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ORIGINAL PAPER

Chaotic synchronization between linearly coupled discrete fractional Hénon maps

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Abstract: Chaotic synchronization between linearly coupled discrete fractional Hénon maps is investigated in this paper. We obtain the numerical formula of discrete fractional Hénon map by utilizing the discrete fractional calculus. We tune the linear coupling parameter and the order parameter of discrete fractional Hénon map to obtain the two discrete fractional Hénon maps in a synchronized regime and analyze the effect of linear coupling on synchronized degree. It demonstrates that the order parameter of discrete fractional Hénon map affects synchronization dynamics and with the increase of linear coupling strength, the effect of synchronization between discrete fractional Hénon maps is enhanced. Further investigation reveals that the transition of synchronization between discrete fractional Hénon maps are related to the critical changes in linearly coupled strength.

Keywords: Discrete fractional calculus; Chaotic synchronization; Fractional Hénon map; Caputo-like delta difference

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1. Introduction

Synchronization, which means adjustment or entrainment of frequencies of periodic oscillators due to a weak interaction, has received much attention during the last century [1, 2], because of its interesting phenomena as well as its wide applications in physics, life sciences, and engineering [3–6]. Since the seminal work by Pecora and Carroll in 1990 [7], chaotic synchronization has been aroused great interest. The dynamics, especially the synchronization, of coupled systems have been studied in different fields [8, 9], such as phase synchronization [10], small world network [11], and so on. Furthermore, chaotic synchronization has been applied to many fields, for example secure communication [12, 13].

However, most of the results associated with the synchronization for coupled systems are based on the continuous integer or fractional differential equations [14, 15], which, of course, does not account the interaction between the fractional difference equations. The fractional difference has a short history. Miller and Ross [18] have begun the theory of the fractional difference. After that, the theory of the fractional difference equations is developing faster and several new applications have been deeply analyzed. For example, Atici and Eloe [19] have discussed the discrete initial value problem and gave the existence results. Holm [20] has proposed the Laplace transform for solving discrete fractional equations in the nabla's sense. Abedel [21, 22] has systemically discussed the Caputo and the Riemann-Liouville fractional differences as well as their properties. Han [23] has discussed the existence and nonexistence of positive solutions to a discrete fractional boundary value problem with a parameter. Wu [24] has analyzed the dynamical behaviors of fractional sine and standard maps.

We wonder whether the synchronization phenomenon exists or not when the discrete fractional maps of couplings have been introduced, which may cause complicated interaction between the discrete fractional maps. In this paper we consider the chaotic synchronization of two linearly coupled the discrete fractional Hénon maps [25], described by

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$$\begin{cases} {}^C\Delta_p^v x_1(t) = 1 - x_1(t+v-1)(1 - a x_1(t+v-1)) + y_1(t+v-1) + \varepsilon(x_2(t+v-1) - x_1(t+v-1)), \\ \quad y_1(n) = b x_1(n-1), \\ {}^C\Delta_p^v x_2(t) = 1 - x_2(t+v-1)(1 - a x_2(t+v-1)) + y_2(t+v-1) + \varepsilon(x_1(t+v-1) - x_2(t+v-1)), \\ \quad y_2(n) = b x_2(n-1), \end{cases} \tag{1}$$

where a, b are the parameters, v is the order of discrete fractional Hénon map, and ε is the coupling intensity. It may be noted that the parameters may influence the dynamics of the systems, not only the regular states, but also the chaotic movements, which therefore affect the evolution the pace of the coupled maps. The main goal of this paper is to investigate the effect of the linearly coupling strength ε on transition to synchronization with different parameter conditions.

2. Basic definitions and preliminaries about discrete fractional calculus

We start with some necessary definitions from discrete fractional calculus theory and review the preliminary results so that paper is self-contained. In the following we use two notations $\mathbb{N}_p$ and Δ^m. $\mathbb{N}_p$ denotes the isolated time scale and $\mathbb{N}_p = \{p, p+1, p+2, \ldots\}, p \in \mathbb{R}$. Δ^m is the mth order forward difference operator

$$(\Delta^m f)(t) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(t+k)\,. \tag{2}$$

When $m = 1, \Delta f(t) = f(t+1) - f(t)$.

