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Any *l*-state solutions of the Schrodinger equation interacting with Hellmann–Kratzer potential model

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Abstract: The approximate analytical solutions of the radial Schrodinger equation have been obtained with a newly proposed potential called Hellmann–Kratzer potential. The potential is a superposition of Hellmann potential and modified Kratzer potential. The Hellmann–Kratzer potential actually comprises of three different potentials which include Yukawa potential, Coulomb potential and Kratzer potential. The aim of combining these potentials is to have a wide application. The energy eigenvalue and the corresponding wave function are calculated in a closed and compact form using the Nikiforov–Uvarov method. The energy equation for some potentials such as Kratzer, Hellmann, Yukawa and Coulomb potentials has also been obtained by varying some potential parameters. Our results excellently agree with the already existing literature. Some numerical results have been computed. We have plotted the behaviour of the energy eigenvalues with different potential parameters and also reported on the numerical result.

1. Introduction

The study of the bound-state process is very fundamental when trying to understand molecular spectrum of diatomic molecules since the beginning of quantum mechanics. The bound-state solutions to the Schrodinger equation (SE) are only possible for some potentials of real interest [1, 2]. Quite recently, numerous researchers have tried to solve the problem of obtaining exact or approximate solutions to the Schrodinger equation for some potential of interest [3, 4]. The exact or approximate solutions of these equations with the central potential play an important role in quantum mechanics [5–9].

The analytical solution with $l = 0$ and $l \neq 0$ for some potentials has been addressed by many researchers in nonrelativistic quantum mechanics and relativistic quantum mechanics for bound and scattering states [10–15]. Some of these potentials include: Deng–Fan potential [16], Morse potential [17], Hellmann potential [18], hyperbolic molecular potential [19], Manning–Rosen potential [20] and the Poschl–Teller-like potential [21]. Also, different methods have been employed to obtain the solutions of the nonrelativistic wave equations with a chosen potential model.

These include: the factorization method [22–24], modified factorization method [25, 26], supersymmetry quantum mechanics (SUSYQM) [27–29], asymptotic iteration method (AIM) [30–32], algebraic approach [33], exact quantization rule [17], Nikiforov–Uvarov method (NU) [10, 11], etc.

The Hellmann potential [34–36] is a superposition of Yukawa and Coulomb potentials, which is given as:

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$$V(r) = -\left(\frac{V_0}{r}\right) + \left(\frac{V_1 e^{-\alpha r}}{r}\right) \qquad (1)$$

where V_0 and V_1 are the potential strength of Coulomb and Yukawa potentials, respectively, α is the screening parameter and r is the distance between the two particles.

The Hellmann potential was first studied by Hellmann [34–36]. Thereafter, various researchers worked on the potential, e.g. [37], used the supersymmetric approach to study the approximate analytic solutions of the three-dimensional Schrodinger equation with this potential by applying a suitable approximation scheme to the centrifugal term. [38] obtained approximate eigensolutions of the DKP and Klein–Gordon equations with the Hellmann potential. [39] solved the approximate bound-state solutions of the Hellmann potential using the generalized parametric Nikiforov–Uvarov method. The Hellmann potential found its applications in the field of atomic and condensed matter physics, *e.g.* electron core [40, 41], electron–ion [42] inner-shell ionization problem, alkali hydride molecules and solid-state physics [43, 44]. In like manner, the modified Kratzer potential is given as:

$$V(r) = D_e\left(\frac{r - r_e}{r}\right)^2 \qquad (2)$$

where D_e is the dissociation energy and r_e is the equilibrium inter-nuclear distance.

The Kratzer potential [45] is mostly applied in atomic and molecular physics and quantum chemistry [46]. It is used to describe the interactions of molecular structure in quantum mechanics. The Kratzer potential is made up of a long-range attraction and a repulsive part. The integration of these parts makes this potential reliable in terms of its vibrational and rotational energy eigenvalues [32, 47, 48]. The Kratzer potential approaches infinity as the inter-nuclear distance approaches zero, due to the repulsion that exists between the molecules of the potential. As the inter-nuclear molecular distance approaches infinity, the potential decomposes to zero [47, 48]. [32] presented an exact analytical solution of the radial Schrodinger equation for the Kratzer potential using the asymptotic iteration method (AIM). The exact bound-state energy eigenvalues (E_{nl}) and corresponding eigenfunctions (R_{nl}) were calculated for various values of n and l quantum numbers for selected diatomic molecules.

