Indian J Phys (July 2015) 89(7):649–656
DOI 10.1007/s12648-014-0629-9

ORIGINAL PAPER

Solutions of D-dimensional Klein–Gordon equation for multiparameter exponential-type potential using supersymmtric quantum mechanics

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Received: 03 September 2014 / Accepted: 05 November 2014 / Published online: 4 December 2014

Abstract: In this paper, we present solutions of the Klein–Gordon equation for the multiparameter potential for arbitrary *l*-state in *D*-dimensions using the super-symmetric quantum mechanics method. We have obtained the energy levels for the multiparameter potential and the corresponding wave functions expressed in terms of hypergeometric function in a closed form for arbitrary *l*-state. We have discussed in detail the special cases of this potential.

Keywords: Klein–Gordon equation; Supersymmetric quantum mechanics; Multiparameter potential

PACS Nos.: 03.65.Ge; 03.65.Pm; 03.65.Ta

1. Introduction

The Klein–Gordon equation (KGE) is the well-known relativistic wave equation that describes spin-zero particles. It is also known that the analytic solutions of the Klein–Gordon equations are only possible in a few cases such as harmonic and Coulomb potentials [1, 2]. However, for arbitrary *l*-states ($l \neq 0$), the KGE does not admit an exact solution. Thus, KGE can be solved approximately using different approximation schemes [3, 4]. The solutions of KGE plays important role in quantum mechanics since its solutions contain all the necessary informations regarding the quantum system such as probability density and entropy. Different potential models have been studied using various approaches such as Nikiforov–Uvarov [5], exact quantization rule [6], supersymmetric quantum mechanics (SUSYQM) [7] and asymptotic iteration method (AIM) [8] to obtained the bound state solutions of Schrödinger, Klein–Gordon and Dirac equations [9–12]. Interestingly, the study of the relativistic wave equation in recent years particularly the Klein–Gordon and Dirac equation have attracted the attention of many authors, because the solutions of these equations play in getting the relativistic effect in nuclear physics and other areas [13]. With the introduction of the SUSYQM and the concept of shape invariance in physics [14], the study of the solvable potential models in both relativistic and non-relativistic quantum mechanics have received a great interest [15]. The concept of SUSYQM allows one to determine the eigenfunctions and eigenvalues analytically for solvable potentials model using algebraic operator formulation without solving the Schrödinger-like differential equation by standard series method [16]. Recently, the KGE in generalized *D*-dimensions for different potentials is getting the attention of researchers [17–20]. This multidimensional space analysis of KGE has also been investigated for different potentials [21].

The purpose of the present paper is to attempt to study the bound state solutions of the KGE with the multiparameter exponential-type potential using the SUSYQM in *D*-dimension [22, 23]. We have determined approximate eigenvalues and the eigenfunction by employing the improved Greene–Aldrich approximation scheme [24].

2. Theoretical consideration

The KGE in higher dimension for spherically symmetric potential reads [25],

$$-\Delta_D \psi_{n,l,m}(r,\Omega_D) = \left\{ [E_{n,l} - V(r)]^2 - [m + S(r)]^2 \right\} \psi_{n,l,m}(r,\Omega_D) \quad (1)$$

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where $E_{n,l}$, m, $V(r)$ and $S(r)$ are the relativistic energy, rest mass, the repulsive vector potential and the attractive scalar potential respectively and Δ_D is defined as

$$\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_D^2(\Omega_D)}{r^2} \tag{2}$$

The total wave function in D-dimension is written as,

$$\psi_{n,l,m}(r,\Omega_D) = R_{n,l}(r)Y_l^m(\Omega_D) \tag{3}$$

The term $\frac{\Lambda_D^2(\Omega_D)}{r^2}$ is the generalization of the centrifugal term for the higher dimensional space. The eigenvalues of $\Lambda_D^2(\Omega_D)$ are defined by the relation,

$$\Lambda_D^2(\Omega_D)Y_l^m(\Omega_D) = l(l+D-2)Y_l^m(\Omega_D) \tag{4}$$

where $Y_l^m(\Omega_D)$, $R_{n,l}$ and l represent the hyperspherical harmonics, the hyperradial wave function and the orbital angular momentum quantum number respectively. Now substituting ansatz $R_{n,l}(r) = r^{-\frac{(D-1)}{2}}F_{n,l}(r)$ for the wave function into Eq. (3) yields,

$$\left\{\frac{d^2}{dr^2} + \left(E_{n,l} - V(r)\right)^2 - (m+S(r))^2 - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right\}F_{n,l}(r) = 0 \tag{5}$$

