Indian J Phys (October 2020) 94(10):1615–1624
https://doi.org/10.1007/s12648-019-01589-2

ORIGINAL PAPER

Bifurcation and stability analysis of commensurate fractional-order van der Pol oscillator with time-delayed feedback

J Chen[1], Y Shen[2]*, X Li[1], S Yang[2] and S Wen[3]

[1]Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang 050043, People's Republic of China

[2]Department of Mechanical Engineering, Shijiazhuang Tiedao University, Shijiazhuang 050043, People's Republic of China

[3]Transportation Institute, Shijiazhuang Tiedao University, Shijiazhuang 050043, People's Republic of China

Received: 10 November 2018 / Accepted: 15 July 2019 / Published online: 4 October 2019

Abstract: The stability and existence conditions of Hopf bifurcation of a commensurate fractional-order van der Pol oscillator with time-delayed feedback are studied. Firstly, the necessary and sufficient conditions for the asymptotic stability of the equilibrium point of fractional-order van der Pol oscillator with linear displacement feedback are obtained, and it is found that the conditions are not only related to the feedback gain, but also to the fractional order. Secondly, regarding time delay as a bifurcation parameter, the stability of the commensurate fractional-order van der Pol system with time-delayed feedback is investigated based on the characteristic equation. Under some conditions, the critical value of time delay is calculated. The equilibrium point is stable when the parameter is less than the critical value and will be unstable if the parameter is greater than it. Moreover, the conditions for the occurrence of Hopf bifurcation are obtained. Finally, choosing four typical system parameters, some numerical simulations are carried out to verify the correctness of the obtained theoretical results.

Keywords: Fractional-order van der Pol oscillator; Time delay; Stability; Hopf bifurcation

PACS Nos.: 02.30.Hq; 05.45.-a

1. Introduction

Fractional calculus is a branch of applied mathematics that deals with the study on fractional-order integral and derivative operators in real or complex domains. A great number of real-world objects can be generally identified and described by the fractional-order model. The main advantage of the fractional-order model in comparison with the integer-order model is that a fractional-order derivative can provide excellent performance in the description of memory and hereditary properties of various processes. In recent years, the study of oscillatory behaviors in fractional-order systems has received considerable attention in various fields, such as physics, engineering, economics, biology, and materials science [1–24]. Meanwhile, many different types of van der Pol oscillators containing fractional-order derivatives have attracted more and more attention [17, 18, 23–30]. For example, Tavazoei et al. [25] found a simple criterion which determined the oscillation range for a fractional-order van der Pol oscillator. Attari et al. [26] established the boundary between oscillatory and non-oscillatory regions using a describing function method for a fractional-order van der Pol-like oscillator. Guo and Leung et al. [27, 28] studied the oscillatory region and asymptotic solution of the fractional-order van der Pol system via the residue harmonic balance technique. Shen et al. [29] obtained the approximate analytical solution of van der Pol oscillator with two kinds of fractional-order derivatives based on the averaging method. Xiao et al. [30] investigated the Hopf bifurcation control for fractional-order van der Pol oscillator under state feedback schemes.

The time delay is an inherent part of many dynamical systems such as machine tool dynamics, neural networks, biological systems, and chemical or process control systems, and it could lead to the instability of the dynamical system and the damage to the control performance [2, 3]. In many fields of science and engineering, the combination of

*Corresponding author, E-mail: shenyongjun@126.com

the delay with fractional calculus was successfully applied, especially when the researchers tried to use fractional-order models to describe complex systems with memory effects. For example, Deng et al. [31] studied the stability of linear fractional differential equations with multiple time delays. Wen et al. [32], based on the averaging method, investigated the approximate analytical solution of Mathieu–Duffing oscillator under fractional-order delayed feedback. Huang et al. [33] studied the stability and bifurcation of a class of delayed fractional complex-valued neural networks. Mahmoud et al. [34] investigated the control of the chaotic fractional Burke–Shaw system using time-delayed feedback control.

Similar to classical differential systems, the study on stability is always a central task for fractional-order systems. To our best knowledge, the stability of fractional-order van der Pol oscillator under time-delayed feedback is rarely studied. In this paper, choosing time delay as the bifurcation parameter, we will discuss the asymptotical stability of solutions in commensurate fractional-order van der Pol oscillator. The paper is organized as follows. In Sect. 2, some definitions are briefly introduced and the method of stability analysis for the time-delayed fractional-order systems is represented. Section 3 presents the stability of the equilibrium point and the occurrence of Hopf bifurcation of fractional-order van der Pol oscillator with time-delayed feedback. Numerical simulations are carried out to verify the theoretical results in Sect. 4. Finally, the main conclusions are given in Sect. 5.

2. Preliminaries

There are many definitions of fractional-order derivatives. The Grünwald–Letnikov definition, Riemann–Liouville definition and Caputo definition are widely employed [6]. The main advantage of Caputo derivative lies in the initial conditions of fractional-order differential equations having the same forms as those of integer-order ones, so that Caputo definition is adopted in this paper. The α-order derivative of function $g(t)$ in Caputo sense is described as follows

$$ {}_0^C D_t^{\alpha} g(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} g^{(m)}(\tau)\mathrm{d}\tau, \tag{1}$$

where $m-1 \le \alpha < m$, $m \in \mathrm{N}$, and $\Gamma(.)$ is the Gamma function.

The following fractional-order system is studied

$$\frac{\mathrm{d}^{\alpha_i} x_i}{\mathrm{d}t^{\alpha_i}} = f_i(x_1, x_2, \ldots, x_n), \tag{2}$$

where $0<\alpha_i<1$, $\quad i=1,2,\ldots n$, $\alpha_i \in \mathrm{Q}$. The notation $\frac{\mathrm{d}^{\alpha_i}}{\mathrm{d}t^{\alpha_i}}$ is Caputo fractional-order derivative operator of Eq. (1). If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, Eq. (2) is called as a commensurate fractional-order system; otherwise, it is an incommensurate fractional-order one.

The following commensurate fractional-order system is discussed

$$\frac{\mathrm{d}^{\alpha} x}{\mathrm{d}t^{\alpha}} = f(x), \tag{3}$$

where $0<\alpha<1$ and $x \in \mathrm{R}^n$. It is concluded that the equilibrium point x_0 of Eq. (3) is locally asymptotically stable if and only if all the eigenvalues λ of the Jacobian matrix $J = \frac{\partial f}{\partial x}\Big|_{x_0}$ satisfy $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ [25].

