KdV equation along with an extra term responsible for the effect of linear Landau damping of electrons. In the present paper, we have considered a fully ionized collisionless unmagnetized plasma consisting of warm adiabatic ions, isothermal and nonthermal [9] electrons. So, to describe the effect of linear Landau damping of electrons on the nonlinear behaviour of IA waves propagating along x-axis, we consider the Vlasov–Poisson model for two different electron species and the fluid model for ions. The Vlasov–Poisson model for two electron species at different temperatures can be written in the following form:

$$\sqrt{\frac{m_e}{m}}\frac{\partial f_{ce}}{\partial t}+v_{||}\frac{\partial f_{ce}}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial f_{ce}}{\partial v_{||}}=0, \tag{1}$$

$$\sqrt{\frac{m_e}{m}}\frac{\partial f_{se}}{\partial t}+v_{||}\frac{\partial f_{se}}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial f_{se}}{\partial v_{||}}=0, \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2}=n_{ce}+n_{se}-n, \tag{3}$$

where

$$n_{ce}=\int_{-\infty}^{\infty} f_{ce}\mathrm{d}v_{||}, n_{se}=\int_{-\infty}^{\infty} f_{se}\mathrm{d}v_{||}. \tag{4}$$

The above equations along with the equation of continuity of ions and the equation of motion for ion fluid form a system of coupled equations. The continuity equation and the momentum equation for ion fluid can be taken as

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}=0, \tag{5}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\sigma}{n}\frac{\partial p}{\partial x}=-\frac{\partial \phi}{\partial x}. \tag{6}$$

In the momentum equation (6), the pressure term has been included to get the effect of ion temperature. To make a closed system of equations, we take the following adiabatic pressure law:

$$p=n^{\gamma}, \tag{7}$$

where we have neglected the effects of viscosity, thermal conductivity and energy transfer due to collisions.

In Eqs. (1)–(7), f_{ce}, f_{se}, n, u, $v_{||}$, p, ϕ, x and t are the velocity distribution function of nonthermal electrons, the velocity distribution function of isothermal electrons, the ion number density, the ion fluid velocity, the velocity of electrons in phase space, the ion pressure, the electrostatic potential, the spatial variable and time, respectively, and these quantities have been normalized by n_0 (unperturbed ion number density), n_0, n_0, $c_s(=\sqrt{K_B T_{ef}/m})$, $V_{te}(=\sqrt{K_B T_{ef}/m_e})$, $n_0 K_B T_i$, $K_B T_{ef}/e$, $\lambda_D(=\sqrt{K_B T_{ef}/4\pi n_0 e^2})$ and $\omega_{pi}^{-1}(=\sqrt{m/4\pi n_0 e^2})$, where $\sigma=T_i/T_{ef}$ and $\gamma(=3)$ is the adiabatic index. Again, K_B is the Boltzmann constant, m is the mass of an ion, m_e is the mass of an electron, $-e$ is the charge of an electron, T_i is the average ion temperature and T_{ef} is given by the following equation [10]:

$$\frac{n_{c0}+n_{s0}}{T_{ef}}=\frac{n_{c0}}{T_{ce}}+\frac{n_{s0}}{T_{se}}, \tag{8}$$

where n_{c0}, n_{s0}, T_{ce} and T_{se} are, respectively, unperturbed nonthermal electron number density, unperturbed isothermal electron number density, average temperature of nonthermal electrons and average temperature of isothermal electrons.

On the basis of the above-mentioned normalization of the independent and dependent variables, the unperturbed velocity distribution functions of nonthermal Cairns [9] distributed electrons and isothermal electrons can be written in the following form:

$$f_{c0}=\bar{n}_{c0}\sqrt{\frac{\sigma_c}{2\pi}}\left(\frac{1+\alpha_e\sigma_c^2 v_{||}^4}{1+3\alpha_e}\right)\exp\left[-\frac{\sigma_c v_{||}^2}{2}\right], \tag{9}$$

$$f_{s0}=\bar{n}_{s0}\sqrt{\frac{\sigma_s}{2\pi}}\exp\left[-\frac{\sigma_s v_{||}^2}{2}\right], \tag{10}$$

where $\alpha_e(\geq 0)$ is the nonthermal parameter associated with the Cairns model [9] for electron species and the expressions of $\bar{n}_{c0}$, $\bar{n}_{s0}$, σ_c and σ_s are given by

$$\bar{n}_{c0}=\frac{n_{c0}}{n_0}, \bar{n}_{s0}=\frac{n_{s0}}{n_0}, \sigma_c=\frac{T_{ef}}{T_{ce}}, \sigma_s=\frac{T_{ef}}{T_{se}}. \tag{11}$$

Using (11), Eq. (8) and the unperturbed charge neutrality condition ($n_{c0}+n_{s0}=n_0$) can be written as

$$\bar{n}_{c0}\sigma_c+\bar{n}_{s0}\sigma_s=1, \bar{n}_{c0}+\bar{n}_{s0}=1. \tag{12}$$

Following Dalui et al. [10] and using Eq. (12), we can write the expressions of $\bar{n}_{c0}$, $\bar{n}_{s0}$, σ_c and σ_s in the following form:

$$\bar{n}_{s0}=\frac{n_{sc}}{1+n_{sc}}, \bar{n}_{c0}=\frac{1}{1+n_{sc}}, \tag{13}$$

$$\sigma_s=\frac{1+n_{sc}}{\sigma_{sc}+n_{sc}}, \sigma_c=\sigma_{sc}\frac{1+n_{sc}}{\sigma_{sc}+n_{sc}}, \tag{14}$$

where $n_{sc}=\frac{n_{s0}}{n_{c0}}$ and $\sigma_{sc}=\frac{T_{se}}{T_{ce}}$.

If we neglect the electron-to-ion mass ratio, then (1) and (2) assume the following form:

$$v_{||}\frac{\partial f_{ce}}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial f_{ce}}{\partial v_{||}}=0, \tag{15}$$

$$v_{||}\frac{\partial f_{se}}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial f_{se}}{\partial v_{||}}=0. \tag{16}$$

The solutions of (15) and (16) can be written as follows:

$$f_{ce}=f_{c0}(v_{||}^2-2\phi), \tag{17}$$

$$f_{se}=f_{s0}(v_{||}^2-2\phi). \tag{18}$$

Substituting (17) and (18) into the expressions n_{ce} and n_{se} as given in the first and second equations of (4), we get the following expressions for n_{ce} and n_{se}:

$$n_{ce}=\bar{n}_{c0}(1-\beta_e\sigma_c\phi+\beta_e\sigma_c^2\phi^2)\exp[\sigma_c\phi], \tag{19}$$

$$n_{se}=\bar{n}_{s0}\exp[\sigma_s\phi]. \tag{20}$$

The linearized dispersion relation of the IA wave obtained from a set of Eqs. (5), (6), (7) and the Poisson equation (3) can be written as

$$\frac{\omega}{k}=M_s\sqrt{\frac{(M_s^2-\gamma\sigma)^{-1}+\frac{\gamma\sigma}{M_s^2}k^2}{(M_s^2-\gamma\sigma)^{-1}+k^2}}, \tag{21}$$

where ω is the normalized wave frequency and k is the normalized wave number and we have used Eqs. (19) and (20) to describe n_{ce} and n_{se} in Eq. (3). The expression of M_s is given by

$$M_s=\sqrt{\gamma\sigma+\frac{1}{\bar{n}_{so}\sigma_s+(1-\beta_e)\bar{n}_{co}\sigma_c}}. \tag{22}$$

Now, for long-wavelength plane wave perturbation, i.e. for $k\rightarrow 0$, from the linear dispersion relation (21), we have

$$\lim_{k\to 0}\frac{\omega}{k}=M_s \text{ and } \lim_{k\to 0}\frac{\partial\omega}{\partial k}=M_s. \tag{23}$$

Therefore, for long-wavelength plane wave perturbation (for small value of k), the phase of the wave can be written as

$$kx-\omega t=k(x-M_st)+\left\{\frac{M_s^2-\gamma\sigma}{\sqrt{2M_s}}\right\}^2k^3t+O(k^5). \tag{24}$$

This equation suggests to choose the stretched space coordinate and stretched time as

$$\xi=\epsilon^{\frac{1}{2}}(x-M_st),\tau=\epsilon^{\frac{3}{2}}t, \tag{25}$$

where $k=\epsilon^{\frac{1}{2}}$ and consequently, ϵ measures the weakness of dispersion. Since, we have considered the weakly nonlinear and weakly dispersive IA wave, then ϵ also measures the weakness of nonlinearity if we assume that the weakness of nonlinearity is of the same order of weakness of dispersion. Therefore, ϵ measures the weakness of dispersion as well as the weakness of nonlinearity.

In the present paper, our main aim is to consider the effect of linear Landau damping of electrons on IA solitary waves. Now, if we neglect the electron-to-ion mass ratio, then the nonlinear behaviour of the IA wave can be expressed by a set hydrodynamic equations (5), (6), (7) and (3) along with equations (19) and (20). From these hydrodynamic equations, one can analyse the nonlinear behaviour of the small amplitude IA wave with the help of usual KdV or modified KdV equations. So, to include the kinetic effect of electrons or to study the effect of linear Landau damping of electrons on IA solitary wave, we cannot neglect electron-to-ion mass ratio. But we have assumed that the effect of electron Landau damping on the nonlinear behaviour of IA wave is small and the effect of linear Landau damping of electrons on the nonlinear behaviour of IA wave is of the same order of nonlinearity, i.e. dispersion, nonlinearity and the effect of linear Landau damping of electrons are small but of the same order of magnitude. Therefore, following Ott and Sudan [33], we replace $\sqrt{m_e/m}$ by $\epsilon\alpha_1$ in Eqs. (1) and (2), and consequently these two equations can be written in the following form:

$$\alpha_1\epsilon\frac{\partial f_{ce}}{\partial t}+v_{||}\frac{\partial f_{ce}}{\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial f_{ce}}{\partial v_{||}}=0, \tag{26}$$

$$\alpha_1\epsilon\frac{\partial f_{se}}{\partial t}+v_{||}\frac{\partial f_{se}}{\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial f_{se}}{\partial v_{||}}=0. \tag{27}$$

Now using (7), the momentum equation (6) can be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\gamma\sigma n^{\gamma-2}\frac{\partial n}{\partial x}=-\frac{\partial\phi}{\partial x}. \tag{28}$$

Again, using (4), the Poisson equation (3) can be written as

$$\frac{\partial^2\phi}{\partial x^2}=\int_{-\infty}^{\infty}f_{ce}\mathrm{d}v_{||}+\int_{-\infty}^{\infty}f_{se}\mathrm{d}v_{||}-n. \tag{29}$$

Therefore, Eqs. (26), (27), (29), (5) and (28) are the basic equations to derive Korteweg–de Vries (KdV) equation and different modified Korteweg–de Vries equations including the effect of linear Landau damping of electrons. Finally, we have solved the different macroscopic nonlinear evolution equations including the kinetic effect of electrons on IA waves by considering appropriate initial and boundary conditions.