Recently, there has been great interest in combination of two or more potentials in both the relativistic and nonrelativistic regime. The essence of combining two or more physical potential models is to have a wider range of applications [49]. For example, Hellmann [34] studied Schrodinger equation with a superposition of Coulomb potential and Yukawa potential; this potential named as Hellmann potential. His result is applicable in the area

where both Coulomb potential and Yukawa potential, respectively, find applications. Bearing this in mind, we attempt to study the Schrodinger equation with a newly proposed potential obtained from a combination of Hellmann potential [Eq. (1)] and modified Kratzer potential [Eq. (2)]. The proposed potential is of the form:

$$V(r) = -\left(\frac{V_0}{r}\right) + \left(\frac{V_1 e^{-\alpha r}}{r}\right) + D_e\left(\frac{r - r_e}{r}\right)^2 \qquad (3)$$

If $D_e = 0$, Eq. (3) reduces to the Hellmann potential, if $V_0 = V_1 = 0$, the potential reduces to the modified Kratzer potential, if $V_0 = D_e = 0$, it reduces to the Yukawa or modified Coulomb potential, and if $V_1 = D_e = 0$, it reduces to the Coulomb potential.

This paper is organized as follows: in Sect. 2, we shall briefly introduce the basic concept of the Nikiforov–Uvarov method. Section 3 is focused primarily on the approximate solution of the Schrodinger equation for the Hellmann–modified Kratzer potential system using the NU method. In Sect. 4, we shall discuss special cases of the potential under consideration. In Sect. 5, we discuss results and in Sect. 6, we give a brief concluding remark.

2. Review of the Nikiforov–Uvarov method

The Nikiforov–Uvarov (NU) method [10, 11] is based on solving the hypergeometric-type second-order differential equations by means of the special orthogonal functions [50]. The main equation which is closely associated with the method is given in the following form [51]

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \qquad (4)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second-degree, $\tilde{\tau}(s)$ is a first-degree polynomial, and $\psi(s)$ is a function of the hypergeometric type.

The exact solution of Eq. (4) can be obtained by using the transformation

$$\psi(s) = \phi(s)y(s) \qquad (5)$$

This transformation reduces Eq. (4) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \qquad (6)$$

The function $\phi(s)$ can be defined as the logarithm derivative

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \qquad (7)$$

where $\pi(s) = \frac{1}{2}[\tau(s) - \tilde{\tau}(s)] \qquad (8)$

with $\pi(s)$ being at most a first-degree polynomial. The

second $\psi(s)$ being $y_n(n)$ in Eq. (5) is the hypergeometric function with its polynomial solution given by Rodrigues relation

$$y^{(n)}(s) = \frac{N_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n \rho(s)] \tag{9}$$

where N_n is the normalization constant and $\rho(s)$ is the weight function which must satisfy the condition

$$(\sigma(s)\rho(s))' = \sigma(s)\tau(s) \tag{10}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{11}$$

It should be noted that the derivative of $\tau(s)$ with respect to s should be negative. The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{12}$$

where $k = \lambda - \pi'(s)$ (13)

The value of k can be obtained by setting the discriminant of the square root in Eq. (12) equal to zero. As such, the new eigenvalue equation can be given as:

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \quad n = 0, 1, 2, \ldots, \tag{14}$$

3. Bound-state solutions

The radial Schrodinger equation [52, 53] can be given as:

$$\frac{d^2 R_{nl}}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}\right]R_{nl} = 0 \tag{15}$$

where μ is the reduced mass, E_{nl} is the energy spectrum, $\hbar$ is the reduced Planck's constant, and n and l are the radial and orbital angular momentum quantum numbers, respectively (or vibration–rotation quantum number in quantum chemistry). Substituting Eq. (3) into Eq. (15) gives:

$$\frac{d^2 R_{n\ell}(r)}{dr^2} + \left[\frac{2\mu E_{nl}}{\hbar^2} - \frac{2\mu D_e}{\hbar^2} + \left(\frac{4\mu D_e r_e}{\hbar^2}\right)\left(\frac{1}{r}\right)\right.$$
$$- \left(\frac{2\mu D_e r_e^2}{\hbar^2}\right)\left(\frac{1}{r^2}\right) + \frac{2\mu V_0}{\hbar^2}\left(\frac{1}{r}\right) - \frac{2\mu V_1}{\hbar^2}\left(\frac{e^{-\alpha r}}{r}\right) \tag{16}$$
$$\left. - \frac{\ell(\ell+1)}{r^2}\right]R_{n\ell}(r) = 0$$