Definition 2.1 [19] Let $v > 0$. The vth fractional sum f is defined by

$$\Delta_p^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=p}^{t-v} (t-s-1)^{(v-1)} f(s)\,, \tag{3}$$

where p is the starting point, f is defined for $s = p$ mod 1 and $\Delta^{-v} f$ is defined for $t = (p+v)$ mod 1, in fact, Δ^{-v} maps functions defined on $\mathbb{N}_p$ to functions defined on $\mathbb{N}_{p+v}$. In addition,

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t-v+1)}. \tag{4}$$

Definition 2.2 [26] Let $v > 0$ and $m-1 < v < m$, where m denotes a positive integer, $m = \lceil \mu \rceil, \lceil \cdot \rceil$ denotes the ceiling of number. The vth fractional Caputo-like difference is defined as

$$\begin{aligned} {}^C\Delta_p^v f(t) &= \Delta^{v-m}(\Delta^m f(t)) \\ &= \frac{1}{\Gamma(m-v)} \sum_{s=p}^{t-(m-v)} (t-s-1)^{(m-v-1)} (\Delta^m f)(s), \\ & t \in \mathbb{N}_{p+m-v}. \end{aligned} \tag{5}$$

Remark 2.1 If $0 < v < 1$ and the starting point $p = 0$, then the μth fractional Caputo-like difference is defined as

$$\begin{aligned} {}^C\Delta_0^v f(t) &= \Delta^{v-1}(\Delta f(t)) \\ &= \frac{1}{\Gamma(1-v)} \sum_{s=0}^{t-(1-v)} (t-s-1)^{(-v)} (\Delta f)(s), \quad t \in \mathbb{N}_{1-v}. \end{aligned} \tag{6}$$

Theorem 2.1 [27] *For the delta fractional difference equation*

$$\begin{cases} {}^C\Delta_p^v x(t) = f(t+v-1, \ x(t+v-1)), \\ \Delta^k x(p) = x_k, \quad m = [v]+1, k = 0,1,2,\ldots,m-1, \end{cases} \tag{7}$$

the equivalent discrete integral equation can be obtained as

$$\begin{aligned} x(t) = x_0(t) + \frac{1}{\Gamma(v)} \sum_{s=p+m-v}^{t-v} (t-s-1)^{(v-1)} \\ \times f(s+v-1, x(s+v-1)), \quad t \in \mathbb{N}_{p+m}, \end{aligned} \tag{8}$$

where the initial iteration reads

$$x_0(t) = \sum_{k=0}^{m-1} \frac{(t-p)^{(k)}}{k!} \Delta^k x(p). \tag{9}$$

Remark 2.2 If $0 < v < 1$ and the starting point $p = 0$, then Eq. (8) can be written as

$$\begin{aligned} x(t) = x(0) + \frac{1}{\Gamma(v)} \sum_{s=1-v}^{t-v} (t-s-1)^{(v-1)} \\ \times f(s+v-1, x(s+v-1)), \quad t \in \mathbb{N}_1. \end{aligned} \tag{10}$$

3. Chaotic synchronization of two linearly coupled discrete fractional Hénon maps

For $0 < v < 1$, the following equivalent discrete integral form of Eq. (1) can be obtained by using Theorem 2.1,

$$\begin{cases} x_1(t) = x_1(p) + \dfrac{1}{\Gamma(v)} \displaystyle\sum_{s=p+1-v}^{t-v} \dfrac{\Gamma(t-s)}{\Gamma(t-s+1-v)} (1 - x_1(s+v-1)(1 - ax_1(s+v-1)) + y_1(s+v-1) \\ \qquad + \varepsilon(x_2(s+v-1) - x_1(s+v-1))), \\ y_1(n) = bx_1(n-1), \\ x_2(t) = x_2(p) + \dfrac{1}{\Gamma(v)} \displaystyle\sum_{s=p+1-v}^{t-v} \dfrac{\Gamma(t-s)}{\Gamma(t-s+1-v)} (1 - x_2(s+v-1)(1 - ax_2(s+v-1)) + y_2(s+v-1) \\ \qquad + \varepsilon(x_1(s+v-1) - x_2(s+v-1))), \\ y_2(n) = bx_2(n-1), \end{cases} \tag{11}$$

