Indian J Phys (January 2018) 92(1):81–90
https://doi.org/10.1007/s12648-017-1078-z

ORIGINAL PAPER

Milne problem for non-absorbing medium with extremely anisotropic scattering kernel in the case of specular and diffuse reflecting boundaries

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Received: 21 October 2016 / Accepted: 08 June 2017 / Published online: 9 August 2017

Abstract: The Milne problem is studied in one speed neutron transport theory using the linearly anisotropic scattering kernel which combines forward and backward scatterings (extremely anisotropic scattering) for a non-absorbing medium with specular and diffuse reflection boundary conditions. In order to calculate the extrapolated endpoint for the Milne problem, Legendre polynomial approximation (P_N method) is applied and numerical results are tabulated for selected cases as a function of different degrees of anisotropic scattering. Finally, some results are discussed and compared with the existing results in literature.

Keywords: P_N-method; Milne problem; Extrapolated endpoint; Non-absorbing medium; Specular and diffuse reflectivities

PACS Nos.: 28.20.Cz; 28.20.Gd; 95.30.Jx

1. Introduction

The Milne problem is a well known problem for both the radiative transfer field and neutron transport theory. The problem is to obtain the angular density distribution function everywhere in the half-space $0 \le x \le \infty$ with zero incident flux. Neutrons diffuse from a source at $x = +\infty$. There is a vacuum or reflecting medium in the region $x < 0$. The extrapolation endpoint of neutrons leaving from the boundary at $x = 0$ is determined with Milne problem (Case and Zweifel [1]). Placzek and Seidel [2] and Noble [3] used the Wigner-Hopf technique for solving the Milne problem with vacuum boundary condition. The extrapolated endpoint of the half space was obtained by LeCaine [4] and Marshak [5] with variational method. The formulations of the problem for the isotropic and the linearly anisotropic cases were considered by McCormick and Kuščer [6], McCormick [7] and Case and Zweifel [1]. Shure and Natelson [8] used the Case singular eigenfunction method to calculate extrapolated endpoint for the special cases of absorbing and non-absorbing media. Using the Weigner-Hopf method, Williams [9] solved the problem analytically with the diffuse reflecting boundary condition. The problem with the reflecting boundary condition for non-absorbing medium was solved by Razi [10] and AbdelKrim and Degheidy [11] using variational approach. Tezcan [12], used the P_N method, and calculated extrapolated endpoint in the case of extremely anisotropic scattering kernel for non-absorbing medium. Atalay [13, 14] solved the same problem including absorbing medium by using the singular eigenfunction method for linearly anisotropic scatterig. Loyalka and Naz [15] calculated the linear extrapolation distance for conservative case of isotropic scattering (c=1) using Gaussian quadratures method. Degheidy and El-Shahat [16] studied the problem by a technique based on constructing integral equations. Using the variational principle, Grzesik [17] solved the Milne problem with linear anisotropic scattering.

In this paper, we considered the Milne problem for non-absorbing medium under the specular and diffuse boundary condition. We obtained the extrapolation distance z_0 by

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using Legendre polynomial approximation in the case of linearly and extremely anisotropic scattering.

In the plane geometry, the neutron transport equation with one speed, time independent and source free is given by Case and Zweifel [1], as follows

$$\mu \frac{\partial \Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = 2\pi c \int_{-1}^{+1} f(\mu,\mu')\Psi(x,\mu')d\mu', \quad (1)$$

where $\Psi(x,\mu)$ is the angular distribution of neutrons, c is the number of secondary neutrons per collisions, μ is the cosine of the angle between the direction of the neutron velocity and the positive x axis and $f(\mu,\mu')$ is the scattering kernel which is assumed to be of the form of the combination of linearly anisotropic scattering and extremely anisotropic scattering (Siewert [18]), (Razi [19]), (de Azevedo et al. [20]), (Degheidy et al. [21]), (Degheidy et al. [22]). Explicitly, the function $f(\mu,\mu')$ is given by,

$$f(\mu,\mu') = \frac{1}{4\pi} a(1 + 3f_1\mu\mu') + \frac{b}{2\pi}\delta(\mu' - \mu) + \frac{d}{2\pi}\delta(\mu' + \mu), \quad (2)$$

where f_1 represents linearly anisotropic parameter $(-1 \le 3f_1 \le +1)$. Setting $a = 1 - \alpha$, $b = \alpha$ and $d = 0$ $f(\mu,\mu')$ represents the forward scattering plus linearly anisotropic scattering. Also setting $a = 1 - \alpha$, $b = 0$ and $d = \alpha$ it represents the backward scattering plus linearly anisotropic scattering. By normalizing the scattering kernel, one obtain that α is a real constant $(0 \le \alpha \le 1)$ and

$$a + b + d = 1, \quad (3)$$

The boundary conditions are given by the following expressions for the specular and diffuse reflecting boundary

$$\Psi(0,\mu) = \rho^s \Psi(0,-\mu) + 2\rho^d \int_{0}^{+1} \mu' \Psi(0,\mu')d\mu', \qquad \mu > 0 \quad (4)$$

$$\lim_{x\to\infty} \Psi(x,\mu) = 0 \quad (5)$$

where ρ^s $(0 \le \rho^s \le 1)$ and ρ^d $(0 \le \rho^d \le 1)$ are specular and diffuse reflectivities of the boundary, respectively.