The radial Schrodinger equation for this potential can be solved exactly for $l = 0$ (s-wave) but cannot be solved for this potential for $l \neq 0$. To obtain the solution for $l \neq 0$, we employ the approximation scheme proposed by Greene

and Aldrich [54] to deal with the centrifugal term, which is given as:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1 - e^{-\alpha r})^2} \tag{17}$$

It is noted that for a short-range potential, the relation Eq. (17) is a good approximation to $\frac{1}{r^2}$, as proposed by Greene and Aldrich [55, 56]. This implies that Eq. (17) is not a good approximation to the centrifugal barrier when the screening parameter α becomes large. Thus, the approximation is valid when $\alpha \ll 1$. Substituting the approximation Eq. (17) into Eq. (16), we obtain an equation of the form:

$$\frac{d^2 R_{n\ell}(r)}{dr^2}$$
$$+ \left[\begin{array}{c} \frac{2\mu}{\hbar^2}(E_{nl} - D_e) + \left(\frac{4\mu D_e r_e}{\hbar^2}\right)\left(\frac{\alpha}{1-e^{-\alpha r}}\right) - \left(\frac{2\mu D_e r_e^2}{\hbar^2}\right)\left(\frac{\alpha^2}{(1-e^{-\alpha r})^2}\right) \\ + \frac{2\mu V_0}{\hbar^2}\left(\frac{\alpha}{1-e^{-\alpha r}}\right) - \frac{2\mu V_1}{\hbar^2}\left(\frac{\alpha e^{-\alpha r}}{1-e^{-\alpha r}}\right) - \frac{\alpha^2 \ell(\ell+1)}{(1-e^{-\alpha r})^2} \end{array}\right]$$
$$R_{n\ell}(r) = 0$$

(18)

Equation (18) can be simplified by introducing the following dimensionless abbreviations

$$\begin{cases} \varepsilon_n = -\frac{2\mu}{\hbar^2 \alpha^2}(E_{n\ell} - D_e) \\ \beta = \frac{4\mu D_e r_e}{\hbar^2 \alpha} \\ \gamma = -\frac{2\mu V_1}{\hbar^2 \alpha} \\ \chi = \frac{2\mu V_0}{\hbar^2 \alpha} \\ \delta = \left(\frac{2\mu D_e r_e^2}{\hbar^2} + \ell(\ell+1)\right) \end{cases} \tag{19}$$

$$\frac{d^2 R_{n\ell}(r)}{dr^2} + \frac{1}{(1-e^{-\alpha r})^2}$$
$$[-\varepsilon_n(1-e^{-\alpha r})^2 + \beta(1-e^{-\alpha r}) + \chi(1-e^{-\alpha r})$$
$$+ \gamma e^{-\alpha r}(1-e^{-\alpha r}) - \delta]R_{n\ell}(r) = 0 \tag{20}$$

Using a transformation $s = e^{-\alpha r}$ so as to enable us to apply the NU method as a solution of the hypergeometric type

$$\frac{d^2 R_{n\ell}(r)}{dr^2} = \alpha^2 s^2 \frac{d^2 R_{n\ell}(s)}{ds^2} + \alpha^2 s \frac{dR_{n\ell}(s)}{ds}, \tag{21}$$

we obtain the differential equation

$$\frac{d^2 R_{n\ell}}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dR_{n\ell}}{ds} + \frac{1}{s^2(1-s)^2}$$
$$[-(\varepsilon_n + \gamma)s^2 + (2\varepsilon_n - \beta - \chi + \gamma)s - (\varepsilon_n - \beta - \chi + \delta)]$$
$$R_{n\ell}(s) = 0$$

(22)

Comparing Eq. (22) and Eq. (4), we have the following parameters

$$\left\{\begin{array}{c}\tilde{\tau}(s) = 1 - s \\ \sigma(s) = s(1-s) \\ \tilde{\sigma}(s) = -(\varepsilon_n + \gamma)s^2 + (2\varepsilon_n - \beta - \chi + \gamma)s - (\varepsilon_n - \beta - \chi + \delta)\end{array}\right\}$$
$$(23)$$