Next, the n-dimensional linear fractional-order system with multiple time delays is considered as follows

$$\begin{cases} \frac{\mathrm{d}^{\alpha_1} x_1}{\mathrm{d}t^{\alpha_1}} = a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1n}x_n(t-\tau_{1n}) \\ \frac{\mathrm{d}^{\alpha_2} x_2}{\mathrm{d}t^{\alpha_2}} = a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2n}x_n(t-\tau_{2n}) \\ \vdots \\ \frac{\mathrm{d}^{\alpha_n} x_n}{\mathrm{d}t^{\alpha_n}} = a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \cdots + a_{nn}x_n(t-\tau_{nn}) \end{cases}, \tag{4}$$

where $0<\alpha_i<1$, $\quad i=1,2,\ldots n$, $\alpha_i \in \mathrm{Q}$. The initial values $x_i(t) = \varphi_i(t)$ are given for $-\max_{i,j}(\tau_{ij}) = -\tau_{\max} \le t \le 0$, $i=1,2,\ldots n$. $A=(a_{i,j})_{n\times n}$ is the coefficient matrix for Eq. (4).

Taking Laplace transform on both sides of Eq. (4) and arranging them, one could obtain the following matrix

$$\Delta(s) = \begin{pmatrix} s^{\alpha_1} - a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}} \\ -a_{21}e^{-s\tau_{21}} & s^{\alpha_2} - a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^{\alpha_n} - a_{nn}e^{-s\tau_{nn}} \end{pmatrix}. \tag{5}$$

Hence, the stability of Eq. (4) can be entirely determined by the distribution of the roots of $\det(\Delta(s)) = 0$.

Lemma 1 ([31]) *Supposing* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = q \in (0,1)$ *and* $s^q = \lambda$, *the zero solution of Eq.* (4) *is Lyapunov globally asymptotically stable, if all the eigenvalues of A satisfy* $|\arg(\lambda)| > \frac{q\pi}{2}$, *and the characteristic equation* $\det(\Delta(s)) = 0$ *has no purely imaginary roots for any* $\tau_{i,j} \ge 0$, $i,j = 1, 2, \ldots, n$.

Remark For nonlinear fractional-order systems with time delay, these systems can be changed into the linear time-delayed fractional-order systems by using transformation techniques and Taylor formula. Then, the above lemma can be adopted to discuss easily the stability and bifurcation of nonlinear fractional delayed systems [25, 31].

Since it is transcendental, the characteristic equation of $\det(\Delta(s)) = 0$ is difficult to find its eigenvalues analytically. In order to discuss the distribution of the roots of the exponential polynomial, the following conclusion in [35] is presented.

Lemma 2 *Consider the exponential polynomial*

$$P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m},$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$, $p_j^{(i)}$ $(i = 0, 1, \ldots, m;\ j = 1, 2, \ldots, n)$ *are constants, and the sum of the orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *on the open right half plane can be changed only if a zero appears on or crosses the imaginary axis when* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary.*

3. Stability analysis of fractional-order van der Pol oscillator with time-delayed feedback

Recently, the following fractional-order van der Pol system was introduced, and the stability and bifurcation control of the system under nonlinear feedback were investigated [25, 30]

$$\begin{cases} \dfrac{d^q x_1(t)}{dt^q} = x_2(t) \\ \dfrac{d^q x_2(t)}{dt^q} = -x_1(t) - a(x_1^2(t) - 1)x_2(t) \end{cases}, \tag{6}$$

where $0 < q < 1$ and $a > 0$.

In this paper, the commensurate fractional-order van der Pol system is considered with the delayed feedback controller

$$\begin{cases} \dfrac{d^q x_1(t)}{dt^q} = x_2(t) \\ \dfrac{d^q x_2(t)}{dt^q} = -x_1(t) - a(x_1^2(t) - 1)x_2(t) - kx_1(t-\tau) \end{cases}, \tag{7}$$

where $\tau > 0$ and $k > 0$. The unique equilibrium point of Eq. (7) is $(x_1, x_2) = (0, 0)$, and the corresponding linearized model is

$$\begin{cases} \dfrac{d^q x_1(t)}{dt^q} = x_2(t) \\ \dfrac{d^q x_2(t)}{dt^q} = -x_1(t) + ax_2(t) - kx_1(t-\tau) \end{cases}. \tag{8}$$

The coefficient matrix A is described by

$$A = \begin{pmatrix} 0 & 1 \\ -1-k & a \end{pmatrix}.$$

Its characteristic equation is

$$\det(\Delta(s)) = \begin{vmatrix} s^q & -1 \\ 1 + ke^{-s\tau} & s^q - a \end{vmatrix} = s^{2q} - as^q + 1 + ke^{-s\tau} = 0. \tag{9}$$

Using Lemma 1 and the method in the literature [25], one could obtain the following result.

Theorem 1 *If* $\tau = 0$, *for each fixed* q, *the equilibrium* $(0, 0)$ *of Eq. (7) is asymptotically stable if and only if* $0 < a < a_c$, *where*

$$a_c = 2\sqrt{1+k}\cos\frac{q\pi}{2}. \tag{10}$$

Then, for any $\tau > 0$, the conditions that the characteristic equation $\det(\Delta(s)) = 0$ has no purely imaginary roots would be investigated.

Supposing $s = i\omega = \omega\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ $(\omega > 0)$ be a root of Eq. (9), one could obtain

$$\omega^{2q}(\cos q\pi + i\sin q\pi) - a\omega^q\left(\cos\frac{q\pi}{2} + i\sin\frac{q\pi}{2}\right) + 1 + k(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{11}$$

Separating real and imaginary parts of Eq. (11), one could get

$$\omega^{2q}\cos q\pi - a\omega^q\cos\frac{q\pi}{2} + 1 = -k\cos\omega\tau, \tag{12a}$$

$$\omega^{2q}\sin q\pi - a\omega^q\sin\frac{q\pi}{2} = k\sin\omega\tau. \tag{12b}$$

Adding up the squares of both equations, one could obtain

$$\omega^{4q} + A_3\omega^{3q} + A_2\omega^{2q} + A_1\omega^q + A_0 = 0. \tag{13}$$

where $A_3 = -2a\cos\frac{q\pi}{2}$, $A_2 = a^2 + 2\cos q\pi$, $A_1 = -2a\cos\frac{q\pi}{2}$, and $A_0 = 1 - k^2$.