2. P_N method and calculations

Inserting Eq. (2) into (1), the time independent source free one speed transport equation for the forward, backward and linearly anisotropic scattering may be rewritten as

$$\begin{aligned} \mu \frac{\partial \Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = {} & \frac{ca}{2} \int_{-1}^{+1} (1 + 3f_1\mu\mu')\Psi(x,\mu')d\mu' \\ & + b \int_{-1}^{+1} \Psi(x,\mu')\delta(\mu' - \mu)d\mu' \\ & + d \int_{-1}^{+1} \Psi(x,\mu')\delta(\mu' + \mu)d\mu', \end{aligned} \quad (6)$$

For the solution of Eq. (6), the angular distribution and Dirac Delta function can be expanded in terms of Legendre polynomials as follows:

$$\Psi(x,\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \psi_n(x) P_n(\mu), \quad (7a)$$

$$\delta(\mu - \mu') = \sum_{n=0}^{\infty} a_n P_n(\mu), \quad (7b)$$

$$\delta(\mu + \mu') = \sum_{n=0}^{\infty} b_n P_n(\mu) \quad (7c)$$

where

$$\begin{aligned} \psi_n(x) &= \int_{-1}^{+1} P_n(\mu)\Psi(x,\mu)d\mu, \qquad n = 1, 3, \ldots, \infty, \\ a_n &= \frac{2n+1}{2} P_n(\mu') \\ b_n &= (-1)^n a_n. \end{aligned} \quad (8)$$

Multiplying Eq. (6) by $(2m+1)P_m(\mu)$ and integrated both side of Eq. (6) over μ on the interval $[-1, +1]$, and using the following orthogonality and recurrence relations for $P_m(\mu)$

$$\begin{aligned} \int_{-1}^{+1} P_n(\mu)P_m(\mu)d\mu &= \frac{2}{2m+1}\delta_{nm}, \\ \mu P_m(\mu) &= \frac{1}{2m+1}[(m+1)P_{m+1}(\mu) + mP_{m-1}(\mu)], \end{aligned} \quad (9)$$

the following moment equation for linearly plus extremely anisotropic scattering kernel which consist of the infite sets of coupled differantial equations with $n = 0, 1, \ldots$ is obtained as

$$\begin{aligned} & n\frac{d\psi_{n-1}(x)}{dx} + (n+1)\frac{d\psi_{n+1}(x)}{dx} \\ & + (2n+1)(1 - c(b + (-1)^n d)) \\ & \left[1 - \frac{ac}{1 - c(b+d)}(\delta_{n0} + \frac{\gamma a}{3(1 - cb + cd)}\delta_{n1})\right]\psi_n(x) = 0, \end{aligned} \quad (10)$$

where $\Psi_{-1}(x)=0$ and δ_{kl} is the kronecker delta function. Substituting the followings

$$c' = \frac{ac}{1-c(b+d)}, \tag{11a}$$

$$\gamma' = \frac{\gamma a}{3(1-cb+cd)}, \tag{11b}$$

$$\gamma = 3f_1, \tag{11c}$$

$$z = ((1-bc)^2 - c^2d^2)^{1/2}x, \tag{11d}$$

$$\phi_n(z) = (1-bc-(-1)^n cd)^{1/2}\psi_n(x), \tag{11e}$$

into Eq. (10) one obtaines

$$n\frac{\phi_{n-1}(z)}{dz} + (n+1)\frac{d\phi_{n+1}(z)}{dz} + (2n+1) \\ (1-c'(\delta_{n0}+\gamma'\delta_{n1}))\phi_n(z) = 0. \tag{12}$$

Equation (12) is similar to the equation for the linearly anisotropic scattering. Consequently, Eq. (10) for linearly plus extremely anisotropic scattering kernels is transformed to Eq. (12) for linearly anisotropic scattering, where Ψ_m, x and c are replaced by Φ_m, z, c', respectively. In the P_N method, in order to terminate and to solve the infinite set of coupled equations for ϕ_n in Eq. (12) it is sufficient to get $d\phi_{n+1}(z)/dz = 0$ and $n = 0, 1, \ldots, N$. As a result of this, $N+1$ coupled equations with $N+1$ unknowns can be found. To obtain the solutions to the these $N+1$ coupled equations, the boundary condition given in Eq. (4) should be rearranged by using Marshak boundary condition (Marshak [23]):

$$\int_0^1 \mu^m \Psi(0,\mu)d\mu = 0, \qquad m = 1,3,\ldots,N \tag{13}$$

Using Eq. (4) in (13), the boundary condition for specular and diffuse reflecting in P_N method can be written as

$$\int_0^1 \mu^m \Psi(0,\mu)d\mu - \rho^s \int_0^1 \mu^m \Psi(0,-\mu)d\mu - \frac{2\rho^d}{m+1} \\ \int_0^1 \mu'\Psi(0,\mu')d\mu' = 0, \quad m = 1,3,\ldots,N. \tag{14}$$