Substituting these polynomials into Eq. (12), we get $\pi(s)$ to be

$$\pi(s) = -\frac{s}{2} \pm \sqrt{(a-k)s^2 + (b+k)s + c} \qquad (24)$$

where

$$\left\{\begin{array}{c}a = \frac{1}{4} + \varepsilon_n + \gamma \\ b = -2\varepsilon_n + \beta + \chi - \gamma \\ c = \varepsilon_n - \beta - \chi + \delta\end{array}\right\} \qquad (25)$$

To find the constant k, the discriminant of the expression under the square root of Eq. (24) should be equal to zero. As such, we have that

$$k\pm = -(2\delta - \chi - \beta - \gamma) \pm 2\sqrt{\varepsilon_n - \beta - \chi + \delta}\sqrt{\frac{1}{4} + \delta} \qquad (26)$$

Substituting Eq. (26) into Eq. (24) yields

$$\pi = -\frac{s}{2} \pm \left\{\begin{array}{ll}(\sqrt{a_1} - \sqrt{a_3})s - \sqrt{a_1}; & \text{for } k_+ = -(a_2) + 2\sqrt{a_1}\sqrt{a_3} \\ (\sqrt{a_1} - \sqrt{a_3})s + \sqrt{a_1}; & \text{for } k_- = -(a_2) - 2\sqrt{a_1}\sqrt{a_3}\end{array}\right\} \qquad (27)$$

where

$$\left\{\begin{array}{c}a_1 = \varepsilon_n - \beta - \chi + \delta \\ a_2 = 2\delta - \chi - \beta - \gamma \\ a_3 = \left(\frac{1}{4} + \delta\right)\end{array}\right\} \qquad (28)$$

From the knowledge of NU method, we choose the expression $\pi(s)_-$ in which the function $\tau(s)$ has a negative derivative. This is given by

$$k_- = -(2\delta - \chi - \beta - \gamma) - 2\sqrt{\varepsilon_n - \beta - \chi + \delta}\sqrt{\frac{1}{4} + \delta} \qquad (29)$$

with $\tau(s)$ being obtained as

$$\tau(s) = 1 - 2s - 2\left(\sqrt{\varepsilon_n - \beta - \chi + \delta} - \sqrt{\frac{1}{4} + \delta}\right)s + 2\sqrt{\varepsilon_n - \beta - \chi + \delta} \qquad (30)$$

Referring to Eq. (13), we define the constant λ as

$$\lambda = -(2\delta - \beta - \gamma) - 2\sqrt{\varepsilon_n - \beta - \chi + \delta}\sqrt{\frac{1}{4} + \delta} - \frac{1}{2} - \left(\sqrt{\varepsilon_n - \beta - \chi + \delta} - \sqrt{\frac{1}{4} + \delta}\right) \qquad (31)$$

Taking the derivative of Eq. (30) with respect to s, we have

$$\tau'(s) = -2 - 2\left(\sqrt{\frac{1}{4} + \delta} + \sqrt{\varepsilon_n - \beta - \chi + \delta}\right) \qquad (32)$$

And also taking the derivative of $\sigma(s)$ with respect to s from Eq. (23), we have

$$\sigma''(s) = -2 \qquad (33)$$

Substituting Eqs. (29) and (30) into Eq. (14), we obtain

$$\lambda_n = n^2 + n + 2n\left(\sqrt{\frac{1}{4} + \delta} + \sqrt{\varepsilon_n - \beta - \chi + \delta}\right) \qquad (34)$$

By comparing Eqs. (31) and (34), the exact energy eigenvalue equation is obtained as

$$\varepsilon_n = \beta - \delta + \chi + \frac{1}{4}\left[\frac{\gamma + \beta + \chi - \delta - (n+\zeta)^2}{(n+\zeta)}\right]^2 \qquad (35)$$

$$\zeta = \frac{1}{2}\left(1 + \sqrt{1+4\delta}\right) \qquad (36)$$

Substituting Eq. (23) and Eq. (36) into Eq. (35) yields the energy eigenvalue equation of the Hellmann potential plus modified Kratzer potential in the form

$$E_{nl} = \frac{\hbar^2\alpha^2}{2\mu}\left[\delta - \frac{4\mu D_e r_e}{\hbar^2\alpha} - \right] - V_0\alpha + D_e$$
$$- \frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\left(\frac{4\mu D_e r_e + 2\mu V_0 - 2\mu V_1}{\hbar^2\alpha}\right) - \delta}{2(n+\zeta)} - \frac{(n+\zeta)}{2}\right]^2 \qquad (37)$$