3. Derivation of different evolution equations

To derive different nonlinear evolution equations including the kinetic effect of electrons on IA waves propagating along x-axis, we consider the following stretching of the space coordinate and time:

$$\xi = \epsilon^{\frac{1}{2}}(x - Vt), \tau = \epsilon^{\frac{3}{2}}t, \tag{30}$$

where V is a constant and ϵ is a small parameter.

3.1. KdV equation including the effect Landau damping

To derive the KdV equation including the effect of linear Landau damping of electrons, we take the following perturbation expansions of the dependent variables:

$$\Lambda = \Lambda^{(0)} + \sum_{i=1}^{\infty} \epsilon^i \Lambda^{(i)}(\xi, \tau), \tag{31}$$

where $\Lambda = n,\ u,\ \phi,\ f_{ce}$ and f_{se} with $(n^{(0)}, u^{(0)},\ \phi^{(0)},\ f_{ce}^{(0)},\ f_{se}^{(0)}) = (1, 0, 0, f_{c0}, f_{s0})$.

Substituting (30) and (31) into Eqs. (26), (27), (29), (5) and (28) and collecting the terms of different powers of ϵ on both sides of each equation, we get a sequence of equations and from this sequence of equations, we get the following nonlinear evolution equation:

$$\frac{\partial\phi^{(1)}}{\partial\tau} + AB_1\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi} + \frac{1}{2}A\frac{\partial^3\phi^{(1)}}{\partial\xi^3} + \frac{1}{2}AE\alpha_1\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial\phi^{(1)}}{\partial\xi'}\frac{d\xi'}{\xi-\xi'} = 0, \tag{32}$$

where we have used the same procedure of Bandyopadhyay and Das [37] to derive Eq. (32).

The coefficients A, B_1, E are given by

$$A = \frac{1}{V}(V^2 - \sigma\gamma)^2, \tag{33}$$

$$B_1 = \frac{1}{2}\left[\frac{3V^2 + \sigma\gamma(\gamma - 2)}{(V^2 - \sigma\gamma)^3} - \left(\bar{n}_{c0}\sigma_c^2 + \bar{n}_{s0}\sigma_s^2\right)\right], \tag{34}$$

$$E = \frac{V}{\sqrt{2\pi}}\left[\bar{n}_{c0}\sigma_c^{3/2}\left(1 - \frac{3}{4}\beta_e\right) + \bar{n}_{s0}\sigma_s^{3/2}\right]. \tag{35}$$

The constant V is given by

$$(V^2 - \sigma\gamma)(1 - \bar{n}_{c0}\sigma_c\beta_e) = 1, \tag{36}$$

where $\beta_e = \frac{4\alpha_e}{1+3\alpha_e}$ and the physically admissible range of β_e is $0 \le \beta_e \le \frac{4}{7}$. The physically admissible range of β_e is pointed out by Verheest and Pillay [44]. The calculation regarding the physically admissible range of β_e has been given by Debnath et al. [45], although, mathematically, β_e is restricted by the inequality: $0 \le \beta_e < \frac{4}{3}$.

If we neglect electron-to-ion mass ratio, i.e. if we set $\alpha_1 = 0$, then the nonlinear evolution equation (32) simply reduces to the well-known KdV equation.

Equation (32) describes the propagation of weakly nonlinear and weakly dispersive IA solitary waves in a multi-species collisionless unmagnetized plasma consisting of nonthermal and isothermal electrons including the effect of linear Landau damping of electrons.

From Eq. (32), we see that the nonlinearity of the IA wave is only due to the second term of (32), i.e. AB_1 is responsible for the nonlinearity of the system. When $AB_1 = 0$, i.e. $B_1 = 0$ (as $A \neq 0$ for any set of physically admissible values of the parameters of the system), it is not possible to discuss the nonlinear behaviour of IA waves with the help of the evolution equation (32).

In Fig. 1, B_1 is plotted against σ_{sc} for $\gamma = 3$, $\sigma = 0.001$ and for (a) $n_{sc} = 0.05$, (b) $n_{sc} = 0.2$, (c) $n_{sc} = 0.3$ and (d) $n_{sc} = 0.5$. Here, red, black, green and blue curves of each figure correspond to $\beta_e = 0$, $\beta_e = 0.2$, $\beta_e = 0.4$ and $\beta_e = 0.57$ respectively. From Fig. 1(a), (b) and (c), we see that there exists a value $\sigma_{sc}^{(c)}$ of σ_{sc} such that $B_1 = 0$ at $\sigma_{sc} = \sigma_{sc}^{(c)}$, and more specifically, $B_1 < 0$ for $\sigma_{sc} < \sigma_{sc}^{(c)}$ and $B_1 > 0$ for $\sigma_{sc} > \sigma_{sc}^{(c)}$. Again, from Fig. 1(d), we see that $B_1 > 0$ for all values of β_e. From Fig. 1, it is evident that there exists a region $R_I = \{(n_{sc}, \sigma_{sc}, \beta_e) : B_1(n_{sc}, \sigma_{sc}, \beta_e) \neq 0\}$ such that each point of R_I satisfies the condition $B_1(n_{sc}, \sigma_{sc}, \beta_e) \neq 0$. On the other hand, there must exist a collection of points from the entire parameter space such that every point of the collection must satisfy equation $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$ and consequently for these values of the parameters n_{sc}, σ_{sc} and β_e we cannot use the KdV-like evolution equation to investigate the effect of linear Landau damping of electrons on IA solitary waves. To confirm the existence of a region $R_{II} =$

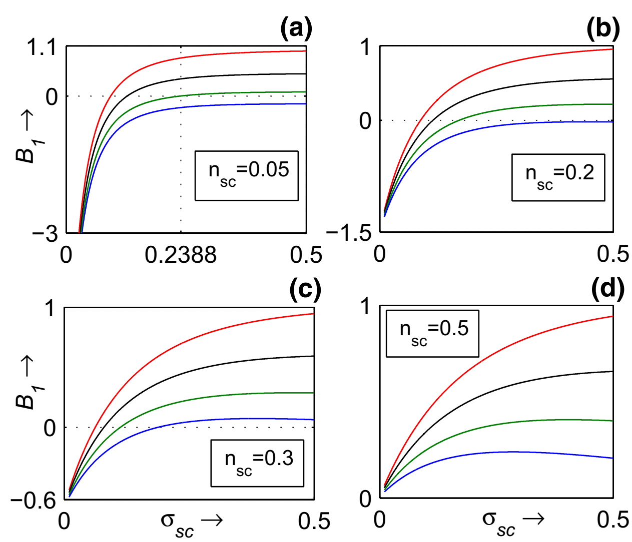

Fig. 1 B_1 is plotted against σ_{sc} for $\gamma = 3$, $\sigma = 0.001$ and for **(a)** $n_{sc} = 0.05$, **(b)** $n_{sc} = 0.2$, **(c)** $n_{sc} = 0.3$ and **(d)** $n_{sc} = 0.5$. Red, black, green and blue curves of each figure correspond to $\beta_e = 0$, $\beta_e = 0.2$, $\beta_e = 0.4$ and $\beta_e = 0.57$ respectively. **(a)**–**(c)** show the existence of points $\sigma_{sc}^{(c)}$ such that $B_1 = 0$ for some values of β_e whereas **(d)** shows that $B_1 > 0$ for all values of β_e and for all σ_{sc} lying within the interval (0, 0.5). In particular, for $n_{sc} = 0.05$, $\beta_e = 0.4$, the value of $\sigma_{sc}^{(c)}$ is 0.2388 (approx.) (colour figure online)

$\{(n_{sc}, \sigma_{sc}, \beta_e) : B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0\}$ in the entire parameter space, we consider the following figures in different parameter planes.

Now, it is simple to check that B_1 is a function of n_{sc}, σ_{sc} and β_e for any prescribed value of σ and γ, i.e. $B_1 = B_1(n_{sc}, \sigma_{sc}, \beta_e)$. Throughout this paper, we take $\gamma = 3$ and $\sigma = 0.001$, then all the coefficients A, B_1, E can be regarded as functions of n_{sc}, σ_{sc} and β_e. Therefore, B_1 is a function of σ_{sc} and n_{sc} for any given value of β_e, and consequently, $B_1 = 0$ gives a functional relationship between σ_{sc} and n_{sc}. This functional relationship between σ_{sc} and n_{sc} is plotted in Fig. 2 when $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$ for different values of β_e. Here, red, black, green and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.4$, $\beta_e = 0.5$ and $\beta_e = 0.57$ respectively. From this figure, we see that the interval of existence of σ_{sc} increases with increasing β_e when $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$.

Again, from the equation $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$, we get a functional relationship between n_{sc} and β_e for any given value of σ_{sc}. In Fig. 3, n_{sc} is plotted against β_e when $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$ for (a) $\sigma_{sc} = 0.05$, (b) $\sigma_{sc} = 0.1$, (c) $\sigma_{sc} = 0.2$ and (d) $\sigma_{sc} = 0.4$. From this figure, we see that the interval of existence of β_e decreases with increasing σ_{sc} when $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$.

Similarly, when $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$, we get a functional relationship between σ_{sc} and β_e for any fixed value of n_{sc}. When $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$, then the functional relation between σ_{sc} and β_e is plotted in Fig. 4 for different values of n_{sc} with $\sigma = 0.001$. Red, black, green and blue curves correspond to $n_{sc} = 0.1$, $n_{sc} = 0.2$, $n_{sc} = 0.3$ and $n_{sc} = 0.4$ respectively. From this figure, we see that the interval of existence of β_e increases with increasing n_{sc} whereas σ_{sc} decreases with increasing n_{sc} for any fixed β_e.

So, Figs. 1, 2, 3 and 4 confirm the existence of a region $R_{II} = \{(n_{sc}, \sigma_{sc}, \beta_e) : B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0\}$ in the parameter space such that each point of R_{II} satisfies the equation $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$. Therefore, for $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$ or for $(n_{sc}, \sigma_{sc}, \beta_e) \in R_{II}$, it is necessary to modify the KdV-like evolution equation to investigate the effect of linear Landau damping of electrons on IA solitary waves.

