Effect of correlation in FitzHugh–Nagumo model with non-Gaussian noise and multiplicative signal

X L Li and L J Ning*

College of Mathematics and Information Science, Shaanxi Normal University, Xian 710062, China

Abstract: The effect of correlation in FitzHugh–Nagumo neural model induced by non-Gaussian noise and multiplicative signal is studied. Based on the corresponding Fokker–Planck equation, the explicit expressions of the stationary probability distribution function, the mean first passage time and signal-to-noise ratio are obtained, respectively. By analyzing the influence of different parameters, we observe that the system undergoes a succession of phase transition-like phenomena as correlation strength λ, correlation time τ and multiplicative noise intensity D are increased. And the mean first passage time exhibits a maximum, which identifies the noise-enhanced stability effect when correlation strength $\lambda < 0$. Furthermore, inhibition phenomenon and double stochastic resonance occur in FitzHugh–Nagumo neural model under different values of system parameters.

1. Introduction

FitzHugh–Nagumo (FHN) neural model is one of the simplified modifications of the widely known Hodgkin–Huxley model [1], which describes neuron dynamics and in general the dynamics of excitable systems in different fields, such as the kinetics of chemical reactions and solid state physics [2–5]. Brownian diffusion, also in periodic potential, is an appropriate model to describe fluctuations of neuronal activity [6–8]. As a simple but representative example of an excitable system, the dynamics of the FHN model in a noisy environment has attracted considerable interest. For example, the dynamics of a FHN system subjected to autocorrelated noise, has been investigated [9], where the role of colored noise on the phenomena of noise-enhanced stability and resonant activation has been analyzed. Lindner et al. [10] have investigated FHN model under the influence of white Gaussian noise in the excitable regime, where coherence resonance phenomenon was observed. Toral et al. [11] have studied the existence of a system size coherence resonance effect in the coupled FHN

models. Stochastic resonance (SR) in a FHN system with time-delayed feedback has been analyzed by Wu and Zhu [12]. They have revealed that SR of the system is a non-monotonic function of the noise intensity and the signal period and variation of the time-delayed feedback can induce periodic SR in the system. The effect of SR has also been studied in a FHN neuronal model driven by colored noise [13] and in an extended FHN system with a field-dependent activator diffusion [14].

However, previous research is in the case of Gaussian noise, but an experimental research shows that some noise in the nervous, biological and physical systems tend to non-Gaussian distribution [15, 16]. Because non-Gaussian noise leads to a non-Markov process and the mathematical expression is complex, studies of non-Gaussian noise are rare. Zhao et al. [17] showed the stationary and transient properties of a non-Gaussian noise-driven FHN model. Zhang et al. [18] have studied the SR in FHN neural system driven by non-Gaussian noise. They have observed that the presence of non-Gaussian noise is conductive to the enhancement of the response to the output signal of the FHN neural system. And noise-enhanced stability (NES) and SR in the presence of colored non-Gaussian noise have also been studied [19, 20]. More recently, SR in FHN model driven by multiplicative signal and non-Gaussian

*Corresponding author, E-mail: ninglijuan@snnu.edu.cn

noise has also been investigated [21]. But a system has always been disturbed simultaneously by external environmental perturbations and intrinsic thermal fluctuations, and there are certain situations, where the strong external perturbations can lead to some changes in the internal structure of the system and thus may be correlated with each other as well. Thus, the effect of correlation in FHN neural model with non-Gaussian and multiplicative signal needs to be investigated.

In this paper, we introduce noise correlation to FHN model with non-Gaussian noise and multiplicative signal. The approximate Fokker–Planck equation is obtained by the path integral approach and unified colored noise approximation. Based on the corresponding Fokker–Planck equation, explicit expressions of the stationary probability distribution function (SPDF), the mean first passage time (MFPT) and signal-to-noise ratio (SNR) are obtained by using the theory of SNR in the adiabatic limit.

2. FHN neural model

2.1. Stationary properties of the model

The dynamical equation of FHN model is given by [22]

$$\frac{dv}{dt} = v(a - v)(v - 1) - w, \tag{1}$$

$$\frac{dw}{dt} = bv - rw, \tag{2}$$

where, in the neural context, v is a fast variable denoting the neuron membrane voltage and w is a slow or recovery variable, which is related to the time-dependent conductance of the potassium channels in the membrane; $0 < a < 1$ is essentially the threshold value; b and r are positive constants. For the sake of simplicity, we have taken $r = 1$. By means of the adiabatic elimination method, the one-dimensional Langevin equation for the FHN model can be obtained as [22]