The corresponding wave functions can be evaluated by substituting $\pi(s)_-$ and $\sigma(s)$ from Eq. (27) and Eq. (23), respectively, into Eq. (7) and solving the first-order differential equation. This gives

$$\Phi(s) = s^{\sqrt{\varepsilon_n - \beta - \chi + \delta}}(1-s)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \delta}} \qquad (38)$$

The weight function $\rho(s)$ from Eq. (10) can be obtained as

$$\rho(s) = s^{2\sqrt{\varepsilon_n - \beta - \chi + \delta}}(1-s)^{2\sqrt{\frac{1}{4} + \delta}} \qquad (39)$$

From the Rodrigues relation of Eq. (9), we obtain

$$y_n(s) \equiv N_{n,l}P_n^{\left(2\sqrt{\varepsilon_n - \beta - \chi + \delta}, 2\sqrt{\frac{1}{4} + \delta}\right)}(1 - 2s) \qquad (40)$$

where $P_n^{(\theta, \vartheta)}$ is the Jacobi polynomial.

Substituting $\Phi(s)$ and $y_n(s)$ from Eq. (38) and Eq. (40), respectively, into Eq. (5), we obtain

$$\Psi_n(s) = N_{n,l} s^{\sqrt{\varepsilon_n - \beta - \chi + \delta}} (1-s)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \delta}} P_n^{\left(2\sqrt{\varepsilon_n - \beta - \chi + \delta},\, 2\sqrt{\frac{1}{4} + \delta}\right)} (1 - 2s) \tag{41}$$

From the definition of the Jacobi polynomials [57],

$$P_n^{\left(2\eta,\, 2\sqrt{\frac{1}{4} + \delta}\right)}(1 - 2s) = \frac{\Gamma(n + 2\eta + 1)}{n! \Gamma(2\eta + 1)}\, {}_2$$

$$F_1\left(-n, 2\eta + 2\sqrt{\frac{1}{4} + \delta} + n + 1, 2\eta + 1; s\right) \tag{42}$$

$$\eta = \sqrt{\varepsilon_n - \beta - \chi + \delta} \tag{43}$$

In terms of hypergeometric polynomials, Eq. (41) can be written as

$$\psi(s) = N_{n,l} s^{\eta} (1 - s)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \delta}} \frac{\Gamma(n + 2\eta + 1)}{n! \Gamma(2\eta + 1)}\, {}_2$$

$$F_1\left(-n, 2\eta + 2\sqrt{\frac{1}{4} + \delta} + n + 1, 2\eta + 1; s\right) \tag{44}$$

Using the normalization condition, we obtain the normalization constant as follows:

$$\int_0^\infty R_{n,\ell}(r) \times R_{n,\ell}(r)^* \, dr = 1 \tag{45}$$

$$-\frac{1}{\alpha} \int_1^0 |R_{n,\ell}(s)|^2 \frac{ds}{s} = 1, s = e^{-\alpha r} \tag{46}$$

$$\frac{1}{2\alpha} \int_{-1}^1 |R_{n,\ell}(z)|^2 \frac{2}{1 - z} dz = 1, z = 1 - 2s \tag{47}$$

Substituting Eq. (41) into Eq. (47), we have

$$\frac{N_{n\ell}^2}{2\alpha} \int_{-1}^1 \left(\frac{1 - z}{2}\right)^{2\eta - 1} \left(\frac{1 + z}{2}\right)^{2\phi} \left[P_n^{(2\eta,\, 2\phi - 1)}(z)\right]^2 dz = 1, \tag{48}$$

where

$$\phi = \frac{1}{2} + \sqrt{\frac{1}{4} + \delta}, \tag{49}$$

$$\eta = \sqrt{\varepsilon_n - \beta - \chi + \delta} \tag{50}$$

Comparing Eq. (48) with the integral of the form

$$\int_1^{-1} \left(\frac{1 - p}{2}\right)^x \left(\frac{1 + p}{2}\right)^y \left[P_n^{(2x,\, 2y - 1)}(p)\right]^2 dp$$

$$= \frac{2\Gamma(x + n + 1)\Gamma(y + n + 1)}{n! \times \Gamma(\alpha + \beta + n + 1)} \tag{51}$$