where $t \in \mathbb{N}_{p+1}$. As a result, the numerical formula with the starting point $p = 0$ can be presented explicitly

$$\begin{cases} x_1(n) = x_1(0) + \dfrac{1}{\Gamma(v)} \displaystyle\sum_{j=1}^{n} \dfrac{\Gamma(n-j+v)}{\Gamma(n-j+1)} (1 - x_1(j-1)(1 - ax_1(j-1)) + y_1(j-1) + \varepsilon(x_2(j-1) - x_1(j-1))), \\ y_1(n) = bx_1(n-1), \\ x_2(n) = x_2(0) + \dfrac{1}{\Gamma(v)} \displaystyle\sum_{j=1}^{n} \dfrac{\Gamma(n-j+v)}{\Gamma(n-j+1)} (1 - x_2(j-1)(1 - ax_2(j-1)) + y_2(j-1) + \varepsilon(x_1(j-1) - x_2(j-1))), \\ y_2(n) = bx_2(n-1), \qquad n \geq 1. \end{cases} \tag{12}$$

Obviously, for coupled strength $\varepsilon = 0$, the coupled map (1) is regarded as the discrete fractional Hénon map. For different values of a, b and v, the discrete fractional Hénon map may be chaotic, intermittent, or converge to a periodic orbit. In this paper, all quantities are dimensionless. When the parameters are fixed at $b = 0.3, v = 0.7$, the bifurcation diagram with the variation of parameter a is plotted in Fig. 1 to reveal the dynamical behaviors of the discrete fractional Hénon map. Cascading of period-doubling bifurcations to chaos can be found, which implies, for certain parametric condition, the original regular movement with period of T may switch to chaos. For example, as shown in Fig. 2, a chaotic state can be observed for $a = 1.0, b = 0.3, v = 0.7$.

To explore the synchronization of two linearly coupled discrete fractional Hénon maps, we set $a = 1.0, b = 0.3, v = 0.7$ and the initial conditions $x_1(0) = y_1(0) = 0.0, x_2(0) = y_2(0) = 0.1$. From Eq. (12), we know that $x(n)$, $y(n)$ depend on the past information $x(0), x(1), x(2), \ldots, x(n-1)$ and $y(0), y(1), y(2), \ldots, y(n-1)$, which is called discrete memory effect of the system. The memory effects of the discrete map means that their present state of evolution depends on all past states. For $v = 1$, it shrinks to the classical Hénon map among which the $x(n)$ depends on $x(n-1)$. The map of integer order does not

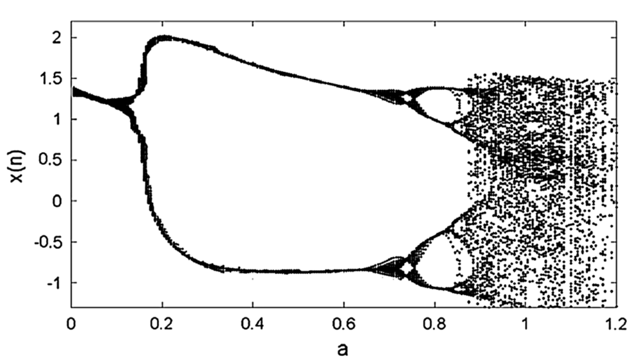

Fig. 1 Bifurcation diagram of the discrete fractional Hénon map for $b = 0.3, v = 0.7$