Letting $\omega^q = z$, Eq. (13) becomes

$$z^4 + A_3z^3 + A_2z^2 + A_1z + A_0 = 0. \tag{14}$$

Lemma 3 ([35]) *If* $A_0 < 0$, *Eq. (14) has at least one positive root.*

Denoting $H(z) = z^4 + A_3z^3 + A_2z^2 + A_1z + A_0$, we have $H'(z) = 4z^3 + 3A_3z^2 + 2A_2z + A_1$ and set

$$4z^3 + 3A_3z^2 + 2A_2z + A_1 = 0. \tag{15}$$

Letting $z = y - \frac{A_3}{4}$ and substituting it into Eq. (15), one could obtain

$$y^3+p_1y+q_1=0. \tag{16}$$

where $p_1=\frac{A_2}{2}-\frac{3}{16}A_3^2$ and $q_1=\frac{A_3^3}{32}-\frac{A_2A_3}{8}+\frac{A_1}{4}$.

Using Cardano formula, we have

$$y_1=\sqrt[3]{-\frac{q_1}{2}+\sqrt{\Delta}}+\sqrt[3]{-\frac{q_1}{2}-\sqrt{\Delta}},$$

$$y_2=\sigma\cdot\sqrt[3]{-\frac{q_1}{2}+\sqrt{\Delta}}+\sigma^2\cdot\sqrt[3]{-\frac{q_1}{2}-\sqrt{\Delta}},$$

$$y_3=\sigma^2\cdot\sqrt[3]{-\frac{q_1}{2}+\sqrt{\Delta}}+\sigma\cdot\sqrt[3]{-\frac{q_1}{2}-\sqrt{\Delta}}.$$

where $\Delta=\frac{q_1^2}{4}+\frac{p_1^3}{27}$ and $\sigma=\frac{-1+\sqrt{3}i}{2}$.

Therefore, the zeros of $H'(z)$ are $z_i=y_i-\frac{A_3}{4}$, $i=1,2,3$.

Lemma 4 ([35]) *Supposing $A_0\geq 0$, one could establish the following conclusions.*

(i) When $\Delta\geq 0$, Eq. (14) has positive roots if and only if $z_1>0$ and $H(z_1)<0$;

(ii) When $\Delta<0$, Eq. (14) has positive roots if and only if there exists at least one $z^*\in\{z_1, z_2, z_3\}$, satisfying $z^*>0$ and $H(z^*)\leq 0$.

Now we consider the case that Eq. (14) has positive roots. Without loss of generality, we assume that it has two positive roots, denoted by $z_i^*, i=1,2$. Then, Eq. (13) has two positive roots written as $\omega_i=(z_i^*)^{\frac{1}{q}}$, $i=1,2$.

Letting

$$\tau_k^{(j)}=\frac{1}{\omega_k}\left[\arctan\left(\frac{a\omega_k^q\sin\frac{q\pi}{2}-\omega_k^{2q}\sin q\pi}{\omega_k^{2q}\cos q\pi-a\omega_k^q\cos\frac{q\pi}{2}+1}\right)+\pi j\right],$$
$$k=1,\ 2;\ j=0,\ 1,\ 2,\cdots \tag{17}$$

then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq. (9) with $\tau=\tau_k^{(j)}, k=1,\ 2;\ j=0,\ 1,\ 2,\ldots$. Obviously, $\lim_{j\to\infty}\tau_k^{(j)}=\infty, k=1,\ 2$.

Thus, we can define

$$\tau_0=\tau_{k_0}^{(j_0)}=\min_{j\geq 0}\{\tau_1^{(j)},\ \tau_2^{(j)}\},\quad \omega_0=\omega_{k_0},\ z_0=z_{k_0}^*. \tag{18}$$

Lemma 5 *Supposing $0<a<a_c$, the following conclusions could be established.*

i. If one of the following three conditions holds

 (a) $A_0<0$;
 (b) $A_0\geq 0$, $\Delta\geq 0$, $z_1>0$, and $H(z_1)<0$;
 (c) $A_0\geq 0$, $\Delta<0$, and there exists a $z^*\in\{z_1, z_2, z_3\}$ meeting $z^*>0$ and $H(z^*)\leq 0$;
 all roots of Eq. (9) have negative real parts when $\tau\in[0,\ \tau_0)$;

ii. If the conditions (a)–(c) of (i) are not satisfied, all the roots of Eq. (9) have negative real parts for all $\tau\geq 0$.

Proof When $\tau=0$, from Theorem 1, we know that all roots of Eq. (9) have negative real parts if $0<a<a_c$.

From Lemmas 3 and 4, we know that Eq. (9) has no roots with zero real part for all $\tau\geq 0$ if (a)–(c) are not satisfied.

If one of (a)–(c) holds, when $\tau\neq\tau_k^{(j)}, k=1,\ 2;\ j=0,\ 1,\ 2,\ldots$, Eq. (9) has no roots with zero real part.

Moreover, τ_0 is the minimum value of τ so that Eq. (9) has purely imaginary root. Using Lemma 2, we can draw the conclusions.

Letting

$$s(\tau)=\alpha(\tau)+i\omega(\tau) \tag{19}$$

be the root of Eq. (9) satisfying $\alpha(\tau_0)=0$, $\omega(\tau_0)=\omega_0$, one could obtain the following results.

Lemma 6 *Supposing $H'(\omega_0^q)\neq 0$, $\pm i\omega_0$ is a pair of simple purely imaginary roots of Eq. (9) if $\tau=\tau_0$. Moreover, $\left.\frac{d(\mathrm{Re}s(\tau))}{d\tau}\right|_{\tau=\tau_0}\neq 0$, and $\left.\frac{d(\mathrm{Re}s(\tau))}{d\tau}\right|_{\tau=\tau_0}$ has the same sign as $\omega_0^qH'(\omega_0^q)$.*

Proof If $i\omega_0$ is not a simple root of Eq. (9), it must satisfy

$$\left.\frac{d}{ds}[s^{2q}-as^q+1+ke^{-s\tau}]\right|_{s=i\omega_0}=0.$$

Separating real and imaginary parts, one could obtain

$$2q\omega_0^{2q-1}\cos\left(\frac{2q-1}{2}\pi\right)-aq\omega_0^{q-1}\cos\left(\frac{q-1}{2}\pi\right)$$
$$=k\tau\cos\omega_0\tau, \tag{20a}$$

$$2q\omega_0^{2q-1}\sin\left(\frac{2q-1}{2}\pi\right)-aq\omega_0^{q-1}\sin\left(\frac{q-1}{2}\pi\right)$$
$$=-k\tau\sin\omega_0\tau. \tag{20b}$$