P_1 **Approximation**: In P_1 approximation, taking $N=1$, i.e. $n=0$ and $n=1$, two coupled moment equations ϕ_0 and ϕ_1 can be obtained using Eq. (12) as

$$\frac{d\phi_1(z)}{dz} + (1-c')\phi_0(z) = 0, \tag{15a}$$

$$\frac{d\phi_0(z)}{dz} + 3(1-\gamma')\phi_1(z) = 0. \tag{15b}$$

If we set $c=1$ for non-absorbing medium, we obtain $c'=1$ from Eq. (11a), therefore $\phi_1(z)$ can be found as a constant

$$\phi_1(z) = -(1-b+d)^{1/2}. \tag{16}$$

Substituting Eq. (16) into (15b) we obtain

$$\phi_0(z) = 3(1-\gamma')(1-b+d)^{1/2}z + A_0. \tag{17}$$

Using Eqs. (16, 17) in (11d, 11e), the moment solutions for linearly anisotropic scattering are

$$\psi_1(x) = -1, \tag{18}$$

$$\psi_0(x) = (3(1-b+d)-\gamma a)x + \frac{A_0}{(1-b-d)^{1/2}}. \tag{19}$$

Here, A_0 is constant. It can be obtained from Eq. (7a) for $x=0$ and $n=0,1$,

$$\psi(0,\mu) = \frac{1}{2}\psi_0(0)P_0(\mu) + \frac{3}{2}\psi_1(0)P_1(\mu), \tag{20}$$

and using boundary condition given by Eq. (14), A_0 is therefore found as

$$A_0 = 2\frac{(1-b-d)^{1/2}(1+\rho^s+\rho^d)}{1-\rho^s-\rho^d}. \tag{21}$$

The extrapolated endpoint z_0 is defined as a distance from a vacuum boundary to where the asymptotic intensity vanishes, and it can be written mathematically as $\Psi_0(x)\big|_{x=-z_0} = 0$ in which exponential terms are eliminated. From the definitions one can get

$$z_0 = \frac{A_0}{(3(1-b+d)-\gamma a)(1-b-d)^{1/2}}, \tag{22}$$

and substituting Eq. (21) in (22), the extrapolated endpoint for the specular and diffuse reflectivities can be obtained as

$$z_0 = \frac{2(1+\rho^s+\rho^d)}{(3(1-b+d)-\gamma a)(1-\rho^s-\rho^d)}. \tag{23}$$

P_3 **Approximation**: In P_3 approximation, i.e. $n=0,1,2,3$, we get $c=1$, the moment equations from Eq. (12) can be written as

$$\frac{d\phi_1(z)}{dz} = 0, \tag{24a}$$

$$2\frac{d\phi_2(z)}{dz} + \frac{d\phi_0(z)}{dz} + 3(1-\gamma')\phi_1(z) = 0, \tag{24b}$$

$$3\frac{d\phi_3(z)}{dz} + 2\frac{d\phi_1(z)}{dz} + 5\phi_2(z) = 0, \tag{24c}$$

$$3\frac{d\phi_2(z)}{dz} + 7\phi_3(z) = 0. \tag{24d}$$

From Eqs. (24c) and (24d), a second order differential equation is obtained for $\phi_2(z)$

$$-\frac{9}{7}\frac{d^2\phi_2(z)}{dz^2} + 5\phi_2(z) = 0. \tag{25}$$

Table 1 The extrapolated endpoint for the diffuse reflection ($\rho^s = 0$) and isotropic scattering ($f_1 = 0$)

ρ^d	P_1	P_3	P_5	P_7	P_9	P_{21}	P_{23}	P_{25}	P_{27}	P_{29}	Exact
0	0.66667	0.70509	0.70821	0.70920	0.70964	0.71028	0.71030	0.71032	0.71034	0.71035	0.7104
0.1	0.81481	0.85324	0.85635	0.85734	0.85779	0.85842	0.85845	0.85847	0.85849	0.85850	0.8585
0.2	1.00000	1.03384	1.04154	1.04253	1.04298	1.04361	1.04366	1.04366	1.04367	1.04369	1.0437
0.3	1.23810	1.27652	1.27963	1.28063	1.28107	1.28170	1.28173	1.28175	1.28177	1.28178	1.2818
0.4	1.55556	1.59398	1.59709	1.59809	1.59853	1.59916	1.59919	1.59921	1.59923	1.59924	1.5993
0.5	2.00000	2.03842	2.04154	2.04253	2.04298	2.04361	2.04364	2.04366	2.04367	2.04369	2.0437
0.6	2.66667	2.70509	2.70821	2.70920	2.70964	2.71028	2.71030	2.71032	2.71034	2.71035	2.7104
0.7	3.77778	3.81621	3.81932	3.82031	3.82076	3.82139	3.82141	3.82143	3.82145	3.82146	3.8214
0.8	6.00000	6.03843	6.04154	6.04253	6.04298	6.04361	6.04363	6.04366	6.04367	6.04369	6.0436
0.9	12.6667	12.70509	12.70821	12.70920	12.70964	12.71028	12.71030	12.71032	12.71034	12.71035	12.7104
0.99	132.667	132.70509	132.70821	132.70920	132.70965	132.71028	132.71030	132.71032	132.71034	132.71035	–
0.999	1332.670	1332.705	1332.708	1332.709	1332.709	1332.710	1332.710	1332.710	1332.710	1332.710	–