$$\frac{dv}{dt} = v(a - v)(v - 1) - bv + \xi(t). \tag{3}$$

The potential function,

$$U(v) = \frac{1}{4}v^4 - \frac{a+1}{3}v^3 + \frac{a+b}{2}v^2 \tag{4}$$

Corresponding to Eq. (3), there are two stable states: $v_1 = 0$, which represents the neurons of cells in the resting state. $v_2 = \frac{a+1+\sqrt{(a-1)^2-4b}}{2}$, which represents the neurons of cells in the excited state and an unstable state: $v_u = \frac{a+1-\sqrt{(a-1)^2-4b}}{2}$. In its bistable regime, i.e., $b < \frac{(a-1)^2}{4}$. If we consider the external environmental fluctuation and

the intrinsic thermal fluctuation, the dimensionless form of the one-dimensional Langevin equation (Eq. 3) reads:

$$\frac{dv}{dt} = v(a - v)(v - 1) - bv + vA\cos(\omega t) + v\eta(t) + \xi(t). \tag{5}$$

where A and ω are amplitude and frequency of the periodic signal, respectively. And all variables are normalized. $\eta(t)$ is non-Gaussian white noise and its statistical properties are described by [23]

$$\frac{d\eta(t)}{dt} = -\frac{1}{\tau}\frac{dV_q(\eta)}{d\eta} + \frac{1}{\tau}\varepsilon(t). \tag{6}$$

Here

$$V_q(\eta) = \frac{D}{\tau(q-1)}\ln\left(1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right), \langle\eta(t)\rangle = 0, \tag{7}$$

$$\langle\eta^2(t)\rangle = \begin{cases} \frac{2D}{\tau(5-3q)}, & q < \frac{5}{3} \\ \infty, & \frac{5}{3} \le q < 3 \end{cases}$$

where $\varepsilon(t)$ and $\xi(t)$ are Gaussian white noise and their statistical properties are given by:

$$\langle\varepsilon(t)\rangle = \langle\xi(t)\rangle = 0, \langle\varepsilon(t)\varepsilon(t')\rangle = 2D\delta(t - t'), \tag{8}$$

$$\langle\xi(t)\xi(t')\rangle = 2Q\delta(t - t'), \langle\xi(t)\varepsilon(t')\rangle = 2\lambda\sqrt{DQ}\delta(t - t'). \tag{9}$$

Here Q is the intensity of the additive noise $\xi(t)$; D is the intensity of $\varepsilon(t)$; λ is the correlation strength between multiplicative and additive noise; and τ is the correlation time of non-Gaussian noise $\eta(t)$. Using the path integral approach [24], the non-Gaussian noise can be written as

$$\frac{d\eta(t)}{dt} = -\frac{1}{\tau_{eff}}\eta(t) + \frac{1}{\tau_{eff}}\varepsilon_1(t), \tag{10}$$

Here, $\varepsilon_1(t)$ is Gaussian white noise and its statistical property is as follows:

$$\langle\varepsilon_1(t)\rangle = 0, \langle\varepsilon_1(t)\varepsilon_1(t')\rangle = 2D_{eff}\delta(t - t'), \tag{11}$$

where τ_{eff} is an effective correlation time of noise and D_{eff} is an effective intensity of noise:

$$\tau_{eff} = \frac{2(2-q)}{5-3q}\tau, \quad D_{eff} = \frac{2(2-q)^2}{5-3q}D. \tag{12}$$

Parameter q is the deviation of non-Gaussian noise from Gaussian behavior. When $q \to 1$, $\eta(t)$ approximates as colored Gaussian noise that its associated time is τ_{eff} and noise intensity is D_{eff}. Using the unified colored noise approximation [25], Eq. (5) takes the form:

$$\frac{\partial\rho(v,t)}{\partial t} = -\frac{\partial}{\partial v}[A(v)\rho(v,t)] + \frac{\partial^2}{\partial v^2}[B(v)\rho(v,t)]. \tag{13}$$

in which

$$B(v) = \left[\frac{g(v)}{c(v)}\right]^2,$$

$$A(v) = \frac{f(v)}{c(v)} + \sqrt{B(v)}\left(\sqrt{B(v)}\right)',$$

$$f(v) = v(a-v)(v-1) - bv + vA\cos(\omega t),$$

$$g(v) = \sqrt{v^2 D_{\text{eff}} + 2\lambda\sqrt{D_{\text{eff}} Q}v + Q},$$

$$c(v) = 1 - \tau_{\text{eff}}[-2v^2 + (a+1)v].$$

Thus, SPDF $\rho_{st}(v)$ can be derived from Eq. (13) in the adiabatic limit as

$$\rho_{\text{st}}(v) = \frac{N}{\sqrt{B(v)}} e^{-\frac{V(v)}{D_{\text{eff}}}}. \tag{14}$$

where N is normalization constant. $V(v)$ is the effective potential function and its form follows