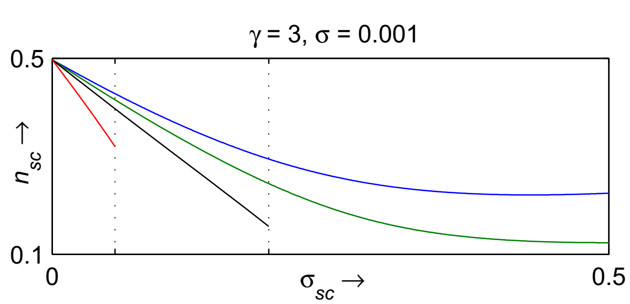

Fig. 2 n_{sc} is plotted against σ_{sc} when $B_1 = 0$ for different values of β_e. For every value of β_e, we have a curve in the $\sigma_{sc} - n_{sc}$ parameter plane and at every point on this curve we get a value of σ_{sc} as well as a value of n_{sc}, and finally for these values of β_e, σ_{sc} and n_{sc}, the equation $B_1 = 0$ holds good. Red, black, green and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.4$, $\beta_e = 0.5$ and $\beta_e = 0.57$ respectively (colour figure online)

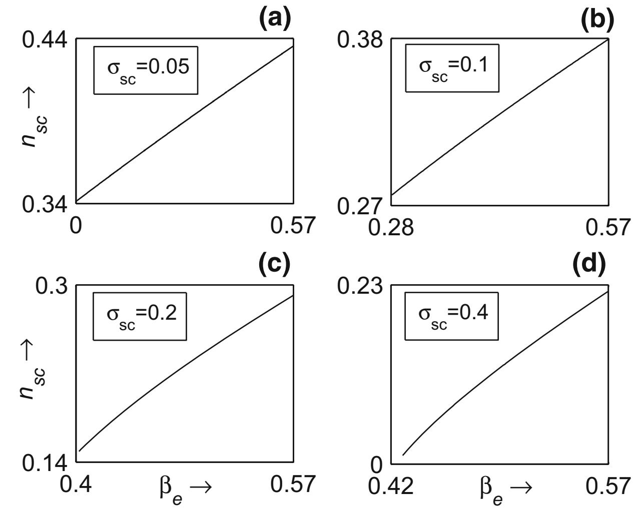

Fig. 3 n_{sc} is plotted against β_e when $B_1 = 0$ for $\sigma = 0.001$ and for different values of σ_{sc}. For every value of σ_{sc}, we have a curve in the $\beta_e - n_{sc}$ parameter plane and at every point on this curve we get a value of β_e as well as a value of n_{sc}, and finally for these values of σ_{sc}, β_e and n_{sc}, the equation $B_1 = 0$ holds good (colour figure online)

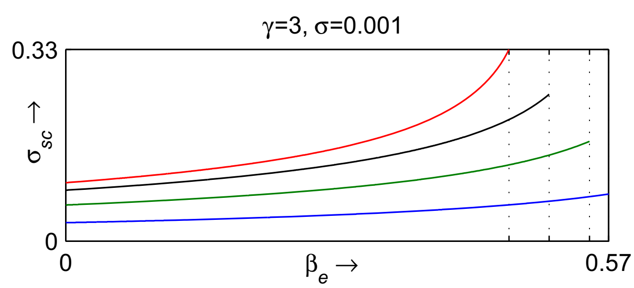

Fig. 4 σ_{sc} is plotted against β_e when $B_1 = 0$ for different values of n_{sc}. For a fixed value of n_{sc}, we have a curve in the $\beta_e - \sigma_{sc}$ parameter plane and at every point on this curve we get a value of β_e as well as a value of σ_{sc}, and finally for these values of n_{sc}, β_e and σ_{sc}, the equation $B_1 = 0$ holds good. Red, black, green and blue curves correspond to $n_{sc} = 0.1$, $n_{sc} = 0.2$, $n_{sc} = 0.3$ and $n_{sc} = 0.4$ respectively (colour figure online)

3.2. MKdV equation including the Landau damping effect

When $B_1 = 0$, we take the following perturbation expansions of the dependent variables:

$$\Lambda = \Lambda^{(0)} + \sum_{i=1}^{\infty} \epsilon^{\frac{i}{2}} \Lambda^{(i)}(\xi, \tau), \tag{37}$$

where $\Lambda = n$, u, ϕ, f_{ce} and f_{se} with $(n^{(0)}, u^{(0)}, \phi^{(0)}, f_{ce}^{(0)}, f_{se}^{(0)}) = (1, 0, 0, f_{c0}, f_{s0})$.

Substituting (30) and (37) into Eqs. (26), (27), (29), (5) and (28) and collecting the terms of different powers of ϵ, we get a sequence of equations and from this sequence of equations, following Bandyopadhyay and Das [37], we get the following nonlinear evolution equation:

$$\frac{\partial\phi^{(1)}}{\partial\tau}+AB_2[\phi^{(1)}]^2\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{1}{2}A\frac{\partial^3\phi^{(1)}}{\partial\xi^3}+\frac{1}{2}AE\alpha_1\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial\phi^{(1)}}{\partial\xi'}\frac{d\xi'}{\xi-\xi'}=0. \tag{38}$$

Here, it is important to mention that the condition $B_1=0$ has been used to eliminate the term $AB_1\frac{\partial(\phi^{(1)}\phi^{(2)})}{\partial\xi}$ from the final form of (38). The expressions of A, E, V are given by (33), (35), (36), respectively, and the expression of B_2 can be written as

$$B_2=\frac{1}{4}\left[H_2-\left(\bar{n}_{c0}\sigma_c^3(1+3\beta_e)+\bar{n}_{s0}\sigma_s^3\right)\right], \tag{39}$$

where

$$H_2=\frac{1}{(V^2-\sigma\gamma)^5}\left[15V^4+\sigma\gamma(\gamma^2+13\gamma-18)V^2+\sigma^2\gamma^2(\gamma-2)(2\gamma-3)\right]. \tag{40}$$

If $\alpha_1=0$, then the nonlinear evolution equation (38) simply reduces to the well-known MKdV equation.

From Eq. (38), we see that the nonlinearity of the IA wave is only due to the second term of (38). So, Eq. (38) describes the nonlinear dynamics of IA waves when $B_1=0$ and $B_2\neq 0$.

Now, in Fig. 5, B_2 is plotted against β_e when $B_1=0$ for $\gamma=3$ and $\sigma=0.001$, and for different values of n_{sc}. In fact, for given values of γ, σ and n_{sc}, B_1 is a function of σ_{sc} and β_e only and consequently if we solve the equation $B_1=0$ with respect to the unknown σ_{sc}, we get σ_{sc} as a function of β_e. If we put all the values of γ, σ, n_{sc} and σ_{sc} in the expression of B_2, we get B_2 as a function of β_e. This B_2 is plotted against β_e in Fig. 5. Here, red, black and blue curves correspond to $n_{sc}=0.02$, $n_{sc}=0.05$ and $n_{sc}=0.08$ respectively. This figure clearly shows that there exists a value $\beta_e^{(c)}$ of β_e such that $B_2=0$ at $\beta_e=\beta_e^{(c)}$ and more specifically, $B_2<0$ for $\beta_e<\beta_e^{(c)}$, $B_2>0$ for $\beta_e>\beta_e^{(c)}$ and $B_2=0$ at $\beta_e=\beta_e^{(c)}$. In particular, for $n_{sc}=0.05$, the value of $\beta_e^{(c)}$ is approximately equal to 0.2847. Therefore, there exist points $(n_{sc},\sigma_{sc},\beta_e)$ in the parameter space such that $B_1=B_2=0$. So, now it is necessary to divide the region R_{II} into two regions $R_{II}^{(a)}$ and $R_{II}^{(b)}$ such that $R_{II}^{(a)}=\{(n_{sc},\sigma_{sc},\beta_e):B_1=0 \text{ and } B_2\neq 0\}$ and $R_{II}^{(b)}=\{(n_{sc},\sigma_{sc},\beta_e):B_1=B_2=0\}$.

We see that Eq. (38) is free from any nonlinear effect when $B_1=B_2=0$, i.e. if $(n_{sc},\sigma_{sc},\beta_e)\in R_{II}^{(b)}$. To explain the existence of the region $R_{II}^{(b)}$, we consider Fig. 6. Now, it is simple to check that B_1 and B_2 are the functions of n_{sc}, σ_{sc} and β_e for any prescribed value of σ and γ, i.e. $B_1=B_1(n_{sc},\sigma_{sc},\beta_e)$ and $B_2=B_2(n_{sc},\sigma_{sc},\beta_e)$ for any given value of σ and γ. We have mentioned earlier that throughout this paper we take $\gamma=3$ and $\sigma=0.001$. Now, for given values of β_e and σ_{sc}, $B_1(n_{sc},\sigma_{sc},\beta_e)=0$ gives an equation for the unknown n_{sc} and consequently, $B_1=0$ gives a real solution for n_{sc}. Let $n_{sc}=n_{sc}(\beta_e,\sigma_{sc})$ be the physically admissible real solution of the equation $B_1(n_{sc},\sigma_{sc},\beta_e)=0$, i.e. the physically admissible real solution n_{sc} of the equation $B_1(n_{sc},\sigma_{sc},\beta_e)=0$ can be considered as a function of β_e and σ_{sc}. If we put this value of $n_{sc}(\beta_e,\sigma_{sc})$ in the expression of $B_2(n_{sc},\sigma_{sc},\beta_e)$ then the function B_2 is reduced to a function of β_e and σ_{sc} only, i.e. $B_2=B_2(\beta_e,\sigma_{sc})$. Again, $B_2=B_2(\beta_e,\sigma_{sc})=0$ gives a functional relationship between σ_{sc} and β_e. This functional relationship between σ_{sc} and β_e is plotted in Fig. 6, for fixed values of the other parameters, i.e. in Fig. 6, σ_{sc} is plotted against the nonthermal parameter β_e when $B_1=0$ and $B_2=0$. Figure 6 shows a variation of σ_{sc} against the