We have the normalization constant as

$$N_{n\ell} =$$

$$\sqrt{\frac{n! 2\left(\sqrt{\varepsilon_n - \beta - \chi + \delta}\right)\alpha\Gamma\left(2\left(\sqrt{\varepsilon_n - \beta - \chi + \delta}\right) + \sqrt{1 + 4\delta} + n + 2\right)}{\Gamma\left(2\left(\sqrt{\varepsilon_n - \beta - \chi + \delta}\right) + n + 1\right)\Gamma\left(n + 2 + \sqrt{1 + 4\delta}\right)}} \tag{52}$$

4. Special cases

In this section, we take adjustments of some potential parameters in Eqs. (3) and (37) to have the following cases

4.1. Hellmann potential

If $D_e = 0$, Eq. (3) reduces to the Hellmann potential

$$V(r) = -\left(\frac{V_0}{r}\right) + \left(\frac{V_1 e^{-\alpha r}}{r}\right) \tag{53}$$

and the energy equation (Eq. 37) becomes

$$E_{n\ell} = \left[\frac{\hbar^2 \alpha^2 \ell(\ell + 1)}{2\mu} - V_0\alpha\right]$$

$$-\frac{\hbar^2}{2\mu} \left[\frac{\left(\frac{2\mu V_0 - 2\mu V_1}{\hbar^2}\right) - \alpha\ell(\ell + 1) - \alpha(n + \ell + 1)^2}{2(n + \ell + 1)}\right]^2 \tag{54}$$

Equation (54) is in full agreement with the results in Refs. [37, 39, 52, 53].

4.2. Yukawa potential

$V_0 = D_e = 0$, Eq. (3) reduces to the Yukawa or modified Coulomb potential

$$V(r) = \frac{V_1 e^{-\alpha r}}{r} \tag{55}$$

and the energy equation (Eq. 37) becomes

$$E_{n\ell} = \left[\frac{\hbar^2 \alpha^2 \ell(\ell + 1)}{2\mu}\right]$$

$$-\frac{\hbar^2}{2\mu} \left[\frac{\left(\frac{2\mu V_1}{\hbar^2}\right) - \alpha\ell(\ell + 1) - \alpha(n + \ell + 1)^2}{2(n + \ell + 1)}\right]^2 \tag{56}$$

Equation (56) is in full agreement with the results in Refs. [52].

4.3. Kratzer potential

If $V_0 = V_1 = 0$ and $\alpha \to 0$, the potential reduces to the modified Kratzer potential

$$V(r) = D_e \left(\frac{r - r_e}{r} \right)^2 \tag{57}$$

and the energy equation (Eq. 37) becomes

$$E_{n\ell} = D_e - \frac{\hbar^2}{2\mu} \left[\frac{\left(\frac{4\mu D_e r_e}{\hbar^2} \right)}{2n + 1 + \sqrt{1 + 4\left(\frac{2\mu D_e r_e^2}{\hbar^2} + \ell(\ell+1) \right)}} \right]^2 \tag{58}$$

Equation (58) is in full agreement with the results in Eq. (14) of Ref. [58] and Eq. (125) of Ref. [59]

4.4. Coulomb potential

If $V_1 = D_e = 0$, Eq. (3) reduces to the Coulomb potential

$$V(r) = - \left(\frac{V_0}{r} \right) \tag{59}$$

and the energy equation (Eq. 37) becomes

$$E_{n\ell} = \left[\frac{\hbar^2 \alpha^2 \ell(\ell+1)}{2\mu} - V_0 \alpha \right] - \frac{\hbar^2}{2\mu} \left[\frac{\left(\frac{2\mu V_0}{\hbar^2} \right) - \alpha\ell(\ell+1) - \alpha(n+\ell+1)^2}{2(n+\ell+1)} \right]^2 \tag{60}$$

If $\alpha \to 0$

$$E_{n\ell} = - \frac{\mu V_0^2}{2\hbar^2 (n+\ell+1)^2} \tag{61}$$

The result of Eq. (61) is very consistent with the result obtained in Eq. (101) of Ref. [59].