Here, we consider the solvable multiparameter exponential-type potential [26, 27],

$$V(r) = \frac{Ae^{-2\eta r}}{(1-e^{-2\eta r})} + \frac{Be^{-2\eta r}}{(1-e^{-2\eta r})^2} + \frac{Ce^{-4\eta r}}{(1-e^{-2\eta r})^2} \tag{6}$$

where A, B and C are the potential parameter and η is the screening parameter. Now substituting Eq. (6) into Eq. (5) and considering the particular case $V(r) = S(r)$, we obtain the second order Schrodinger-like equation

$$\left\{\frac{d^2}{dr^2} + E_{n,l}^2 - m^2 - 2(E_{n,l}+m)\left(\frac{Ae^{-2\eta r}}{(1-e^{-2\eta r})} + \frac{Be^{-2\eta r}}{(1-e^{-2\eta r})^2} + \frac{Ce^{-4\eta r}}{(1-e^{-2\eta r})^2}\right) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right\}F_{n,l}(r) = 0 \tag{7}$$

3. Solutions of Klein–Gordon equation in D-dimension using SUSYQM

It is well-known that Eq. (7) is exactly solvable only when $l = 0$. However, in the presence of the centrifugal term ($l \neq 0$), one can only obtain an approximate solution of Eq. (7). When $\eta r < 1$, we invoke an improved Greene–Aldrich approximation scheme [24] to deal with the centrifugal term as,

$$\frac{1}{r^2} \approx 4\eta^2\left[c_0 + \frac{e^{-2\eta r}}{(1-e^{-2\eta r})^2}\right] \tag{8}$$

where $c_0 = \frac{1}{12}$ is a dimensionless constant obtained from the Taylor expansion of Eq. (8). Substitution of Eq. (8) into Eq. (7) yields,

$$-\frac{d^2F_{n,l}}{dr^2} + \left(\frac{2(E_{n,l}+m)(C-A)e^{-4\eta r} + \left(2(E_{n,l}+m)(A+B) + \eta^2(D+2l-1)(D+2l-3)\right)e^{-2\eta r}}{(1-e^{-2\eta r})^2}\right)F_{n,l}(r) = \tilde{E}_{n,l}F_{n,l} \tag{9}$$

where,

$$\tilde{E}_{n,l} = E_{n,l}^2 - m^2 - \eta^2(D+2l-1)(D+2l-3)c_0 \tag{10}$$

In the SUSYQM formulation, the ground-state wave function $F_{0,l}(r)$ is given by [7, 14, 15] (see "Appendix").

$$F_{0,l}(r) = \exp\left(-\int W(r)dr\right), \tag{11}$$

in which the integrand is called the superpotential and the Hamiltonian is composed of the raising and lowering operators

$$H_- = \hat{A}^+\hat{A} = -\frac{d^2}{dr^2} + V_-(r), \tag{12}$$

$$H_+ = \hat{A}\hat{A}^+ = -\frac{d^2}{dr^2} + V_+(r), \tag{13}$$

with

$$\hat{A} = \frac{d}{dr} - W(r), \tag{14}$$

$$\hat{A}^+ = -\frac{d}{dr} - W(r), \tag{15}$$

and

$$V_{\pm}(r) = W^2(r) \mp W'(r) \tag{16}$$

The superpotential obeys the associated Riccati equation:

$$W^2(r) \mp W'(r) = V_{eff}(r) - \tilde{E}_{0,l}, \tag{17}$$

Based on the SUSYQM,we choose the superpotential in the form,

$$W(r) = \frac{-Q_1 e^{-2\eta r}}{(1 - e^{-2\eta r})} + Q_2 \tag{18}$$

where,

We construct the pair of supersymmetric partner potentials $V_+(r)$ and $V_-(r)$ as follows,