Exact, Williams [9]

Table 2 The extrapolated endpoint for the specular reflection ($\rho^d = 0$) and isotropic scattering ($f_1 = 0$)

ρ^d	P_1	P_3	P_5	P_7	P_9	P_{21}	P_{23}	P_{25}	P_{27}	P_{29}	Exact
0	0.66667	0.70509	0.70821	0.70920	0.70964	0.71028	0.71030	0.71032	0.71034	0.71035	0.7104
0.1	0.81481	0.85692	0.86054	0.86171	0.86225	0.86301	0.86304	0.86307	0.86309	0.86311	0.8632
0.2	1.00000	1.04575	1.04991	1.05129	1.05192	1.05283	1.05287	1.05289	1.05292	1.05294	1.0531
0.3	1.23810	1.28747	1.29220	1.29380	1.29454	1.29560	1.29565	1.29568	1.29571	1.29573	1.2959
0.4	1.55556	1.60852	1.61387	1.61570	1.61655	1.61778	1.61784	1.61788	1.61791	1.61794	1.6181
0.5	2.00000	2.05653	2.06252	2.06461	2.06558	2.06699	2.06706	2.06710	2.06714	2.06717	2.0674
0.6	2.66667	2.72674	2.73341	2.73576	2.73686	2.73848	2.73855	2.73860	2.73865	2.73868	2.7389
0.7	3.77778	3.84136	3.84875	3.85138	3.85262	3.85445	3.85453	3.85459	3.85464	3.85468	3.8550
0.8	6.00000	6.06706	6.07521	6.07814	6.07953	6.08158	6.08167	6.08174	6.08180	6.08185	6.0822
0.9	12.66667	12.73719	12.74613	12.74937	12.75092	12.75321	12.75331	12.75339	12.75345	12.75351	12.7539
0.99	132.66667	132.7403	132.7500	132.7535	132.7552	132.7577	132.7578	132.7579	132.7580	132.7580	132.758
0.999	1332.6667	1332.741	1332.750	1332.754	1322.756	1332.758	1332.758	1332.758	1332.758	1332.759	–

Exact, Williams [9]

The solution of Eq. (25) is given by

$$\phi_2(z) = A_1 exp(-\alpha_1 z); \qquad \alpha_1 = (35/9)^{1/2} \tag{26}$$

where A_1 is constant. Inserting Eq. (26) into (24d), $\phi_3(z)$ is found as

$$\phi_3(z) = \frac{3}{7} A_1 \alpha_1 exp(-\alpha_1 z). \tag{27}$$

Using Eqs.(26, 27) and substituting $\phi_1(z) = -(1 - b + d)^{1/2}$ in Eq. (24b), we obtain

$$\phi_0(z) = 3(1-\gamma')(1-b+d)^{1/2} z + A_0 - 2A_1 exp(-\alpha_1 z). \tag{28}$$

Using Eqs. (11d) and (11e), we obtain

$$\psi_3(x) = \frac{3}{7} \frac{A_1 \alpha_1 e^{-\alpha_1((1-b)^2-d^2)^{1/2} x}}{(1-b+d)^{1/2}}, \tag{29a}$$

$$\psi_2(x) = \frac{A_1 e^{-\alpha_1((1-b)^2-d^2)^{1/2} x}}{(1-b-d)^{1/2}}, \tag{29b}$$

$$\psi_1(x) = -1, \tag{29c}$$

$$\psi_0(x) = (3(1-b+d) - \gamma a)x + \frac{2A_1 e^{-\alpha_1((1-b)^2-d^2)^{1/2} x}}{(1-b-d)^{1/2}} + \frac{A_0}{(1-b-d)^{1/2}}, \tag{29d}$$

where, A_0 and A_1 are constants. By using the lowest order of Marshak boundary condition given in Eq. (14) the constants A_0, A_1 can be found as follows

$$A_0 = \frac{3}{4} A_1 + 2\left(\frac{1+\rho^s-\rho^d}{1-\rho^s-\rho^d}\right)(1-b-d)^{1/2}, \tag{30a}$$

$$A_1 = \frac{56(1-b-d)^{1/2}(2+(-3+\rho^d)\rho^d - 11\rho^d\rho^s - 2(\rho^s)^2)}{(-1+\rho^d+\rho^s)\{175(-2+\rho^d+2\rho^s) + 96\alpha_1[\rho^d - 2(1+\rho^s)]\}}. \tag{30b}$$

Finally, the extrapolated endpoint z_0 is calculated from Eq. (22) using A_0 in Eq. (30a) for P_3 approximation.