$$V(v) = -\int \frac{1}{v^2 + 2\lambda\sqrt{\frac{Q}{D_{\text{eff}}}}v + \frac{Q}{D_{\text{eff}}}}[v(a-v)(v-1)$$

$$- bv + vA\cos(\omega t)]\left[1 - \tau_{\text{eff}}\left(-2v^2 + (a+1)v\right)\right]dv$$

$$= A_1 v^4 - A_2 v^3 + A_3 v^2 + A_4 v + A_5 \ln\left[(m+v)^2 + n\right]$$

$$+ A_6 \arctan\left(\frac{m+v}{\sqrt{n}}\right)$$

$$- \left[B_1 v^2 - B_2 v + B_3 \ln\left((m+v)^2 + n\right)\right.$$

$$\left. + B_4 \arctan\left(\frac{m+v}{\sqrt{n}}\right)\right] A\cos(\omega t),$$

in which $d = \frac{Q}{D_{\text{eff}}}, m = \lambda\sqrt{d}, n = d(1-\lambda^2), c_1 = a+1, \ c_2 = a+b, k_1 = 2\tau_{\text{eff}}, k_2 = \tau_{\text{eff}} c_1, k_3 = 1 + \tau_{\text{eff}} c_1^2 + 2\tau_{\text{eff}} \quad c_2, k_4 = c_1 + \tau_{\text{eff}} c_1 c_2, k_5 = c_2, A_1 = \frac{k_1}{4}, A_2 = \frac{2mk_1}{3} + k_2, A_3 = \frac{k_3 - (n-3m^2)k_1}{2} + 3mk_2, A_4 = 4mk_1(n-m^2) + 3k_2(n-3m^2) \quad -2mk_3 - k_4, A_5 = \frac{k_1(5m^4 - 10m^2 n + n^2) + k_3(3m^2 - n) + k_5}{2} - 6k_2 m(n - m^2) + k_4, A_6 = \frac{-k_1 m(m^4 - 10m^2 n + 5n^2) - 3k_2(m^4 - 6m^2 n + n^2) + k_3 m(3n - m^4) + k_4(n - m^2) - k_5 m}{\sqrt{n}},$ $B_1 = \frac{k_1}{2}, B_2 = 2mk_1 + k_2, B_3 = \frac{k_1(3m^2 - n) + 1}{2} + k_2, B_4 = \frac{k_1 m(3n - m^2) + k_2(n - m^2) - m}{\sqrt{n}}.$

2.2. MFPT of the model

In this section, we are interested in the time from one stable state to the other stable state under the influences of environmental random perturbations. Random fluctuations present in this system can induce transitions between these two states. From this point of view, it is interesting to study the MFPT of transition from one state (v_2) to the other (v_1). MFPT is one of the basic quantities, which describe the escaping problem. It is the average time that the system set

out from a steady state, passes through the potential barrier and enters another potential well. Using the steepest-descent approximation [26, 27], the explicit expression for MFPT of the process $v(t)$ to reach the state v_2 with the initial condition $v(t = 0) = v_1$ is given by

$$T = \frac{2\pi}{\sqrt{|U''(v_u)U''(v_1)|}}\exp\left[\frac{V(v_1) - V(v_u)}{D_{\text{eff}}}\right]$$

$$= \frac{2\pi}{\left[3v_u^2 - 2(a+1)v_u + (a+b)\right]\left[3v_1^2 - 2(a+1)v_1 + (a+b)\right]}$$

$$\times \exp\left[\frac{V(v_1) - V(v_u)}{D_{\text{eff}}}\right] \tag{15}$$

Note that the above result is valid only when the intensity of both types of noise, measured by D_{eff} and Q, is small in comparison with the energy barrier height [28, 29], that is

$$D_{\text{eff}}, \ Q < |V(v_u) - V(v_1)|.$$

It provides the restriction on the parameters (such as $D_{\text{eff}}, Q\ldots$). The following results in this study are restricted to valid regions.

2.3. SNR of the model

To obtain the expression of SNR in terms of the output signal power spectrum, the key problem is to calculate the transition rate. In this paper, we consider the adiabatic approximation [30, 31]. Therefore, we can obtain the transition rates [26, 27]

$$W_{v_1 \to v_2} = W_1$$
$$= \frac{\sqrt{|U''(v_u)U''(v_1)|}}{2\pi}\exp\left[\frac{V(v_1, t) - V(v_u, t)}{D_{\text{eff}}}\right], \tag{16}$$

$$W_{v_2 \to v_1} = W_2$$
$$= \frac{\sqrt{|U''(v_u)U''(v_2)|}}{2\pi}\exp\left[\frac{V(v_2, t) - V(v_u, t)}{D_{\text{eff}}}\right], \tag{17}$$

where U'' is the second derivative of U with respect to v. Equations (15) and (16) are valid for asymmetric cases.