$$V_-(r) = W^2 - \frac{dW}{dr} = \frac{Q_1(Q_1 - 2\eta)e^{-2\eta r}}{(1 - e^{-2\eta r})^2} + \frac{Q_1^2 e^{-4\eta r} - 2Q_1 Q_1 e^{-2\eta r} + 2Q_1 Q_1 e^{-4\eta r} - Q_1^2 e^{-2\eta r}}{(1 - e^{-2\eta r})^2} + Q_2^2. \tag{23}$$

$$Q_1 = -\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}, \tag{19}$$

$$Q_2 = \frac{2(E_{n,l} + m)(C - A) - \left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}\right)^2}{2\left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}\right)} \tag{20}$$

$$\tilde{E}_{0,l} = -\left[\frac{2(E_{n,l} + m)(C - A) - \left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}\right)^2}{2\left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}\right)}\right]^2 \tag{21}$$

Using Eq. (58), we obtained the partner potentials as follows:

$$V_+(r) = W^2 + \frac{dW}{dr} = \frac{Q_1(Q_1 + 2\eta)e^{-2\eta r}}{(1 - e^{-2\eta r})^2} + \frac{Q_1^2 e^{-4\eta r} - 2Q_1 Q_1 e^{-2\eta r} + 2Q_1 Q_1 e^{-4\eta r} - Q_1^2 e^{-2\eta r}}{(1 - e^{-2\eta r})^2} + Q_2^2, \tag{22}$$

It is not difficult to see that the partner potentials are shape invariant via mapping of the form $Q_1 \to Q_1 + 2\eta$. Also, it is easy to check the shape-invariance condition.

$$V_+(r, \rho_0) = V_-(r, \rho_i) + R(\rho_i) \tag{24}$$

which holds via the mapping $Q_1 \to Q_1 + 2\eta$. In our study $\rho_0 = Q_1$ and ρ_i is a function of ρ_0, i.e. $\rho_1 = f(\rho_0) = \rho_0 + 2\eta$. Thus, $\rho_n = \rho_0 + 2n\eta$ and from Eq. (24), we write

$$\begin{aligned}\tilde{E}^-_{n,l} = \sum_{k=1}^{n} R(\rho_k) = &\left(\frac{2(E_{n,l} + m)(C - A) - \left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}\right)}{2\left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)}\right)}\right)^2 \\ &- \left(\frac{2(E_{n,l} + m)(C - A) - \left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)} + 2n\eta\right)}{2\left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l} + m)(B + C) + \eta^2(D + 2l - 1)(D + 2l - 3)} + 2n\eta\right)}\right)^2\end{aligned} \tag{25}$$

The complete energy spectrum is given as,

$$\tilde{E}_{n,l} = \tilde{E}_{0,l} + \tilde{E}^{-}_{n,l} = -\left(\frac{2(E_{n,l}+m)(C-A) - \left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l}+m)(B+C) + \eta^2(D+2l-1)(D+2l-3)} + 2n\eta\right)}{2\left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l}+m)(B+C) + \eta^2(D+2l-1)(D+2l-3)} + 2n\eta\right)}\right)^2 \quad (26)$$

Using Eq. (10), we have obtained the required relativistic energy spectrum for the multiparameter exponential-type potential as,

$$E^2_{n,l} - m^2 = \eta^2(D+2l-1)(D+2l-3)c_0 - \left(\frac{2(E_{n,l}+m)(C-A) - \left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l}+m)(B+C) + \eta^2(D+2l-1)(D+2l-3)} + 2n\eta\right)}{2\left(-\eta \pm \sqrt{\eta^2 + 2(E_{n,l}+m)(B+C) + \eta^2(D+2l-1)(D+2l-3)} + 2n\eta\right)}\right)^2 \quad (27)$$

To find the wave function, we have used change of variable $z = e^{-2\eta r}$ in Eq. (9) and we get

$$\frac{d^2F_{n,l}}{dz^2} + \frac{1-z}{z(1-z)}\frac{dF_{n,l}}{dz} + \frac{1}{z^2(1-z)^2}\left\{-Wz^2 - fz + g\right\}F_{n,l}(z) = 0 \quad (28)$$

where,

$$W = \frac{2E_{n,l}C}{4\eta^2} + \frac{2mC}{4\eta^2} - \frac{2E_{n,l}A}{4\eta^2} - \frac{2mA}{4\eta^2} - \frac{E^2_{n,l}}{4\eta^2} + \frac{m^2}{4\eta^2} + \frac{(D+2l-1)(D+2l-3)c_0}{4} \quad (29)$$