P_N **Approximation**: In P_N approximation for $c = 1$, the moment equations are obtained from Eq. (12) as

$$\frac{d\phi_1(z)}{dz} = 0,$$
$$2\frac{d\phi_2(z)}{dz} + \frac{d\phi_0(z)}{dz} + 3(1-\gamma')\phi_1(z) = 0,$$
$$3\frac{d\phi_3(z)}{dz} + 2\frac{d\phi_1(z)}{dz} + 5\phi_2(z) = 0,$$
$$\vdots$$
$$N\frac{d\phi_N(z)}{dz} + (N-1)\frac{d\phi_{N-2}(z)}{dz} + (2N-1)\phi_{N-1}(z) = 0,$$
$$N\frac{d\phi_{N-1}(z)}{dz} + (2N+1)\phi_N(z) = 0. \tag{31}$$

After some algebric manipulations, (N−1)th order differential equation is obtained from Eq. (31) and the solution of this equation is

$$\phi_{N-1}(z) = \sum_{k=1}^{\frac{N-1}{2}} A_k exp(-\alpha_k z). \tag{32}$$

The other solutions of moment equations can be found as

$$\phi_N(z) = -\frac{N}{2N+1}\frac{d\phi_{N-1}(z)}{dz}$$
$$\phi_{k-2}(z) = -\frac{2k-1}{k-1}\int \phi_{k-1}(z)dz - \frac{k}{k-1}\phi_k(z), \quad k = 4, 5, \ldots, N$$
$$\phi_1(z) = -(1-b+d)^{1/2}$$
$$\phi_0(z) = -3(1-\gamma')(1-b+d)^{1/2} z - \frac{2}{3}\phi_2(z) + A_0. \tag{33}$$

After these moment solutions $\phi_n(z)$ are transformed into $\psi_n(x)$ for $n = 0, 1, 2, \ldots, N$ using Eqs. (11d, 11e), the constants A_0 and A_k can be computed by using Marshak

Table 3 The extrapolated endpoint for the different values of ρ^s, ρ^d and isotropic scattering case ($f_1 = 0$) in P_{29} approximation

ρ^s/ρ^d	0.1		0.2		0.3		0.4		0.5	
	A	B	A	B	A	B	A	B	A	B
0.1	1.04829	1.0484	1.29104	1.2912	1.61319	1.6133	2.06238	2.0626	2.73384	2.7341
0.2	1.28639	1.2865	1.60850	1.6086	2.05764	2.0578	2.72905	2.7292	3.84495	3.8452
0.3	1.60385	1.6039	2.05294	2.0531	2.72431	2.7245	3.84016	3.8453	6.06717	6.0763
0.4	2.04829	2.0484	2.71961	2.7197	3.83542	3.8356	6.06238	6.0626	12.73384	12.7341
0.5	2.71496	2.7151	3.83072	3.8309	6.05764	6.0578	12.72905	12.7292	–	–

A; our result, B; Degheidy [16]

Table 4 The extrapolated endpoint for the different specular and diffuse reflection coefficients in linearly anisotropic scattering for P_{29} approximation

$\rho^s or \rho^d$	$f_1 = 0.1$				$f_1 = 0.2$				$f_1 = 0.3$			
	Diffuse		Specular		Diffuse		Specular		Diffuse		Specular	
	A	B	A	B	A	B	A	B	A	B	A	B
0	0.78928	0.78939	0.78928	0.78939	0.88794	0.88806	0.88794	0.88806	1.01479	1.01492	1.01479	1.01492
0.1	0.95389	0.95677	0.95901	0.95916	1.07313	1.07637	1.07888	1.07906	1.22643	1.23014	1.23301	1.23321
0.2	1.15965	1.16601	1.16994	1.17023	1.30461	1.31176	1.31618	1.31651	1.49098	1.49915	1.50420	1.50458
0.3	1.42420	–	1.43970	–	1.60223	–	1.61967	–	1.83112	–	1.85105	–
0.4	1.77694	–	1.79771	–	1.99905	–	2.02242	–	2.28463	–	2.31134	–
0.5	2.27076	2.29588	2.29686	2.29891	2.55461	2.52829	2.58397	2.58627	2.91955	2.95185	2.95311	2.95574
0.6	3.01150	–	3.04298	–	3.38794	–	3.42335	–	3.87193	–	3.91240	–
0.7	4.24607	–	4.28298	–	4.77683	–	4.81835	–	5.45923	–	5.50669	–
0.8	6.71521	–	6.75761	–	7.55461	–	7.60231	–	8.63384	–	8.68835	–
0.9	14.12262	14.3479	14.17056	14.2089	15.88794	16.14140	15.94188	15.9850	18.15765	18.4473	18.21929	18.2686
0.99	147.45595	149.933	147.5089	147.993	165.88794	168.6740	165.9476	166.4920	189.58622	192.771	189.6543	190.277
0.999	1480.7893	–	1480.843	–	1665.8879	–	1665.948	–	1903.8719	–	1903.941	–

A; our result, B; Atalay [14]

boundary condition given in Eq. (14). Finally, the extrapolated endpoint z_0 can be calculated from $\Psi_0(x)\big|_{x=-z_0} = 0$.