Denoting

$$U_1=2q\omega_0^{2q-1}\cos\left(\frac{2q-1}{2}\pi\right)-aq\omega_0^{q-1}\cos\left(\frac{q-1}{2}\pi\right),$$

and

$$V_1=2q\omega_0^{2q-1}\sin\left(\frac{2q-1}{2}\pi\right)-aq\omega_0^{q-1}\sin\left(\frac{q-1}{2}\pi\right),$$

one could get

$$-\cot\omega_0\tau=\frac{U_1}{V_1}. \tag{21}$$

Denoting

$$U_2=\omega_0^{2q}\cos q\pi-a\omega_0^q\cos\frac{q\pi}{2}+1,$$

and

$$V_2 = \omega_0^{2q} \sin q\pi - a\omega_0^q \sin\frac{q\pi}{2},$$

one could obtain

$$-\cot\omega_0\tau = \frac{U_2}{V_2}. \tag{22}$$

From Eqs. (21) and (22), we have $\frac{U_1}{V_1} = \frac{U_2}{V_2}$, which means $U_1V_2 - U_2V_1 = 0$.

Thus, one could get

$$2\omega_0^{3q} - 3a\cos\frac{q\pi}{2}\omega_0^{2q} + (a^2 + 2\cos q\pi)\omega_0^q - a\cos\frac{q\pi}{2} = 0.$$

Recalling $\omega_0^q = z_0$, we have

$$4z_0^3 - 6a\cos\frac{q\pi}{2}z_0^2 + 2(a^2 + 2\cos q\pi)z_0 - 2a\cos\frac{q\pi}{2} = 0. \tag{23}$$

Noticing that $A_3 = -2a\cos\frac{q\pi}{2}$, $A_2 = a^2 + 2\cos q\pi$, $A_1 = -2a\cos\frac{q\pi}{2}$, $A_0 = 1 - k^2$, and $H'(z) = 4z^3 + 3A_3z^2 + 2A_2z + A_1$, it could be found that Eq. (23) is in contradiction with condition $H'(\omega_0^q) \neq 0$. Therefore, $\pm i\omega_0$ is a pair of simple purely imaginary roots of Eq. (9).

Differentiating Eq. (9) with respect to τ, one could obtain

$$(2q\,s^{2q-1} - aq\,s^{q-1} - k\tau\,e^{-s\tau})\frac{ds}{d\tau} - kse^{-s\tau} = 0, \tag{24}$$

which means

$$\left(\frac{ds}{d\tau}\right)^{-1} = \frac{2qs^{2q-1} - aqs^{q-1}}{kse^{-s\tau}} - \frac{\tau}{s}. \tag{25}$$

When $\tau = \tau_0$ and $s = i\omega_0$, Eq. (25) becomes

$$\begin{aligned}
\left(\frac{ds}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0} &= -\frac{1}{i\omega_0}\left[\frac{2q(i\omega_0)^{2q-1} - aq(i\omega_0)^{q-1}}{-ke^{-i\omega_0\tau}}\right] - \frac{\tau_0}{i\omega_0} \\
&= \frac{i}{\omega_0}\left[\frac{2q(i\omega_0)^{2q-1} - aq(i\omega_0)^{q-1}}{(i\omega_0)^{2q} - a(i\omega_0)^q + 1}\right] + \frac{i\tau_0}{\omega_0} \\
&= \frac{i}{\omega_0}\frac{U_1 + iV_1}{U_2 + iV_2} + \frac{i\tau_0}{\omega_0} \\
&= \frac{i}{\omega_0}\frac{1}{U_2^2 + V_2^2}[(U_1U_2 + V_1V_2) \\
&\quad - i(U_1V_2 - U_2V_1)] + \frac{i\tau_0}{\omega_0}
\end{aligned}$$

Accordingly, one could get

$$\begin{aligned}
\mathrm{Re}\left(\frac{ds}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0} &= \frac{U_1V_2 - U_2V_1}{\omega_0(U_2^2 + V_2^2)} = \frac{U_1V_2 - U_2V_1}{\omega_0k^2} \\
&= \frac{q\omega_0^{q-1}}{\omega_0k^2}\frac{H'(\omega_0^q)}{2} = \frac{q\omega_0^{q-2}}{2k^2}H'(\omega_0^q).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{sign}\left\{\frac{d(\mathrm{Res}(\tau))}{d\tau}\Bigg|_{\tau=\tau_0}\right\} &= \mathrm{sign}\left\{\left(\frac{d(\mathrm{Res}(\tau))}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0}\right\} \\
&= \mathrm{sign}\{\omega_0^q H'(\omega_0^q)\}.
\end{aligned}$$

We complete the proof of Lemma 6.

Based on Lemmas 3–6, the following theorem is available.

Theorem 2 *Letting ω_0, z_0, τ_0, and $s(\tau)$ be defined by Eqs.* (18) and (19) *and assuming $0 < a < a_c$, the following results could be obtained.*

I. If the conditions (a)–(c) of Lemma 5 are not satisfied, the zero equilibrium point of Eq. (7) is asymptotically stable for all $\tau \geq 0$;
II. If one of the conditions (a), (b), and (c) is satisfied, the zero equilibrium point of Eq. (7) is asymptotically stable when $\tau \in [0, \tau_0)$.
III. If the condition of (II.) is satisfied, and $H'(\omega_0^q) \neq 0$, Eq. (7) undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$.

4. Results and discussion

In this section, we select four typical system parameters to simulate the fractional-order van der Pol oscillator, respectively, to verify the correctness of the above theoretical results. The simulation results are all based on the power series expansion method [6, 23], and step-length is $h = 0.005$.

(1) Choosing $q = 0.8$ and $k = 0.5$, we have $a_c = 0.7569$ from Eq. (10). Selecting $a = 0.6 < a_c$, one can get $A_0 = 0.75 > 0$, $\Delta = -0.0046 < 0$, and there exists $z_1 = 0.9999 > 0$ satisfying $H(z_1) = -0.2497 < 0$. Thus, the condition (c) of Lemma 5 is satisfied. At this point, Eq. (14) has two positive roots $z_1^* = \omega_1^q = 1.2345$ and $z_2^* = \omega_2^q = 0.69$, so that $\omega_1 = 1.3013$ and $\omega_2 = 0.6289$. By using Eq. (17), one could obtain $\tau_1 = 0.3018$ and $\tau_2 = 0.3654$. Therefore, we get $\tau_0 = 0.3018$, $\omega_0 = 1.3013$, and $H'(\omega_0^q) = 2.3532 > 0$.