$$f = \frac{2E_{n,l}A}{4\eta^2} + \frac{2mA}{4\eta^2} + \frac{2E_{n,l}B}{4\eta^2} + \frac{2mB}{4\eta^2} + \frac{2E^2_{n,l}}{4\eta^2} - \frac{2m^2}{4\eta^2} - \frac{(D+2l-1)(D+2l-3)c_0}{2} + \frac{(D+2l-1)(D+2l-3)}{4} \quad (30)$$

$$g = \frac{E^2_{n,l}}{4\eta^2} - \frac{m^2}{4\eta^2} - \frac{(D+2l-1)(D+2l-3)c_0}{4} \quad (31)$$

The corresponding wave function is determined from Eq. (28) as follows:

$$F_{n,l} = \frac{\Gamma\left(n+1+2\sqrt{g}\right)}{n!\Gamma\left(1+2\sqrt{g}\right)}\left(e^{-2\eta r}\right)^{\sqrt{g}}\left(1-e^{-2\eta r}\right)^{1/2+\sqrt{\frac{1}{4}+W+g-f}} {}_2F_1\left(-n, n+2\sqrt{g}+2\sqrt{\frac{1}{4}+W+g-f}+1; 2\sqrt{g}+1; e^{-2\eta r}\right) \quad (32)$$

4. Results and discussion

In this section we have investigated the energy eigenvalues and the corresponding eigenfunctions of Hulthen, Manning–Rosen, Eckart and Deng–Fan potentials as special cases of multiparameter exponential-type potential.

4.1. Hulthen potential

The Hulthén potential is very important in atomic and molecular fields [28]. This potential has been used to explain the electronic properties of F-colour centre in alkali halides [29]. In this special case, we choose $B = C = 0$, $A = -Ze^2\delta$, $\eta = \delta$, where δ is the screening parameter and the multiparameter exponential-type potential turns into the Hulthen potential as,

$$V(r) = \frac{-Ze^2\delta e^{-2\delta r}}{1-e^{-2\delta r}} \quad (33)$$

The energy spectra of the Hulthen potential is obtained from the energy equation of the multiparameter exponential-type potential Eq. (27) by substituting the above parameters as,

$$E_{n,l}^2 - m^2 = \delta^2(D+2l-1)(D+2l-3)c_0 - \left(\frac{2(E_{n,l}+m)(Ze^2\delta) - \left(-\delta \pm \sqrt{\delta^2+\delta^2(D+2l-1)(D+2l-3)} + 2n\delta\right)}{2\left(-\delta \pm \sqrt{\delta^2+\delta^2(D+2l-1)(D+2l-3)} + 2n\delta\right)}\right)^2 \tag{34}$$

The corresponding wave function of the Hulthen potential is obtained from Eq. (32) as follows:

$$F_{n,l} = \frac{\Gamma\left(n+1+2\sqrt{g^H}\right)}{n!\,\Gamma\left(1+2\sqrt{g^H}\right)}\left(e^{-2\eta r}\right)^{\sqrt{g^H}} (1-e^{-2\eta r})^{1/2+\sqrt{\frac{1}{4}+W^H+g^H-f^H}} {}_2F_1\left(-n, n+2\sqrt{g^H}+2\sqrt{\frac{1}{4}+W^H+g^H-f^H} +1; 2\sqrt{g^H}+1; e^{-2\eta r}\right) \tag{35}$$

where,

$$W^H = \frac{2E_{n,l}Ze^2}{4\delta} + \frac{2mZe^2}{4\delta} - \frac{E_{n,l}^2}{4\delta^2} + \frac{m^2}{4\delta^2} + \frac{(D+2l-1)(D+2l-3)c_0}{4} \tag{36}$$

$$f^H = -\frac{2E_{n,l}Ze^2}{4\delta} - \frac{2mZe^2}{4\delta} + \frac{2E_{n,l}^2}{4\delta^2} - \frac{2m^2}{4\delta^2} - \frac{(D+2l-1)(D+2l-3)c_0}{2} + \frac{(D+2l-1)(D+2l-3)}{4} \tag{37}$$

$$g^H = \frac{E_{n,l}^2}{4\delta^2} - \frac{m^2}{4\delta^2} - \frac{(D+2l-1)(D+2l-3)c_0}{4} \tag{38}$$

This result is in good agreement with that obtained by Agboola [30].