3. Numerical results

Numerical values of the extrapolated endpoint are given for isotropic scattering for diffuse reflection in Table 1 and for specular reflection in Table 2. They are compared with the exact values given in Williams's paper [9]. Table 3 shows the extrapolated endpoints for different values of ρ^s and ρ^d for isotropic scattering and they are compared with Degheidy [16]. It can be seen from these tables that the distance z_0 at which the flux drops off the zero, increases with increasing the specular and diffuse components of reflectivity. In Tables 4 and 5, the numerical values of z_0 are given for different values of ρ^s and ρ^d for the linearly anisotropic scattering and compared with Atalay [14]. In these tables, the extrapolated endpoint values increase with increasing linearly anisotropic scattering coefficient f_1. Table 6 shows the emergent angular distribution $\Psi(0, -\mu)$ for various values of ρ^s and ρ^d for isotropic scattering ($f_1 = 0$). The results are in good agreement with the literature for higher order P_N approximation. In Tables 7, 8, the extrapolated endpoint values are given for the extremely anisotropic scattering kernel. Figures 1, 2, 3 and 4 show the behaviors of the extrapolated endpoint as a function of b and d for different values of ρ^s and ρ^d. From these figures, it is seen that when forward scattering coefficent b increases, z_0 also increases. When backward scattering coefficient d increases, z_0 also decreases. We have carried out all our numerical computations in Mathematica programming.

Table 5 The extrapolated endpoint different values of ρ^s and ρ^d in linearly anisotropic scattering ($f_1 \neq 0$) for P_{29} approximation

ρ^s/ρ^d	0.1	0.2	0.3	0.4	0.5
$f_1 = 0.1$					
0.1	1.16475	1.43446	1.79241	2.29151	3.03757
0.2	1.42930	1.78720	2.28624	3.03225	4.27213
0.3	1.78203	2.28102	3.02698	4.26681	6.74127
0.4	2.27586	3.02176	4.26155	6.73595	14.14871
0.5	3.01662	4.25636	6.73071	14.14339	–
$f_1 = 0.2$					
0.1	1.31034	1.61377	2.01646	2.57794	3.41726
0.2	1.60796	2.01060	2.57202	3.41128	4.80615
0.3	2.00479	2.56615	3.40535	4.80017	7.583931
0.4	2.56034	3.39949	4.79424	7.57794	14.91730
0.5	3.39370	4.78840	7.57205	15.91131	–
$f_1 = 0.3$					
0.1	1.49754	1.84431	2.30453	2.94622	3.90544
0.2	1.83767	2.29782	2.93945	3.89860	5.49274
0.3	2.29119	2.93275	3.89183	5.48590	8.66735
0.4	2.92611	3.88513	5.47913	8.66051	18.19120
0.5	3.87851	5.47246	8.65377	18.18436	–

4. Conclusions

The Milne problem with the specular and diffuse reflecting boundary conditions is solved using Legendre polynomial approximation for a non-absorbing half space medium. In this work, the combination of the linearly and extremely anisotropic scattering functions are considered. The moment equations, infinite set of coupled differential equations which are obtained from the neutron transport equation for linearly plus extremely anisotropic scattering kernel are transformed into the equations written for linearly anisotropic scattering kernel. After that, applying N th order approximation in the P_N method, the set of N+1 coupled differential equations are obtained and reduced to one (N−1) th order differential equation for $\Psi_{N-1}(x)$. The remaining N moments for $\Psi_N(x)$ are derived after finding the solution of $\Psi_{N-1}(x)$. The constants of these solutions

Table 6 The emergent angular distribution $\Psi(0, -\mu)$ for the selected values of ρ^s and ρ^d in isotropic scattering ($f_1 = 0$)