When time delay $\tau = 0.26 < \tau_0$ is taken, the zero solution of Eq. (7) is asymptotically stable, as shown in Fig. 1(a, b). When $\tau = 0.34 > \tau_0$ is chosen, the zero solution of Eq. (7) is unstable and the periodic solution appears, as shown in Fig. 1(c, d). When $\tau = 0.38 > \tau_2 > \tau_0$ is selected, the zero solution of Eq. (7) is still unstable, as shown in Fig. 1(e, f). Therefore, Eq. (7) undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$, which is completely consistent with the conclusion of Theorem 2.

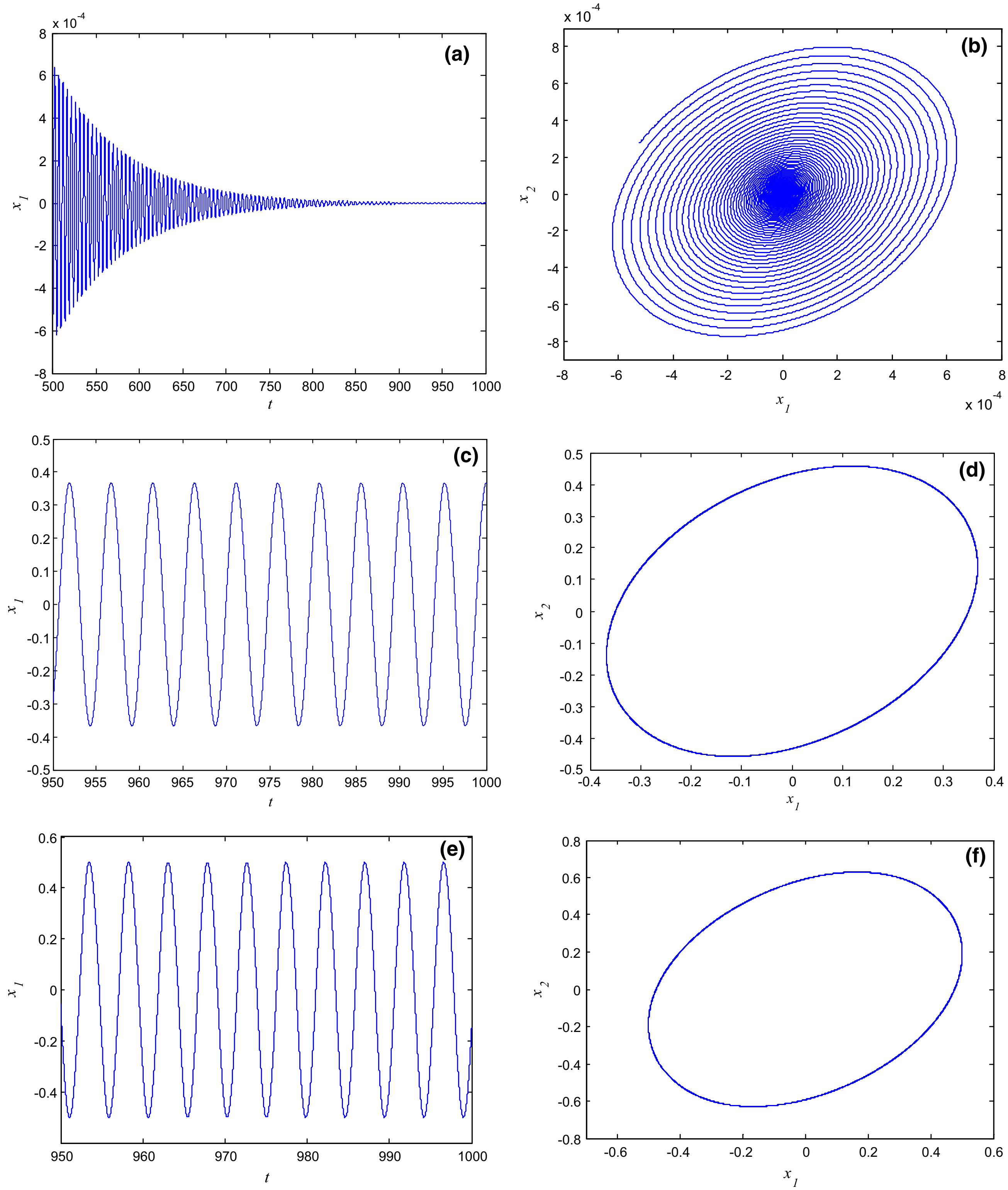

Fig. 1 Time histories and phase diagrams of Eq. (7) with $q = 0.8$ and $k = 0.5$, where the subfigures (**a**), (**c**), and (**e**) denote time histories, and (**b**), (**d**), and (**f**) denote phase diagrams for $\tau = 0.26$, 0.34, and 0.38, respectively

(2) Choosing $q = 0.8$ and $k = 0.9$, it could be obtained $a_c = 0.8519$ from Eq. (10). Selecting $a = 0.75 < a_c$, one can get $A_0 = 0.19 > 0$, $\Delta = 0.0013 > 0$, and $z_1 = 0.995 > 0$ meeting $H(z_1) = -0.7927 < 0$, which satisfy the condition (b) of Lemma 5. Meanwhile, Eq. (14) has two positive roots $z_1^* = \omega_1^q = 1.3912$ and