4.2. Manning–Rosen potential

Manning–Rosen potential is one of the short range potential and it has been used to describe the diatomic molecular vibration [31]. The Manning–Rosen potential has been one of the most useful and elegant potential model for studying the energy eigenvalues of diatomic molecules [32]. As an empirical potential, the Manning–Rosen potential gives an excellent description of the interaction between two atoms in a diatomic molecule, and it is very good for describing such interactions close to the surface [33]. The special case of Manning–Rosen potential is obtained from the multi-parameter potential by considering, $B = 0, A = -\frac{V_0}{b^2}, C = \frac{\alpha(\alpha-1)}{b^2}$ and $\eta = \frac{1}{b}$. Thus, the Manning–Rosen potential becomes,

$$V(r) = \frac{1}{b^2}\left(\frac{\alpha(\alpha-1)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{V_0 e^{-\frac{r}{b}}}{1-e^{-\frac{r}{b}}}\right) \tag{39}$$

The energy level of the Manning–Rosen potential is obtained from the energy equation of the multiparameter exponential-type potential by putting the values of A, B and C given above as,

$$E_{n,l}^2 - m^2 = \frac{1}{b^2}(D+2l-1)(D+2l-3)c_0 - \left(\frac{2(E_{n,l}+m)\left(\frac{\alpha(\alpha-1)}{b^2}+\frac{V_0}{b^2}\right) - \left(-\frac{1}{b} \pm \sqrt{\frac{1}{b^2}+2(E_{n,l}+m)\left(\frac{\alpha(\alpha-1)}{b^2}\right)+\frac{1}{b^2}(D+2l-1)(D+2l-3)} + \frac{2n}{b}\right)}{2\left(-\frac{1}{b} \pm \sqrt{\frac{1}{b^2}+2(E_{n,l}+m)\left(\frac{\alpha(\alpha-1)}{b^2}\right)+\frac{1}{b^2}(D+2l-1)(D+2l-3)} + \frac{2n}{b}\right)}\right)^2 \tag{40}$$

The corresponding wave function for the Manning–Rosen potential becomes,

$$F_{n,l} = \frac{\Gamma\left(n+1+2\sqrt{g^{MR}}\right)}{n!\Gamma\left(1+2\sqrt{g^{MR}}\right)}\left(e^{-2\eta r}\right)^{\sqrt{g^{MR}}}(1-e^{-2\eta r})^{1/2+\sqrt{\frac{1}{4}+W^{MR}+g^{MR}-f^{MR}}}\,{}_2F_1\left(-n, n+2\sqrt{g^{MR}}+2\sqrt{\frac{1}{4}+W^{MR}+g^{MR}-f^{MR}}+1; 2\sqrt{g^{MR}}+1; e^{-2\eta r}\right) \quad (41)$$

where

$$W^{MR} = \frac{2E_{n,l}\alpha(\alpha-1)}{4}+\frac{2m\alpha(\alpha-1)}{4}+\frac{2E_{n,l}V_0}{4}+\frac{2mV_0}{4}-\frac{E_{n,l}^2b^2}{4}+\frac{m^2b^2}{4}+\frac{(D+2l-1)(D+2l-3)c_0}{4} \quad (42)$$

$$f^{MR} = -\frac{2E_{n,l}V_0}{4}-\frac{2mV_0}{4}+\frac{2E_{n,l}^2b^2}{4}-\frac{2m^2b^2}{4}-\frac{(D+2l-1)(D+2l-3)c_0}{2}+\frac{(D+2l-1)(D+2l-3)}{4} \quad (43)$$

$$g^{MR} = \frac{E_{n,l}^2b^2}{4}-\frac{m^2b^2}{4}-\frac{(D+2l-1)(D+2l-3)c_0}{4} \quad (44)$$

It is consistent with the result reported by Chen et al. [34].