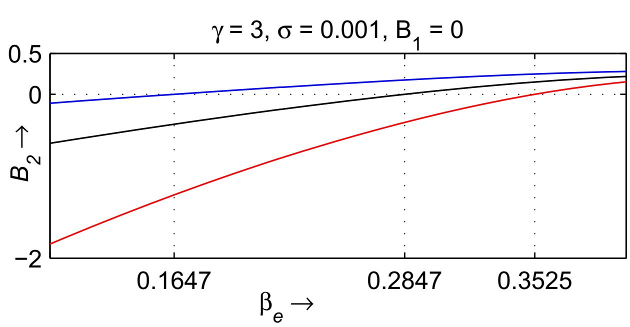

Fig. 5 B_2 is plotted against β_e when $B_1=0$ for different values of n_{sc}, i.e. the solution of the equation $B_1=0$ for the unknown σ_{sc} gives σ_{sc} as a function of β_e and consequently one can express B_2 as a function of β_e, this B_2 is plotted against β_e. Red, black and blue curves correspond to $n_{sc}=0.02$, $n_{sc}=0.05$ and $n_{sc}=0.08$ respectively. This figure shows the existence of a point $\beta_e^{(c)}$ of β_e where $B_2=0$. In particular, for $n_{sc}=0.05$, the value of $\beta_e^{(c)}$ is approximately 0.2847 (colour figure online)

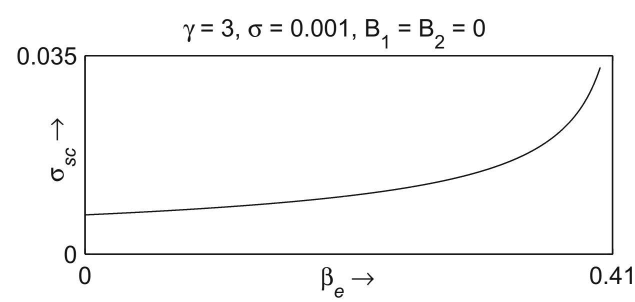

Fig. 6 σ_{sc} is plotted against β_e when $B_1=B_2=0$. Here, B_1 and B_2 both are the functions of n_{sc}, σ_{sc} and β_e. Therefore, solving the equation $B_1=0$ for the unknown n_{sc}, we get n_{sc} as a function of σ_{sc} and β_e. If we put this solution for n_{sc} in the expression of B_2, we get B_2 as a function of σ_{sc} and β_e. Finally, the equation $B_2=0$ gives σ_{sc} as a function of β_e. We plot this solution σ_{sc} against β_e and along this curve, we have $B_1=B_2=0$

nonthermal parameter β_e in β_e-σ_{sc} parameter plane when $B_1 = B_2 = 0$. This figure shows the existence of a curve in the β_e-σ_{sc} parameter plane along which $B_1 = 0$ and $B_2 = 0$. Different values of σ will give different curves in the β_e-σ_{sc} parameter plane along which $B_1 = 0$ and $B_2 = 0$. Therefore, the existence of region $R_{II}^{(b)}$ in the parameter space is confirmed, and consequently, in this region of parameter space, it is not possible to describe the nonlinear dynamics of IA waves either by the KdV-like Eq. (32) or by the MKdV-like Eq. (38). Therefore, for $B_1 = B_2 = 0$, a further modification of the evolution equation (38) is necessary to study the effect of linear Landau damping of electrons on IA solitary waves. In the next subsection, we have derived a new evolution equation when the conditions $B_1 = 0$ and $B_2 = 0$ hold simultaneously.

3.3. FMKdV equation including the Landau damping effect

To derive the FMKdV equation including the effect of linear Landau damping of electrons when the conditions $B_1 = 0$ and $B_2 = 0$ hold simultaneously, we take the following perturbation expansions of the dependent variables:

$$\Lambda = \Lambda^{(0)} + \sum_{i=1}^{\infty} \epsilon^{\frac{i}{3}} \Lambda^{(i)}(\xi, \tau), \tag{41}$$

where $\Lambda = n,\ u,\ \phi,\ f_{ce}$ and f_{se} with $(n^{(0)}, u^{(0)},\ \phi^{(0)},\ f_{ce}^{(0)},\ f_{se}^{(0)}) = (1, 0, 0, f_{c0}, f_{s0})$.

Substituting (30) and (41) into Eqs. (26), (27), (29), (5) and (28) and collecting the terms of different powers of ϵ on both sides of each equation, we get a sequence of equations.

3.3.1. *Equations for ion fluid at the order $\epsilon^{5/6}$*

At the order $\epsilon^{5/6}$, solving the equation of continuity and the equation of motion of ion fluid for the unknowns $n^{(1)}$ and $u^{(1)}$, we get

$$n^{(1)} = \frac{1}{V^2 - \sigma\gamma} \phi^{(1)}, u^{(1)} = \frac{V}{V^2 - \sigma\gamma} \phi^{(1)}. \tag{42}$$

3.3.2. *Vlasov–Boltzmann equation at the order $\epsilon^{5/6}$*

The Vlasov–Boltzmann equation of nonthermal electrons at the order $\epsilon^{5/6}$ is

$$v_{||} \frac{\partial f_{ce}^{(1)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{c0}}{\partial v_{||}} = 0. \tag{43}$$

The above equation does not have a unique solution and consequently to get the unique solution of Eq. (43), we follow the method of Ott and Sudan [33]. This method suggests to add an extra higher-order time derivative term $\epsilon^{17/6} \alpha_1 \frac{\partial f_{ce}^{(1)}}{\partial \tau}$ with the Vlasov–Boltzmann equation at the order $\epsilon^{5/6}$. So, Eq. (43) can be written in the following form:

$$\alpha_1 \epsilon^2 \frac{\partial f_{ce\epsilon}^{(1)}}{\partial \tau} + v_{||} \frac{\partial f_{ce\epsilon}^{(1)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{c0}}{\partial v_{||}} = 0, \tag{44}$$

where $f_{ce}^{(1)}$ is replaced by $f_{ce\epsilon}^{(1)}$ and one can get $f_{ce}^{(1)}$ from the solution of the above equation by considering the following relation for $j = 1$.

$$f_{ce}^{(j)} = \lim_{\epsilon \to 0} f_{ce\epsilon}^{(j)}, \quad j = 1, 2, 3, \ldots. \tag{45}$$

To solve (44), we have assumed that the time dependence of any perturbed quantity is of the form $\exp(i\omega\tau)$ and we can write Eq. (44) as

$$i\alpha_1 \omega \epsilon^2 f_{ce\epsilon}^{(1)} + v_{||} \frac{\partial f_{ce\epsilon}^{(1)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{c0}}{\partial v_{||}} = 0. \tag{46}$$

Now, taking the Fourier transform of this equation with respect to ξ, we get

$$\widetilde{f}_{ce\epsilon}^{(1)} = -2 \frac{\partial f_{c0}}{\partial v_{||}^2} \frac{s v_{||}}{s v_{||} + \alpha_1 \omega \epsilon^2} \widetilde{\phi}^{(1)}, \tag{47}$$

where the Fourier transform of g with respect to ξ is defined as

$$\widetilde{g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{-is\xi} d\xi. \tag{48}$$

Again, using the Landau prescription to resolve the singularities involved, Eq. (47) can be written as

$$\widetilde{f}_{ce\epsilon}^{(1)} = -2 \frac{\partial f_{c0}}{\partial v_{||}^2} \left\{ s v_{||} \mathcal{P} \frac{1}{s v_{||} + \alpha_1 \omega \epsilon^2} + i\pi s v_{||} \delta(s v_{||} + \alpha_1 \omega \epsilon^2) \right\} \widetilde{\phi}^{(1)}. \tag{49}$$

Taking limit $\epsilon \to 0$, we get

$$\widetilde{f}_{ce}^{(1)} = -2 \frac{\partial f_{c0}}{\partial v_{||}^2} \left\{ s v_{||} \mathcal{P} \frac{1}{s v_{||}} + i\pi s v_{||} \delta(s v_{||}) \right\} \widetilde{\phi}^{(1)}, \tag{50}$$

where we have used the relation (45) for $j = 1$.

Now, using the relations $x\mathcal{P}(1/x) = 1$ and $x\delta(x) = 0$, Eq. (50) can be simplified as

$$\widetilde{f}_{ce}^{(1)} = -2 \frac{\partial f_{c0}}{\partial v_{||}^2} \widetilde{\phi}^{(1)}. \tag{51}$$

Taking Fourier inversion of the above equation, we get

$$f_{ce}^{(1)} = -2 \frac{\partial f_{c0}}{\partial v_{||}^2} \phi^{(1)}. \tag{52}$$

Similarly, considering the Vlasov–Boltzmann equation of isothermal electrons at the order $\epsilon^{5/6}$, we get

$$f_{\rm se}^{(1)} = -2\frac{\partial f_{\rm s0}}{\partial v_{\|}^2}\phi^{(1)}. \tag{53}$$

3.3.3. Poisson equation at the order $\epsilon^{\frac{1}{3}}$

From the Poisson equation at the order $\epsilon^{\frac{1}{3}}$, we get

$$n^{(1)} = \int_{-\infty}^{\infty} f_{\rm ce}^{(1)} {\rm d}v_{\|} + \int_{-\infty}^{\infty} f_{\rm se}^{(1)} {\rm d}v_{\|}. \tag{54}$$

Using (52) and (53), the above equation can be written in the following form:

$$n^{(1)} = (1 - \bar{n}_{\rm c0}\sigma_{\rm c}\beta_{\rm e})\phi^{(1)}. \tag{55}$$

Using this equation and the first equation of (42), we get Eq. (36). Therefore, the Poisson equation at the order $\epsilon^{\frac{1}{3}}$ gives the dispersion relation (36) which determines the constant V.