We state by considering a system described by a discrete random dynamical variable v that adopts two possible values: v_1 and v_2, with probabilities $n_{1,2}$, respectively. Such probabilities satisfy the condition $n_1 + n_2 = 1$. The master equation governing the evolution of n_1 (or similarly for $n_2 = 1 - n_1$) is

$$\frac{dn_1}{dt} = -\frac{dn_2}{dt} = W_2(t)n_2(t) - W_1(t)n_1(t)$$
$$= W_2(t) - [W_2(t) + W_1(t)]n_1(t), \tag{18}$$

where $W_{1,2}(t)$ are the transition rates out of the $v_{1,2}$ states. Since we assume that the signal amplitude is small enough
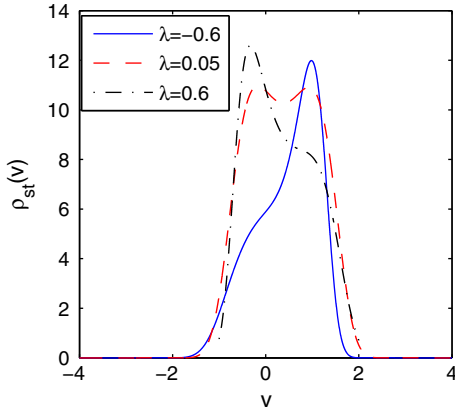
Fig. 1 Stationary probability distribution ρ_{st} versus normalized v for $D = 0.5$, $Q = 0.5$, $q = 0.5$, $\tau = 0.1$
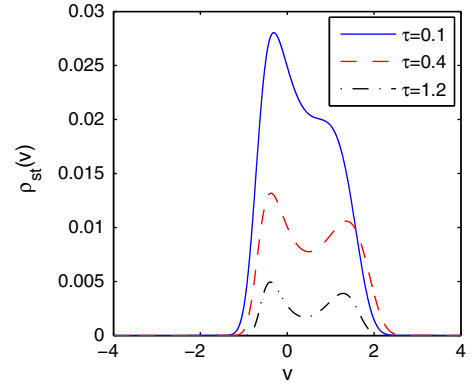

Fig. 2 Stationary probability distribution ρ_{st} versus normalized v for $D = 0.5$, $Q = 0.5$, $q = 0.5$, $\lambda = 0.5$
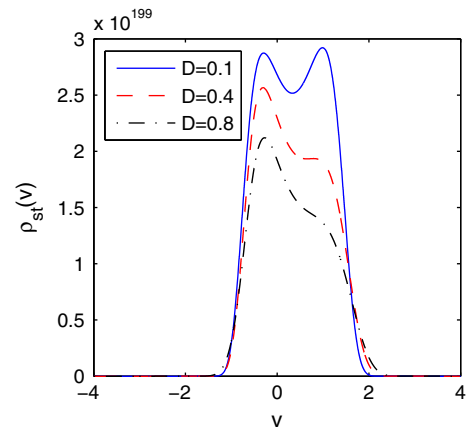

Fig. 3 Stationary probability distribution ρ_{st} versus normalized v for $Q = 0.5$, $q = 0.5$, $\lambda = 0.5$, $\tau = 0.1$
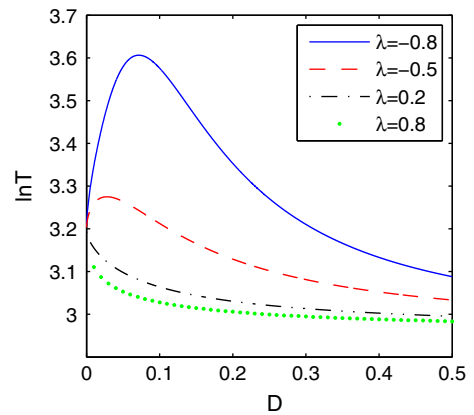

Fig. 4 MFPT versus normalized multiplicative noise intensity D for different values of λ with $\tau = 0.1$, $q = 0.5$, $Q = 0.001$

(i.e., $A \ll 1$), the transition rates $W_{1,2}(t)$ can be expanded up to the first order of A as

$$W_1(t) = \mu_1 - \Delta_1 A \cos(\omega t), \tag{19}$$

$$W_2(t) = \mu_2 + \Delta_2 A \cos(\omega t), \tag{20}$$

where $\mu_1 = W_1|_{A\cos(\omega t)=0}, \mu_2 = W_2|_{A\cos(\omega t)=0}, \quad \Delta_1 = -\frac{dW_1}{d[A\cos(\omega t)]}\Big|_{A\cos(\omega t)=0}$ and $\Delta_2 = -\frac{dW_2}{d[A\cos(\omega t)]}\Big|_{A\cos(\omega t)=0}$. Within the framework of the theory [30, 31], the expression of SNR in terms of the output signal power spectrum can be given by

$$SNR = \frac{A^2\pi(\mu_1\Delta_2 + \mu_2\Delta_1)^2}{4\mu_1\mu_2(\mu_1 + \mu_2)}. \tag{21}$$

Here we take $a = 0.5$, $b = 0.01$.