4.3. Eckart potential

The Eckart potential is one of the solvable exponential–type potential in quantum mechanics [35]. Eckart potential is one of most important potential models in physics and chemical physics [36] and the bound state solution of the Schrödinger equation [37] and the scattering states [38] of this potential has been investigated. The Eckart potential is obtained from the multiparameter potential by the setting $A=-\alpha$, $B=\beta$, $C=0$ and $\eta=\frac{2}{a}$ as,

$$V(r) = -\frac{\alpha e^{-\frac{r}{a}}}{1-e^{-\frac{r}{a}}}+\frac{\beta e^{-\frac{r}{a}}}{(1-e^{-\frac{r}{a}})^2} \quad (45)$$

The energy eigenvalues for the Eckart potential from Eq. (27) becomes,

$$E_{n,l}^2-m^2 = \frac{4}{a^2}(D+2l-1)(D+2l-3)c_0 - \left(\frac{2(E_{n,l}+m)\alpha-\left(-\frac{2}{a}\pm\sqrt{\frac{4}{a^2}+2(E_{n,l}+m)\beta+\frac{4}{a^2}(D+2l-1)(D+2l-3)}+\frac{4n}{a}\right)}{2\left(-\frac{2}{a}\pm\sqrt{\frac{4}{a^2}+2(E_{n,l}+m)\beta+\frac{4}{a^2}(D+2l-1)(D+2l-3)}+\frac{4n}{a}\right)}\right)^2, \quad (46)$$

and wave function as,

$$F_{n,l} = \frac{\Gamma\left(n+1+2\sqrt{g^{Ec}}\right)}{n!\Gamma\left(1+2\sqrt{g^{Ec}}\right)}\left(e^{-2\eta r}\right)^{\sqrt{g^{Ec}}}(1-e^{-2\eta r})^{1/2+\sqrt{\frac{1}{4}+W^{Ec}+g-f}}\,{}_2F_1\left(-n, n+2\sqrt{g^{Ec}}+2\sqrt{\frac{1}{4}+W^{Ec}+g^{Ec}-f^{Ec}}+1; 2\sqrt{g^{Ec}}+1; e^{-2\eta r}\right) \quad (47)$$

with

$$W^{Ec} = 2E_{n,l}\alpha a^2+2m\alpha a^2-E_{n,l}^2a^2+m^2a^2+\frac{(D+2l-1)(D+2l-3)c_0}{4} \quad (48)$$

$$f^{Ec} = -2E_{n,l}\alpha a^2-2m\alpha a^2+2E_{n,l}\beta a^2+2m\beta a^2+E_{n,l}^2a^2-m^2a^2-\frac{(D+2l-1)(D+2l-3)c_0}{2}+\frac{(D+2l-1)(D+2l-3)}{4} \quad (49)$$

$$g^{Ec} = E_{n,l}^2a^2-m^2a^2-\frac{(D+2l-1)(D+2l-3)c_0}{4} \quad (50)$$

and it is in agreement with that of Akpan et al. [39] for D = 3 and q = 1.

4.4. Deng–Fan potential

The Deng–Fan potential [40, 41] is the simplest modified form of Morse potential and is related to the Manning–Rosen and Eckart potentials. This potential is used to describe diatomic molecular energy spectra and electromagnetic transition and is usually regarded as the true internuclear potential in diatomic molecules. In this case, the choice $A = -2bD_e$, $B = 0$, $C = D_e b^2$ and $\eta = \frac{\alpha}{2}$, where D_e is the dissociation energy. With these parameters, we have obtained the Deng–Fan potential from Eq. (6) as,

$$V(r) = \frac{-2bD_e e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{D_e b^2 e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}, \tag{51}$$

and from Eq. (27), the energy spectra for the Deng–Fan potential becomes,

$$E_{n,l}^2 - m^2 = \frac{\alpha^2}{4}(D + 2l - 1)(D + 2l - 3)c_0 - \left(\frac{2(E_{n,l} + m)(D_e b^2 + 2bD_e) - \left(-\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + 2(E_{n,l} + m)(D_e b^2) + \frac{\alpha^2}{4}(D + 2l - 1)(D + 2l - 3)} + n\alpha \right)}{2\left(-\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + 2(E_{n,l} + m)(D_e b^2) + \frac{\alpha^2}{4}(D + 2l - 1)(D + 2l - 3)} + n\alpha \right)} \right)^2 \tag{52}$$