5. Results and discussion

In order to test the accuracy of our work, we compute numerical values for energy spectrum and graphical solutions. The energy eigenvalues of the Hellmann potential plus modified Kratzer potential are computed using Eq. (37). The explicit values of these energies for different vibrational and rotational quantum numbers are given in Table 1.

We have plotted the shape of the Hellmann potential plus modified Kratzer potential in Fig. 1. This figure gives an insight into the behaviour of the combined potential. Also, the variation of the energy eigenvalues with different parameters of the combined potential such as D_e, r_e, V_0, V_1

Table 1 Energy eigenvalues (in eV) of Hellmann potential plus modified Kratzer potential for different values of n and l. We choose $\hbar = \mu = 1$, $\alpha = 0.1$, $D_e = 5$ and $r_e = 0.5$

| n	l	E_{nl} $V_0 = 1$, $V_1 = 2$	E_{nl} $V_0 = 2$, $V_1 = 1$	E_{nl} $V_0 = 1$, $V_1 = 1$
0	1	3.6098162360	2.2796306770	3.0643666800
1		4.0680211760	3.3626175540	3.8022514920
2		4.2584808060	3.8223529290	4.1132507880
3		4.3500319940	4.0539643880	4.2674991130
0	2	4.0719078040	3.3067322330	3.7811328920
1		4.2748679510	3.8144702010	4.1198274230
2		4.3695016860	4.0623536930	4.2827117330
3		4.4158953190	4.1965038950	4.3681882100
0	3	4.3280575940	3.8742497370	4.1776921370
1		4.4125722780	4.1128007210	4.3302170000
2		4.4526026280	4.2399312150	4.4087038460
3		4.4693698690	4.3106957100	4.4493119800
0	4	4.4727133300	4.1936321330	4.4011779700
1		4.5019773290	4.3048219030	4.4661193210
2		4.5119882390	4.3653425830	4.4981264210
3		4.5103332090	4.3970090220	4.5109823540
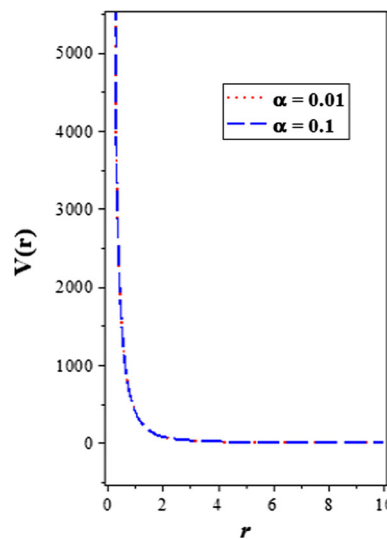
Fig. 1 Shape of Hellmann potential plus modified Kratzer potential for different values of the screening parameter (α). We choose $V_0 = 1$, $V_2 = 2$, $D_e = 5$, and $r_e = 0.5$

and α is shown in Figs. 2, 3, 4, 5 and 6, respectively, for various values of n and l. In Fig. 2, it is seen that there is increase in energy eigenvalues as the dissociation energy (D_e) increases. In Figs. 3, 4, 5 and 6, it is seen that energy eigenvalues decrease as parameters (r_e, V_0, V_1 and α) increase. In Figs. 4, 5 and 6, the decrease in energy tends to spread out from zero position down for different
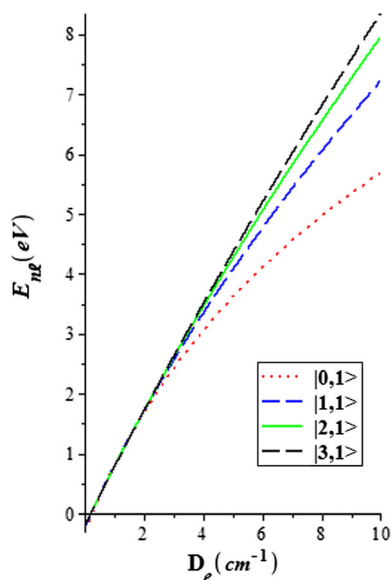
Fig. 2 Energy eigenvalues variation with dissociation energy for various vibrational quantum numbers