3. Results and discussion

In Fig. 1, we study the effect of correlation strength λ on SPDF. As we can see, for a negative λ value, the curve shows a single peak region near $v = 1$, which illustrates that probability of distribution of membrane variable voltage is mostly in $v = 1$. However, when the value of correlation strength is increased, the curves show a double-peak structure ($\lambda = 0.05$), which just are the positions of two states in potential function $U(v)$. As the value of correlation strength continues to increase, the right peak disappears. It is seen that correlation strength causes the system to change from one peak to two peaks and then to one peak; namely, the system undergoes a succession of two phase transition-like phenomena as correlation strength λ is increased.

Figure 2 shows the effect of correlation time τ on SPDF. As shown in Fig. 2, for a small τ, the curve shows a single peak region near $v = 0$. As τ increases, the height of peak

near $v = 0$ decreases. At the same time, a new peak appears near $v = 1$. It can be said that correlation time causes the system to change from one peak to two peaks;
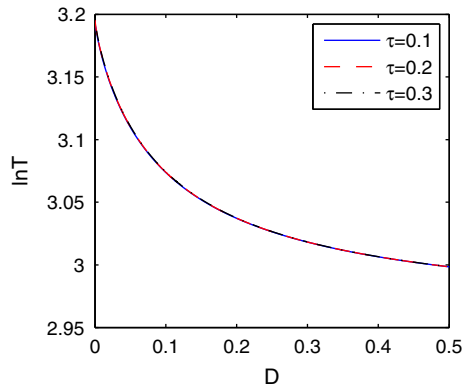
Fig. 5 MFPT versus normalized multiplicative noise intensity D for different values of τ with $\lambda = 0.5$, $q = 0.5$, $Q = 0.001$
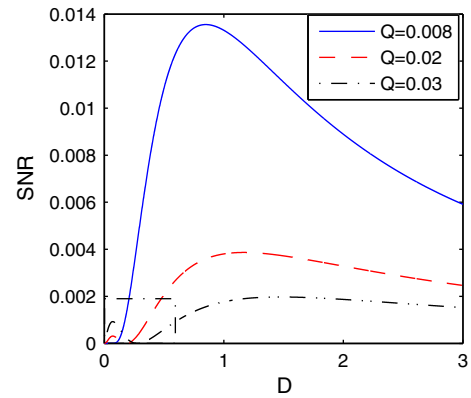

Fig. 7 SNR versus normalized multiplicative noise intensity D for different values of Q with $\tau = 0.1$, $\lambda = 0.5$, $q = 0.5$, $A = 0.05$
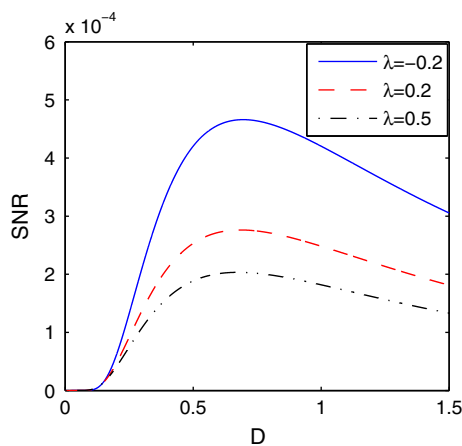

Fig. 6 SNR versus normalized multiplicative noise intensity D for different values of λ with $\tau = 0.1$, $q = 0.5$, $Q = 0.001$, $A = 0.05$
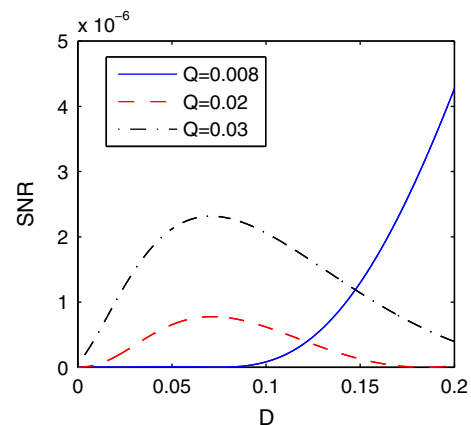

Fig. 8 SNR versus normalized multiplicative noise intensity D for different values of Q with $\tau = 0.1$, $\lambda = 0.5$, $q = 0.5$, $A = 0.05$

namely, the system undergoes a phase transition-like phenomenon as τ is increased. This peculiar role of noise correlation time on the SPDF represents a new feature due to the presence of noise correlation, with respect to the case of absence of noise correlation, where the increasing of τ cannot induce a phase transition-like phenomenon [17]. In addition, as τ continues increasing, one can see that the heights of two peaks are close to the same. The result shows that the channel is open, which is conducive to the transfer of information between the resting state and the excited state.