3.3.4. Equations for ion fluid at the order $\epsilon^{7/6}$

At the order $\epsilon^{7/6}$, solving the continuity equation and the momentum equation of ion fluid for the unknowns $n^{(2)}$ and $u^{(2)}$, we get

$$n^{(2)} = \frac{\phi^{(2)}}{(V^2 - \sigma\gamma)} + \frac{3V^2 + \sigma\gamma(\gamma - 2)}{2(V^2 - \sigma\gamma)^3}[\phi^{(1)}]^2, \tag{56}$$

$$u^{(2)} = \frac{V\phi^{(2)}}{(V^2 - \sigma\gamma)} + \frac{V(V^2 + \sigma\gamma^2)}{2(V^2 - \sigma\gamma)^3}[\phi^{(1)}]^2. \tag{57}$$

3.3.5. Vlasov–Boltzmann equation at the order $\epsilon^{7/6}$

At the order $\epsilon^{7/6}$, the Vlasov–Boltzmann equation for nonthermal and isothermal electrons are

$$v_{\|}\frac{\partial f_{\rm ce}^{(2)}}{\partial \xi} + \frac{\partial \phi^{(2)}}{\partial \xi}\frac{\partial f_{\rm c0}}{\partial v_{\|}} + \frac{\partial \phi^{(1)}}{\partial \xi}\frac{\partial f_{\rm ce}^{(1)}}{\partial v_{\|}} = 0, \tag{58}$$

$$v_{\|}\frac{\partial f_{\rm se}^{(2)}}{\partial \xi} + \frac{\partial \phi^{(2)}}{\partial \xi}\frac{\partial f_{\rm s0}}{\partial v_{\|}} + \frac{\partial \phi^{(1)}}{\partial \xi}\frac{\partial f_{\rm se}^{(1)}}{\partial v_{\|}} = 0. \tag{59}$$

Following exactly the same analysis as given in Sect. 3.3.2, the solutions of (58) and (59) can be written as follows:

$$f_{\rm ce}^{(2)} = -2\frac{\partial f_{\rm c0}}{\partial v_{\|}^2}\phi^{(2)} - 2\frac{\partial g_{\rm c0}}{\partial v_{\|}^2}\psi^{(2)}, \tag{60}$$

$$f_{\rm se}^{(2)} = -2\frac{\partial f_{\rm s0}}{\partial v_{\|}^2}\phi^{(2)} - 2\frac{\partial g_{\rm s0}}{\partial v_{\|}^2}\psi^{(2)}, \tag{61}$$

where

$$\psi^{(2)} = -(\phi^{(1)})^2, g_{\rm c0} = \frac{\partial f_{\rm c0}}{\partial v_{\|}^2}, g_{\rm s0} = \frac{\partial f_{\rm s0}}{\partial v_{\|}^2}. \tag{62}$$

3.3.6. Poisson equation at the order $\epsilon^{\frac{2}{3}}$

It is simple to check that the Poisson equation at the order $\epsilon^{\frac{2}{3}}$ is identically satisfied due to the dispersion relation (36) and the condition $B_1 = 0$.

3.3.7. Equations for ion fluid at the order $\epsilon^{9/6}$

Again, at the order $\epsilon^{9/6}$, solving the continuity equation of ions and the momentum equation of ions for the unknowns $n^{(3)}$ and $u^{(3)}$, we obtain the following equations:

$$n^{(3)} = \frac{\phi^{(3)}}{V^2 - \sigma\gamma} + \frac{3V^2 + \sigma\gamma(\gamma - 2)}{(V^2 - \sigma\gamma)^3}\phi^{(1)}\phi^{(2)} + \frac{1}{6}H_2[\phi^{(1)}]^3, \tag{63}$$

$$u^{(3)} = \frac{V\phi^{(3)}}{V^2 - \sigma\gamma} + \frac{V(V^2 + \sigma\gamma^2)}{(V^2 - \sigma\gamma)^3}\phi^{(1)}\phi^{(2)} + G_2[\phi^{(1)}]^3, \tag{64}$$

where

$$G_2 = \frac{V}{6}\left[\frac{3V^4 + \sigma\gamma^2(\gamma + 7)V^2 + \sigma^2\gamma^3(2\gamma - 1)}{(V^2 - \sigma\gamma)^5}\right]. \tag{65}$$

3.3.8. Vlasov–Boltzmann equation at the order $\epsilon^{9/6}$

Following exactly the same analysis as given in Sect. 3.3.2, the solutions of the Vlasov–Boltzmann equations of non-thermal and isothermal electrons at the order $\epsilon^{9/6}$ can be written as follows:

$$f_{\rm ce}^{(3)} = -2\frac{\partial f_{\rm c0}}{\partial v_{\|}^2}\phi^{(3)} - 2\frac{\partial g_{\rm c0}}{\partial v_{\|}^2}\psi^{(3)} - 2\frac{\partial h_{\rm c0}}{\partial v_{\|}^2}\chi^{(3)}, \tag{66}$$

$$f_{\rm se}^{(3)} = -2\frac{\partial f_{\rm s0}}{\partial v_{\|}^2}\phi^{(3)} - 2\frac{\partial g_{\rm s0}}{\partial v_{\|}^2}\psi^{(3)} - 2\frac{\partial h_{\rm s0}}{\partial v_{\|}^2}\chi^{(3)}, \tag{67}$$

where

$$\psi^{(3)} = -2\phi^{(1)}\phi^{(2)}, \chi^{(3)} = \frac{2}{3}(\phi^{(1)})^3, h_{\rm c0} = \frac{\partial g_{\rm c0}}{\partial v_{\|}^2}, h_{\rm s0} = \frac{\partial g_{\rm s0}}{\partial v_{\|}^2}. \tag{68}$$

3.3.9. Poisson equation at the order ϵ

It is simple to check that the Poisson equation at the order ϵ is also identically satisfied due to the dispersion relation (36) and the conditions $B_1 = 0$ and $B_2 = 0$.

3.3.10. Equations for ion fluid at the order $\epsilon^{11/6}$

At the order $\epsilon^{11/6}$, solving the continuity equation and the momentum equation of ions, $\frac{\partial n^{(4)}}{\partial \xi}$ and $\frac{\partial u^{(4)}}{\partial \xi}$ can be expressed as functions of $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$ and $\phi^{(4)}$ along with their different derivatives with respect to ξ and τ. In particular, $\frac{\partial n^{(4)}}{\partial \xi}$ can be written as

$$\begin{aligned}\frac{\partial n^{(4)}}{\partial \xi} =& \frac{1}{(V^2-\sigma\gamma)}\frac{\partial \phi^{(4)}}{\partial \xi} + \frac{2V}{(V^2-\sigma\gamma)^2}\frac{\partial \phi^{(1)}}{\partial \tau} \\ &+ \frac{3V^2+\sigma\gamma(\gamma-2)}{(V^2-\sigma\gamma)^3}\frac{\partial}{\partial \xi}[\phi^{(1)}\phi^{(3)}+\frac{1}{2}(\phi^{(2)})^2] \\ &+ \frac{H_2}{2}\frac{\partial}{\partial \xi}[(\phi^{(1)})^2\phi^{(2)}] + \frac{H_3}{6}[\phi^{(1)}]^3\frac{\partial \phi^{(1)}}{\partial \xi},\end{aligned} \tag{69}$$

where

$$\begin{aligned}H_3 =& \frac{1}{(V^2-\sigma\gamma)^7}[105V^6+\sigma\gamma(\gamma^3+21\gamma^2+161\gamma-174)V^4 \\ &+\sigma^2\gamma^2(8\gamma^3+53\gamma^2-162\gamma+108)V^2 \\ &+\sigma^3\gamma^3(\gamma-2)(2\gamma-3)(3\gamma-4)].\end{aligned} \tag{70}$$

3.3.11. *Vlasov–Boltzmann equation at the order $\epsilon^{11/6}$*

At the order $\epsilon^{11/6}$, the Vlasov–Boltzmann equation of nonthermal electrons is

$$\begin{aligned}&v_{||}\frac{\partial f_{ce}^{(4)}}{\partial \xi} + \frac{\partial \phi^{(4)}}{\partial \xi}\frac{\partial f_{c0}}{\partial v_{||}} + \frac{\partial \psi^{(4)}}{\partial \xi}\frac{\partial g_{c0}}{\partial v_{||}} + \frac{\partial \chi^{(4)}}{\partial \xi}\frac{\partial h_{c0}}{\partial v_{||}} \\ &+ \frac{\partial \kappa^{(4)}}{\partial \xi}\frac{\partial k_{c0}}{\partial v_{||}} + 2\alpha_1 V\frac{\partial f_{c0}}{\partial v_{||}^2}\frac{\partial \phi^{(1)}}{\partial \xi} = 0,\end{aligned} \tag{71}$$

where we have used Eqs. (52), (60) and (66) to get Eq. (71) and in this equation, we have used the following notations:

$$\begin{aligned}&\psi^{(4)} = -2\phi^{(1)}\phi^{(3)} - (\phi^{(2)})^2, \chi^{(4)} = 2(\phi^{(1)})^2\phi^{(2)}, \\ &\kappa^{(4)} = -\frac{1}{3}(\phi^{(1)})^4, k_{c0} = \frac{\partial h_{c0}}{\partial v_{||}^2}.\end{aligned} \tag{72}$$

Including an extra higher-order time derivative term $\epsilon^{23/6}\alpha_1\frac{\partial f_{ce}^{(4)}}{\partial \tau}$, Eq. (71) can be written as

$$\begin{aligned}&\alpha_1\epsilon^2\frac{\partial f_{ce\epsilon}^{(4)}}{\partial \tau} + v_{||}\frac{\partial f_{ce\epsilon}^{(4)}}{\partial \xi} + \frac{\partial \phi^{(4)}}{\partial \xi}\frac{\partial f_{c0}}{\partial v_{||}} + \frac{\partial \psi^{(4)}}{\partial \xi}\frac{\partial g_{c0}}{\partial v_{||}} \\ &+ \frac{\partial \chi^{(4)}}{\partial \xi}\frac{\partial h_{c0}}{\partial v_{||}} + \frac{\partial \kappa^{(4)}}{\partial \xi}\frac{\partial k_{c0}}{\partial v_{||}} \\ &+ 2\alpha_1 V\frac{\partial f_{c0}}{\partial v_{||}^2}\frac{\partial \phi^{(1)}}{\partial \xi} = 0,\end{aligned} \tag{73}$$

where $f_{ce}^{(4)}$ is replaced by $f_{ce\epsilon}^{(4)}$ and $f_{ce}^{(4)}$ can be obtained from the unique solution of Eq. (73) by considering the relation (45) for $j=4$.

Now, assuming the τ dependence of the perturbed quantities is of the form $\exp(i\omega\tau)$ and taking the Fourier transform with respect to ξ, we get the following equation from Eq. (73):

$$\begin{aligned}i\widetilde{f}_{ce\epsilon}^{(4)} =& -\frac{2}{s}\left[\frac{\partial f_{c0}}{\partial v_{||}^2}\widetilde{\phi}_{\xi}^{(4)} + \frac{\partial g_{c0}}{\partial v_{||}^2}\widetilde{\psi}_{\xi}^{(4)} + \frac{\partial h_{c0}}{\partial v_{||}^2}\widetilde{\chi}_{\xi}^{(4)} + \frac{\partial k_{c0}}{\partial v_{||}^2}\widetilde{\kappa}_{\xi}^{(4)}\right] \\ &\times\frac{sv_{||}}{sv_{||}+\alpha_1\omega\epsilon^2} - 2\alpha_1 V\frac{\partial f_{c0}}{\partial v_{||}^2}\frac{1}{sv_{||}+\alpha_1\omega\epsilon^2}\widetilde{\phi}_{\xi}^{(1)},\end{aligned} \tag{74}$$

where $\widetilde{\phi}_{\xi}^{(4)}$, $\widetilde{\psi}_{\xi}^{(4)}$, $\widetilde{\chi}_{\xi}^{(4)}$, $\widetilde{\kappa}_{\xi}^{(4)}$ and $\widetilde{\phi}_{\xi}^{(1)}$ are, respectively, the Fourier transform of $\phi_{\xi}^{(4)}$, $\psi_{\xi}^{(4)}$, $\chi_{\xi}^{(4)}$, $\kappa_{\xi}^{(4)}$ and $\phi_{\xi}^{(1)}$.