Fig. 4 Energy eigenvalues variation with parameter "V_0" for various vibrational quantum numbers. We choose $\hbar = \mu = 1$, $V_2 = 2$, $\alpha = 0.1$, $D_e = 5$ and $r_e = 0.5$
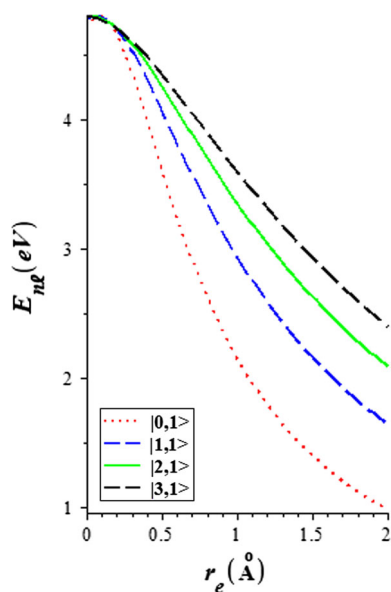

Fig. 3 Energy eigenvalues variation with equilibrium bond length for various vibrational quantum numbers. We choose $\hbar = \mu = V_0 = 1$, $V_2 = 2$, $D_e = 5$ and $\alpha = 0.1$

Fig. 5 Energy eigenvalues variation with parameter "V_1" for various vibrational quantum numbers. We choose $\hbar = \mu = V_0 = 1$, $\alpha = 0.1$, $D_e = 5$ and $r_e = 0.5$

vibrational quantum numbers. We have also observed a uniform increase in energy as the dissociation energy (D_e) increases for the different vibrational quantum numbers.

6. Conclusion

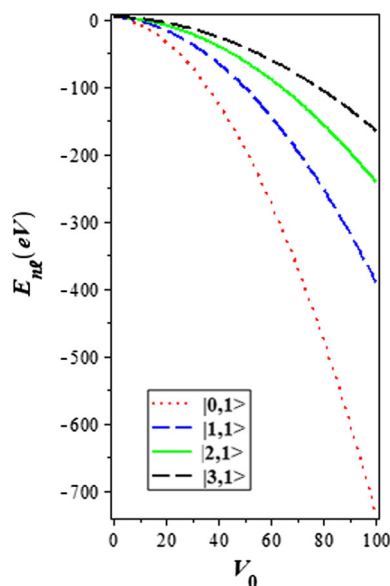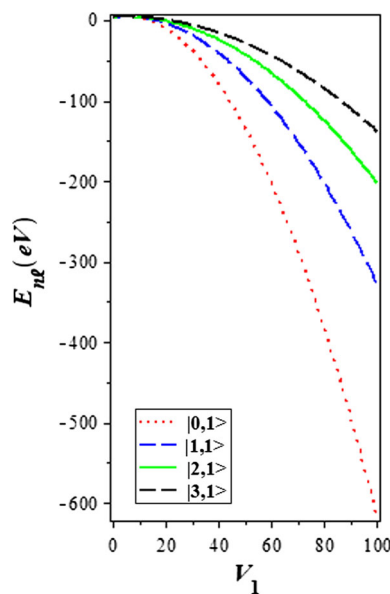In this paper, we obtained solutions of the Schrodinger equation with a combination of three different potentials (Coulomb potential, Yukawa potential and modified Kratzer potential) which we called Hellmann–Kratzer potential using Nikiforov–Uvarov method. The eigensolutions are obtained for Hellmann–Kratzer potential and other useful potentials like the Coulomb potential, Yukawa potential and modified Kratzer potential. Our results find applications where the component potentials are useful.
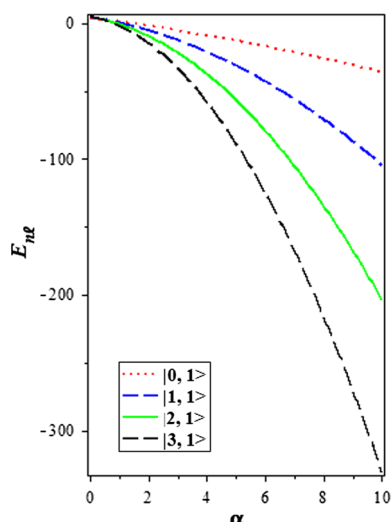
Fig. 6 Energy eigenvalues variation with screening parameter (α) for various vibrational quantum numbers. We choose $\hbar = \mu = V_0 = 1$, $V_1 = 2$, $D_e = 5$ and $r_e = 0.5$

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