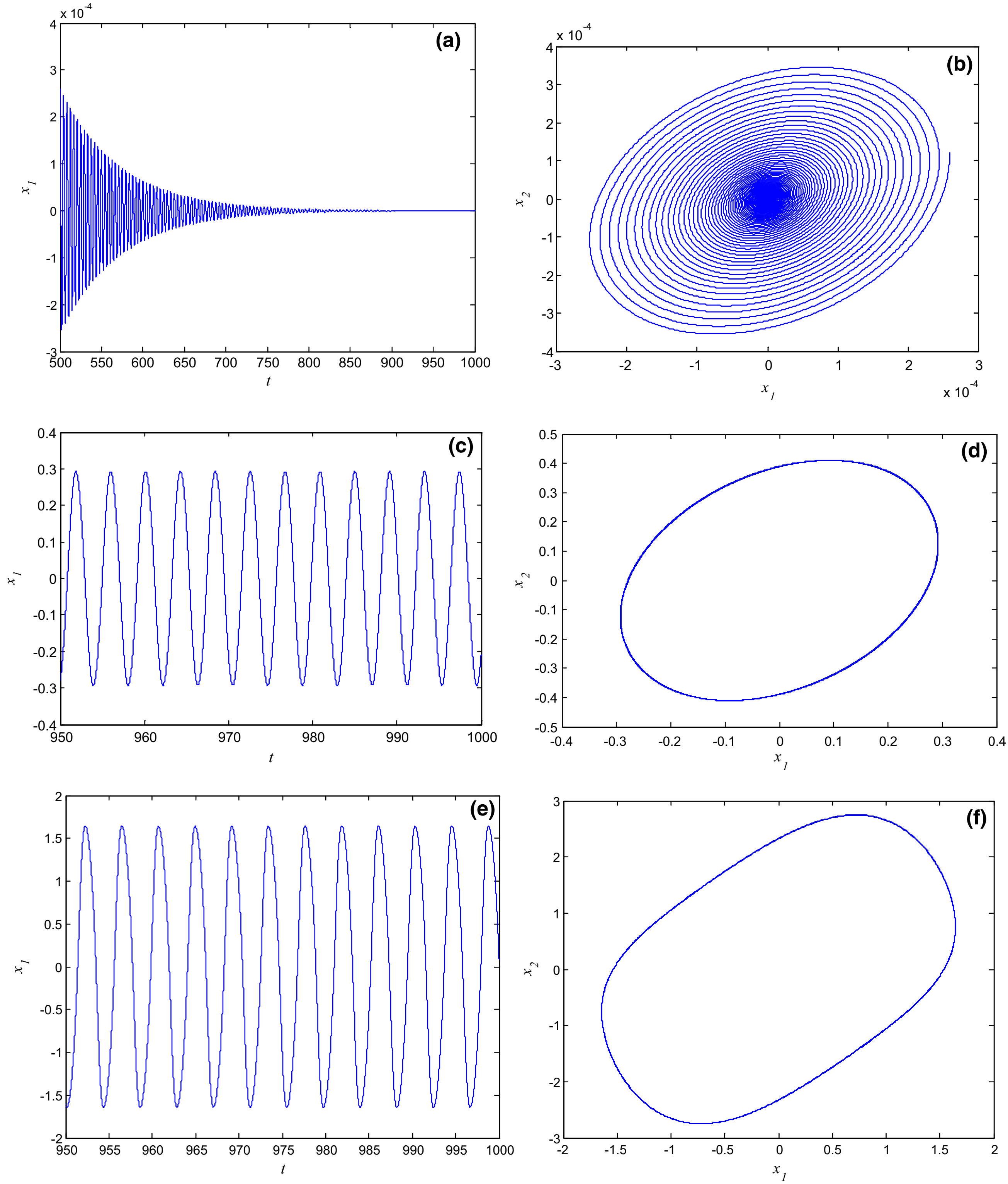

Fig. 2 Time histories and phase diagrams of Eq. (7) with $q = 0.8$ and $k = 0.9$, where the subfigures (**a**), (**c**), and (**e**) denote time histories, and (**b**), (**d**), and (**f**) denote phase diagrams for $\tau = 0.08$, 0.12, and 0.9, respectively

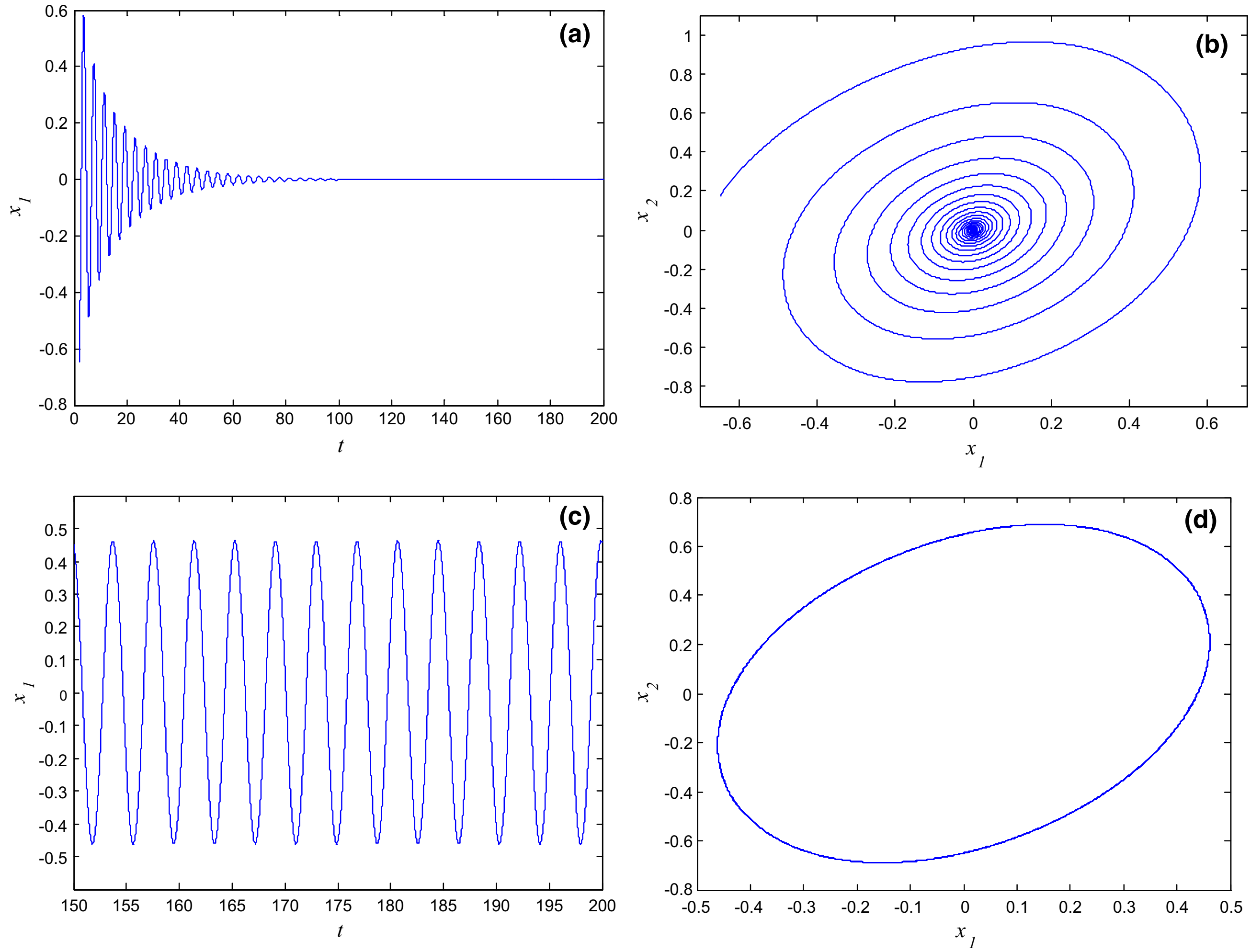

Fig. 3 Time histories and phase diagrams of Eq. (7) with $q = 0.8$ and $k = 1.1$ where the subfigures (**a**) and (**c**) denote time history diagrams, and (**b**) and (**d**) denote phase diagrams for $\tau = 0.1$ and 0.2, respectively