Now, making $\epsilon\to 0$ and using the relations $x\mathcal{P}(1/x)=1$, $x\delta(x)=0$ and $s\delta(sv_{||}) = \text{sgn}(s)\delta(v_{||})$, we get the following expression of $\widetilde{f}_{ce}^{(4)}$:

$$\begin{aligned}is\widetilde{f}_{ce}^{(4)} =& -2\left[\frac{\partial f_{c0}}{\partial v_{||}^2}\widetilde{\phi}_{\xi}^{(4)} + \frac{\partial g_{c0}}{\partial v_{||}^2}\widetilde{\psi}_{\xi}^{(4)} + \frac{\partial h_{c0}}{\partial v_{||}^2}\widetilde{\chi}_{\xi}^{(4)} + \frac{\partial k_{c0}}{\partial v_{||}^2}\widetilde{\kappa}_{\xi}^{(4)}\right] \\ &-2\alpha_1 V\frac{\partial f_{c0}}{\partial v_{||}^2}\left[sP\left(\frac{1}{sv_{||}}\right) + i\pi\,\text{sgn}(s)\delta(v_{||})\right]\widetilde{\phi}_{\xi}^{(1)}.\end{aligned} \tag{75}$$

Integrating (75) over the velocity space, we get

$$\begin{aligned}is\int_{-\infty}^{+\infty}\widetilde{f}_{ce}^{(4)}\mathrm{d}v_{||} =& -2\left[F_{c0}\widetilde{\phi}_{\xi}^{(4)} + G_{c0}\widetilde{\psi}_{\xi}^{(4)} + H_{c0}\widetilde{\chi}_{\xi}^{(4)} + K_{c0}\widetilde{\kappa}_{\xi}^{(4)}\right] \\ &- 2i\pi\alpha_1 VZ_{c0}\,\text{sgn}(s)\widetilde{\phi}_{\xi}^{(1)},\end{aligned} \tag{76}$$

where F_{c0}, G_{c0}, H_{c0}, K_{c0}, Z_{c0} are given in Appendix 1.

Taking Fourier inversion of (76), we get

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(\int_{-\infty}^{+\infty}f_{ce}^{(4)}\mathrm{d}v_{||}\right) =& -2\left[F_{c0}\phi_{\xi}^{(4)} + G_{c0}\psi_{\xi}^{(4)} + H_{c0}\chi_{\xi}^{(4)} + K_{c0}\kappa_{\xi}^{(4)}\right] \\ &+ 2\alpha_1 VZ_{c0}\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial \phi^{(1)}}{\partial \xi'}\frac{\mathrm{d}\xi'}{\xi-\xi'},\end{aligned} \tag{77}$$

where we have used the convolution theorem of Fourier transform to find the inverse Fourier transform of $\text{sgn}(s)\widetilde{\phi}_{\xi}^{(1)}$. Here, $\frac{\partial \phi^{(1)}}{\partial \xi'}$ is the value of $\frac{\partial \phi^{(1)}}{\partial \xi}$ at $\xi=\xi'$.

Similarly, considering the Vlasov–Boltzmann equation of the isothermal electrons at the order $\epsilon^{11/6}$, we get

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(\int_{-\infty}^{+\infty}f_{se}^{(4)}\mathrm{d}v_{||}\right) =& -2\left[F_{s0}\phi_{\xi}^{(4)} + G_{s0}\psi_{\xi}^{(4)} + H_{s0}\chi_{\xi}^{(4)} + K_{s0}\kappa_{\xi}^{(4)}\right] \\ &+ 2\alpha_1 VZ_{s0}\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial \phi^{(1)}}{\partial \xi'}\frac{\mathrm{d}\xi'}{\xi-\xi'},\end{aligned} \tag{78}$$

where F_{s0}, G_{s0}, H_{s0}, K_{s0}, Z_{s0} are given in Appendix 2.

3.3.12. *Poisson equation at the order $\epsilon^{4/3}$*

From the Poisson equation at the order $\epsilon^{\frac{4}{3}}$, we get

$$n^{(4)} = \int_{-\infty}^{\infty} f_{ce}^{(4)} dv_{\parallel} + \int_{-\infty}^{\infty} f_{se}^{(4)} dv_{\parallel} - \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}. \tag{79}$$

Differentiating this equation with respect to ξ, using equations (77) and (78) in the resulting equation, we get the following expression of $\frac{\partial n^{(4)}}{\partial \xi}$ as follows:

$$\begin{aligned}\frac{\partial n^{(4)}}{\partial \xi} =& -\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + (1 - \bar{n}_{c0}\sigma_c \beta_e)\frac{\partial \phi^{(4)}}{\partial \xi} \\ &+ \left(\bar{n}_{c0}\sigma_c^2 + \bar{n}_{s0}\sigma_s^2\right)\frac{\partial}{\partial \xi}[\phi^{(1)}\phi^{(3)} + \frac{1}{2}(\phi^{(2)})^2] \\ &+ \frac{1}{2}\left[\bar{n}_{c0}\sigma_c^3(1+3\beta_e) + \bar{n}_{s0}\sigma_s^3\right]\frac{\partial}{\partial \xi}[(\phi^{(1)})^2\phi^{(2)}] \\ &+ \frac{1}{6}\left[\bar{n}_{c0}\sigma_c^4(1+8\beta_e) + \bar{n}_{s0}\sigma_s^4\right][\phi^{(1)}]^3\frac{\partial \phi^{(1)}}{\partial \xi} \\ &- \alpha_1 E \mathcal{P}\int_{-\infty}^{\infty}\frac{\partial \phi^{(1)}}{\partial \xi'}\frac{d\xi'}{\xi - \xi'}.\end{aligned} \tag{80}$$

Now, eliminating $\frac{\partial n^{(4)}}{\partial \xi}$ from Eqs. (69) and (80), we get

$$\begin{aligned}&\frac{\partial \phi^{(1)}}{\partial \tau} + AB_3[\phi^{(1)}]^3\frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2}A\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{1}{2}AE\alpha_1\mathcal{P} \\ &\int_{-\infty}^{\infty}\frac{\partial \phi^{(1)}}{\partial \xi'}\frac{d\xi'}{\xi - \xi'} = 0,\end{aligned} \tag{81}$$

where we have used the dispersion relation (36), conditions $B_1 = 0$ and $B_2 = 0$ to eliminate the terms $\frac{\partial \phi^{(4)}}{\partial \xi}$, $AB_1\frac{\partial}{\partial \xi}[\phi^{(1)}\phi^{(3)} + \frac{1}{2}(\phi^{(2)})^2]$ and $AB_2\frac{\partial}{\partial \xi}[(\phi^{(1)})^2\phi^{(2)}]$ respectively, to simplify Eq. (81).

Here, B_3 is given by

$$B_3 = \frac{1}{12}\left[H_3 - \left(\bar{n}_{c0}\sigma_c^4(1+8\beta_e) + \bar{n}_{s0}\sigma_s^4\right)\right], \tag{82}$$

where H_3 is given by Eq. (70).

Therefore, the Poisson equation at the order $\epsilon^{\frac{4}{3}}$ gives a FMKdV equation including the effect of Landau damping which describes the nonlinear behaviour of IA waves when $B_1 = 0$, $B_2 = 0$ but $B_3 \neq 0$.

4. Solitary wave solution

In more compact form, we can write the KdV equation, MKdV equation and FMKdV equation as

$$\begin{aligned}&\frac{\partial \phi^{(1)}}{\partial \tau} + AB_r[\phi^{(1)}]^r\frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2}A\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{1}{2}AE\alpha_1\mathcal{P} \\ &\int_{-\infty}^{\infty}\frac{\partial \phi^{(1)}}{\partial \xi'}\frac{d\xi'}{\xi - \xi'} = 0,\end{aligned} \tag{83}$$

where $r = 1, 2, 3$.

If we put $\alpha_1 = 0$ in Eq. (83), then Eq. (83) reduces to a KdV equation for $r = 1$, an MKdV equation for $r = 2$ and a FMKdV equation for $r = 3$.

For a solitary wave solution of (83) with $\alpha_1 = 0$, we consider the following transformation of the independent variables:

$$X = \xi - U\tau, \tau' = \tau. \tag{84}$$

Under the above transformation of independent variables, Eq. (83) with $\alpha_1 = 0$ assumes the following form:

$$\frac{\partial \phi^{(1)}}{\partial \tau} - U\frac{\partial \phi^{(1)}}{\partial X} + AB_r[\phi^{(1)}]^r\frac{\partial \phi^{(1)}}{\partial X} + \frac{1}{2}A\frac{\partial^3 \phi^{(1)}}{\partial X^3} = 0, \tag{85}$$

where we drop the prime on the independent variable τ to simplify the notation.