and wave function is given by

$$F_{n,l} = \frac{\Gamma\left(n + 1 + 2\sqrt{g^{DF}}\right)}{n!\Gamma\left(1 + 2\sqrt{g^{DF}}\right)} \left(e^{-2\eta r}\right)^{\sqrt{g^{DF}}} (1 - e^{-2\eta r})^{1/2 + \sqrt{\frac{1}{4} + W^{DF} + g^{DF} - f^{DF}}} {}_2F_1\left(-n, n + 2\sqrt{g^{DF}} + 2\sqrt{\frac{1}{4} + W^{DF} + g^{DF} - f^{DF}} + 1; 2\sqrt{g^{DF}} + 1; e^{-2\eta r} \right) \tag{53}$$

where,

$$W^{DF} = \frac{2E_{n,l} D_e b^2}{\alpha^2} + \frac{2mD_e b^2}{\alpha^2} + \frac{E_{n,l} bD_e}{\alpha^2} + \frac{mD_e b}{\alpha^2} - \frac{E_{n,l}^2}{\alpha^2} + \frac{m^2}{\alpha^2} + \frac{(D + 2l - 1)(D + 2l - 3)c_0}{4} \tag{54}$$

$$f^{DF} = -\frac{2E_{n,l} D_e b^2}{\alpha^2} - \frac{2mD_e b^2}{\alpha^2} + \frac{2E_{n,l}^2}{\alpha^2} - \frac{2m^2}{\alpha^2} - \frac{(D + 2l - 1)(D + 2l - 3)c_0}{2} + \frac{(D + 2l - 1)(D + 2l - 3)}{4} \tag{55}$$

$$g^{DF} = \frac{E_{n,l}^2}{\alpha^2} - \frac{m^2}{\alpha^2} - \frac{(D + 2l - 1)(D + 2l - 3)c_0}{4} \tag{56}$$

It is the same as the one obtained by Dong et al. [3].

5. Conclusions

The success in the determination of the bound state solutions of Klein–Gordon equation is a big task that must be done since this solution brings information about quantum state of the system under investigation. In this work, we have considered the Klein–Gordon equation with a multi-parameter exponential interaction and reported the approximate analytical solutions of the problem via the super-symmetric quantum mechanics (SUSYQM) formalism. The special cases, Hulthen, Manning–Rosen, Eckart and Deng–Fan potentials are consistent with those reported in the literature [3, 30, 34, 39]. The approximate analytical solution is obtained by invoking the improved Greene–Aldrich approximation scheme to the centrifugal term. Finally, the results of our work are both interesting for theoretical and experimental physicists.

Appendix: Supersymmetry quantum mechanics

In the SUSYQM, we normally deal with the partner Hamiltonians

$$H_\pm = \frac{p^2}{2m} + V_\pm(x), \tag{57}$$

where

$$V_{\pm}(x) = \Phi^2(x) \pm \Phi'(x). \tag{58}$$

In the case of good SUSY, i.e. $E_0 = 0$, the ground state of the system is obtained via

$$\phi_0^-(x) = Ce^{-U}, \tag{59}$$

where C is a normalization constant and

$$U(x) = \int_{x_0}^{x} dz\Phi(z). \tag{60}$$

Next, if the shape invariant condition

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1), \tag{61}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F{:}a_0 \mapsto a_1 = F(a_0)$ and $R(a_1)$ does not include x, the higher state solutions are obtained via

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{62}$$

$$\phi_n^-(a_0, x) = \prod_{s=0}^{n-1} \left(\frac{A^{\dagger}(a_s)}{[E_n - E_s]^{1/2}} \right) \phi_0^-(a_n, x), \tag{63}$$

$$\phi_0^-(a_n, x) = C \exp\left\{ -\int_0^x dz\Phi(a_n, z) \right\}, \tag{64}$$

where

$$A_s^{\dagger} = -\frac{\partial}{\partial x} + \Phi(a_s, x). \tag{65}$$

Therefore, this condition determines the spectrum of the bound states of the Hamiltonian

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s, x) + E_s. \tag{66}$$

and the energy eigenfunctions of

$$H_s\phi_{n-s}^-(a_s, x) = E_n\phi_{n-s}^-(a_s, x), \quad n \geq s \tag{67}$$

are related via [27–30]

$$\phi_{n-s}^-(a_s, x) = \frac{A^{\dagger}}{[E_n - E_s]^{1/2}} \phi_{n-(s+1)}^-(a_{s+1}, x). \tag{68}$$

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