$-\mu$	$\rho^d = 0.0$	$\rho^d = 0.2$	$\rho^d = 0.4$	$\rho^d = 0.6$	$\rho^d = 0.8$
$\rho^s = 0$					
0.1	1.18027	1.68027	2.51360	4.18027	9.18027
0.2	1.19913	1.69913	2.53246	4.19913	9.19913
0.3	1.46110	1.96110	2.79444	4.46110	9.46110
0.4	1.55339	2.05339	2.88673	4.55339	9.55339
0.5	1.76704	2.26704	3.10037	4.76704	9.76704
0.6	1.88438	2.38438	3.21771	4.88438	9.88438
0.7	2.05383	2.55383	3.38717	5.05383	10.0538
0.8	2.22933	2.72933	3.56266	5.22933	10.2293
0.9	2.38267	2.88267	3.7160	5.38267	10.3827
1.0	2.59895	3.09895	3.93228	5.59895	10.5989
$-\mu$	$\rho^s = 0.0$	$\rho^s = 0.2$	$\rho^s = 0.4$	$\rho^s = 0.6$	$\rho^s = 0.8$
$\rho^d = 0$					
0.1	1.18027	1.68504	2.50759	4.19368	9.38526
0.2	1.19913	1.67880	2.57907	4.13763	9.07603
0.3	1.46110	1.96261	2.78528	4.46511	9.59919
0.4	1.55339	2.04421	2.93737	4.52577	9.48316
0.5	1.76704	2.27007	3.09607	4.77591	9.88266
0.6	1.88438	2.38146	3.26017	4.87573	9.86080
0.7	2.05383	2.55470	3.41334	5.05653	10.08230
0.8	2.22933	2.73506	3.55956	5.24654	10.34169
0.9	2.38267	2.88912	3.71281	5.40209	10.49565
1.0	2.59895	3.11776	3.82831	5.65495	10.97911

Table 7 The extrapolated endpoint for the different values of b and using P_{29} approximation in the case of $f_1 = 0$, $d = 0$

ρ^s/ρ^d	0	0.1	0.2	0.3	0.4	0.5
$b = 0.1$						
0	0.7892815	0.9538906	1.159652	1.424202	1.776936	2.270763
0.1	0.9590067	1.64768	1.429318	1.782052	2.275879	3.016620
0.2	1.169935	1.434485	1.787219	2.281046	3.0211787	4.256355
0.3	1.439704	1.792437	2.286265	3.027005	4.261573	6.730709
0.4	1.797709	2.291536	3.032276	4.266844	6.735980	14.14339
0.5	2.296860	3.037601	4.272169	6.741305	14.14871	–
$b = 0.5$						
0	1.420707	1.717003	2.087373	2.563564	3.198485	4.087373
0.1	1.726212	2.096582	2.572773	3.207694	4.096582	5.429916
0.2	2.105883	2.582073	3.216994	4.105883	5.439216	7.661438
0.3	2.591467	3.226387	4.115276	5.448610	7.670832	12.11528
0.4	3.235875	4.124764	5.458098	7.680320	12.12476	25.45810
0.5	4.134348	5.467682	7.689904	12.13435	25.46768	–
$b = 0.9$						
0	7.103534	8.585015	10.43687	12.81782	15.99242	20.43673
0.1	8.631060	10.48291	12.86386	16.03847	20.48291	27.14958
0.2	10.52941	12.91037	16.08497	20.52941	27.19608	38.30719
0.3	12.95733	16.13194	20.57638	27.24305	38.35416	60.57638
0.4	16.17938	20.62382	27.29049	38.40160	60.62382	127.2905
0.5	20.67174	27.33841	38.44952	60.67174	127.3384	–

Table 8 The extrapolated endpoint for the different values of d and using P_{29} approximation in the case of $f_1 = 0$, $b = 0$

ρ^s/ρ^d	0	0.1	0.2	0.3	0.4	0.5
$d = 0.1$						
0	0.6476648	0.7823449	0.9506951	1.167145	1.455746	1.859786
0.1	0.7869615	0.9553117	1.171762	1.460362	1.864403	2.470463
0.2	0.9600215	1.176472	1.465072	1.869112	2.475173	3.485274
0.3	1.181278	1.469878	1.873918	2.479979	3.490080	5.510282
0.4	1.474783	1.878824	2.484884	3.494985	5.515187	11.57579
0.5	1.883831	2.489892	3.499993	5.520195	11.58080	–
$d = 0.5$						
0	0.4808969	0.5796623	0.7031191	0.8618492	1.073489	1.369786
0.1	0.5846012	0.7080580	0.8667882	1.078428	1.374725	1.819169
0.2	0.7133265	0.8720567	1.083697	1.379993	1.824438	2.565178
0.3	0.8776890	1.089329	1.385626	1.830070	2.570811	4.052292
0.4	1.095365	1.391661	1.836105	2.576846	4.058327	8.502772
0.5	1.398144	1.842588	2.583329	4.064811	8.509255	–
$d = 0.9$						
0	0.3871759	0.4651486	0.5626145	0.6879278	0.8550122	1.088930
0.1	0.4715786	0.5690445	0.6943578	0.8614421	1.095360	1.446237
0.2	0.5764445	0.7017578	0.8688422	1.102760	1.453638	2.038433
0.3	0.7103654	0.8774497	1.111368	1.462245	2.047040	3.216631
0.4	0.8875868	1.121505	1.472382	2.057177	3.226768	6.735540
0.5	1.133619	1.484496	2.069292	3.238882	6.747654	–