Furthermore, we study the effect of multiplicative noise intensity D on SPDF in Fig. 3. From Fig. 3, it is seen that for small value of D, the curve exhibits a symmetric bimodal structure. As D increases, the heights of two peaks decrease and the right peak disappears. In other words, the multiplicative noise intensity D can also induce a phase transition-like phenomenon.

Figure 4 displays the effect of the noise correlation strength λ on MFPT. As we can see, when the noise

correlation strength $\lambda > 0$ (i.e., $\lambda = 0.2$ and 0.8 in Fig. 4), it is shown that the MFPT is monotonic and decreases with increasing multiplicative noise intensity D. However, when the noise correlation strength is smaller (i.e., $\lambda = -0.8$, and -0.5 in Fig. 4), the MFPT first increases, reaches a maximum and then decreases with increasing multiplicative noise intensity D. This maximum for MFPT identifies the NES effect. Moreover, the maximum of MFPT increases with decreasing noise correlation strength λ, i.e., the decreasing λ intensifies the NES effect. Our results showed that, under smaller noise strength λ, a critical multiplicative noise intensity D exists at which the MFPT induced by noise is a maximum. The NES effect has been shown in previous investigations [32–35], which implies that the stability of metastable or unstable states can be enhanced by the noise and the mean lifetime of the metastable state is longer than the deterministic decay time. This resonance-like behavior contradicts the monotonic behavior predicted by Kramers theory [36, 37].
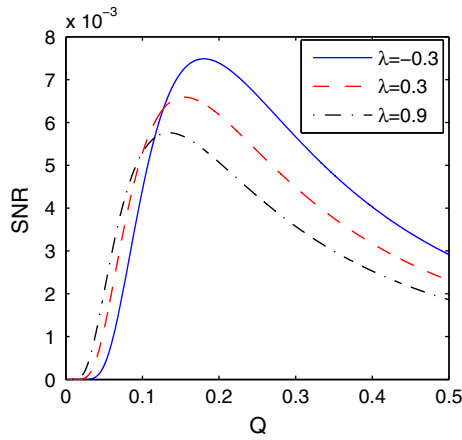
Fig. 9 SNR versus normalized addictive noise intensity Q for different values of λ with $D = 0.01$, $q = 0.1$, $\tau = 0.1$, $A = 0.05$

Fig. 11 SNR versus normalized addictive noise strength λ for different values of τ with $Q = 0.1$, $q = 0.1$, $D = 0.1$, $A = 0.05$

Fig. 10 SNR versus normalized addictive noise intensity Q for different values of D with $\lambda = 0.5$, $q = 0.1$, $\tau = 0.1$, $A = 0.05$
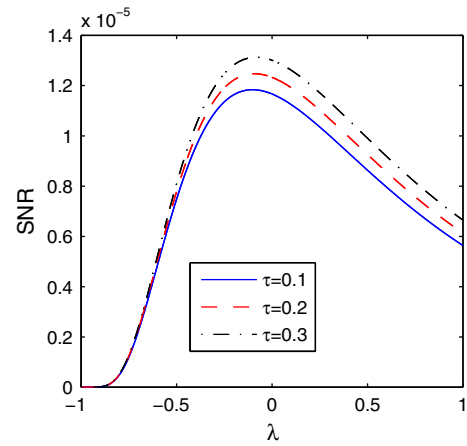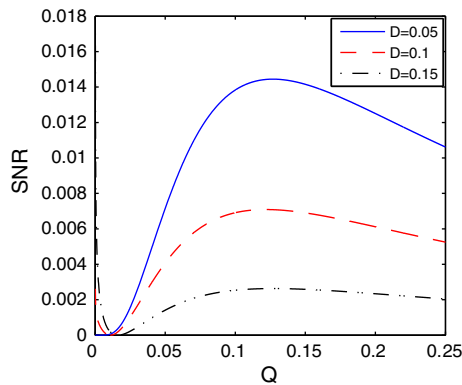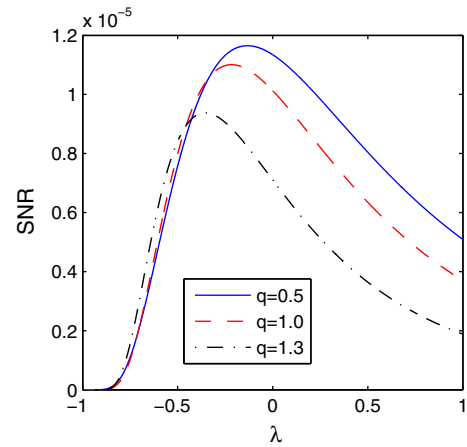

Fig. 12 SNR versus normalized addictive noise strength λ for different values of q with $\tau = 0.1$, $Q = 0.1$, $D = 0.1$, $A = 0.05$

In Fig. 5, we have shown the effect of correlation time τ on MFPT. We can see that the curves are monotonic and decrease as multiplicative noise intensity D increases. Meanwhile, when fixed the multiplicative noise intensity D, the curves stay the same with increasing correlation time τ.