$z_2^* = \omega_2^q = 0.2547$, so that one could get $\omega_1 = 1.5109$ and $\omega_2 = 0.1810$. According to Eq. (17), one could find $\tau_1 = 0.1073$ and $\tau_2 = 0.8852$. Therefore, we have $\tau_0 = 0.1073, \omega_0 = 1.5109$, and $H'(\omega_0^q) = 4.6785 > 0$.

When time delay $\tau = 0.08 < \tau_0$ is chosen, the zero solution of Eq. (7) is asymptotically stable, as shown in Fig. 2(a) and Fig. 2(b). When $\tau = 0.12 > \tau_0$ is taken, the zero solution of Eq. (7) is unstable and the periodic solution will appear, as shown in Fig. 2(c, d). When $\tau = 0.9 > \tau_2 > \tau_0$ is selected, the zero solution of Eq. (7) is still unstable, as shown in Fig. 2(e, f). Therefore, Hopf bifurcation occurs at the origin when $\tau = \tau_0$.

(3) Choosing $q = 0.8$ and $k = 1.1$, we get $a_c = 0.8956$ from Eq. (10). Taking $a = 0.7 < a_c$, one could obtain $A_0 = -0.21 < 0$. Thus, the condition (a) of Lemma 5 is satisfied. At this point, Eq. (14) has only one positive root $z_1^* = \omega_1^q = 1.4681$, so $\omega_1 = 1.6160$. Based on Eq. (17), one could get $\tau_1 = 0.1648$. Therefore, we obtain $\tau_0 = 0.1648$, $\omega_0 = 1.6160$, and $H'(\omega_0^q) = 6.1148 > 0$.

When time delay $\tau = 0.1 < \tau_0$ is taken, the zero solution of Eq. (7) is asymptotically stable, as shown in Fig. 3(a, b). When $\tau = 0.2 > \tau_0$ is chosen, the zero solution of Eq. (7) is unstable and the periodic solution appears, as shown in Fig. 3(c, d). Therefore, Eq. (7) undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$.

(4) Choosing $q = 0.8$ and $k = 0.1$, we have $a_c = 0.6482$ from Eq. (10). Selecting $a = 0.4 < a_c$, one could get $A_0 = 0.99 > 0$, $\Delta = -0.0121573 < 0$, and $H(z)$ has a unique stationary point $z_1 = 0.995063 > 0$, but $H(z_1) = 0.0369 > 0$. So, the conditions (a)–(c) of Lemma 5 are not satisfied. It can be seen from Fig. 4 that the zero solution of Eq. (7) is asymptotically stable for any $\tau \geq 0$, which means Hopf bifurcation does not occur.