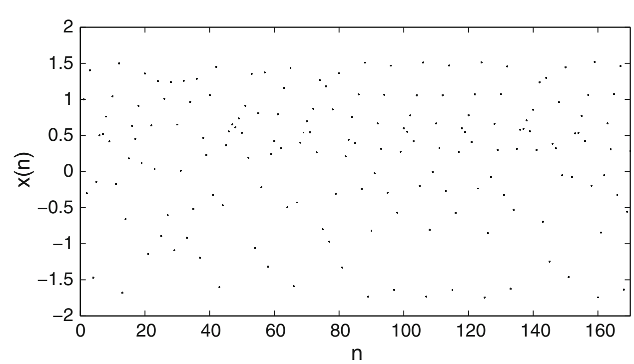

Fig. 2 Time series of the discrete fractional Hénon map for $a = 1.0, b = 0.3, v = 0.7$

hold such memory. This is one of the major differences between the integer map and the fractional one.

The synchronization can be obtained by the diffuse clouds as shown in Figs. 3, 4, 5. For $\varepsilon = 0.5$, the two linearly coupled discrete fractional Hénon maps are synchronized after $n \geq 20$ in Figs. 3 and 4. The phenomena can also be proved by the error diagram plotted in Figs. 6 and 7. It is implied the difference in state between the two maps tends to zero with the increase of n, which demonstrates a synchronized state between two rhythms.

The rhythm relationship between the two discrete fractional Hénon maps may be affected by the linear coupling strength and the order of discrete fractional Hénon map, measured by ε and v. In the following, we investigate the influence of the parameters ε and v on the synchronization. More information about the range of parameters that can be achieved in the two linearly coupled discrete fractional Hénon maps as given in Fig. 8. From Fig. 8, one can see that for a certain the order parameter, there exists a critical

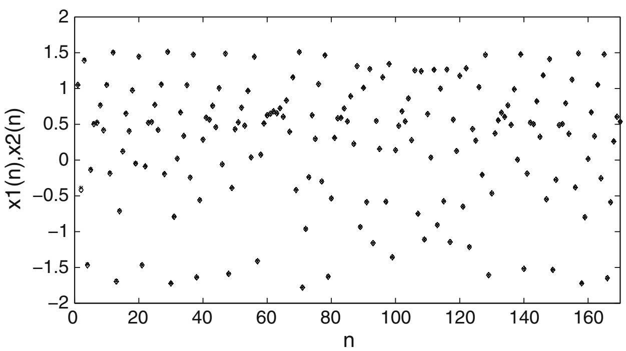

Fig. 3 Chaotic synchronization for $a = 1.0, b = 0.3, v = 0.7, d = 0.5$, the symbol *diamond* denotes first system, the symbol *asterisk* denotes second system

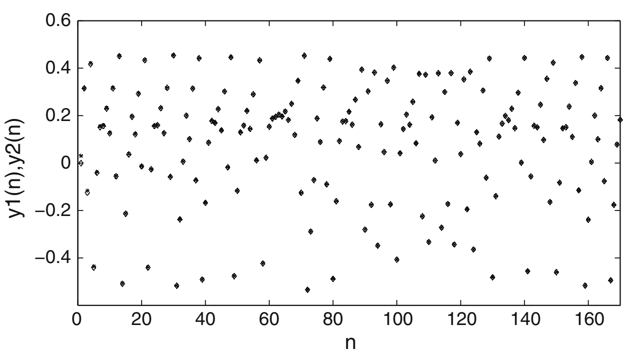

Fig. 4 Chaotic synchronization for $a = 1.0, b = 0.3, v = 0.7, d = 0.5$, the symbol *diamond* denotes first system, the symbol *asterisk* denotes second system

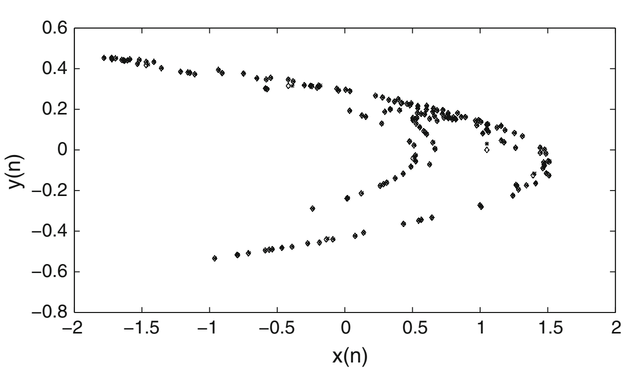

Fig. 5 Chaotic synchronization for $a = 1.0, b = 0.3, v = 0.7, d = 0.5$, the symbol *diamond* denotes first system, the symbol *asterisk* denotes second system