In Fig. 6, we present the effects of correlation strength λ between the additive and multiplicative noises on SNR as a function of multiplicative noise intensity D when all other parameters are fixed. The existence of the maximum in these curves is the identifying characteristic of SR phenomenon induced by the multiplicative noise. One can clearly see from Fig. 6 that the peak of SNR is decreased as λ increases, which means that the increasing of λ can weaken SR effect. In comparison with the case of absence of non-Gaussian noise in which there is a critical effect on SR phenomenon [38], one can see that the increasing λ plays a different role for the cases of non-Gaussian and Gaussian noises.

Figures 7 and 8 display the effect of the additive noise intensity Q on SNR as a function of the multiplicative noise intensity D when all other parameters are fixed. The curves of Fig. 8 are amplification of curves in Fig. 7. From Figs. 7 and 8, we can see that SNR shows a single peak structure when $Q = 0.008$. However, as additive noise intensity Q increases, it appears a double-peak structure, which denotes SR effect in a broad sense [39]. More interestingly, as Q increases, the height of left peak of SNR increases and right peak of SNR decreases.

The effect of correlation strength λ on SNR is shown in Fig. 9. It is shown that the existence of a maximum in these curves is the identifying characteristic of SR phenomenon induced by the additive noise. One can see that the increasing λ plays a similar role on SR phenomenon induced by the additive noise as that induced by the multiplicative noise. That is, the decreasing of λ can weaken SR effect. But the position of the maximum shifts to a smaller value of Q as λ increases.
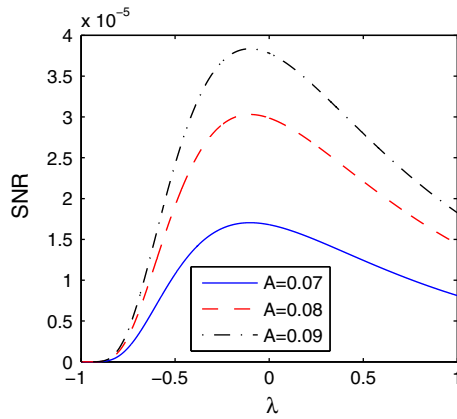
Fig. 13 SNR versus normalized addictive noise strength λ for different values of A with $\tau = 0.1$, $Q = 0.1$, $D = 0.1$, $q = 0.1$

In Fig. 10, we present the effects of multiplicative noise intensity D on SNR as a function of addictive noise intensity Q when all other parameters are fixed. One can see that there is a single peak when $D = 0.05$. As the multiplicative noise intensity D increases, it shows a maximum value and a minimum value. The minimum value forms a suppression platform, which shows inhibition phenomenon. However, the maximum value corresponds to SR. Meanwhile, the height of resonance peak decreases with D increasing, implying that the increasing of D weakens SR. Compared with the case of absence of noise correction [21], one can find that the increasing D represents a new feature of SR phenomenon due to the presence of noise correction.

We have plotted the SNR as a function of λ in Figs. 11–13. Figure 11 presents the effect of correlation time τ on SNR as a function of correlation strength λ when other parameters are fixed. One can see from Fig. 11 that SNR as a function of λ shows a single peak structure and the height of single peak increases with the increase in τ, which means that the increase in τ enhances the output SNR. The influence of deviation parameter q on SNR is plotted in Fig. 12. The results are totally different from that in Fig. 11. One can see that the increasing q can weaken SNR. Meanwhile, we can also find the position of single peak shifts to a smaller value of λ as the deviation parameter q increases. Figure 13 shows the effect of amplitude A of the multiplicative signal on SNR. We can see that the amplitude A can enhance SR effect.