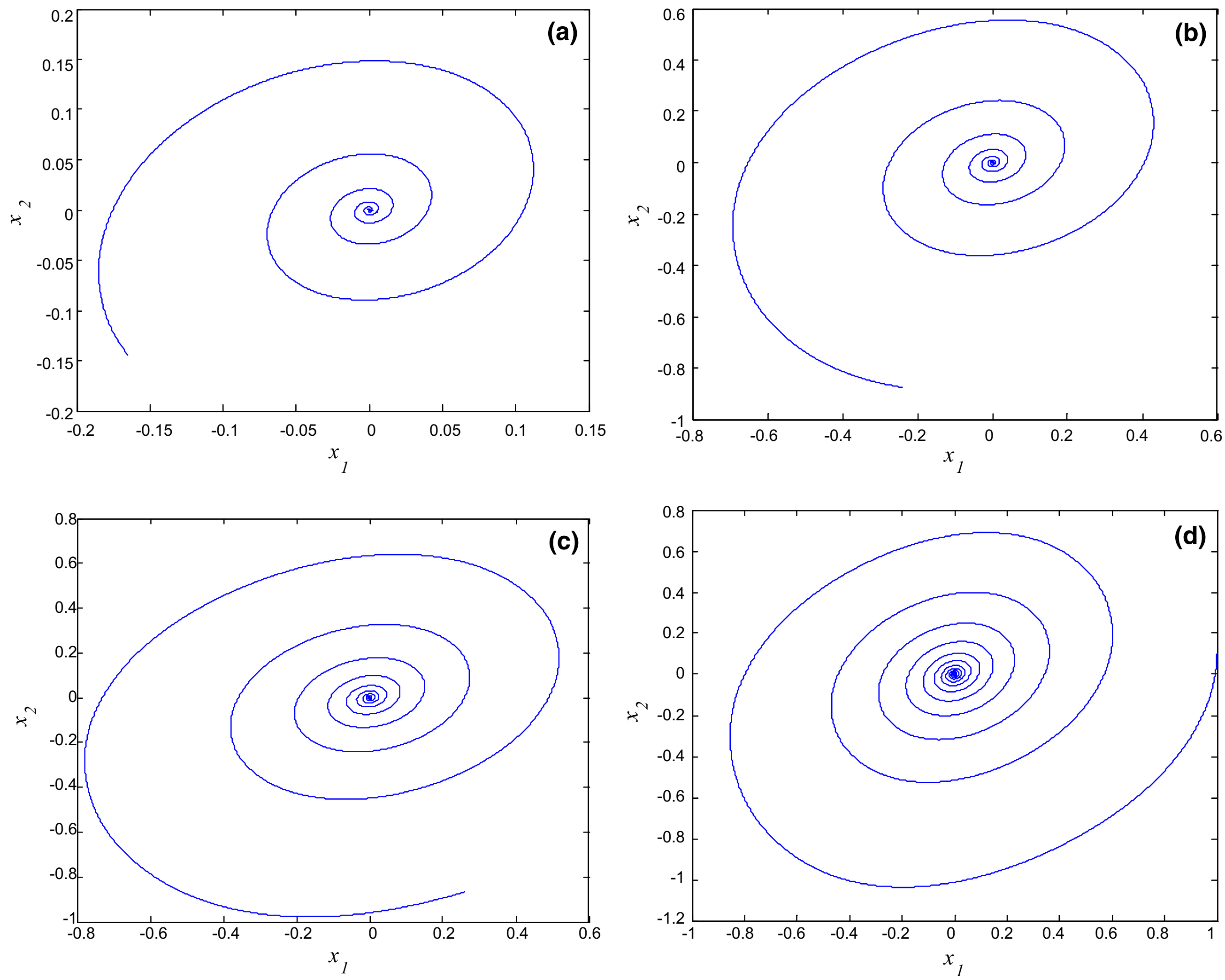

Fig. 4 Phase diagrams of Eq. (7) with $q = 0.8$ and $k = 0.1$ for (**a**) $\tau = 0$, (**b**) $\tau = 0.5$, (**c**) $\tau = 1$, and (**d**) $\tau = 2$

5. Conclusions

The stability and existence conditions of Hopf bifurcation of a commensurate fractional-order van der Pol oscillator with time-delayed feedback are investigated by choosing time delay τ as the bifurcation parameter. When time delay is 0, the fractional-order van der Pol oscillator becomes a fractional-order system only with linear displacement feedback. The necessary and sufficient conditions for the asymptotic stability of the equilibrium point are obtained, which is not only related to the feedback gain, but also to the fractional order. When time delay is larger than 0, the critical value of time delay for the stability of the equilibrium point is calculated based on the characteristic equation, and the generation conditions of the Hopf bifurcation are obtained. Therefore, the stability and bifurcation of fractional-order systems can be easily controlled by adjusting the parameter value of time delay.

Acknowledgements The work reported in this paper was supported by the State Scholarship Fund from the China Scholarship Council (CSC), the National Natural Science Foundation of China (Nos. 11772206, 11672191, 11602152), the Natural Science Foundation of Hebei Province (E2018210056), Hundred Talents Innovation Program of Hebei Province (SLRC2017053), and the Training Program for Leading Talent in University Innovative Research Team in Hebei Province (LJRC018).

References

[1] I Podlubny *Fractional Differential Equations* (San Diego: Academic Press) (1999)
[2] Z H Wang and H Y Hu *J. Sound Vib.* **233** 215 (2000)
[3] H Y Hu and Z H Wang *Dynamics of Controlled Mechanical Systems with Delayed Feedback* (New York: Springer) (2002)
[4] C P Li and G J Peng *Chaos Solit. Fract.* **22** 443 (2004)
[5] E Ahmed, A M A El-Sayed and H A A El-Saka *Phys. Lett. A* **358** 1 (2006)
[6] I Petráš *Fractional-Order Nonlinear System* (Beijing: Higher Education Press) (2011)

[7] C Donato and G Giuseppe *Int. J. Bifurc. Chaos* **18** 1845 (2008)
[8] K H Sun, X Wang and J C Sprott *Int. J. Bifurc. Chaos* **20** 1209 (2010)
[9] Y J Shen, S P Yang and H J Xing *Acta Phys. Sinica* **61** 110505 (2012)
[10] Y J Shen, S P Yang, H J Xing and H X Ma *Commun. Nonlinear Sci.* **17** 3092 (2012)
[11] Y J Shen, S P Yang, H J Xing and H X Ma *Int. J. Non-Linear Mech.* **47** 975 (2012)
[12] J J Sun and Q Ding *Advances in Analysis and Control of Time-Delayed Dynamical Systems* (Beijing: Higher Education Press) (2013)
[13] V E Tarasov *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* (Beijing: Higher Education Press) (2013)
[14] V V Uchaikin *Fractional Derivatives for Physicists and Engineers* (Beijing: Higher Education Press) (2013)
[15] C P Li and F H Zeng *Int. J. Bifurc. Chaos* **22** 1230014 (2012)
[16] L C Chen, W H Wang, Z S Li and W Q Zhu *Int. J. Non–Linear Mech.* **48** 44 (2013)
[17] Y J Shen, P Wei and S P Yang *Nonlinear Dyn.* **77** 1629 (2014)
[18] L C Chen and W Q Zhu *Theor. Appl. Mech. Lett.* **4** 013010 (2014)
[19] Y G Yang, W Xu, X D Gu and Y H Sun *Chaos Solit. Fract.* **77** 190 (2015)
[20] Y J Shen, S F Wen, X H Li, S P Yang and H J Xing *Nonlinear Dyn.* **85** 1457 (2016)
[21] Y Xu, Y G Li and D Liu *Nonlinear Dyn.* **83** 2311 (2016)
[22] J Y Hou, X H Li and J F Chen *J. Vibroeng.* **18** 4812 (2016)
[23] J F Chen, X H Li, J H Tang and Y F Liu *Shock Vib.* 5975329 (2017)
[24] Y Y Ma and L J Ning *Indian J. Phys.* **93** 61 (2019)
[25] M S Tavazoei, M Haeri, M Attari, S Bolouki and M Siami *J. Vib. Control* **15** 803 (2009)
[26] M Attari, M Haeri and M S Tavazoei *Nonlinear Dyn.* **61** 265(2010)
[27] Z J Guo, A Y T Leung and H X Yang *Appl. Math. Model.* **35** 3918 (2011)
[28] A Y T Leung, H X Yang and Z J Guo *J. Sound Vib.* **331** 111 (2012)
[29] Y J Shen, S P Yang and C Y Sui *Chaos Solit. Fract.* **67** 94 (2014)
[30] M Xiao, G P Jiang, W X Zheng, S L Yan, Y H Wan and C X Fan *Asian J. Control* **17** 1756 (2015)
[31] W H Deng, C P Li and J H Lü *Nonlinear Dyn.* **48** 409 (2007)
[32] S F Wen, Y J Shen, S P Yang and J Wang *Chaos Solit. Fract.* **94** 54 (2017)
[33] C D Huang, J D Cao, M Xiao, A Alsaedi and T Hayat *Appl. Math. Comput.* **292** 210 (2017)
[34] G M Mahmoud, A A Arafa, T M Abed-Elhameed and E E Mahmoud *Chaos Solit. Fract.* **104** 680 (2017)
[35] X L Li and J J Wei *Chaos Solit. Fract.* **26** 519 (2005)