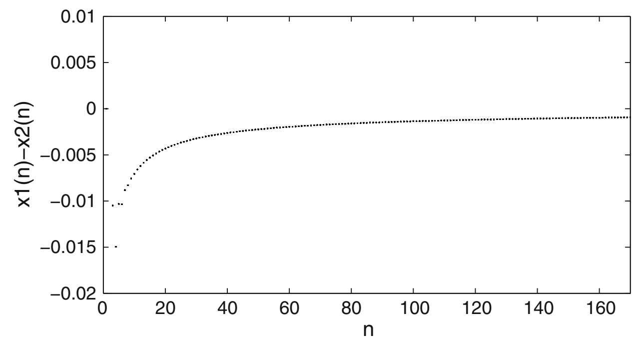

Fig. 6 Chaotic synchronization for $a = 1.0, b = 0.3, v = 0.7, d = 0.5$

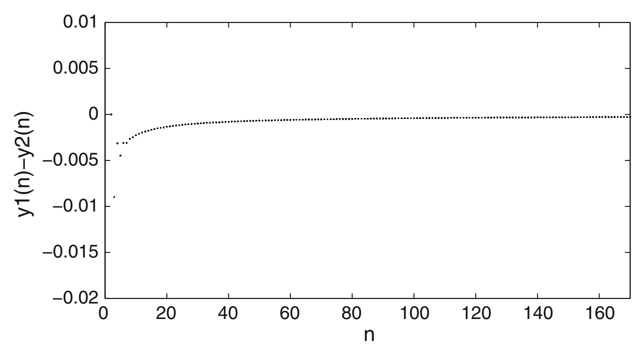

Fig. 7 Chaotic synchronization for $a = 1.0, b = 0.3, v = 0.7, d = 0.5$

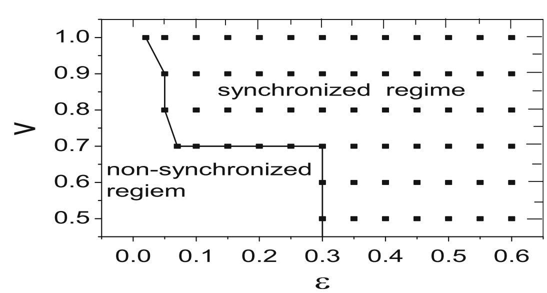

Fig. 8 Chaotic synchronization regime for $a = 1.0, b = 0.3$

linear coupling parameter, up which synchronization will occur. With the increase of the order parameter, the synchronized regime increases, that is to say, the larger the order parameter, the smaller the critical value of linear coupling parameter. Furthermore, with the increase of coupling strength, much stronger interaction between the two chaotic discrete fractional Hénon maps results in the state error between two discrete fractional Hénon maps approaching zero, i.e., the two rhythms of two discrete fractional Hénon maps may change from nonsynchronization to synchronization, which can be verified from Fig. 8.

4. Conclusions

As we all know, the chaotic synchronization of discrete maps has appeared in differential fields. It has played a crucial role in the nonlinear science. The fractional maps have recently appeared as a new topic. This paper aims to investigate the synchronization phenomenon between linearly coupled discrete fractional Hénon maps with the variation of coupling parameter and the order parameter of discrete fractional Hénon map. The occurrence of synchronization is related to the order parameter of discrete fractional Hénon map, which influences the structures of chaotic attractors. Furthermore, with the increase of coupling strength, the effect of synchronization between two discrete fractional Hénon maps is enhanced. The result can be further applied to secure communications. It provides a possible tool to deeply understand these nonlinear phenomena.

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