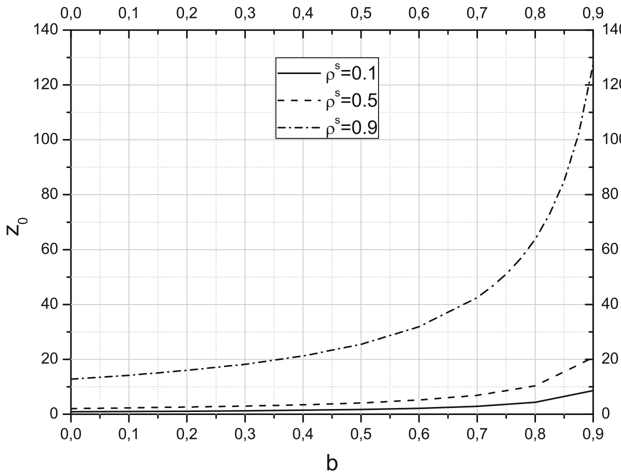

Fig. 1 The behaviors of the extrapolated endpoint as a function of b for different values of ρ^s

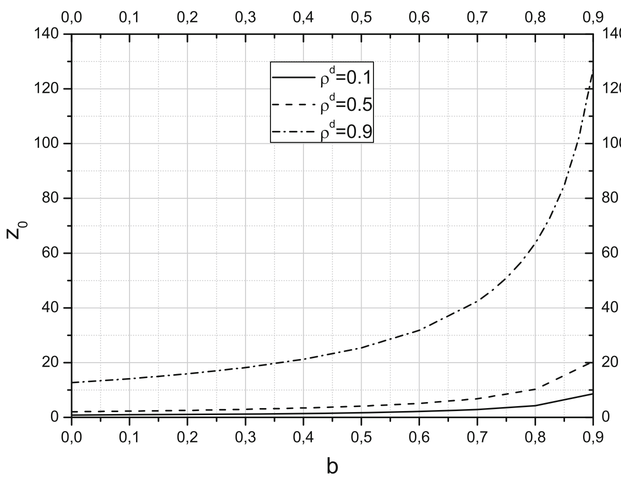

Fig. 3 The behaviors of the extrapolated endpoint as a function of b for different values of ρ^d

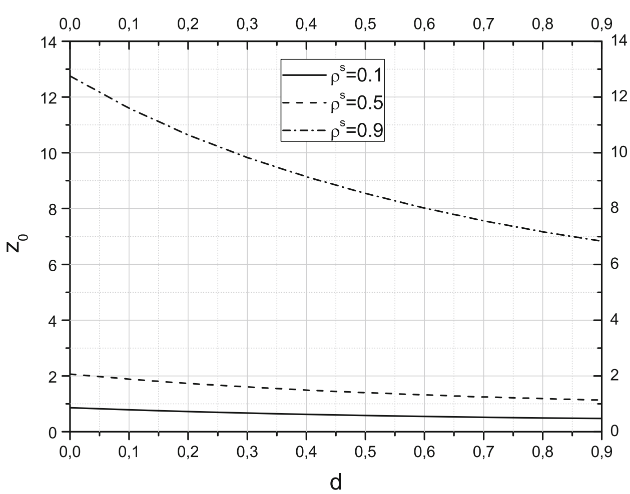

Fig. 2 The behaviors of the extrapolated endpoint as a function of d for different values of ρ^s

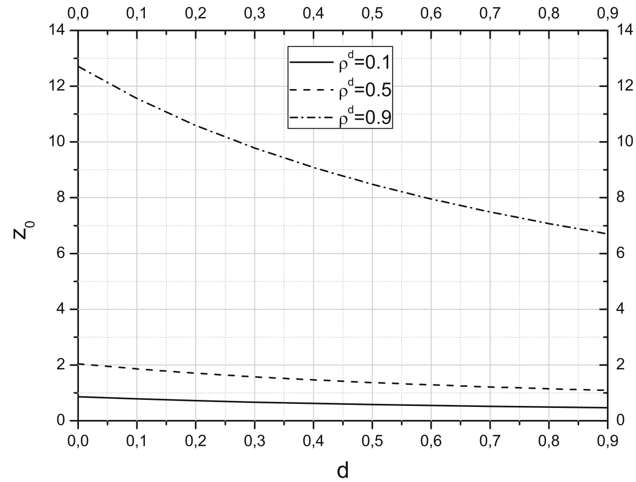

Fig. 4 The behaviors of the extrapolated endpoint as a function of d for different values of ρ^d

are calculated using Marshak boundary condition with specular and diffuse reflectivities of the boundary. And then the distance z_0 is obtained using A_0 constant for each N th order approximation. Finally some results for the distance z_0 and the emergent angular distributions are tabulated in tables and compared with the literature (Williams [9]), (Degheidy and El-Shahat [16]), (Atalay [14]). The comparisons with the present numerical results in given tables show that the P_N method agrees with the available results reported in the literatures. Furthermore, the numerical values of extrapolated endpoint z_0 are calculated using forward and backward anisotropic scattering kernel for the non-absorbing medium for the first time in the literature.

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