4. Conclusions

In this paper, we have discussed the effect of correlation in FHN neural model with non-Gaussian noise and a multiplicative signal. Firstly, Fokker–Planck equation is calculated by using the approximation of the probability density approach and then, the explicit expression of SPDF is obtained. The effects of correlation strength λ, correlation time τ and multiplicative noise intensity D on SPDF are studied. It is shown that correlation strength λ can induce a succession of two phase transition-like phenomena and correlation time and multiplicative noise intensity can induce a (single) phase transition-like phenomenon. Secondly, we obtain the explicit expression for MFPT. We find that MFPT exhibits a maximum, which identifies the NES effect when correlation strength $\lambda < 0$. At last, we obtain an explicit expression for SNR by using the existing theory [17, 18]. The existence of a maximum in the SNR is the identifying characteristic of SR phenomenon. We observe that correlation strength λ weakens SR effect. Meanwhile, we also find that the inhibition phenomenon and double SR occur under different noise strengths. Moreover, we find that the changes of multiplicative noise intensity and additive noise intensity bring different effects, which make the system generate a variety of phenomena.

References

[1] R Fitzhugh *J. Gen. Physiol.* **43** 867 (1960)
[2] A S Pikovsky and J Kurths *Phys. Rev. Lett.* **78** 78775 (1997)
[3] V A Makarov, V I Nekorkin and M G Velarde *Phys. Rev. Lett.* **86** 3431 (2001)
[4] B Lindner, J Garcia-Ojalvo, A Neiman, L Schimansky-Geier *Phys. Rep.* **392** 32 (2004)
[5] Evgeniya V Pankratova, A V Polovinkin and B Spagnolo *Phys. Lett. A* **344** 43 (2005)
[6] K Wiesenfeld, D Pantazelou, C Dames and F Moss *Phys. Rev. Lett.* **72** 2125 (1994)
[7] C H Kurrer and K Schulten *Phys. Rev. E* **51** 6214 (1995)
[8] A A Dubkov and B Spagnolo *Phys. Rev. E* **72** 041101 (2005)
[9] D Valenti, G Augello and B Spagnolo *Eur. Phys. J. B* **65** 443 (2008)
[10] B Lindner and L Schimansky-Geier *Phys. Rev. E* **60** 7270 (1999)
[11] R Toral, C R Mirasso and J D Gunton *Europhys. Lett.* **61** 162 (2003)
[12] D Wu and S Q Zhu *Phys. Lett. A* **372** 5299 (2008)
[13] D Nozaki and Y Yamamoto *Phys. Lett. A* **243** 281 (1998)
[14] C J Tessone and H S Wio *Phys. A* **374** 46 (2007)
[15] S M Bezrukov and I Vodyanoy *Nature* **385** 319 (1997)
[16] I Goychuk and P Hanggi *Phys. Rev. E* **61** 4272 (2000)
[17] Y Zhao, W Xu and S C Sou *Acta Phys. Sin.* **58** 1396 (2009)
[18] J J Zhang and Y F Jin *Acta Phys. Sin.* **61** 13 (2012)
[19] A Fiasconaro and B Spagnolo *Phys. Rev. E* **80** 041110 (2009)
[20] R N Mantegna and B Spagnolo *Nuovo Cimento D* **17** 873 (1995)
[21] X L Li and L J Ning *Indian J. Phys.* **89** 189 (2015)
[22] T Alarcon, A Perez-Madid, J M Rubi *Phys. Rev. E* **57** 4979 (1998)
[23] L Borland *Phys. Lett. A* **245** 67 (1998)
[24] M A Fuentes, R Toral and H S Wio *Phys. A* **295** 114 (2001)

[25] G Hu *Stochastic Force and Nonlinear Systems* (Shanghai: Shanghai Scientific and Technological Education Publishing House) (1944)

[26] P Hanggi, F Marchesoni and P Grigolini *Z. Phys. B* **56** 333 (1984)

[27] R F Fox *Phys. Rev. A* **33** 467 (1986)

[28] Y Jia, S N Yu and J R Li *Phys. Rev. E* **62** 1869 (2000)

[29] Q Liu and Y Jia *Phys. Rev. E* **70** 041907 (2004)

[30] S Bouzat and H S Wio *Phys. Rev. E* **59** 5142 (1999)

[31] H S Wio and S Bouzat *Braz. J. Phys.* **29** 136 (1999)

[32] B Spagnolo et al. *Acta Phys. Pol. B* **38** 1925 (2007)

[33] N V Agudov and B Spagnolo *Phys. Rev. E* **64** 035102 (2001)

[34] A A Dubkov, N V Agudov and B Spagnolo *Phys. Rev. E* **69** 061103 (2004)

[35] R N Mantegna and B Spagnolo *Phys. Rev. Lett.* **76** 563 (1996)

[36] H A Kramers *Physica* **7** 284 (1940)

[37] P Hanggi, P Talkner and M Borkovec *Rev. Mod. Phys.* **62** 251 (1990)

[38] Z L Jia and D C Mei *Chin. J. Phys.* **49** 6 (2011)

[39] V Barzykin and K Seki *Europhys. Lett.* **57** 6555 (1997)