For the travelling wave solution of (85), we take

$$\phi^{(1)} = \phi_0(X). \tag{86}$$

Substituting (86) into (85), we get the following ordinary differential equation of ϕ_0:

$$-U\frac{d\phi_0}{dX} + AB_r[\phi_0]^r\frac{d\phi_0}{dX} + \frac{1}{2}A\frac{d^3\phi_0}{dX^3} = 0. \tag{87}$$

To get the solitary wave solution of (87), we use the boundary conditions: ϕ_0, $\frac{d^n\phi_0}{dX^n} \to 0$ as $|X| \to \infty$ for $n = 1$, 2, 3, ... and using these conditions, the solitary wave solution of (87) can be written as

$$\phi_0 = a \ \operatorname{sech}^{\frac{2}{r}}[WX], \tag{88}$$

where the amplitude (a) and width ($\frac{1}{W}$) are given by

$$a^r = \frac{(r+1)(r+2)U}{2AB_r} \quad \text{and} \quad W^2 = \frac{r^2 U}{2A}. \tag{89}$$

Now, using (89), Eq. (88) can be written as

$$\phi_0 = a \ \operatorname{sech}^{\frac{2}{r}}\left[\sqrt{\frac{r^2 a^r B_r}{(r+1)(r+2)}}\left\{\xi - \frac{2AB_r a^r \tau}{(r+1)(r+2)}\right\}\right]. \tag{90}$$

Again, multiplying Eq. (83) by $\phi^{(1)}$ and then integrating the resulting equation with respect to ξ within the interval $(-\infty, \infty)$, and finally, using the boundary conditions: $\phi^{(1)}$, $\frac{\partial^n \phi^{(1)}}{\partial \xi^n} \to 0$ as $|\xi| \to \infty$ for $n = 1, 2, 3, \ldots$, we get the following equation:

$$\begin{aligned}&\frac{\partial}{\partial \tau}\int_{-\infty}^{\infty}\left(\phi^{(1)}\right)^2 d\xi = -AE\alpha_1\int_{-\infty}^{\infty}\phi^{(1)} \\ &\left[\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial \phi^{(1)}}{\partial \xi'}\frac{d\xi'}{\xi - \xi'}\right]d\xi.\end{aligned} \tag{91}$$

If we neglect the electron-to-ion mass ratio, then Eq. (91) reduces to the following equation: $\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}(\phi^{(1)})^2 d\xi = 0$. This equation shows that the wave energy is conserved. On the other hand, if $\alpha_1 \neq 0$ and if the initial perturbation is of the form (90), then the integral appearing in the right-hand side of (91) is positive for $r = 1, 2, 3$ and consequently from (91), we have the inequality : $\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}\left(\phi^{(1)}\right)^2 d\xi < 0$ for any values of the parameters of the system, because A, E, α_1 are all strictly positive. This inequality shows that the initial perturbation of the form (90) will decay to zero. This phenomenon suggests that the amplitude of the solitary wave solution of the form (90) is not a constant but decreases slowly with time.

Now for $\alpha_1 \neq 0$, to get a solitary wave solution of Eq. (83), we shall follow the method of Ott and Sudan [33]. So, using the prescription of Ott and Sudan [33], we have introduced the following space coordinate:

$$X = \sqrt{\frac{r^2 a^r B_r}{(r+1)(r+2)}}\left[\xi - \frac{2AB_r}{(r+1)(r+2)}\int_0^\tau a^r d\tau\right], \tag{92}$$

where the amplitude (a) is a slowly varying function of time. Therefore, considering $\phi^{(1)}$ as a function of X and τ, i.e. $\phi^{(1)} = \phi^{(1)}(X,\tau)$, Eq. (83) can be written as

$$\begin{aligned}&\frac{\partial\phi^{(1)}}{\partial\tau} + \left[-\frac{2AB_r a^r W}{(r+1)(r+2)} + \frac{rX}{2a}\frac{\partial a}{\partial\tau}\right]\frac{\partial\phi^{(1)}}{\partial X}\\ &+ AB_r W\left(\phi^{(1)}\right)^r\frac{\partial\phi^{(1)}}{\partial X} + \frac{1}{2}AW^3\frac{\partial^3\phi^{(1)}}{\partial X^3}\\ &+ \frac{1}{2}AEW\alpha_1\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial\phi^{(1)}}{\partial X'}\frac{dX'}{X-X'} = 0,\end{aligned} \tag{93}$$

where $\frac{\partial\phi^{(1)}}{\partial X'} = \frac{\partial\phi^{(1)}}{\partial X}$ at $X = X'$.

To find the solitary wave solution, we follow the procedure of Ott and Sudan [33] and considering two time scales with respect to α_1 as $\tau_0 = \tau$, $\tau_1 = \alpha_1\tau$, We take the solution of (93) as

$$\phi^{(1)}(X,\tau) = q^{(0)}(X,\tau_0,\tau_1) + \alpha_1 q^{(1)}(X,\tau_0,\tau_1) + O(\alpha_1^2). \tag{94}$$

Substituting (94) into (93) and equating the coefficients of order unity $[(\alpha_1)^0]$ and order α_1 $[(\alpha_1)^1]$ on each side of the resulting equation, we get the following equations:

$$\rho\left[\frac{\partial q^{(0)}}{\partial\tau_0} + \frac{rX}{2a}\frac{\partial a}{\partial\tau_0}\frac{\partial q^{(0)}}{\partial X}\right] + L\left[\frac{\partial q^{(0)}}{\partial X}\right] = 0, \tag{95}$$

$$\rho\left[\frac{\partial q^{(1)}}{\partial\tau_0} + \frac{rX}{2a}\frac{\partial a}{\partial\tau_0}\frac{\partial q^{(1)}}{\partial X}\right] + \frac{\partial[Lq^{(1)}]}{\partial X} = \rho Mq^{(0)}, \tag{96}$$

where

$$\rho = \frac{2}{AW^3}, L = \frac{\partial^2}{\partial X^2} + \frac{2(r+1)(r+2)}{r^2 a^r}(q^{(0)})^r - \frac{4}{r^2}, \tag{97}$$

$$-Mq^{(0)} = \frac{\partial q^{(0)}}{\partial\tau_1} + \frac{rX}{2a}\frac{\partial a}{\partial\tau_1}\frac{\partial q^{(0)}}{\partial X} + \frac{1}{2}AEW\mathcal{P}\int_{-\infty}^{\infty}\frac{\partial q^{(0)}}{\partial X'}\frac{dX'}{X-X'}. \tag{98}$$

Now, in view of initial and boundary conditions: $\phi^{(1)}(X,0) = a_0 \operatorname{sech}^{\frac{2}{r}}X$ and $\phi^{(1)}(\pm\infty,\tau) = 0$, it is simple to check that $q^{(0)} = a \operatorname{sech}^{\frac{2}{r}}[X]$ is the soliton solution of (95) if and only if $\frac{\partial a}{\partial\tau_0} = 0$ and consequently the solution of (95) can be written in the following form: $q^{(0)} = a(\tau_1)\operatorname{sech}^{\frac{2}{r}}[X]$, where $a(\tau_1)$ is an arbitrary function of τ_1 except for the initial condition $a(0) = a_0$. Therefore, Eq. (96) can be written as

$$\rho\frac{\partial q^{(1)}}{\partial\tau_0} + \frac{\partial}{\partial X}[Lq^{(1)}] = \rho Mq^{(0)}. \tag{99}$$

Now, for the existence of the solution of (99), we have the following consistency condition:

$$\int_{-\infty}^{\infty}\operatorname{sech}^{\frac{2}{r}}[X]Mq^{(0)}\mathrm{d}X = 0. \tag{100}$$

The above equation states that the right-hand side of (99) is perpendicular to the kernel of adjoint operator of $\frac{\partial}{\partial X}[L]$ and this kernel is $\operatorname{sech}^{\frac{2}{r}}[X]$, which satisfies the boundary conditions at $X = \pm\infty$, i.e. $\operatorname{sech}^{\frac{2}{r}}[X] \to 0$ as $X \to \pm\infty$.

Eq. (100) gives the following differential equation for the solitary wave amplitude a:

$$M_r\frac{\partial}{\partial\tau_1}\left(\frac{a}{a_0}\right) + W_r\left(\frac{a}{a_0}\right)^{\frac{r}{2}+1} = 0, \tag{101}$$

where a_0 is the value of a when $\tau = 0$ and

$$M_r = \int_{-\infty}^{\infty}[\operatorname{sech} X]^{\frac{4}{r}}(1 - X\tanh X)\mathrm{d}X, \tag{102}$$

$$W_r = \frac{1}{2}I_r AE\sqrt{\frac{r^2 a_0^r B_r}{(r+1)(r+2)}}, \tag{103}$$

$$I_r = \mathcal{P}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\operatorname{sech} X]^{\frac{2}{r}}\frac{\partial[\operatorname{sech} X']^{\frac{2}{r}}}{\partial X'}\frac{\mathrm{d}X\mathrm{d}X'}{X-X'}. \tag{104}$$

Now, it is simple to check that $M_1 = 1$, $M_2 = 1$, $M_3 \approx 0.6468$. In Appendix 3, we have generalized the method of Weiland et al. [46] to find I_r. Using this method and MATHEMATICA [47], we get the following numerical values of I_r for $r = 1, 2, 3$: $I_1 \approx 2.9231$, $I_2 \approx 2.7726$, $I_3 \approx 2.6649$.

For $r = 1, 2, 3$ the solution of (101) can be written as

$$a = a_0\left(1 + \frac{\tau}{T_r}\right)^{-\frac{2}{r}}, \quad (105)$$

where T_r is given by the following equation:

$$T_r = \left[\frac{r}{4M_r}AE\alpha_1\sqrt{\frac{r^2 a_0^r B_r}{(r+1)(r+2)}}I_r\right]^{-1}. \quad (106)$$

Eq. (105) shows that the amplitude of solitary wave solution is proportional to $\left(1 + \frac{\tau}{T_r}\right)^{-\frac{2}{r}}$ for $r = 1, 2, 3$.

Therefore, the first-order solitary wave solution of the evolution Eq. (83) can be written in the following form : $\phi^{(1)} = a\ \ \mathrm{sech}^{\frac{2}{r}}X$ for $r = 1$, 2 and 3, where the amplitude (a) of the solitary wave is not a constant but it is a function of time τ and its functional form is given by Eq. (105). From Eq. (105), we see that the amplitude of the solitary wave decreases slowly with time τ.

5. Conclusions

We have considered a collisionless unmagnetized electron–ion plasma consisting of warm adiabatic ions and two distinct populations of electrons at different temperatures—a cooler one is isothermally distributed and follows Maxwell–Boltzmann distribution, whereas the hotter one is nonthermally distributed and obeys the distribution function of Cairns et al. [9].

Considering the Vlasov–Poisson model for two different electron species and the fluid model for ions, we have derived a KdV-like evolution equation including the effect of linear Landau damping of electrons. We have studied the propagation of weakly nonlinear and weakly dispersive IA waves using this KdV-like evolution equation.

We have seen that the coefficient of the nonlinear term of the KdV-like evolution equation vanishes along different family of curves in different parameter planes, viz., $\sigma_{sc} - n_{sc}$, $\beta_e - \sigma_{sc}$, $\beta_e - n_{sc}$. In this situation, to describe the nonlinear behaviour of IA waves, we have derived an MKdV-like evolution equation including the effect of linear Landau damping of electrons having nonlinear term $\left(\phi^{(1)}\right)^2\frac{\partial\phi^{(1)}}{\partial\xi}$ but the term responsible for the effect of linear Landau damping of electrons remains the same in both KdV and MKdV-like evolution equations.

Again, we have seen that the coefficients of the nonlinear terms of both KdV and MKdV-like evolution equations simultaneously vanish along a family of curves for different values of σ. In this situation, for the first time, we have derived a FMKdV-like evolution equation including the effect of linear Landau damping of electrons and this equation efficiently describes the nonlinear behaviour of IA waves. We have found that the nonlinear term of FMKdV-like evolution equation is of the form $\left(\phi^{(1)}\right)^3\frac{\partial\phi^{(1)}}{\partial\xi}$ but the term responsible for the effect of linear Landau damping of electrons remains same in all KdV, MKdV and FMKdV-like evolution equations.

The evolution equations can be written in a more compact form by considering the nonlinear term of the form $\left(\phi^{(1)}\right)^r\frac{\partial\phi^{(1)}}{\partial\xi}$ for $r = 1, 2, 3$. For $r = 1, 2$ and 3, we, respectively, get KdV, MKdV and FMKdV-like evolution equations. Using the multiple time scale analysis with respect to the small parameter α_1, we have generalized the method of Ott and Sudan [33] to solve evolution equation (83).

The solitary wave solution of the evolution equation (83) can be simplified as $\phi^{(1)} = a\ \ \mathrm{sech}^{\frac{2}{r}}X$, where the amplitude a of the solitary wave solution of (83) is a decreasing function of time and its functional form is given by Eq. (105).

For the first time, we have found the solitary wave solution of FMKdV-like evolution equation and we have seen that the amplitude of solitary wave solution of FMKdV-like evolution equation is proportional to $\left(1 + \frac{\tau}{T_3}\right)^{-\frac{2}{3}}$, where T_3 is given by Eq. (106) for $r = 3$.

For $r = 1$, the amplitude a of the KdV soliton is plotted against τ in Fig. 7 for $\gamma = 3$, $\sigma = 0.001$, $\sigma_{sc} = 0.25$ and $n_{sc} = 0.3$ and for different values of β_e. Here, red, black and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.4$ and $\beta_e = 0.57$ respectively. From this figure, we see that the amplitude a of the KdV soliton increases with increasing β_e for any fixed τ. This figure also shows that the amplitude decreases with time.

For $r = 2$, the amplitude a of the MKdV soliton is plotted against τ in Fig. 8 when $B_1 = 0$ for $\gamma = 3$, $\sigma = 0.001$ and $\sigma_{sc} = 0.25$, and for different values of β_e. Here,

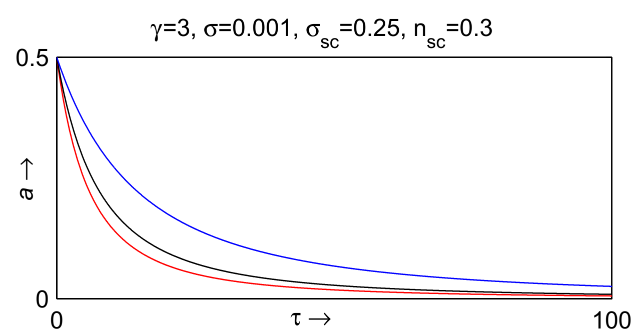

Fig. 7 The amplitude (a) of the KdV soliton is plotted against τ for different values of β_e. Red, black and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.4$ and $\beta_e = 0.57$ respectively. This figure shows that the amplitude of the KdV soliton decreases with increasing time τ for any fixed value of β_e whereas for any fixed value of τ, the amplitude of the KdV soliton increases with the increasing nonthermal parameter β_e (colour figure online)

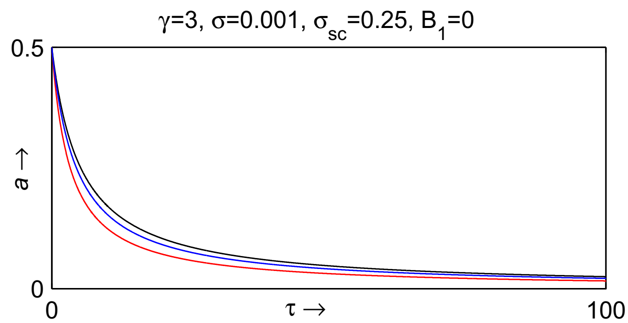

Fig. 8 The amplitude (a) of the MKdV soliton is plotted against τ for different values of β_e when $B_1 = 0$. Red, black and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.45$ and $\beta_e = 0.57$ respectively. This figure shows that the amplitude of the MKdV soliton decreases with increasing time τ for any fixed value of β_e (colour figure online)

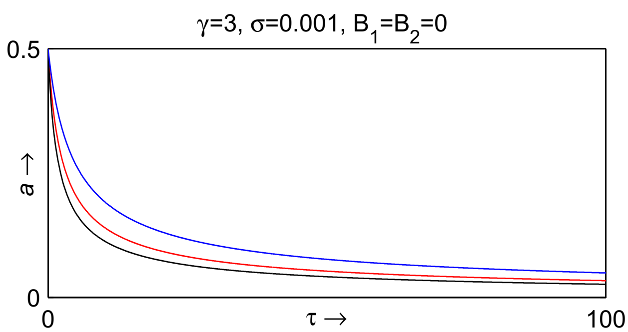

Fig. 9 The amplitude (a) of the FMKdV soliton is plotted against τ for different values of β_e when $B_1 = B_2 = 0$. Red, black and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.352$ and $\beta_e = 0.42$ respectively. This figure shows that the amplitude of the FMKdV soliton decreases with increasing time τ for any fixed value of β_e (colour figure online)

red, black and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.45$ and $\beta_e = 0.57$ respectively. This figure shows that the amplitude decreases with time.

For $r = 3$, the amplitude a of the FMKdV soliton is plotted against τ in Fig. 9 when $B_1 = B_2 = 0$ for $\gamma = 3$ and $\sigma = 0.001$, and for different values of β_e. Red, black and blue curves correspond to $\beta_e = 0$, $\beta_e = 0.352$ and $\beta_e = 0.42$ respectively. This figure shows that the amplitude decreases with time.

Therefore, from Figs. 7, 8 and 9, we can conclude that the amplitude of the IA soliton decreases with time τ for all $r = 1, 2, 3$ if the effect of linear Landau damping of electrons is taken into account.

Finally, it is important to note that if we neglect the effect of linear Landau damping of electrons, then Eqs. (1)–(7) reduce to a full set of hydrodynamic equations and simultaneously the nonlinear evolution equation (83) reduces to KdV and different modified KdV equation for different values of $r = 1, 2$ and 3. These equations can describe the small amplitude solitary wave solutions under different circumstances of the present plasma system, viz., the nonlinear evolution equation is a KdV-like equation if $B_1 \neq 0$ or a modified KdV-like equation if $B_1 = 0$ but $B_2 \neq 0$ or a further modified KdV-like equation if $B_1 = B_2 = 0$ but $B_3 \neq 0$. In fact, here Vlasov–Poisson model of electron species depends on the inertia of electrons, i.e. if we neglect the inertia of electrons, then the system of equations reduces to a system of hydrodynamic equations and all the usual nonlinear evolution equations can be obtained from Eq. (83) by neglecting the effect of linear Landau damping of electrons. Therefore, one can assume that the treatment made in this paper is physically consistent when we are going to consider the effect of linear Landau damping of electrons on IA solitary waves. In fact, VanDam and Taniuti [34] clearly stated that Ott and Sudan [33] considered the electron Landau damping only, being based on an approximation in powers of mass ratio, related to the smallness of electron inertia. Hence, it cannot be applied to treat ion Landau damping. Furthermore, Meiss and Morrison [35] considered nonlinear electron Landau damping on IA solitons. They reported that the theory of Ott and Sudan [33] is valid for time much less than the electron bounce time, i.e. nonlinear effects are important for time greater than electron bounce time. It is also important to note that the last terms of left-hand side of Eqs. (32), (38) and (81) are all equal as these terms are responsible for the effect of linear Landau damping of electrons. But, of course, the more realistic physical situation is to consider nonlinear wave modulation along with nonlinear Landau damping.

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Appendix 1

Coefficients of Eq. (76):

$$J_{c0} = \int_{-\infty}^{+\infty} \frac{\partial j_{c0}}{\partial v_{||}^2} \mathrm{d}v_{||}, \ Z_{c0} = \left.\frac{\partial f_{c0}}{\partial v_{||}^2}\right|_{v_{||}=0}, \tag{107}$$

where $J = F, G, H, K$ for $j = f, g, h, k$, respectively.

Appendix 2

Coefficients of Eq. (78):

$$J_{s0} = \int_{-\infty}^{+\infty} \frac{\partial j_{s0}}{\partial v_{||}^2} \mathrm{d}v_{||}, \ Z_{s0} = \left.\frac{\partial f_{s0}}{\partial v_{||}^2}\right|_{v_{||}=0}, \tag{108}$$

where $J = F, G, H, K$ for $j = f, g, h, k$, respectively.

Appendix 3

Method of finding I_r associated with Eqs. (103) and (104):

$$I_r = \mathcal{P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\text{ sech } X]^{\frac{2}{r}} \frac{\partial [\text{ sech } X']^{\frac{2}{r}}}{\partial X'} \frac{dXdX'}{X - X'}. \tag{109}$$

Now I_r can be written as

$$I_r = -\int_{-\infty}^{\infty} \frac{\partial [\text{ sech } z]^{\frac{2}{r}}}{\partial z} I_{1r} dz, \tag{110}$$

where $X = z'$, $X' = z$ and

$$I_{1r} = \mathcal{P} \int_{-\infty}^{\infty} \frac{[\text{ sech } z']^{\frac{2}{r}}}{z - z'} dz'. \tag{111}$$

Using the following known result

$$\int_{-\infty}^{0} e^{is(z-z')} ds = \pi\delta(z - z') - i\mathcal{P}\frac{1}{z - z'}, \tag{112}$$

form Eq. (112), we get

$$\mathcal{P}\frac{1}{z - z'} = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{s}{|s|} e^{is(z-z')} ds. \tag{113}$$

Using (113), Eq. (111) can be written as

$$I_{1r} = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{s}{|s|} F(s) e^{isz} ds, \tag{114}$$

where

$$F(s) = \int_{-\infty}^{\infty} [\text{ sech } z]^{\frac{2}{r}} e^{-isz} dz. \tag{115}$$

Therefore, Eq. (110) can be written as

$$I_r = \int_{0}^{\infty} s[F(s)]^2 ds. \tag{116}$$

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